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Using the Hilfer–Katugampola fractional derivative in initial-value Mathieu fractional differential equations with application to a particle in the plane

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Abstract

We examine a class of nonlinear fractional Mathieu equations with a damping term. The equation is an important equation of mathematical physics as it has many applications in various fields of the physical sciences. By utilizing Schauder's fixed-point theorem, the existence arises of solutions for the proposed equation with the Hilfer–Katugampola fractional derivative, and an application is additionally examined. Two examples guarantee the obtained results.

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1 Introduction

During recent years, fractional Calculus draws increasing attention due to its applications in various applicable fields such as physics, mechanics, chemistry, engineering, etc. The reader interested in the subject should refer to the papers [1–10]. In the literature, one can find that there are many definitions of fractional derivatives [11–15].

In [16], the authors introduced a new generalized derivative involving exponential functions in their kernels that, upon considering limiting cases, converges to classical derivatives. They solved Cauchy linear fractional-type problems within this derivative. One of the generalizations of the well-known Riemann–Liouville and the Hadamard fractional integrals was introduced by Katugampola ([17–19]) in a new fractional integral operator given by

$$ {}^{\rho}\mathcal{I}_{a+}^{\alpha}[y](t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1} y(\tau)}{(t^{\rho} - \tau^{\rho})^{1-\alpha}} \, \mathrm{d}\tau. $$

Matar *et al.* investigated the existence and uniqueness of solutions for a p-Laplacian boundary value problem defined by a semilinear fractional system that involves Caputo–Katugampola fractional derivatives (C-K$\mathbb{FD}$) [20]. The Mathieu equation (M$\mathbb{E}$) is an important equation of mathematical physics as it has many applications in several fields of the physical sciences [21–31]. In 1868, Émile Léonard Mathieu introduced for the fist time the second-order differential equation and had encountered them while studying vibrating elliptical drumheads, of the form

$$D^2y(t) + \bigl[a - 2b\cos(2t)\bigr]y(t) = 0, \tag{1}$$

where

$$D^2y := \frac{\mathrm{d}y^2}{\mathrm{d}t^2},$$

a and b are real or complex constants [32]. The solution of Equation (1) is built in the form

$$y(t) = \exp(i\sigma t)p(t), \tag{2}$$

where p is a periodical function with period π and σ is the so-called characteristic index depending on the values of a and b. The function $y(t) = \exp(-i\sigma t)p(-t)$, represents the second solution. In 2010, Rand *et al.* studied for the first time $\mathbb{ME}$ in fractional settings and used the method of harmonic balance to obtain both a lower- and a higher-order approximation for the transition curves [33]. Ebaid *et al.* established the approximate analytical solution of the fractional Mathieu equation ($\mathbb{FME}$) by using Adomian decomposition and a series methods [34]. Recently, Harikrishnan *et al.* considered the problems of differential equations with the Hilfer–Katugampola fractional derivative (H-K$\mathbb{FD}$),

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t) + p(t)y(t) = w(t,y(t)), \\ {}^{\rho}\mathcal{I}^{1-\gamma}[y](t)|_{t=a} = \sum_{i=1}^{m} q_i y(e_i), \end{cases}$$

for $t, e_i \in (a,b]$, where ${}^{\rho}\mathcal{D}^{\alpha,\beta}$ is H-K$\mathbb{FD}$ of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$, and ${}^{\rho}\mathcal{I}^{1-\gamma}$ is a generalized fractional derivative of order $(1-\gamma)$ with $\gamma = \alpha + \beta - \alpha\beta$ and $\rho > 0$ [35]. Here, $w : (a,b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, e_i, $(i = 0,1,\dots,m)$ are prefixed points satisfying $a < e_1 \leq e_2 \leq \cdots \leq e_m < b$, and q_i are real numbers. They also established the existence of solutions by using Krasnoselskii's fixed-point theorem.

Our objective in this work is to study the existence and uniqueness of solutions of the Mathieu fractional differential equation ($\mathbb{MFDE}$) with H-K$\mathbb{FD}$,

$${}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t) + p(t)y(t) = w\bigl(t, y(t), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\bigr), \tag{3}$$

for $t \in I_0 = [0,T]$, with the initial condition

$${}^{\rho}\mathcal{I}^{1-\gamma}[y](0) = \sum_{i=1}^{m} q_i y(e_i), \tag{4}$$

for $e_i \in (0,T]$, where $p(t) = a - 2b\cos(2t)$, a, b are real constants, ${}^{\rho}\mathcal{D}^{\alpha,\beta}$ is H-K$\mathbb{FD}$ of order α, type $0 \leq \beta \leq 1$, and ${}^{\rho}\mathcal{I}^{1-\gamma}$ is a generalized fractional derivative of order $(1-\gamma)$, here $\gamma = \alpha + \beta - \alpha\beta$, $\rho > 0$. The map $w : I_0 \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function, e_i for $i = 0,1,\dots,m$ are prefixed points satisfying $0 < e_1 \leq e_2 \leq \cdots \leq e_m < T$, and q_i are real numbers. Also, we consider the existence and uniqueness of Problem (3) and (4).

The plan of the work is as follows. In Sect. 2 we begin with some definitions and lemmas that will be used to prove our main result. In Sect. 3, we prove the existence and uniqueness of the solution. In Sect. 4, we provide two examples to illustrate our main results. We show an application to examine the validity of our theoretical results on the fractional-order representation of the motion of a particle along a straight line in Sect. 5.

2 Essential preliminaries

For the convenience of the reader, we present here some basic definitions and lemmas, which are used throughout this paper.

Definition 1 ([19]) Let $\bar{J} = [a,b]$ be a finite interval on the half-axis $\mathbb{R}^+$, $C(\bar{J})$ be the Banach space of all continuous functions from $\bar{J}$ into $\mathbb{R}^+$ with the norm $\|y\|_C = \max_{t\in\bar{J}} |y(t)|$ and the parameters $\rho > 0$, $0 \leq \gamma < 1$.

(1) The weighted space $C_{\gamma,\rho}(\bar{J})$ of continuous functions y on $(a,b]$ is defined by

$$C_{\gamma,\rho}(\bar{J}) = \left\{ y : (a,b] \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t) \in C(\bar{J}) \right\},$$

with the norm

$$\|y\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t) \right\|_C = \max_{t\in\bar{J}} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t) \right|,$$

where $C_{0,\rho}(\bar{J}) = C(\bar{J})$.

(2) Let $\delta_\rho = (t^\rho \frac{\mathrm{d}}{\mathrm{d}t})$. For $n \in \mathbb{N}$, we denote by $C^n_{\delta_\rho,\gamma}(\bar{J})$ the Banach space of functions y that are continuously differentiable on $\bar{J}$, with operator δ_ρ, up to order $(n-1)$ and that have the derivative $\delta^n_\rho y$ of order n on $(a,b]$ such that $\delta^n_\rho y \in C_{\gamma,\rho}(\bar{J})$, that is,

$$C^n_{\delta_\rho,\gamma}(\bar{J}) = \left\{ y : (a,b] \to \mathbb{R} : \delta^k_\rho \in C(\bar{J}), k = 0,1\ldots,n-1, \delta^n_\rho y \in C_{\gamma,\rho}(\bar{J}) \right\},$$

where $n \in \mathbb{N}$, with the norms

$$\|y\|_{C^n_{\delta_\rho}} = \sum_{k=0}^{n} \max_{t\in\bar{J}} \left|\delta^k_\rho g(x)\right|,$$

$$\|y\|_{C^n_{\delta_\rho,\gamma}} = \sum_{k=0}^{n-1} \left\|\delta^k_\rho g\right\|_C + \left\|\delta^n_\rho y\right\|_{C_{\gamma,\rho}}.$$

For $n = 0$, we have $C^0_{\delta_\rho,\gamma}(\bar{J}) = C_{\gamma,\rho}(\bar{J})$.

Definition 2 ([17, 18]) The generalized left-sided fractional integral ${}^\rho\mathcal{I}^\alpha_{a^+}[y](\cdot)$ of order $\alpha \in \mathbb{C}$, $(Re(\alpha) > 0)$ is defined for $y \in C^1_\gamma(\bar{J})$ by

$${}^\rho\mathcal{I}^\alpha_{a^+}[y](t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(t^\rho - \xi^\rho\right)^{\alpha-1} \xi^{\rho-1} y(\xi)\,\mathrm{d}\xi, \tag{5}$$

for $t > a$ and $\rho > 0$, provided the integral exists. Similarly, the right-sided fractional integral ${}^{\rho}\mathcal{I}^{\alpha}_{b^-}[y](\cdot)$ is defined by

$$ {}^{\rho}\mathcal{I}^{\alpha}_{b^-}[y](t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \left(t^{\rho} - \xi^{\rho}\right)^{\alpha-1} \xi^{\rho-1} y(\xi)\,\mathrm{d}\xi, \quad \forall t < b. \tag{6} $$

Definition 3 ([17, 18]) Let $\alpha \in \mathbb{C}$, with $Re(\alpha) \geq 0$, $n = [Re(\alpha)] + 1$ and $\rho > 0$. The generalized fractional derivatives, corresponding to the generalized fractional integrals (5) and (6), are defined for $0 \leq a < t < b \leq \infty$ and $y \in C^1_{\gamma}(\bar{J})$ by

$$ {}^{\rho}\mathcal{D}^{\alpha}_{a^+}[y](t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(t^{\rho} - \xi^{\rho}\right)^{n-\alpha+1} \xi^{\rho-1} y(\xi)\,\mathrm{d}\xi, \tag{7} $$

and

$$ {}^{\rho}\mathcal{D}^{\alpha}_{b^-}[y](t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(-t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b \left(t^{\rho} - \xi^{\rho}\right)^{n-\alpha+1} \xi^{\rho-1} y(\xi)\,\mathrm{d}\xi, \tag{8} $$

if the integrals exist.

Definition 4 ([36]) Let order α and type β satisfy $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$. The H-K$\mathbb{FD}$ (left sided / right sided), with respect to t, with $\rho > 0$ of a function $y \in C_{1-\gamma,\rho}(\bar{J})$ is defined by

$$ \begin{aligned} {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^{\pm}}[y](t) &= \left(\pm{}^{\rho}\mathcal{I}^{\beta(1-\alpha)}_{a^{\pm}}\left(t^{\rho-1}\frac{\mathrm{d}}{\mathrm{d}t}\right){}^{\rho}\mathcal{I}^{(1-\beta)(1-\alpha)}_{a^{\pm}}[y]\right)(t) \\ &= \left(\pm{}^{\rho}\mathcal{I}^{\beta(1-\alpha)}_{a^{\pm}}\delta_{\rho}{}^{\rho}\mathcal{I}^{(1-\beta)(1-\alpha)}_{a^{\pm}}[y]\right)(t), \end{aligned} \tag{9} $$

where ${}^{\rho}\mathcal{I}^{\eta}_{a^{\pm}}$ is the generalized fractional integral given in Definition 2.

Properties 1 ([17]) We recall some properties of ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}$ as follows:

P1) The operator ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}$ can be written as

$$ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+} = {}^{\rho}\mathcal{I}^{\beta(1-\alpha)}_{a^+}\delta_{\rho}{}^{\rho}\mathcal{I}^{1-\gamma}_{a^+} = {}^{\rho}\mathcal{I}^{\beta(1-\alpha)}_{a^+}{}^{\rho}\mathcal{D}^{\gamma}_{a^+}, $$

where $\gamma = \alpha + \beta(1-\alpha)$.

P2) The fractional derivative ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}$ is an interpolator of the following fractional derivatives:

- *Hilfer* $(\rho \to 1)$,
- *Hilfer–Hadamard* $(\rho \to 0)$,
- generalized $(\beta = 0)$,
- generalized *Caputo*-type $(\beta = 1)$,
- *Riemann–Liouville* $(\beta = 0, \rho \to 1)$,
- *Hadamard* $(\beta = 0, \rho \to 0)$,
- *Caputo* $(\beta = 1, \rho \to 1)$,
- *Caputo–Hadamard* $(\beta = 1, \rho \to 0)$,
- *Liouville* $(\beta = 0, \rho \to 1, a = 0)$,
- *Weyl* $(\beta = 0, \rho \to 1, a = -\infty)$.

First, we state the following key lemma.

Lemma 5 ([37]) *Let ${}^\rho\mathcal{I}^\alpha_{a^+}$ and ${}^\rho\mathcal{D}^\alpha_{a^+}$, as defined in Eqs. (5) and (7), respectively, for $t > a$. Then, for $\alpha \geq 0$ and $\zeta > 0$, we have*

$$ {}^\rho\mathcal{I}^\alpha_{a^+}\left[\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\zeta-1}\right](t) = \frac{\Gamma(\zeta)}{\Gamma(\alpha+\zeta)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\zeta-1}, $$

and

$$ {}^\rho\mathcal{D}^\alpha_{a^+}\left[\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\zeta-1}\right](t) = 0, $$

for almost all $\alpha \in (0,1)$.

Theorem 6 ([6, 17]) *Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$, and $\rho, c \in \mathbb{R}$, $\rho \geq c$, Then, for $y \in C^p_c(J)$,*

$$ {}^\rho\mathcal{I}^\alpha_{a^+}\,{}^\rho\mathcal{I}^\beta_{a^+}[y] = {}^\rho\mathcal{I}^{\alpha+\beta}_{a^+}[y], \qquad {}^\rho\mathcal{D}^\alpha_{a^+}\,{}^\rho\mathcal{D}^\beta_{a^+}[y] = {}^\rho\mathcal{D}^{\alpha+\beta}_{a^+}[y]. $$

Lemma 7 ([36]) *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $y \in C_\gamma(\bar{J})$ and $y \in C^1_\gamma(\bar{J})$, then*

$$ {}^\rho\mathcal{I}^\alpha_{a^+}\,{}^\rho\mathcal{D}^\alpha_{a^+}[y](t) = y(t) - \frac{{}^\rho\mathcal{I}^{1-\alpha}_{a^+}[y](a)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}, \quad \forall x \in J = (a,b). $$

Lemma 8 ([17]) *Let $\alpha > 0$, $0 \leq \gamma < 1$, and $y \in C_\gamma(\bar{J})$. Then,*

$$ {}^\rho\mathcal{D}^\alpha_{a^+}\,{}^\rho\mathcal{I}^\alpha_{a^+}[y](t) = y(t), $$

for each $t \in J$.

Lemma 9 ([36]) *Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$, and $y \in C_{\gamma,\rho}(\bar{J})$. If $\alpha > \gamma$, then ${}^\rho\mathcal{I}^\alpha_{a^+}[y]$ is continuous on $\bar{J}$ and*

$$ {}^\rho\mathcal{I}^\alpha_{a^+}[y](a) = \lim_{t\to a^+} {}^\rho\mathcal{I}^\alpha_{a^+}[y](t) = 0. $$

Throughout the remainder of this paper, we consider the following function spaces defined in [36]. We consider the parameters α, β, γ, and μ satisfying $\gamma = \alpha + \beta - \alpha\beta$, for $0 \leq \mu < 1$,

$$ C^{\alpha,\beta}_{1-\gamma,\rho}(\bar{J}) = \left\{y \in C_{1-\gamma,\rho}(\bar{J}), {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}[y] \in C_{\mu,\rho}(\bar{J})\right\}, \tag{10} $$

$$ C^{\gamma}_{1-\gamma,\rho}(\bar{J}) = \left\{y \in C_{1-\gamma,\rho}(\bar{J}), {}^\rho\mathcal{D}^{\gamma}_{a^+}[y] \in C_{1-\gamma,\rho}(\bar{J})\right\}, \tag{11} $$

and $C^{\gamma}_{1-\gamma,\rho}(\bar{J}) \subset C^{\alpha,\beta}_{1-\gamma,\rho}(\bar{J})$.

Lemma 10 ([36]) *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(1-\alpha)$. If $y \in C^\gamma_{1-\gamma}(\bar{J})$, then*

$$ {}^\rho\mathcal{I}^\gamma_{a^+}\,{}^\rho\mathcal{D}^\gamma_{a^+}[y] = {}^\rho\mathcal{I}^\alpha_{a^+}\,{}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}[y], \tag{12} $$

and

$$ {}^{\rho}\mathcal{D}^{\gamma}_{a^+}\,{}^{\rho}\mathcal{I}^{\alpha}_{a^+}[y] = {}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{a^+}[y]. \tag{13} $$

Lemma 11 ([36]) *Let $g \in L^1(J)$. If ${}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{a^+}[y]$ exists on $L^1(J)$, then*

$$ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}\,{}^{\rho}\mathcal{I}^{\alpha}_{a^+}[y] = {}^{\rho}\mathcal{I}^{\beta(1-\alpha)}_{a^+}\,{}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{a^+}[y]. $$

Lemma 12 ([36]) *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, and $\gamma = \alpha + \beta(1-\alpha)$. If $y \in C_{1-\gamma}(\bar{J})$ and ${}^{\rho}\mathcal{I}^{1-\beta(1-\alpha)}_{a^+} \in C^1_{1-\gamma}(\bar{J})$, then ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}\,{}^{\rho}\mathcal{I}^{\alpha}_{a^+}$ exists on $\bar{J}$ and*

$$ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}\,{}^{\rho}\mathcal{I}^{\alpha}_{a^+}[y](t) = y(t), \quad \forall t \in \bar{J}. \tag{14} $$

The following key theorems are used in the remainder of the paper.

Theorem 13 ([36]) *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$ and $0 \le \beta \le 1$. If $g : (a,b] \times \mathbb{R} \to \mathbb{R}$ is a function such that $g(\cdot, y(\cdot)) \in C_{1-\gamma,\rho}(\bar{J})$ for any $y \in C_{1-\gamma,\rho}$. A function $y \in C^{\gamma}_{1-\gamma,\rho}(\bar{J})$ is the solution of the fractional initial-value problem*

$$ \begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}[y](t) = g(t, y(t)), \\ {}^{\rho}\mathcal{I}^{1-\gamma}_{a^+}[y](a) = c, \end{cases} $$

if and only if y satisfies the following equation

$$ y(t) = \frac{c}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1} \xi^{\rho-1} g\big(\xi, y(\xi)\big)\,\mathrm{d}\xi. $$

Theorem 14 (Banach's fixed-point theorem [38]) *Let $\mathcal{Y}$ be a nonempty closed subset of a Banach space $\mathfrak{X}$ and $\mathcal{F} : \mathcal{Y} \to \mathcal{Y}$ be a contraction operator. Then, there is a unique $y \in \mathcal{Y}$ with $\mathcal{F}(y) = y$.*

Theorem 15 (Schauder's fixed-point theorem [38]) *Let $\mathcal{Y}$ be a nonempty closed subset of a Banach space $\mathfrak{X}$ and $\mathcal{F} : \mathcal{Y} \to \mathcal{Y}$ be a continuous mapping such that $\mathcal{F}(\mathcal{Y}) \subset \mathfrak{X}$ is relatively compact. Then, $\mathcal{F}$ has at least one fixed point in $\mathcal{Y}$.*

Theorem 16 (Arzelà–Ascoli theorem [38]) *A subset $\mathcal{Y}$ of $C(\mathfrak{X})$ is relatively compact iff it is closed, bounded and equicontinuous.*

3 Main result

In the following, we present a significant lemma to show the principal theorems.

Lemma 17 *Let $w : I_0 \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that*

$$ w\big(\cdot, y(\cdot), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\cdot)\big) \in C_{1-\gamma,\rho}(I_0), \quad I_0 = [0, T], $$

for all $y \in C_{1-\gamma,\rho}(I_0)$. We give $y \in C^{\gamma}_{1-\gamma,\rho}(I_0)$, then Problem (3) and (4) is equivalent to the fractional integral equation

$$
\begin{aligned}
y(t) = {}& \frac{\Delta}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\gamma-1}\Bigg[\sum_{i=1}^{m}\omega_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
& \times \left[w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\right) - p(\xi)y(\xi)\right]\mathrm{d}\xi\Bigg] \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
& \times \left[w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}y(\xi)\right) - p(\xi)y(\xi)\right]\mathrm{d}\xi,
\end{aligned}
\tag{15}
$$

where

$$
\Delta = \frac{\rho^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1} - \sum_{i=1}^{m} q_i (e_i^{\rho})^{\gamma-1}}.
\tag{16}
$$

Proof

($\Rightarrow$) We may apply Theorem 13 to reduce the $\mathbb{FME}$ (3) to an equivalent fractional integral equation. Then, we obtain

$$
\begin{aligned}
y(t) = {}& \frac{{}^{\rho}\mathcal{I}^{1-\gamma}[y](0)}{\Gamma(\gamma)}\left(\frac{t^{\rho}}{\rho}\right)^{\gamma-1} - {}^{\rho}\mathcal{I}^{\alpha}[py](t) \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\right)\mathrm{d}\xi.
\end{aligned}
\tag{17}
$$

Now, we replace t by e_i in equation (17) and multiply by q_i, and obtain

$$
\begin{aligned}
q_i y(e_i) = {}& \frac{{}^{\rho}\mathcal{I}^{1-\gamma}[y](0)}{\Gamma(\gamma)} q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1} - q_i {}^{\rho}\mathcal{I}^{\alpha}[py](e_i) \\
& + \frac{1}{\Gamma(\alpha)}\omega_i\int_0^{\xi_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}y(\xi)\right)\mathrm{d}\xi.
\end{aligned}
$$

From ${}^{\rho}\mathcal{I}^{1-\gamma}[y](0) = \sum_{i=1}^{m} q_i y(e_i)$, we write easily

$$
\begin{aligned}
\sum_{i=1}^{m} q_i y(e_i) = {}& \frac{{}^{\rho}\mathcal{I}^{1-\gamma}[y](0)}{\Gamma(\gamma)}\sum_{i=1}^{m} q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1} - \sum_{i=1}^{m} q_i {}^{\rho}\mathcal{I}^{\alpha}[py](e_i) \\
& + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
& \times w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}y(\xi)\right)\mathrm{d}\xi,
\end{aligned}
$$

and

$$\left[1-\frac{1}{\Gamma(\gamma)}\sum_{i=1}^{m}q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1}\right]{}^{\rho}\mathcal{I}^{1-\gamma}[y](0)$$
$$=\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}w\bigl(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\bigr)\,\mathrm{d}\xi$$
$$-\sum_{i=1}^{m}q_i\int_0^{e_i}{}^{\rho}\mathcal{I}^{\alpha}[py](e_i),$$

which implies

$$\begin{aligned}{}^{\rho}\mathcal{I}^{1-\gamma}[y](0)&=\frac{\Gamma(\gamma)\rho^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1}-\sum_{i=1}^{m}q_i(e_i^{\rho})^{\gamma-1}}\\&\quad\times\left[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right.\\&\quad\times w\bigl(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\bigr)\,\mathrm{d}\xi\\&\quad\left.-\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}p(\xi)y(\xi)\,\mathrm{d}\xi\right].\end{aligned}$$

Hence,

$$\begin{aligned}{}^{\rho}\mathcal{I}^{1-\gamma}[y](0)&=\frac{\Delta\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\&\quad\times\bigl[w\bigl(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\bigr)-p(\xi)y(\xi)\bigr]\,\mathrm{d}\xi.\end{aligned}\tag{18}$$

Replacing Eq. (18) in Eq. (17), we obtain Eq. (15).

($\Leftarrow$) Let $y\in C_{1-\gamma}^{\gamma}(I_0)$ satisfy Eq. (15). We prove that y also satisfies the problem (3) and (4). Therefore, we apply the operator ${}^{\rho}\mathcal{D}_{0_+}^{\gamma}$ on both sides of (15), and from Lemmas 5 and 10, we obtain

$${}^{\rho}\mathcal{D}_{0_+}^{\gamma}[y](t)={}^{\rho}\mathcal{D}_{0_+}^{\beta(1-\alpha)}\bigl[w\bigl(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\bigr)-p(t)y(t)\bigr].\tag{19}$$

From the hypothesis $y\in C_{1-\gamma,\rho}^{\gamma}(I_0)$ and (11), we have ${}^{\rho}\mathcal{D}_{0_+}^{\gamma}y\in C_{1-\gamma,\rho}(I_0)$ and

$$\begin{aligned}&{}^{\rho}\mathcal{D}_{0_+}^{\beta(1-\alpha)}\bigl[w\bigl(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\bigr)-p(t)y(t)\bigr]\\&\quad=\delta_{\rho}{}^{\rho}\mathcal{I}_{0_+}^{1-\beta(1-\alpha)}\bigl[w\bigl(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\bigr)-p(t)y(t)\bigr]\\&\quad\in C_{1-\gamma,\rho}(I_0).\end{aligned}\tag{20}$$

From [36, Lemma 5] and

$$\bigl[w\bigl(\cdot,y(\cdot),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\cdot)\bigr)-p(\cdot)y(\cdot)\bigr]\in C_{1-\gamma,\rho}(I_0),$$

it follows that

$$ {}^{\rho}\mathcal{I}_{0_+}^{1-\beta(1-\alpha)}\big(f\big(\cdot,u(\cdot),{}^{\rho}D^{\alpha,\beta}u(\cdot)\big)-p(\cdot)u(\cdot)\big)\in C_{1-\gamma,\rho}(I_0). \tag{21} $$

By Eq. (20), Eq. (21), and Definition 1, we have

$$ {}^{\rho}\mathcal{I}_{0_+}^{1-\beta(1-\alpha)}\big[w\big(\cdot,y(\cdot),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\cdot)\big)-p(\cdot)y(\cdot)\big]\in C^{1}_{1-\gamma,\rho}(I_0). $$

Applying ${}^{\rho}\mathcal{I}_{0_+}^{\beta(1-\alpha)}$ on both sides of Eq. (20) and Lemma 7, we have

$$ \begin{aligned} &{}^{\rho}\mathcal{I}_{0_+}^{\beta(1-\alpha)}\big[{}^{\rho}\mathcal{D}_{0_+}^{\gamma}[y]\big](t)\\ &\quad=\big[w\big(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[u](t)\big)-p(t)y(t)\big]\\ &\qquad-\frac{{}^{\rho}\mathcal{I}_{0_+}^{1-\beta(1-\alpha)}[w(0,y(0),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](0))-p(0)y(0)]}{\Gamma(\beta(1-\alpha))}\\ &\qquad\times\left(\frac{t^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1}. \end{aligned} $$

By Lemma 9 and property (P1) of operator ${}^{\rho}\mathcal{D}_{a^+}^{\alpha,\beta}$, we obtain

$$ {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)=w\big(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\big)-p(t)y(t), $$

that is, equation (3) holds. To this end, applying ${}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}$ of both sides of Eq. (15):

$$ \begin{aligned} {}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}[y](0)={}&\frac{\Delta}{\Gamma(\alpha)}{}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}\left(\frac{t^{\rho}}{\rho}\right)^{\gamma-1}\left[\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\right.\\ &\left.\times\,\xi^{\rho-1}\big[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\big]\,\mathrm{d}\xi\right]\\ &+{}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}\big({}^{\rho}\mathcal{I}_{0_+}^{\alpha}\big[w\big(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\big)-p(t)y(t)\big]\big), \end{aligned} $$

then, applying Lemma 5 and Theorem 6, we obtain

$$ \begin{aligned} {}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}[y](0)={}&\frac{\Delta\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\ &\times\big[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\big]\,\mathrm{d}\xi\\ &+{}^{\rho}\mathcal{I}_{0_+}^{1-\gamma+\alpha}\big[w\big(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\big)-p(t)y(t)\big], \end{aligned} $$

and we can write

$$ \begin{aligned} {}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}[y](0)={}&\frac{\Delta\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\ &\times\big[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\big]\,\mathrm{d}\xi\\ &+{}^{\rho}\mathcal{I}_{0_+}^{1-\beta(1-\alpha)}\big[w\big(t,y(t),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](t)\big)-p(t)y(t)\big]. \end{aligned} $$

By Lemma 9 and since $1-\gamma < 1-\beta(1-\alpha)$, we have

$$\begin{aligned}{}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}[y](0) &= \frac{\Delta\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i \int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi. \end{aligned} \tag{22}$$

Substituting $t=e_i$ and multiplying by q_i in Eq. (15), we obtain

$$\begin{aligned} q_i y(e_i) &= \frac{q_i\Delta}{\Gamma(\alpha)}\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1}\Bigg[\sum_{i=1}^{m} q_i \int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi\Bigg] \\ &\quad + \frac{q_i}{\Gamma(\alpha)}\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi, \end{aligned} \tag{23}$$

then,

$$\begin{aligned} \sum_{i=1}^{m} q_i y(e_i) &= \frac{\Delta}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1}\Bigg[\sum_{i=1}^{m} q_i \int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi\Bigg] \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{m} q_i y(e_i) &= \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\Bigg[\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi\Bigg] \\ &\quad \times \left[1+\Delta\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1}\right], \end{aligned}$$

which implies

$$\begin{aligned} \sum_{i=1}^{m} q_i y(e_i) &= \frac{\Delta\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\int_0^{e_i}\xi^{\rho-1} \\ &\quad \times \left[w\big(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi. \end{aligned} \tag{24}$$

From Eqs. (22) and (24), we obtain

$$
{}^{\rho}\mathcal{I}_{0_+}^{1-\gamma}[y](0) = \sum_{i=1}^{m} q_i[y](e_i).
$$

The proof is completed. □

Now, we will prove our first existence result for the problem (3) and (4) that is based on Shauder's fixed-point theorem.

Theorem 18 *Assume the following hypotheses hold.*

(A1) $w : I_0 \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function such that* $w \in C_{1-\gamma,\rho}$ *for all* $y \in C_{1-\gamma,\rho}(I_0)$.

(A2) *For all* $y, z \in \mathbb{R}$ *there exists a constant* $K > 0$ *such that*

$$
\bigl|w(t,y,z)\bigr| < K, \quad \forall t \in I_0. \tag{25}
$$

Then, the problem (3) *and* (4) *has at least one solution.*

Proof To prove the existence result, we will transform the problem (3) and (4) into a fixed-point problem. We define the operator $\mathcal{F} : C_{1-\gamma,\rho}(I_0) \to C_{1-\gamma,\rho}(I_0)$ by

$$
\begin{aligned}
(\mathcal{F}u)(t) = {}& \frac{\Delta}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\gamma-1}\Biggl[\sum_{i=1}^{m} q_i \int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
& \times \bigl[w\bigl(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\bigr) - p(\xi)y(\xi)\bigr]\,\mathrm{d}\xi\Biggr] \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
& \times \bigl[w\bigl(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\bigr) - p(\xi)y(\xi)\bigr]\,\mathrm{d}\xi.
\end{aligned} \tag{26}
$$

Note that $\mathcal{F}u \in C_{1-\gamma,\rho}(I_0)$, $\forall u \in C_{1-\gamma,\rho}(I_0)$. Since the problem (3) and (4) is equivalent to the fractional integral equation (26), the fixed points of $\mathcal{F}$ are solutions of the problem (3) and (4). We establish that $\mathcal{F}$ satisfies the assumption of Schauder's fixed-point Theorem 15. This could be proved through several steps.

Step 1. We prove that $\mathcal{F}$ is a continuous operator. For any bounded set $\mathcal{Y} \subset C_{1-\gamma,\rho}(I_0)$ there exists $\zeta > 0$ such that

$$
\mathcal{Y} = \bigl\{y \in C_{1-\gamma,\rho}(I_0) : \|y\|_{C_{1-\gamma,\rho}} \le \zeta\bigr\}.
$$

Let $(y_n)_{n\in\mathbb{N}} \in \mathcal{Y}$ be a real sequence such that

$$
\lim_{n\to\infty} \|y_n - y\|_{C_{1-\gamma,\rho}} = 0.
$$

Then, for each $t \in I_0$:

$$
\begin{aligned}
&\left|(\mathcal{F}y_n)(t) - (\mathcal{F}y)(t)\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\quad \leq \frac{\Delta}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i \int_0^{e_i}\left(\frac{e_i^\rho - \xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad \times \left[\left|w\left(\xi, y_n(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}[y]_n(\xi)\right)\right|\right. \\
&\qquad \left. + \left|w\left(\xi, y(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\right)\right| + \left|p(\xi)\right|\left(\left|y_n(\xi)\right| + \left|y(\xi)\right|\right)\right] \mathrm{d}\xi \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^\rho - \xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad \times \left[\left|w\left(\xi, y_n(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}y_n(\xi)\right)\right|\right. \\
&\qquad \left. + \left|w\left(\xi, y(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\right)\right| + \left|p(\xi)\right|\left(\left|y_n(\xi)\right| + \left|y(\xi)\right|\right)\right] \mathrm{d}\xi \\
&\quad \leq \frac{\Delta}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i \int_0^{e_i}\left(\frac{e_i^\rho - \xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad \times \left[2K + |a+2b|\left(\left|y_n(\xi)\right| + \left|y(\xi)\right|\right)\right] \mathrm{d}\xi \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^\rho - \xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad \times \left[2K + |a+2b|\left(\left|y_n(\xi)\right| + \left|y(\xi)\right|\right)\right] \mathrm{d}\xi \\
&\quad \leq \frac{\Delta}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\left[\frac{2K}{\alpha\rho}\left(\frac{e_i^\rho}{\rho}\right)^{\alpha}\right. \\
&\qquad \left. + |a+2b|\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{e_i^\rho}{\rho}\right)^{\alpha+\gamma-1}\left(\|y_n\|_{C_{1-\gamma,\rho}} + \|y\|_{C_{1-\gamma,\rho}}\right)\right] \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\left[\frac{2K}{\alpha\rho}\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right. \\
&\qquad \left. + |a+2b|\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{t^\rho}{\rho}\right)^{\alpha+\gamma-1}\left(\|y_n\|_{C_{1-\gamma,\rho}} + \|y\|_{C_{1-\gamma,\rho}}\right)\right] \\
&\quad \leq \frac{2\Delta}{\Gamma(\alpha)}\sum_{i=1}^{m} q_i\left(\frac{e_i^\rho}{\rho}\right)^{\alpha}\left[\frac{K}{\alpha\rho} + \beta(\gamma,\alpha)\zeta|a+2b|\left(\frac{e_i^\rho}{\rho}\right)^{\gamma-1}\right] \\
&\qquad + \frac{2}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}\left[\frac{K}{\alpha\rho}\left(\frac{T^\rho}{\rho}\right)^{1-\gamma} + \beta(\gamma,\alpha)\zeta|a+2b|\right], \\
&\quad \leq \frac{2}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^\rho}{\rho}\right)^{\alpha}\left(a^\circ + b^\circ\left(\frac{\xi_i^\rho}{\rho}\right)^{\gamma-1}\right)\right. \\
&\qquad \left. + \left(\frac{T^\rho}{\rho}\right)^{\alpha}\left(a^\circ\left(\frac{T^\rho}{\rho}\right)^{1-\gamma} + b^\circ\right)\right],
\end{aligned}
$$

where $a^\circ = \frac{K}{\alpha\rho}$ and

$$
b^\circ = \beta(\gamma,\alpha)\zeta|a+2b|.
$$

Then, Lebesgue's dominated convergence theorem asserts that

$$\left\|(\mathcal{F}y_n)(t)-(\mathcal{F}y)(t)\right\|_{n\to+\infty}\to 0.$$

Consequently, $\mathcal{F}$ is continuous.

Step 2. Let $\zeta \geq \frac{M_0}{1-M_1}$, we will show that $\mathcal{F}(\mathcal{Y})\subset\mathcal{Y}$. From (A2) and for each $t\in I_0$, we have

$$\begin{aligned}
&\left|(\mathcal{F}y)(t)\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\quad\leq \frac{\Delta}{\Gamma(\alpha)}\Bigg[\sum_{i=1}^m q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right|\,\mathrm{d}\xi\Bigg] \\
&\qquad+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right|\,\mathrm{d}\xi \\
&\quad\leq \frac{\Delta}{\Gamma(\alpha)}\Bigg[\sum_{i=1}^m q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad\times\left[\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)\right|+\left|p(\xi)y(\xi)\right|\right]\mathrm{d}\xi\Bigg] \\
&\qquad+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad\times\left[\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)\right|+\left|p(\xi)y(\xi)\right|\right]\mathrm{d}\xi \\
&\quad\leq \frac{\Delta}{\Gamma(\alpha)}\Bigg[\sum_{i=1}^m q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad\times\left[K+|a+2b|\left|y(s)\right|\right]\mathrm{d}\xi\Bigg] \\
&\qquad+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\qquad\times\left[K+|a+2b|\left|y(s)\right|\right]\mathrm{d}\xi \\
&\quad\leq \frac{\Delta}{\Gamma(\alpha)}\Bigg[\sum_{i=1}^m q_i\frac{K}{\alpha\rho}\left(\frac{e_i^\rho}{\rho}\right)^{\alpha} \\
&\qquad+|a+2b|\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{i=1}^m q_i\left(\frac{e_i^\rho}{\rho}\right)^{\alpha+\gamma-1}\|y\|_{C_{1-\gamma,\rho}}\Bigg] \\
&\qquad+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\Bigg[\frac{K}{\alpha\rho}\left(\frac{t^\rho}{\rho}\right)^{\alpha} \\
&\qquad+|a+2b|\Gamma(\alpha)\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{t^\rho}{\rho}\right)^{\alpha+\gamma-1}\|y\|_{C_{1-\gamma,\rho}}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\Delta}{\Gamma(\alpha)} \sum_{i=1}^{m} q_i \left(\frac{e_i^\rho}{\rho}\right)^\alpha \\
&\quad\times \left[\frac{K}{\alpha\rho} + \beta(\gamma,\alpha)\zeta|a+2b|\left(\frac{e_i^\rho}{\rho}\right)^{\gamma-1}\right] \\
&\quad+ \frac{1}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^\alpha \\
&\quad\times \left[\frac{K}{\alpha\rho}\left(\frac{T^\rho}{\rho}\right)^{1-\gamma} + \beta(\gamma,\alpha)\zeta|a+2b|\right] \\
&\leq \frac{|a+2b|\beta(\gamma,\alpha)}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{\xi_i^\rho}{\rho}\right)^{\alpha+\gamma-1} + \left(\frac{T^\rho}{\rho}\right)^\alpha\right]\zeta \\
&\quad+ \frac{K}{\alpha\rho\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^\rho}{\rho}\right)^\alpha + \left(\frac{T^\rho}{\rho}\right)^{\alpha+1-\gamma}\right].
\end{aligned}
$$

We obtain that

$$
\left\|\mathcal{F}(y)\right\|_{C_{1-\gamma,\rho}} \leq M_0 + M_1\zeta \leq \frac{1-M_1}{1-M_1}M_0 + M_1\zeta \leq \zeta,
$$

where

$$
\begin{aligned}
M_0 &= \frac{K}{\alpha\rho\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^\rho}{\rho}\right)^\alpha + \left(\frac{T^\rho}{\rho}\right)^{\alpha+1-\gamma}\right], \\
M_1 &= \frac{|a+2b|\beta(\gamma,\alpha)}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^\rho}{\rho}\right)^{\alpha+\gamma-1} + \left(\frac{T^\rho}{\rho}\right)^\alpha\right].
\end{aligned}
\tag{27}
$$

Then, $\mathcal{F}(\mathcal{Y}) \subset \mathcal{Y}$.

Step 3. We show that $\mathcal{F}$ is uniformly bounded. For any $y \in \mathcal{Y}$, it follows that

$$
\begin{aligned}
&\left|(\mathcal{F}y)(t)\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\quad\leq \frac{\Delta}{\Gamma(\alpha)}\left[\sum_{i=1}^{m} q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right. \\
&\quad\quad\left.\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right|\,\mathrm{d}\xi\right] \\
&\quad\quad+ \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1} \\
&\quad\quad\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right|\,\mathrm{d}\xi \\
&\quad\leq \frac{\Delta}{\Gamma(\alpha)}\left[\sum_{i=1}^{m} q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right. \\
&\quad\quad\left.\times\left[\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)\right|+\left|p(\xi)y(\xi)\right|\right]\mathrm{d}\xi\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\
&\times\left[\left|w\left(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\right)\right|+\left|p(\xi)y(\xi)\right|\right]\mathrm{d}\xi\\
\le{}&\frac{\Delta}{\Gamma(\alpha)}\left[\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right.\\
&\left.\times\left[K+|a+2b|\left|y(s)\right|\right]\mathrm{d}\xi\right]\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\
&\times\left[K+|a+2b|\left|y(s)\right|\right]\mathrm{d}\xi\\
\le{}&\frac{\Delta}{\Gamma(\alpha)}\left[\sum_{i=1}^{m}q_i\frac{K}{\alpha\rho}\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha}\right.\\
&\left.+|a+2b|\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{i=1}^{m}q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha+\gamma-1}\|y\|_{C_{1-\gamma,\rho}}\right]\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\left[\frac{K}{\alpha\rho}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right.\\
&\left.+|a+2b|\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha+\gamma-1}\|y\|_{C_{1-\gamma,\rho}}\right]\\
\le{}&\frac{\Delta}{\Gamma(\alpha)}\sum_{i=1}^{m}q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha}\\
&\times\left[\frac{K}{\alpha\rho}+\beta(\gamma,\alpha)\zeta\,|a+2b|\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1}\right]\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{T^{\rho}}{\rho}\right)^{\alpha}\\
&\times\left[\frac{K}{\alpha\rho}\left(\frac{T^{\rho}}{\rho}\right)^{1-\gamma}+\beta(\gamma,\alpha)\zeta\,|a+2b|\right].\\
\le{}&\frac{1}{\Gamma(\alpha)}\left\{\Delta\sum_{i=1}^{m}q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha}\left[a^{\circ}+b^{\circ}\left(\frac{e_i^{\rho}}{\rho}\right)^{\gamma-1}\right]\right.\\
&\left.+\left(\frac{T^{\rho}}{\rho}\right)^{\alpha}\left[a^{\circ}\left(\frac{T^{\rho}}{\rho}\right)^{1-\gamma}+b^{\circ}\right]\right\},
\end{aligned}
$$

which implies that $\mathcal{F}(\mathcal{Y})$ is uniformly bounded.

Step 4. We prove the equicontinuity of $\mathcal{F}$. Let $y\in\mathcal{Y}$ and $t_1,t_2\in I_0$ with $t_1<t_2$. Therefore,

$$
\begin{aligned}
&\left|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)\right|\\
\le{}&\left|\frac{\Delta}{\Gamma(\alpha)}\left(\frac{t_2^{\rho}}{\rho}\right)^{\gamma-1}\left[\sum_{i=1}^{m}q_i\int_0^{e_i}\left(\frac{e_i^{\rho}-\xi^{\rho}}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right.\right.\\
&\left.\times\left[w\left(\xi,y(\xi),{}^{\rho}\mathcal{D}^{\alpha,\beta}y(\xi)\right)-p(\xi)y(\xi)\right]\mathrm{d}\xi\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left(\frac{t_2^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\
&\times\left[w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi\\
&-\frac{\Delta}{\Gamma(\alpha)}\left(\frac{t_1^\rho}{\rho}\right)^{\gamma-1}\left[\sum_{i=1}^m q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right.\\
&\left.\times\left[w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi\right]\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left(\frac{t_1^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\
&\times\left[w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right]\mathrm{d}\xi\Bigg|\\
\le{}&\frac{\Delta}{\Gamma(\alpha)}\left|\left(\frac{t_2^\rho}{\rho}\right)^{\gamma-1}-\left(\frac{t_1^\rho}{\rho}\right)^{\gamma-1}\right|\\
&\times\left[\sum_{i=1}^m q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right.\\
&\left.\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\big)-p(\xi)y(\xi)\right|\mathrm{d}\xi\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\left(\frac{t_2^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}-\left(\frac{t_1^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\right]\xi^{\rho-1}\\
&\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}y(\xi)\big)-p(\xi)y(\xi)\right|\mathrm{d}\xi\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\\
&\times\left|w\big(\xi,y(\xi),{}^\rho\mathcal{D}^{\alpha,\beta}y(\xi)\big)-p(\xi)y(\xi)\right|\mathrm{d}\xi\\
\le{}&\frac{\Delta}{\Gamma(\alpha)}\left|\left(\frac{t_2^\rho}{\rho}\right)^{\gamma-1}-\left(\frac{t_1^\rho}{\rho}\right)^{\gamma-1}\right|\\
&\times\left[\sum_{i=1}^m q_i\int_0^{e_i}\left(\frac{e_i^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\xi^{\rho-1}\right.\\
&\left.\times\left[K+|a+2b|\big|y(\xi)\big|\right]\mathrm{d}\xi\right]\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\left(\frac{t_2^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}-\left(\frac{t_1^\rho-\xi^\rho}{\rho}\right)^{\alpha-1}\right]\\
&\times\xi^{\rho-1}\left[K+|a+2b|\big|y(\xi)\big|\right]\mathrm{d}\xi\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-q^\rho}{\rho}\right)^{\alpha-1}q^{\rho-1}\\
&\times\left[K+|a+2b|\big|y(\xi)\big|\right]\mathrm{d}\xi\\
\le{}&\frac{\Delta}{\Gamma(\alpha)}\left|\left(\frac{t_2^\rho}{\rho}\right)^{\gamma-1}-\left(\frac{t_1^\rho}{\rho}\right)^{\gamma-1}\right|
\end{aligned}
$$

$$\times \left[\sum_{i=1}^{m} q_i \left(\frac{K}{\alpha\rho} \left(\frac{e_i^\rho}{\rho} \right)^\alpha + |a+2b| \beta(\gamma,\alpha) \left(\frac{e_i^\rho}{\rho} \right)^{\alpha+\gamma-1} \zeta \right) \right]$$
$$+ \frac{K}{\alpha\rho\Gamma(\alpha)} \left[\left| \left(\frac{t_2^\rho}{\rho} \right)^\alpha - \left(\frac{t_1^\rho}{\rho} \right)^\alpha \right| \right]$$
$$+ \frac{\zeta\beta(\gamma,\alpha)}{\Gamma(\alpha)} |a+2b| \left| \left(\frac{t_2^\rho - t_1^\rho}{\rho} \right)^{\alpha+\gamma-1} \right|.$$

We deduce that

$$\left| (\mathcal{F}y)(t_2) - (\mathcal{F}y)(t_1) \right| \to 0,$$

as $|t_2 - t_1| \to 0$, which implies that $\mathcal{F}(\mathcal{Y})$ is equicontinuous.
Thus, by the Arzelà-Ascoli theorem, the operator $\mathcal{F}$ is completely continuous. By the Schauder fixed-point theorem the operator $\mathcal{F}$ has a fixed point $y \in \mathcal{Y}$. Now, we will use the Banach contraction principle to prove the uniqueness of the solution. □

Theorem 19 *Assume that the hypotheses* $(A1)$, $(A2)$, *and*

(A3) *for any* y, z, $\hat{y}, \hat{z} \in \mathbb{R}$ *and* $t \in [0,T]$, *there exist positive constants* $A > 0$, $B < 1$ *such that*

$$\left| w(t,y,z) - w(t,\hat{y},\hat{z}) \right| \le A|y-\hat{y}| + B|z-\hat{z}|$$

hold. Then, the problem (3) *and* (4) *admits a unique solution in* $C_{1-\gamma,\rho}[0,T]$, *whenever*

$$\Sigma = C \frac{\beta(\gamma,\alpha)}{\Gamma(\alpha)} \left[\Delta \sum_{i=1}^{m} q_i \left(\frac{e_i^\rho}{\rho} \right)^{\alpha+\gamma-1} + \left(\frac{T^\rho}{\rho} \right)^\alpha \right] < 1, \tag{28}$$

where $C = \frac{A+|a+2b|}{1-B}$.

Proof Let $y, z \in C_{1-\gamma,\rho} I_0$, be such that

$${}^\rho\mathcal{D}^{\alpha,\beta}[y](t) = w\bigl(t, y(t), {}^\rho\mathcal{D}^{\alpha,\beta}[y](t)\bigr) - p(t)y(t),$$
$${}^\rho\mathcal{D}^{\alpha,\beta}[z](t) = w\bigl(t, z(t), {}^\rho\mathcal{D}^{\alpha,\beta}[z](t)\bigr) - p(t)z(t).$$

Thus, we have

$$\left| \bigl[(\mathcal{F}y)(t) - (\mathcal{F}z)(t) \bigr] \left(\frac{t^\rho}{\rho} \right)^{1-\gamma} \right|$$
$$\le \frac{\Delta}{\Gamma(\alpha)} \left(\frac{t_2^\rho}{\rho} \right)^{\gamma-1} \sum_{i=1}^{m} q_i \left[\int_0^{e_i} \left(\frac{e_i^\rho - \xi^\rho}{\rho} \right)^{\alpha-1} \xi^{\rho-1} \right.$$
$$\times \left| \bigl(w\bigl(\xi, y(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}[y](\xi)\bigr) - p(\xi)y(\xi) \bigr) \right.$$
$$\left. \left. - \bigl(w\bigl(\xi, z(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}[z](\xi)\bigr) - p(\xi)z(\xi) \bigr) \right| \,\mathrm{d}\xi \right]$$
$$+ \frac{1}{\Gamma(\alpha)} \left(\frac{t^\rho}{\rho} \right)^{1-\gamma} \int_0^t \left(\frac{t^\rho - \xi^\rho}{\rho} \right)^{\alpha-1} \xi^{\rho-1}$$

$$\times \left|\left(w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\right) - p(\xi)y(\xi)\right)\right.$$
$$\left. - \left(w\left(\xi, z(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right) - p(\xi)z(\xi)\right)\right| \,\mathrm{d}\xi.$$

Then, for all $t \in I_0$, we have

$$\begin{aligned}
&\left|\left[(\mathcal{F}y)(t) - (\mathcal{F}z)(t)\right]\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\right| \\
&\quad \le \frac{\Delta}{\Gamma(\alpha)}\left(\frac{t_2^{\rho}}{\rho}\right)^{\gamma-1} \sum_{i=1}^{m} q_i \left[\int_0^{e_i} \left(\frac{e_i^{\rho} - \xi^{\rho}}{\rho}\right)^{\alpha-1} \xi^{\rho-1}\right. \\
&\quad\quad \left. \times \left|{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](s) - {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right| \,\mathrm{d}\xi \right] \\
&\quad\quad + \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma} \int_0^{t} \left(\frac{t^{\rho} - \xi^{\rho}}{\rho}\right)^{\alpha-1} \xi^{\rho-1} \\
&\quad\quad \times \left|{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](s) - {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right| \,\mathrm{d}\xi.
\end{aligned}$$

By (A3),

$$\begin{aligned}
&\left|{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi) - {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right| \\
&\quad = \left|\left(w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\right) - p(\xi)y(\xi)\right)\right. \\
&\quad\quad \left. - \left(w\left(\xi, z(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right) - p(\xi)z(\xi)\right)\right| \\
&\quad \le \left|w\left(\xi, y(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi)\right) - w\left(\xi, z(\xi), {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right)\right| \\
&\quad\quad + \left|p(t)\right|\left|y(t) - z(t)\right| \\
&\quad \le A\left|y(t) - z(t)\right| + B\left|{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi) - {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right| \\
&\quad\quad + \left|p(t)\right|\left|y(t) - z(t)\right| \\
&\quad \le \left(A + |a + 2b|\right)\left|y(t) - z(t)\right| \\
&\quad\quad + B\left|{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi) - {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right|.
\end{aligned}$$

Thus,

$$\left|{}^{\rho}\mathcal{D}^{\alpha,\beta}[y](\xi) - {}^{\rho}\mathcal{D}^{\alpha,\beta}[z](\xi)\right| \le C\left|y(t) - z(t)\right|,$$

and

$$\begin{aligned}
&\left|\left((\mathcal{F}y)(t) - (\mathcal{F}z)(t)\right)\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\right| \\
&\quad \le C\frac{\Delta}{\Gamma(\alpha)} \sum_{i=1}^{m} q_i \int_0^{e_i} \left(\frac{e_i^{\rho} - \xi^{\rho}}{\rho}\right)^{\alpha-1} \xi^{\rho-1}\left|y(t) - z(t)\right| \,\mathrm{d}\xi \\
&\quad\quad + \frac{C}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma} \int_0^{t} \left(\frac{t^{\rho} - \xi^{\rho}}{\rho}\right)^{\alpha-1} \xi^{\rho-1}\left|y(t) - z(t)\right| \,\mathrm{d}\xi,
\end{aligned}$$

where $C = \frac{A+|a+2b|}{1-B}$. This gives

$$\begin{aligned}
&\left\|(\mathcal{F}y)-(\mathcal{F}z)\right\|_{C_{1-\gamma,\rho}}\\
&\quad\le C\frac{\beta(\gamma,\alpha)}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m}q_i\left(\frac{e_i^\rho}{\rho}\right)^{\alpha+\gamma-1}+\left(\frac{T^\rho}{\rho}\right)^{\alpha}\right]\|y-z\|_{C_{1-\gamma,\rho}}\\
&\quad\le\Sigma\|y-z\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

Due to Eq. (28), the operator $\mathcal{F}$ is a contraction mapping. Using the Banach contraction mapping theorem, we deduce that $\mathcal{F}$ has a unique fixed point that is the unique solution of the problem (3) and (4). □

4 Some illustrative examples

Now, we illustrate some examples that guarantee our main results. In this case, we use a computational technique for checking the solutions of $\mathbb{MFDE}$s with the H-K$\mathbb{FD}$ problem (3) and (4), and applying a tiny step h by the implicit trapezoidal PI rule, which, as we will see, usually shows excellent accuracy [39].

Example 1 Consider the $\mathbb{MFDE}$ with H-K$\mathbb{FD}$

$$\begin{aligned}
{}^{12}\mathcal{D}^{\frac{3}{7},\frac{5}{8}}[y](t)+p(t)y(t)&=\frac{4}{15+\exp(\frac{t}{\pi})}\left[\frac{\sin^2(y(t))}{10+\sin^2(y(t))}\right.\\
&\quad\left.+\frac{|{}^{12}\mathcal{D}^{\frac{3}{7},\frac{5}{8}}[y](t)|}{45+|{}^{12}\mathcal{D}^{\frac{3}{7},\frac{5}{8}}[y](t)|}\right],
\end{aligned}\tag{29}$$

for $t\in I_0=[0,\pi]$, with the initial condition

$${}^{12}\mathcal{I}^{\frac{3}{14}}[y](0)=\frac{21}{4}y\left(\frac{2\pi}{13}\right)+\frac{19}{5}y\left(\frac{3\pi}{7}\right)+\frac{22}{15}y\left(\frac{8\pi}{11}\right)+\frac{28}{9}y\left(\frac{14\pi}{15}\right),\tag{30}$$

where

$$p(t)=\frac{115}{83}-\frac{2\sqrt{329}}{15}\cos(2t),\quad\forall t\in(0,\pi].$$

Clearly, $\alpha=\frac{3}{7}\in(0,1)$, $\beta=\frac{5}{8}\in[0,1]$,

$$\gamma=\alpha+\beta(1-\alpha)=\frac{11}{14}\in[0,1),$$

$\rho=12>0$, $q_1=\frac{21}{4}$, $q_2=\frac{19}{5}$, $q_3=\frac{22}{15}$, and $q_4=\frac{28}{9}$ are real numbers, $e_1=\frac{2\pi}{13}$, $e_2=\frac{3\pi}{7}$, $e_3=\frac{8\pi}{11}$, $e_4=\frac{14\pi}{15}\in(0,\pi)$, and

$$a=\frac{115}{83},\qquad b=\frac{\sqrt{329}}{15}.$$

We define the map $w:[0,\pi]\times\mathbb{R}^2\to\mathbb{R}$ by

$$w(t,y,z)=\frac{4}{15+\exp(\frac{t}{\pi})}\left[\frac{\sin^2 y}{10+\sin^2 y}+\frac{|z|}{45+|z|}\right]$$

for $y,z \in \mathbb{R}$. Then, by Definition 1, we have

$$C_{\frac{3}{14},12}([0,\pi]) = \left\{y:(a,b]\to\mathbb{R}:\left(\frac{t^{12}-(\frac{115}{83})^{12}}{12}\right)^{\frac{3}{14}}y(t)\in C([0,\pi])\right\},$$

and so $w \in C_{\frac{3}{14},12}([0,\pi])$ for all $y \in C_{\frac{3}{14},12}([0,\pi])$. Also,

$$\begin{aligned}\left|w(t,y,z)\right| &= \left|\frac{4}{15+\exp(\frac{t}{\pi})}\left[\frac{\sin^2 y}{10+\sin^2 y}+\frac{|z|}{45+|z|}\right]\right| \\ &= \left|\frac{4}{15+\exp(\frac{t}{\pi})}\right|\left|\frac{\sin^2 y}{10+\sin^2 y}+\frac{|z|}{45+|z|}\right| \\ &\le \frac{1}{2},\end{aligned}$$

for $t \in [0,\pi]$. Put $K = \frac{1}{2}$. Therefore, w satisfies the conditions (A1) and (A2) of Theorem 18. On the other hand, by employing Eq. (16), we obtain

$$\begin{aligned}\Delta &= \rho^{\gamma-1}\left[\Gamma(\gamma)\rho^{\gamma-1}-\sum_{i=1}^{m}q_i\left(e_i^{\rho}\right)^{\gamma-1}\right]^{-1} \\ &= 12^{(\frac{-3}{14})}\left[12^{(\frac{-3}{14})}\Gamma\left(\frac{11}{14}\right)-\left[\frac{21}{4}\left(\left(\frac{2\pi}{13}\right)^{12}\right)^{(\frac{-3}{14})}+\frac{19}{5}\left(\left(\frac{3\pi}{7}\right)^{12}\right)^{(\frac{-3}{14})}\right.\right. \\ &\quad \left.\left.+\frac{22}{15}\left(\left(\frac{8\pi}{11}\right)^{12}\right)^{(\frac{-3}{14})}+\frac{28}{9}\left(\left(\frac{14\pi}{15}\right)^{12}\right)^{(\frac{-3}{14})}\right]\right]^{-1} \\ &= 12^{(\frac{-3}{14})}\left[12^{(\frac{-3}{14})}\Gamma\left(\frac{11}{14}\right)-\left[\frac{21}{4}\left(\frac{2\pi}{13}\right)^{(\frac{-18}{7})}+\frac{19}{5}\left(\frac{3\pi}{7}\right)^{(\frac{-18}{7})}\right.\right. \\ &\quad \left.\left.+\frac{22}{15}\left(\frac{8\pi}{11}\right)^{(\frac{-18}{7})}+\frac{28}{9}\left(\frac{14\pi}{15}\right)^{(\frac{-18}{7})}\right]\right]^{-1} \\ &\approx -0.01654.\end{aligned}$$

Then, by using Eqs. (27), we obtain

$$\begin{aligned}M_0 &= \frac{K}{\alpha\rho\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m}q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha}+\left(\frac{T^{\rho}}{\rho}\right)^{\alpha+1-\gamma}\right], \\ &= \frac{\frac{1}{2}}{\frac{3}{7}\times 12\Gamma(\frac{3}{7})}\left[\Delta\left[\frac{21}{4}\left(\frac{\frac{2\pi}{13}^{12}}{12}\right)^{(\frac{3}{7})}+\frac{19}{5}\left(\frac{\frac{3\pi}{7}^{12}}{12}\right)^{(\frac{3}{7})}\right.\right. \\ &\quad \left.+\frac{22}{15}\left(\frac{\frac{8\pi}{11}^{12}}{12}\right)^{(\frac{3}{7})}+\frac{28}{9}\left(\frac{\frac{14\pi}{15}^{12}}{12}\right)^{(\frac{3}{7})}\right]+\left.\left(\frac{\pi^{12}}{12}\right)^{(\frac{9}{14})}\right] \\ &\approx 64.87685,\end{aligned}$$

and

$$\begin{aligned} M_1 &= \frac{|a+2b|\beta(\gamma,\alpha)}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^\rho}{\rho}\right)^{\alpha+\gamma-1} + \left(\frac{T^\rho}{\rho}\right)^\alpha\right] \\ &= \frac{|\frac{115}{83}+2\frac{\sqrt{329}}{15}|\beta(\frac{11}{14},\frac{3}{7})}{\Gamma(\frac{3}{7})}\left[\Delta\left[\frac{21}{4}\left(\frac{\frac{2\pi}{13}^{12}}{12}\right)^{(\frac{3}{7})}\right.\right. \\ &\quad + \frac{19}{5}\left(\frac{\frac{3\pi}{7}^{12}}{12}\right)^{(\frac{3}{7})} + \frac{22}{15}\left(\frac{\frac{8\pi}{11}^{12}}{12}\right)^{(\frac{3}{7})} \\ &\quad \left.\left. + \frac{28}{9}\left(\frac{\frac{14\pi}{15}^{12}}{12}\right)^{(\frac{3}{7})}\right] + \left(\frac{\pi^{12}}{12}\right)^{(\frac{9}{14})}\right] \\ &\approx 6775.715045. \end{aligned}$$

Take

$$\zeta \geq \frac{M_0}{1-M_1} \approx -0.0095763.$$

Table 1 shows the numerical results of M_0, M_1, and ζ. These results are shown graphically in Fig. 1. By employing the Algorithm 1, one can obtain all numerical results in this example. Hence, all conditions of Theorem 18 hold, and so the problem has a solution.

Table 1 Numerical results of M_0, M_1, and ζ in Example 1

t	M_0	M_1	ζ
0.00000	−2.4314E−01	−2.5393E+01	−9.2100E−03
0.10472	−2.4314E−01	−2.5393E+01	−9.2100E−03
0.20944	−2.4314E−01	−2.5393E+01	−9.2100E−03
0.31416	−2.4313E−01	−2.5393E+01	−9.2100E−03
0.41888	−2.4312E−01	−2.5392E+01	−9.2100E−03
0.52360	−2.4307E−01	−2.5386E+01	−9.2100E−03
0.62832	−2.4287E−01	−2.5365E+01	−9.2100E−03
0.73304	−2.4227E−01	−2.5302E+01	−9.2100E−03
0.83776	−2.4071E−01	−2.5139E+01	−9.2100E−03
0.94248	−2.3711E−01	−2.4764E+01	−9.2000E−03
1.04720	−2.2955E−01	−2.3974E+01	−9.1900E−03
1.15192	−2.1480E−01	−2.2433E+01	−9.1700E−03
1.25664	−1.8769E−01	−1.9602E+01	−9.1100E−03
1.36136	−1.4032E−01	−1.4655E+01	−8.9600E−03
1.46608	−6.1020E−02	−6.3734E+00	−8.2800E−03
1.57080	6.6950E−02	6.9923E+00	−1.1170E−02
1.67552	2.6702E−01	2.7888E+01	−9.9300E−03
1.78024	5.7122E−01	5.9658E+01	−9.7400E−03
1.88496	1.0225E+00	1.0679E+02	−9.6700E−03
1.98968	1.6775E+00	1.7520E+02	−9.6300E−03
2.09440	2.6098E+00	2.7257E+02	−9.6100E−03
2.19911	3.9136E+00	4.0874E+02	−9.6000E−03
2.30383	5.7082E+00	5.9616E+02	−9.5900E−03
2.40855	8.1425E+00	8.5040E+02	−9.5900E−03
2.51327	1.1401E+01	1.1908E+03	−9.5800E−03
2.61799	1.5712E+01	1.6409E+03	−9.5800E−03
2.72271	2.1349E+01	2.2296E+03	−9.5800E−03
2.82743	2.8646E+01	2.9917E+03	−9.5800E−03
2.93215	3.8001E+01	3.9689E+03	−9.5800E−03
3.03687	4.9891E+01	5.2106E+03	−9.5800E−03

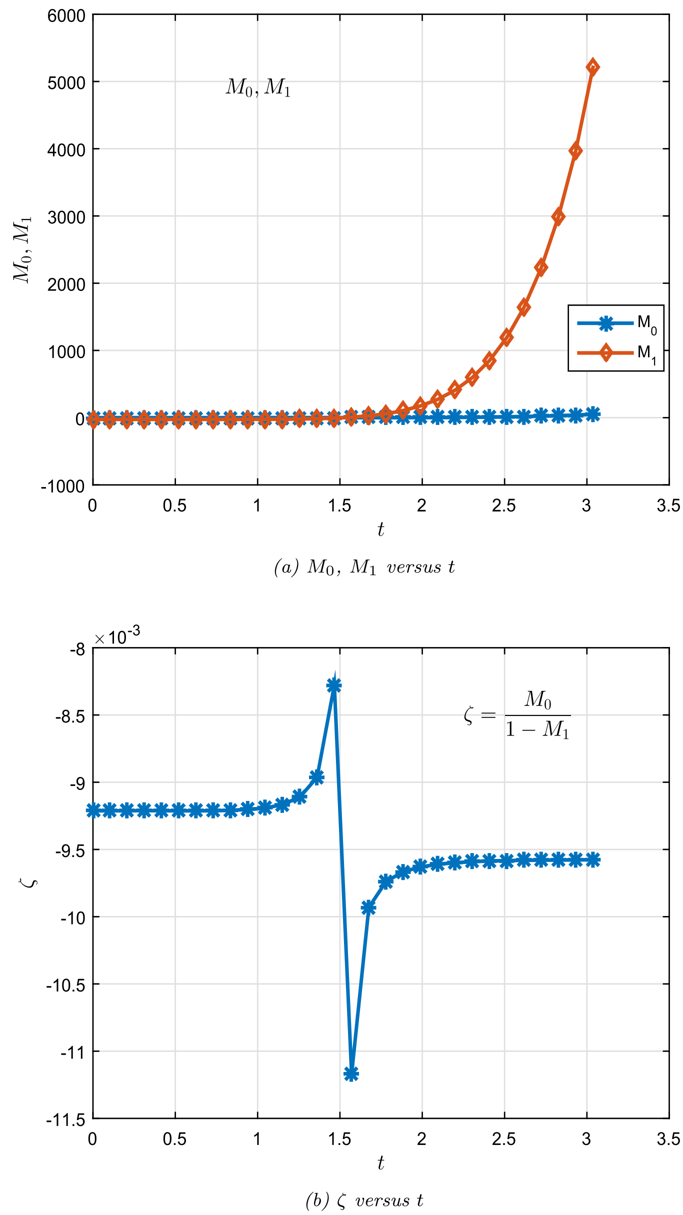

Figure 1 Graphical representation of variables M_0, M_1, and ζ for $t \in (0,\pi]$ in Example 1

Example 2 Consider the $\mathbb{MFDE}$ with H-$\mathbb{KFD}$

$$
{}^{\frac{7}{5}}\mathcal{D}^{\frac{1}{8},\frac{5}{6}}[y](t) + p(t)y(t) = \frac{5}{24 + (1 + \frac{t}{\pi})^2}\left[\frac{|\cos(y(t))|}{16 + |\cos(y(t))|} + \frac{\sin^2({}^{\frac{7}{5}}\mathcal{D}^{\frac{1}{8},\frac{5}{6}}[y](t))}{34 + \sin^2({}^{\frac{7}{5}}\mathcal{D}^{\frac{1}{8},\frac{5}{6}}[y](t))}\right], \tag{31}
$$

for $t \in I_0 = [0,\pi]$, with the initial condition

$$
{}^{\frac{7}{5}}\mathcal{I}^{\frac{7}{48}}[y](0) = \frac{13}{4}y\left(\frac{\pi}{11}\right) + \frac{10}{7}y\left(\frac{3\pi}{8}\right) + \frac{9}{10}y\left(\frac{6\pi}{7}\right), \tag{32}
$$

where

$$p(t) = \frac{\sqrt[3]{75}}{12} - \frac{15}{14}\cos(2t), \quad \forall t \in (0,\pi].$$

Clearly, $\alpha = \frac{1}{8} \in (0,1)$, $\beta = \frac{5}{6} \in [0,1]$, $\gamma = \alpha + \beta(1-\alpha) = \frac{41}{48} \in [0,1)$, $\rho = \frac{7}{5} > 0$, $q_1 = \frac{13}{4}$, $q_2 = \frac{10}{7}$, and $q_3 = \frac{9}{10}$ are real numbers, $e_1 = \frac{\pi}{11}$, $e_2 = \frac{3\pi}{8}$, and $e_3 = \frac{6\pi}{7}$ belong to $(0,\pi)$ and

$$a = \frac{\sqrt[3]{75}}{12}, \qquad b = \frac{15}{14}.$$

We define the map $w : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$w(t,y,z) = \frac{5}{24 + (1+\frac{t}{\pi})^2}\left[\frac{|\cos y|}{16 + |\cos y|} + \frac{\sin^2 z}{34 + \sin^2 z}\right]$$

for $y, z \in \mathbb{R}$. Then, by Definition 1, we have

$$C_{\frac{7}{48},\frac{7}{5}}\bigl([0,\pi]\bigr) = \left\{y : (a,b] \to \mathbb{R} : \left(\frac{t^{\frac{7}{5}} - (\frac{\sqrt[3]{75}}{12})^{\frac{7}{5}}}{\frac{7}{5}}\right)^{\frac{7}{48}} y(t) \in C\bigl([0,\pi]\bigr)\right\},$$

and so $w \in C_{\frac{7}{48},\frac{7}{5}}([0,\pi])$ for all $y \in C_{\frac{7}{48},\frac{7}{5}}([0,\pi])$. Also,

$$\begin{aligned}
\bigl|w(t,y,z) - w(t,\hat{y},\hat{z})\bigr| &= \left|\frac{5}{24 + (1+\frac{t}{\pi})^2}\left[\frac{|\cos y|}{16 + |\cos y|} + \frac{\sin^2 z}{34 + \sin^2 z}\right]\right. \\
&\quad \left. - \frac{5}{24 + (1+\frac{t}{\pi})^2}\left[\frac{|\cos \hat{y}|}{16 + |\cos \hat{y}|} + \frac{\sin^2 \hat{z}}{34 + \sin^2 \hat{z}}\right]\right| \\
&\le \frac{5}{24 + (1+\frac{t}{\pi})^2}\left[\left|\frac{|\cos y|}{16 + |\cos y|} - \frac{|\cos \hat{y}|}{16 + |\cos \hat{y}|}\right|\right. \\
&\quad \left. + \left|\frac{\sin^2 z}{34 + \sin^2 z} - \frac{\sin^2 \hat{z}}{34 + \sin^2 \hat{z}}\right|\right] \\
&\le \frac{1}{85}|y - \hat{y}| + \frac{1}{170}|z - \hat{z}|,
\end{aligned}$$

for $t \in [0,\pi]$. Put $A = \frac{1}{85}$ and $B = \frac{1}{170}$. Therefore, w satisfies the condition (A3) of Theorem 19. Furthermore,

$$C = \frac{A + |a + 2b|}{1 - B} \approx 6.409488.$$

On the other hand, by employing Eq. (16), we obtain

$$\Delta = \rho^{\gamma-1}\left[\Gamma(\gamma)\rho^{\gamma-1} - \sum_{i=1}^{m} q_i\bigl(e_i^{\rho}\bigr)^{\gamma-1}\right]^{-1} \approx -0.181048,$$

Table 2 Numerical results of Σ for $t \in [0, \pi]$ in Example 2

t	Σ
0.00000	−2.8519E+00
0.10472	−1.0688E+00
0.20944	−8.3886E−01
0.31416	−6.9083E−01
0.41888	−5.7925E−01
0.52360	−4.8875E−01
0.62832	−4.1213E−01
0.73304	−3.4542E−01
0.83776	−2.8616E−01
0.94248	−2.3273E−01
1.04720	−1.8399E−01
1.15192	−1.3912E−01
1.25664	−9.7490E−02
1.36136	−5.8640E−02
1.46608	−2.2180E−02
1.57080	1.2190E−02
1.67552	4.4720E−02
1.78024	7.5620E−02
1.88496	1.0505E−01
1.98968	1.3316E−01
2.09440	1.6007E−01
2.19911	1.8590E−01
2.30383	2.1073E−01
2.40855	2.3465E−01
2.51327	2.5772E−01
2.61799	2.8002E−01
2.72271	3.0159E−01
2.82743	3.2248E−01
2.93215	3.4275E−01
3.03687	3.6243E−01

and by using Eqs. (28), we obtain

$$\begin{aligned}
\Sigma &= C\frac{\beta(\gamma,\alpha)}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha+\gamma-1} + \left(\frac{T^{\rho}}{\rho}\right)^{\alpha}\right] \\
&= C\frac{\beta(\frac{41}{48},\frac{1}{8})}{\Gamma(\frac{1}{8})}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^{6}}{6}\right)^{(\frac{1}{8}+\frac{41}{48}-1)} + \left(\frac{\pi^{6}}{6}\right)^{(\frac{1}{8})}\right] \\
&\approx 0.970123 < 1.
\end{aligned}$$

Table 2 shows the numerical results of Σ. These results are shown graphically in Fig. 2. By using the Algorithm 2, we can obtain all numerical results in this example. Therefore, all conditions of Theorem 19 hold, and hence this problem has a solution.

5 An application of a particle in the plane

Linear motion is the most basic of all motions. According to Newton's first law of motion, objects that do not experience any net force will continue to move in a straight line with a constant velocity until they are subjected to a net force.

Here, in this section, we consider an application to examine the validity of our theoretical results on the fractional-order representation of the motion of a particle along a straight line. In this case, we consider a constrained motion of a particle along a straight line re-

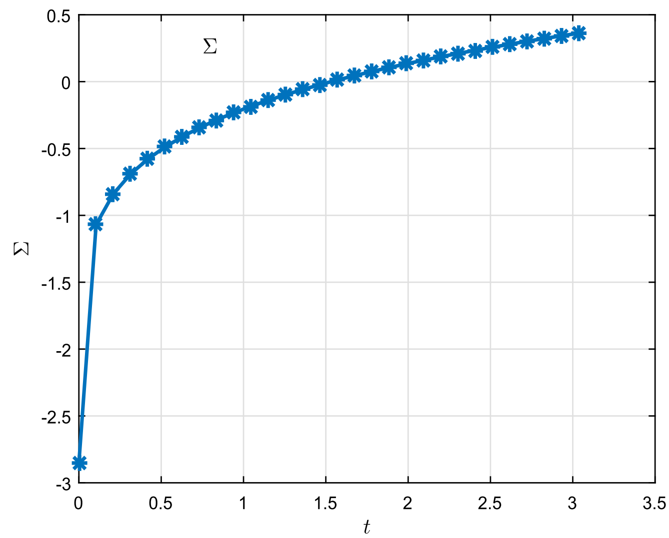

Figure 2 Graphical representation of Σ for $t \in [0,\pi]$ in Example 2

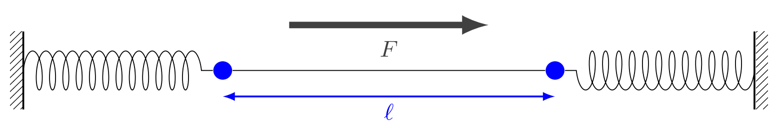

Figure 3 A particle along a straight line restrained by two linear springs with equal spring constants

strained by two linear springs with equal spring constants (stiffness coefficient) under an external force and fractional damping along the t-axis (Fig. 3).

The springs, unless subjected to a force, are assumed to have free length (unstretched length) and resist a change in length. The motion of the system along the t-axis is independent of the initial spring tension. The springs are anchored on the t-axis at $t = -1$ and $t = 1$, and the vibration of the particle in this example is restricted to the t-axis only.

The vibration of the system is represented by a system of equations with the first equation having a similar form to simple harmonic oscillator, which cannot produce instability. Hence, the existence solution of the system depends on the following equation represented as $\mathbb{MFDE}$s with H-$\mathbb{KFD}$

$$\begin{aligned} &{}^{\rho}\mathcal{D}^{0.75,0.4}[y](t) + \frac{1}{8}\big[2 - 2\ell - \theta^2\ell - \theta^2\ell\cos t\big]y(t) \\ &\quad = \nu_1{}^{\rho}\mathcal{D}^{0.75,0.4}[y](t) - \nu_2\sin\big(y(t)\big), \end{aligned} \tag{33}$$

for $t \in I_0 = [0,2]$, here θ, ν_1, ν_2 are constants and ℓ is the unstretched length of the spring, with the initial condition

$${}^{\rho}\mathcal{I}^{0.15}[y](0) = \frac{1}{5}y(0.1) + \frac{1}{5}y(0.4) + \frac{1}{5}y(0.9) + \frac{1}{5}y(1.3) + \frac{1}{5}y(1.6) + \frac{1}{5}y(1.9), \tag{34}$$

where

$$p(t) = \frac{1}{8}\big(2 - 2\ell - \theta^2\ell - \theta^2\ell\cos t\big),$$

for $t \in (0,2]$. Consider particular values of the parameters $\ell = 1.5$ m and $\theta = 0.5$. It is clear that $\rho = 3 > 0$, $\alpha = 0.75 \in (0,1)$, $\beta = 0.4 \in [0,1]$,

$$\gamma = \alpha + \beta(1-\alpha) = 0.85 \in [0,1),$$
$$a = \frac{1}{8}\left(2 - 2\ell - \theta^2\ell\right) = -0.171875, \qquad b = \frac{1}{16}\theta^2\ell = 0.0234375,$$

and

$$q_1 = \frac{32}{5}, \qquad q_2 = \frac{26}{5}, \qquad q_3 = \frac{25}{6}, \qquad q_4 = \frac{24}{7}, \qquad q_5 = \frac{22}{9}, \qquad q_6 = \frac{19}{10},$$
$$e_1 = 0.1, \qquad e_2 = 0.4, \qquad e_3 = 0.9, \qquad e_4 = 1.3, \qquad e_5 = 1.6, \qquad e_6 = 1.9.$$

The general integral solution of (33) is the fractional integral equation

$$\begin{aligned} y(t) = {} & \frac{\Delta}{\Gamma(0.75)}\left(\frac{t^\rho}{\rho}\right)^{-0.15}\Bigg[\sum_{i=1}^{6}\omega_i\int_0^{e_i}\left(\frac{e_i^\rho - \xi^\rho}{\rho}\right)^{-0.25}\xi^{\rho-1} \\ & \times \left[w\left(\xi, y(\xi), {}^\rho\mathcal{D}^{0.75,0.4}[y](\xi)\right) - p(\xi)y(\xi)\right]\mathrm{d}\xi\Bigg] \\ & + \frac{1}{\Gamma(0.75)}\int_0^t\left(\frac{t^\rho - \xi^\rho}{\rho}\right)^{-0.25}\xi^{\rho-1} \\ & \times \left[w\left(\xi, y(\xi), {}^\rho\mathcal{D}^{\alpha,\beta}y(\xi)\right) - p(\xi)y(\xi)\right]\mathrm{d}\xi, \end{aligned} \tag{35}$$

where

$$\Delta = \frac{\rho^{-0.15}}{\Gamma(0.85)\rho^{-0.15} - \sum_{i=1}^{7} q_i(e_i^\rho)^{-0.15}}, \tag{36}$$

e_i, for $i = 0, 1, \ldots, 6$ are prefixed points satisfying

$$0 < e_1 \le e_2 \le e_3 \le e_4 \le e_5 \le e_6 < 2,$$

and q_i are real numbers. We define the map $w : [0,2] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$w(t, y, z) = \frac{\nu_1}{35(\nu_1 + \nu_2)}y - \frac{\nu_2}{15(\nu_1 + \nu_2)}\sin z,$$

for $y, z \in \mathbb{R}$, where nu_1 and nu_1 are positive constants. Then, by Definition 1, we have

$$\begin{aligned} & C_{0.85,3}\left([0,2]\right) \\ & \quad = \left\{y : \left(\frac{1}{8}\left(2 - 2\ell - \theta^2\ell\right), \frac{1}{16}\theta^2\ell\right] \to \mathbb{R} : \right. \\ & \qquad \left.\left(\frac{t^3 - (\frac{1}{8}(2 - 2\ell - \theta^2\ell))^3}{3}\right)^{0.85} y(t) \in C\left([0,2]\right)\right\} \\ & \quad = \left\{y : (-0.17187, 0.02343] \to \mathbb{R} : \left(\frac{t^3 - (-0.171875)^3}{3}\right)^{0.85} y(t) \in C\left([0,2]\right)\right\}, \end{aligned}$$

and so $w \in C_{0.85,3}([0,2])$ for all $y \in C_{0.85,3}([0,2])$. Also,

$$\begin{aligned}\left|w(t,y,z) - w(t,\hat{y},\hat{z})\right| &= \left|\frac{\nu_1}{35(\nu_1+\nu_2)}y - \frac{\nu_2}{15(\nu_1+\nu_2)}\sin z\right.\\ &\quad \left. - \frac{\nu_1}{35(\nu_1+\nu_2)}\hat{y} - \frac{\nu_2}{15(\nu_1+\nu_2)}\sin \hat{z}\right|\\ &\leq \left|\frac{\nu_1}{35(\nu_1+\nu_2)}\right||y-\hat{y}| + \left|\frac{\nu_2}{15(\nu_1+\nu_2)}\right||z-\hat{z}|,\end{aligned}$$

for $t \in [0,2]$. Put $A = \frac{\nu_1}{35(\nu_1+\nu_2)}$ and $B = \frac{\nu_2}{15(\nu_1+\nu_2)}$. Therefore, w satisfies the conditions (A3) of Theorem 19. Furthermore,

$$\begin{aligned}C &= \frac{A + |a+2b|}{1-B}\\ &= \left[\frac{\nu_1}{35(\nu_1+\nu_2)} + |-0.17187 + 2\times 0.02343|\right]\left[\frac{15\nu_1 + 14\nu_2}{15(\nu_1+\nu_2)}\right]^{-1}\\ &= \left[\frac{\nu_1}{35(\nu_1+\nu_2)} + 0.125\right]\left[\frac{15\nu_1 + 14\nu_2}{15(\nu_1+\nu_2)}\right]^{-1}\\ &= \left[\frac{5.375\nu_1 + 4.375\nu_2}{35(\nu_1+\nu_2)}\right]\left[\frac{15(\nu_1+\nu_2)}{15\nu_1 + 14\nu_2}\right]\\ &= \frac{16.125\nu_1 + 13.125\nu_2}{105\nu_1 + 98\nu_2}.\end{aligned}$$

We consider particular values of the parameters $\nu_1 = 7.25$ and $\nu_2 = 0.3$. On the other hand, by employing Eq. (16), we obtain

$$\begin{aligned}\Delta &= \rho^{\gamma-1}\left[\Gamma(\gamma)\rho^{\gamma-1} - \sum_{i=1}^{m} q_i\left(e_i^{\rho}\right)^{\gamma-1}\right]^{-1}\\ &= 3^{-0.15}\left[\Gamma(0.85)3^{-0.15} - \sum_{i=1}^{6} q_i\left(e_i^{3}\right)^{-0.15}\right]^{-1}.\end{aligned}$$

and by using Eq. (28), we obtain

$$\begin{aligned}\Sigma &= C\frac{\beta(\gamma,\alpha)}{\Gamma(\alpha)}\left[\Delta\sum_{i=1}^{m} q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{\alpha+\gamma-1} + \left(\frac{T^{\rho}}{\rho}\right)^{\alpha}\right]0\\ &= C\frac{\beta(0.85,0.75)}{\Gamma(0.75)}\left[\Delta\sum_{i=1}^{6} q_i\left(\frac{e_i^{\rho}}{\rho}\right)^{0.75+0.85-1} + \left(\frac{T^{3}}{3}\right)^{0.75}\right].\end{aligned}$$

Table 3 shows the numerical results of Σ for $\ell = 1, 1.25, 1.5$, and 1.75. These results are shown graphically in Fig. 4. Therefore, all conditions of Theorem 19 hold. Thus, the $\mathbb{MFDE}$ with H-$\mathbb{KFD}$ (33) has a solution.

6 Conclusion

Over the last several years, the study of $\mathbb{FME}$ has drawn increasing attention due to its applications in various fields of the physical sciences, in applied mathematics, and in many

Table 3 Numerical results of Σ for $\ell = 1, 1.25, 1.5$, and 1.75, and $t \in [0, 2]$

t	Σ			
	$\ell = 1.0$	$\ell = 1.25$	$\ell = 1.5$	$\ell = 1.75$
0.00000	−9.1500E−03	−2.9990E−02	−5.0830E−02	−7.1660E−02
0.06667	−9.1100E−03	−2.9880E−02	−5.0640E−02	−7.1400E−02
0.13333	−8.9900E−03	−2.9460E−02	−4.9930E−02	−7.0400E−02
0.20000	−8.7500E−03	−2.8670E−02	−4.8590E−02	−6.8520E−02
0.26667	−8.3800E−03	−2.7470E−02	−4.6560E−02	−6.5650E−02
0.33333	−7.8800E−03	−2.5830E−02	−4.3780E−02	−6.1730E−02
0.40000	−7.2400E−03	−2.3720E−02	−4.0200E−02	−5.6690E−02
0.46667	−6.4400E−03	−2.1120E−02	−3.5800E−02	−5.0480E−02
0.53333	−5.5000E−03	−1.8010E−02	−3.0530E−02	−4.3050E−02
0.60000	−4.3900E−03	−1.4380E−02	−2.4370E−02	−3.4370E−02
0.66667	−3.1100E−03	−1.0210E−02	−1.7300E−02	−2.4390E−02
0.73333	−1.6700E−03	−5.4800E−03	−9.2800E−03	−1.3090E−02
0.80000	−5.0000E−05	−1.7000E−04	−3.0000E−04	−4.2000E−04
0.86667	1.7400E−03	5.7100E−03	9.6800E−03	1.3640E−02
0.93333	3.7200E−03	1.2190E−02	2.0650E−02	2.9120E−02
1.00000	5.8800E−03	1.9270E−02	3.2660E−02	4.6050E−02
1.06667	8.2300E−03	2.6960E−02	4.5700E−02	6.4440E−02
1.13333	1.0760E−02	3.5290E−02	5.9810E−02	8.4330E−02
1.20000	1.3500E−02	4.4250E−02	7.4990E−02	1.0574E−01
1.26667	1.6430E−02	5.3850E−02	9.1270E−02	1.2869E−01
1.33333	1.9560E−02	6.4100E−02	1.0865E−01	1.5320E−01
1.40000	2.2890E−02	7.5020E−02	1.2716E−01	1.7929E−01
1.46667	2.6420E−02	8.6610E−02	1.4680E−01	2.0699E−01
1.53333	3.0160E−02	9.8870E−02	1.6758E−01	2.3630E−01
1.60000	3.4110E−02	1.1182E−01	1.8953E−01	2.6724E−01
1.66667	3.8270E−02	1.2547E−01	2.1266E−01	2.9985E−01
1.73333	4.2650E−02	1.3981E−01	2.3696E−01	3.3412E−01
1.80000	4.7240E−02	1.5486E−01	2.6247E−01	3.7008E−01
1.86667	5.2050E−02	1.7062E−01	2.8918E−01	4.0775E−01
1.93333	5.7080E−02	1.8710E−01	3.1712E−01	4.4714E−01

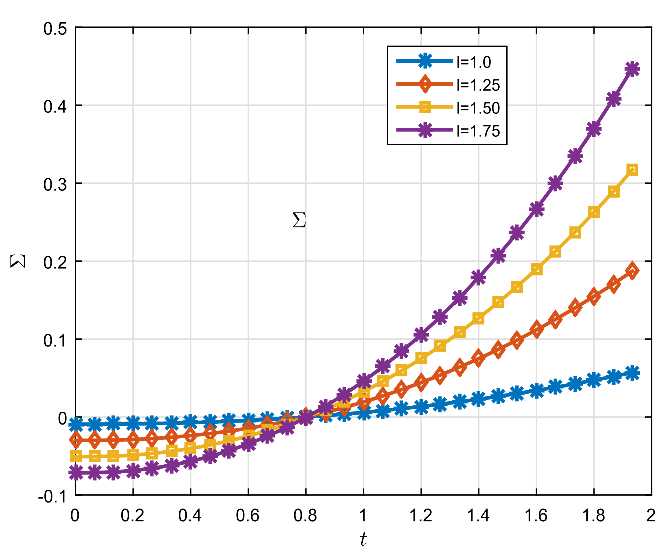

Figure 4 Graphical representation of Σ for $t \in [0, 2]$ and $\ell = 1.0, 1.25, 1.5$, and 1.75

engineering fields. To the best of the authors' knowledge, there are no paper studies on $\mathbb{ME}$ with H-$\mathbb{KFD}$. Motivated by the importance of these equations, we investigated the existence and uniqueness of solutions for $\mathbb{MFDE}$s associated to H-$\mathbb{KFD}$s. The Schauder fixed-point theorem was the key of our analysis to establish the existence of solutions. However, by adding un extra condition, we succeeded in obtaining a unique solution by using the Banach fixed-point theorem. Finally, we present two examples with application to validate our main theoretical results. We were able to produce a computational technique for checking our problem and two algorithms for numerical approximation of solutions with excellent accuracy.

Appendix: Supporting information

Algorithm 1 MATLAB code for Example 1

```
clear;
format long;
alpha = 1/8;
betavar = 5/6;
gammavar= alpha + betavar * ( 1 - alpha);
rho = 12;
q_1= 1/16; q_2= 23/18; q_3= 16/15; q_4= 3/19;
e_1= pi/18; e_2 = 7*pi/18; e_3 =13*pi/18;  e_4 =17*pi/18;
a=115/83; b=sqrt(329)/15;
Result(1,1)= rho^(gammavar-1)/...
     (gamma(gammavar) * rho^(gammavar-1)- ...
     ( q_1 * (e_1^(rho))^(gammavar-1) ...
     + q_2 * (e_2^(rho))^(gammavar-1) ...
     + q_3 * (e_3^(rho))^(gammavar-1) ...
     + q_4 * (e_4^(rho))^(gammavar-1) ));
Result(1,2)= 0.5/(alpha * rho* gamma(alpha)) *(Result(1,1) ...
     * ( q_1 * ( e_1^(rho)/rho)^(alpha) ...
     + q_2 * ( e_2^(rho)/rho)^(alpha) ...
     + q_3 * ( e_3^(rho)/rho)^(alpha) ...
     + q_4 * ( e_4^(rho)/rho)^(alpha) ) ...
     + (pi^(rho)/rho)^(alpha + 1 - gammavar) );
Result(1,3)= abs(a+2*b) *beta(gammavar, alpha)/gamma(alpha) ...
     * (Result(1,1) ...
     * ( q_1 * ( e_1^(rho)/rho)^(alpha) ...
     + q_2 * ( e_2^(rho)/rho)^(alpha) ...
     + q_3 * ( e_3^(rho)/rho)^(alpha) ...
     + q_4 * ( e_4^(rho)/rho)^(alpha) ) ...
     + (pi^(rho)/rho)^(alpha + 1 - gammavar) );
Result(1,4)= Result(1,2)/(1-Result(1,3));
```

Algorithm 2 MATLAB code for Example 2

```
clear;
format long;
alpha = 1/8;
betavar = 5/6;
gammavar= alpha + betavar * ( 1 - alpha);
rho = 1.4;
q_1= 13/4; q_2= 10/7; q_3= 9/10;
e_1= pi/11; e_2 = 3*pi/8; e_3 =6*pi/7;
a=75^(1/3); b=15/14;
Result(1,1)= rho^(gammavar-1)/...
    (gamma(gammavar) * rho^(gammavar-1)- ...
    ( q_1 * (e_1^(rho))^(gammavar-1) ...
    + q_2 * (e_2^(rho))^(gammavar-1) ...
    + q_3 * (e_3^(rho))^(gammavar-1) ...
     ));
A=1/85;
B=1/170;
C= (A+abs(a+2*b))/(1-B);
Result(1,2)= C*  beta(gammavar, alpha)/gamma(alpha)...
    * (Result(1,1) ...
    * ( q_1 * ( e_1^(rho)/rho)^(alpha+gammavar-1) ...
    + q_2 * ( e_2^(rho)/rho)^(alpha+gammavar-1) ...
    + q_3 * ( e_3^(rho)/rho)^(alpha+gammavar-1) ...
    ) + (pi^(rho)/rho)^(alpha) );
```

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Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AB: Actualization, methodology, formal analysis, validation, investigation, and initial draft. *NT*: Actualization, methodology, formal analysis, validation, investigation, initial draft, and supervision of the original draft and editing. *JA*: Actualization, validation, methodology, formal analysis, investigation, and initial draft. *MES*: Actualization, methodology, software, simulation, initial draft, and was a major contributor in writing the manuscript. The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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