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An integral equation representation for American better-of option on two underlying assets

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Abstract

In this paper, we study the problem for pricing of American better-of option on two assets. Due to two correlated underlying assets and early-exercise feature which requires two free boundaries to be determined for the option price, this problem is a complex. We propose a new and efficient approach to solve this problem. Mellin transform methods are mainly used to find the pricing formula, and explicit formula for the option price is derived as an integral equation representation. The formula has two free boundaries which are represented by the coupled integral equations. We propose the numerical scheme based on recursive integration method to implement the integral equations and show that our approach with the proposed numerical scheme is accurate and efficient in computing the prices. In addition, we illustrate significant movements on the option prices and two free boundaries with respect to the selected parameters.

Keywords: American better-of option; Mellin transforms; Integral equation; Double exercise regions

1 Introduction

The problem of option pricing has received a lot of attention because the option is one of the most popular derivatives in the financial market. Black and Scholes [1] first solved the option pricing problem when the underlying asset follows a geometric Brownian motion and provided the closed-form solutions for European option prices. Since the Black–Scholes model was proposed, various option pricing problems have arisen with the development of the financial market. Among the option pricing problems, American option problem has been widely studied by many researchers over past three decades. The main reason is the feature that American option can be exercised at any time before maturity unlike the European options which can be exercised only at maturity. Because of this feature, it is well known that there does not exist the closed-form pricing formulas for the American options. To provide the prices of American options without closed-form pricing formulas, several numerical approaches and analytical pricing formulas have been proposed. For the American option valuation, various numerical methods such as lattice methods [2–4], finite difference methods (FDM) [5, 6], analytical approximation methods [7, 8], Monte Carlo (MC) simulation methods [9, 10], integral representation methods [11, 12], and hybrid methods [13–15] have been developed. These methods have often been used to price American options. In this paper, we consider a type of American option and study the valuation of option on multi-assets.

The options which have multi-assets have been popular with investors in the market because the multi-asset options are useful for hedging or diversification in practice [16]. In fact, there exist various kinds of multi-asset options exchange options [17–19], spread options [20, 21], quanto options [22, 23], basket options [24, 25], rainbow options [26, 27], etc. Among the multi-asset options, we focus on better-of option which is one of rainbow options. The better-of option, which is called "option on the maximum of two risky assets", was first introduced by Stulz [28]. Stulz provided a closed-form pricing formula of European better-of option under the Black–Scholes model [28]. However, there has been no closed-form pricing formula of American better-of option because of the features of American style option. Recently, Gao et al. [29] studied the pricing of an American better-of option using the numerical method. They proposed a primal-dual active-set (PDAS) to solve numerically the discrete linear complementarity problem arising from the pricing of American better-of option. We also deal with the valuation of American better-of option in this paper. Specifically, we derive the analytical pricing formula of American better-of option as an integral equation based on the partial differential equation (PDE) approach.

The main contribution of this paper is to present a new approach for pricing American better-of option. To the best of our knowledge, there is no explicit pricing formula for American better-of option. To solve the PDE for American better-of option price, we adopt Mellin transforms as the main approach. Mellin transform approaches have been employed widely for PDEs in option pricing. Using the properties of Mellin transforms, the PDEs for some options can be replaced by the simple ordinary differential equation (ODE). The applications of the Mellin transforms on the option pricing were first considered by Panini and Srivastav [30], and they provided the solutions for prices of European options and American options. After this pioneer work, various types of options including the standard options have been studied based on the Mellin transform approaches. For instance, American lookback options [31], barrier options [32–34], Russian options [35], basket options [36], vulnerable options [37–40], etc. In line with this research, we propose an efficient approach using the properties of Mellin transforms to obtain a pricing formula for American better-of option and provide the explicit solution as the integral equation.

This paper is organized as follows. In Sect. 2, we formulate the pricing problem for the American better-of option on two correlated assets. In Sect. 3, we study the valuation of the option based on the PDE approach and analyze two free boundaries of American better-of option. Using the Mellin transforms, we provide the explicit pricing formula of American better-of option as an integral representation. In Sect. 4, we propose the numerical scheme for the implementation of the integral equation for the option price and show some numerical results to show the accuracy and efficiency of our approach and the properties of free boundaries and option prices with respect to some parameters. In Sect. 5, we present concluding remarks as well as direction for future work.

2 Model formulation

Under the risk-neutral measure $\mathbb{P}$, we assume that the dynamics of the correlated underlying assets S_1 and S_2 are given by

$$\begin{aligned} dS_{1,t} &= S_{1,t}\big((r-q_1)\,dt + \sigma_1\,dB_{1,t}\big), \\ dS_{2,t} &= S_{2,t}\big((r-q_2)\,dt + \sigma_2\,dB_{2,t}\big), \end{aligned} \tag{1}$$

where $r > 0$ is the constant risk-free interest rate, $q_i > 0$ and $\sigma_i > 0$ ($i = 1, 2$) are dividend rate and volatility of ith underlying asset S_i, respectively. B_1 and B_2 are the standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the natural filtration generated by $(B_{1,t})_{t=0}^{T}$ and $(B_{2,t})_{t=0}^{T}$. We assume

$$d\langle B_1, B_2\rangle_t = \rho\,dt \qquad -1 < \rho < 1.$$

We now consider an *American better-of option on two assets* with a given maturity of the option $T > 0$. In absence of arbitrage opportunities, the price $V(t, s_1, s_2)$ of American better-of option is expressed as the following *optimal stopping problem*:

$$V(t,s_1,s_2) = \sup_{\tau\in\mathcal{U}_{t,T}} \mathbb{E}^{\mathbb{P}}\big[e^{-r(\tau-r)}\max\{S_{1,\tau}, S_{2,\tau}\}|S_{1,t} = s_1,\ S_{2,t} = s_2\big], \tag{2}$$

where $\mathcal{U}_{t,T}$ is the set of all $\mathcal{F}$-stopping times taking values in $[t, T]$.

By a standard approach for the optimal stopping problem (see Peskir and Shiryaev [41].), $V(t, s_1, s_2)$ satisfies the following two-dimensional parabolic variational inequality:

$$\begin{cases} \partial_t V + \mathcal{L}_2 V \le 0 & \text{for } V(t,s_1,s_2) = \max\{s_1,s_2\} \text{ and } (t,s_1,s_2) \in \mathcal{D}_T^2, \\ \partial_t V + \mathcal{L}_2 V = 0 & \text{for } V(t,s_1,s_2) > \max\{s_1,s_2\} \text{ and } (t,s_1,s_2) \in \mathcal{D}_T^2, \\ V(T,s_1,s_2) = \max\{s_1,s_2\} & \text{for } 0 < s_1, s_2 < \infty, \end{cases} \tag{3}$$

where the domain $\mathcal{D}_T^2$ and the operator $\mathcal{L}_2$ are given by

$$\mathcal{D}_T^2 = \big\{(t,s_1,s_2)|0 \le t < T, 0 < s_1, s_2 < \infty\big\}$$

and

$$\mathcal{L}_2 = \frac{\sigma_1^2}{2}s_1^2\frac{\partial}{\partial s_1^2} + \frac{\sigma_2^2}{2}s_2^2\frac{\partial}{\partial s_2^2} + \rho\sigma_1\sigma_2 s_1 s_2\frac{\partial^2}{\partial s_1\partial s_2} + (r-q_1)s_1\frac{\partial}{\partial s_1} + (r-q_2)s_2\frac{\partial}{\partial s_2} - r.$$

Let us consider the following transformation:

$$\mathcal{P}(t,z) = \frac{V(t,s_1,s_2)}{s_2} \quad \text{with } z = \frac{s_1}{s_2}. \tag{4}$$

In terms of the value function $\mathcal{P}(t,z)$, $\mathcal{P}(t,z)$ satisfies

$$\begin{cases} \partial_t \mathcal{P} + \mathcal{L}_1 \mathcal{P} \le 0 & \text{for } \mathcal{P}(t,z) = \max\{z,1\} \text{ and } (t,z) \in \mathcal{D}_T^1, \\ \partial_t \mathcal{P} + \mathcal{L}_1 \mathcal{P} = 0 & \text{for } \mathcal{P}(t,z) > \max\{z,1\} \text{ and } (t,z) \in \mathcal{D}_T^1, \\ \mathcal{P}(T,z) = \max\{z,1\} & \text{for } 0 < z < \infty, \end{cases} \tag{5}$$

where the domain $\mathcal{D}_T^1$ and the operator $\mathcal{L}_1$ are given by

$$\mathcal{D}_T^1 = \{(t,z)|0 \le t < T, 0 < z < \infty\}$$

and

$$\mathcal{L}_1 = \frac{\sigma_z^2}{2}z^2\frac{\partial}{\partial z^2} + (q_2 - q_1)z\frac{\partial}{\partial z} - q_2 \quad \text{with } \sigma_z^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

Then, we can define the continuation region $\mathbf{CR}_z$ and the exercise region $\mathbf{ER}_z$ as follows:

$$\begin{aligned} \mathbf{CR}_z &= \{(t,z) \in \mathcal{D}_T^1|\mathcal{P}(t,z) > \max\{z,1\}\}, \\ \mathbf{ER}_z &= \{(t,z) \in \mathcal{D}_T^1|\mathcal{P}(t,z) = \max\{z,1\}\}. \end{aligned} \tag{6}$$

According to Theorem 7.2 in [42], there exist two free boundaries $\xi_{\text{low}}(t)$ and $\xi_{\text{up}}(t)$ such that

(a) $\xi_{\text{low}}(t)$ is nondecreasing and $\xi_{\text{up}}(t)$ is nonincreasing in $t \in [0,T)$ with $\xi_{\text{low}}(T) = \xi_{\text{up}}(T) = 1$,

(b) The two regions $\mathbf{CR}_z$ and $\mathbf{ER}_z$ are rewritten as

$$\mathbf{CR}_z = \{(t,z) \in \mathcal{D}_T^1|\xi_{\text{low}}(t) < z < \xi_{\text{up}}(t)\}, \qquad \mathbf{ER}_z = \mathbf{ER}_{\text{up}} \cup \mathbf{ER}_{\text{low}}, \tag{7}$$

where

$$\begin{aligned} \mathbf{ER}_{\text{up}} &= \{(t,z) \in \mathcal{D}_T^1|\mathcal{P}(t,z) = z\} = \{(t,z) \in \mathcal{D}_T^1|\xi_{\text{up}}(t) \le z < \infty\}, \\ \mathbf{ER}_{\text{low}} &= \{(t,z) \in \mathcal{D}_T^1|\mathcal{P}(t,z) = 1\} = \{(t,z) \in \mathcal{D}_T^1|0 < z \le \xi_{\text{up}}(t)\}, \end{aligned}$$

(c) The following *smooth-pasting* conditions are established:

$$\frac{\partial \mathcal{P}}{\partial z}\big(t, \xi_{\text{low}}(t)\big) = 1, \qquad \frac{\partial \mathcal{P}}{\partial z}\big(t, \xi_{\text{up}}(t)\big) = z. \tag{8}$$

(d) The optimal stopping time τ^* solution to (2) is given by

$$\tau^* = \inf\{u \ge t|S_{1,u}/S_{2,u} \ge \xi_{\text{up}}(u) \text{ or } S_{1,u}/S_{2,u} \le \xi_{\text{low}}(u)\}. \tag{9}$$

Thus, we can deduce that $\mathcal{P}(t,z)$ satisfies the following inhomogeneous parabolic partial differential equation (PDE):

$$\begin{cases} \partial_t \mathcal{P} + \mathcal{L}_1 \mathcal{P} = -H(t,z) & \text{for } (t,z) \in \mathcal{D}_T^1, \\ \mathcal{P}(t,z) = G(z) & \text{for } 0 < z < \infty, \end{cases} \tag{10}$$

where $h(t,z)$ and $g(z)$ are given by

$$H(t,z) = q_2 \mathbf{1}_{\{0<z\le\xi_{\text{low}}(t)\}} + q_1 z \mathbf{1}_{\{\xi_{\text{up}}(t)\le z<\infty\}} \quad \text{and} \quad G(z) = \max\{z,1\}.$$

The continuation region, the exercise region, and the free boundaries are illustrated in Fig. 1.

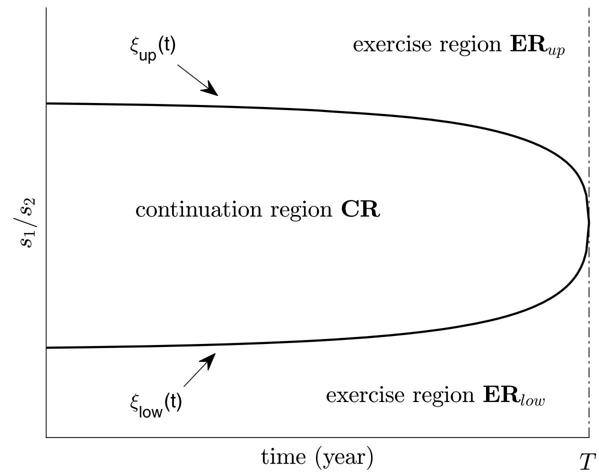

Figure 1 The continuation region **CR**, the exercise region **ER** = $\mathbf{ER}_{up} \cup \mathbf{ER}_{low}$, and the free boundaries $\xi_{up}(t)$, $\xi_{low}(t)$ of $\mathcal{P}(t,z)$

3 Valuation of American better-of option on two assets

In this section, we present the main results of this paper. Specifically, we derive the explicit analytic formulas for the value function $\mathcal{P}(t)$ and two free boundaries $\xi_{\mathrm{up}}(t)$ and $\xi_{\mathrm{low}}(t)$. The main idea to derive the analytic formulas is applying the Mellin transform to the inhomogeneous PDE (10).

Let $\mathcal{P}_M(t,x)$, $H_M(t,x)$, and $G_M(x)$ be the Mellin transforms of $\mathcal{P}(t,z)$, $H(t,z)$, and $G(z)$, respectively, i.e.,

$$\mathcal{P}_M(t,x) = \int_0^\infty \mathcal{P}(t,z)z^{x-1}\,dz, \qquad H_M(t,x) = \int_0^\infty H(t,z)z^{x-1}\,dz, \quad \text{and}$$
$$G_M(x) = \int_0^\infty G(z)z^{x-1}\,dz.$$

By utilizing the Mellin transform to the inhomogeneous PDE (10), we obtain the inhomogeneous ODE as follows:

$$\begin{cases} \frac{d\mathcal{P}_M}{dt} + \frac{\sigma^2}{2}\mathfrak{B}(x)\mathcal{P}_M = H_M(t,x), & \mathcal{P}_M(T,x) = G_M(x), \\ \mathfrak{B}(x) = x^2 + (1-\zeta_2)w - \zeta_1 & \text{with } \zeta_1 = 2(q_2-q_1)/\sigma_z^2, \zeta_2 = 2q_2/\sigma_z^2. \end{cases} \tag{11}$$

Then, we can easily have the solution of inhomogeneous ODE (11) as follows:

$$\mathcal{P}_M(t,z) = e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)(T-t)}G_M(x) - \int_t^T e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)(\eta-t)}H_M(\eta,x)\,d\eta. \tag{12}$$

Applying the inverse Mellin transform to both sides of (12), we obtain

$$\begin{aligned}\mathcal{P}(t,z) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{P}_M(t,x)z^{-x}\,dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)(T-t)}G_M(x) - \int_t^T e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)(\eta-t)}H_M(\eta,x)\,d\eta\right]z^{-x}\,dx\end{aligned}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)(T-t)} G_M(x) z^{-x}\,dx - \int_t^T \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)(\xi-t)} H_M(\eta,x) z^{-x}\,dx\right] d\eta. \tag{13}$$

Let us denote $\mathcal{Q}(t,z)$ by

$$\mathcal{Q}(t,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)t} z^{-x}\,dx. \tag{14}$$

Since $\mathcal{Q}(t,z)$ is the inverse Mellin transform of $\exp\{\frac{\sigma_z^2}{2}\mathfrak{B}(x)t\}$, we deduce that $\exp\{\frac{\sigma_z^2}{2} \times \mathfrak{B}(x)t\}$ is the Mellin transform of $\mathcal{Q}(t,z)$, i.e.,

$$\mathcal{Q}_M(t,x) = \int_0^\infty \mathcal{Q}(t,z) z^{x-1}\,dz = e^{\frac{\sigma_z^2}{2}\mathfrak{B}(x)t}.$$

It follows from the Mellin convolution theorem (see Proposition 3.1 in [35]) that

$$\begin{aligned}
\mathcal{P}(t,z) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_M(x)\mathcal{Q}_M(T-t,x) z^{-x}\,dx \\
&\quad - \int_t^T \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(\eta,x)\mathcal{Q}_M(\eta-t,x)\,dx\right] d\eta \\
&= \int_0^\infty G(\nu)\mathcal{Q}\left(T-t,\frac{z}{\nu}\right)\frac{d\nu}{\nu} - \int_t^T\int_0^\infty H(\eta,\nu)\mathcal{Q}\left(\eta-t,\frac{z}{\nu}\right)\frac{d\nu}{\nu}\,d\eta \\
&= \int_0^\infty \bigl(\max\{\nu,1\}\bigr)\mathcal{Q}\left(T-t,\frac{z}{\nu}\right)\frac{d\nu}{\nu} \\
&\quad + \int_t^T\int_0^\infty \bigl(q_2\mathbf{1}_{\{0<\nu\le\xi_{\mathrm{low}}(\eta)\}} + q_1\nu\mathbf{1}_{\{\xi_{\mathrm{up}}(\eta)\le\nu<\infty\}}\bigr)\mathcal{Q}\left(\eta-t,\frac{z}{\nu}\right)\frac{d\nu}{\nu}\,d\eta.
\end{aligned} \tag{15}$$

Lemma 1 *Let A be an arbitrary real number and B be a positive constant. Then the following equalities hold*:

$$\int_0^B \nu^{-A}\mathcal{Q}\left(t,\frac{z}{\nu}\right)\frac{1}{\nu}\,d\nu = z^{-A}e^{-\frac{1}{2}\{\zeta_1-(1-\zeta_2)A-A^2\}\sigma_z^2 t}\mathcal{N}\left(\frac{-\log\frac{z}{B}+(\frac{1-\zeta_2}{2}+A)\sigma_z^2 t}{\sigma\sqrt{t}}\right),$$
$$\int_B^\infty \nu^{-A}\mathcal{Q}\left(t,\frac{z}{\nu}\right)\frac{1}{\nu}\,d\nu = z^{-A}e^{-\frac{1}{2}\{\zeta_1-(1-\zeta_2)A-A^2\}\sigma_z^2 t}\mathcal{N}\left(\frac{\log\frac{z}{B}-(\frac{1-\zeta_2}{2}+A)\sigma_z^2 t}{\sigma\sqrt{t}}\right),$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function. Proof
First, we consider

$$\begin{aligned}
&\int_0^B \nu^{-A}\mathcal{Q}\left(t,\frac{z}{\nu}\right)\frac{1}{\nu}\,d\nu \\
&= \int_0^B \nu^{-A}e^{-\frac{1}{2}\{(\frac{1-\zeta_2}{2})^2+\zeta_1\}\sigma_z^2 t}\cdot\frac{(\frac{z}{\nu})^{\frac{1-\zeta_2}{2}}}{\sigma_z\sqrt{2\pi t}}e^{-\frac{1}{2}(\frac{\log(z/\nu)}{\sigma_z\sqrt{t}})^2}\frac{1}{\nu}\,d\nu \\
&= -z^{-A}e^{-\frac{1}{2}\{(\frac{1-\zeta_2}{2})^2+\zeta_1\}\sigma_z^2 t}\int_\infty^{\log\frac{z}{B}} e^{Aw}\frac{e^{(\frac{1-\zeta_2}{2})w}}{\sigma_z\sqrt{2\pi t}}e^{-\frac{1}{2}\frac{w^2}{\sigma_z^2 t}}\,dw \quad (w=\log z/\nu)
\end{aligned}$$

$$
\begin{aligned}
&= -z^{-A} e^{-\frac{1}{2}\{(\frac{1-\zeta_2}{2})^2+\zeta_1-(\frac{1-\zeta_2}{2}+A)^2\}\sigma^2 t} \\
&\quad \times \int_{\infty}^{\log \frac{z}{B}} \frac{1}{\sigma_z\sqrt{2\pi t}} \exp\left\{-\frac{1}{2}\left(\frac{w-\sigma^2 t(\frac{1-\zeta_2}{2}+A)}{\sigma_z\sqrt{t}}\right)^2\right\} dw \\
&= z^{-A} e^{-\frac{1}{2}\{\zeta_1-(1-\zeta_2)A-A^2\}\theta^2 t} \mathcal{N}\left(\frac{-\log \frac{z}{B}+\sigma_z^2 t(\frac{1-\zeta_2}{2}+A)}{\sigma_z\sqrt{t}}\right),
\end{aligned}
$$

where the second equality is obtained from the transformation $w = \log(s/u)$. *Similarly, we obtain*

$$
\int_B^{\infty} v^{-A} \mathcal{Q}\left(t, \frac{z}{v}\right) \frac{1}{v}\, dv = z^{-A} e^{-\frac{1}{2}\{\zeta_1-(1-\zeta_2)A-A^2\}\sigma_z^2 t} \mathcal{N}\left(\frac{\log \frac{z}{B} - (\frac{1-\zeta_2}{2}+A)\sigma_z^2 t}{\sigma_z\sqrt{t}}\right).
$$

From (15) and Lemma 1, we have the following proposition.

Proposition 1

$$
\begin{aligned}
\mathcal{P}(t,z) &= z e^{-q_1(T-t)} \mathcal{N}\big(d_1(T-t,z)\big) + e^{-q_2(T-t)} \mathcal{N}\big(-d_2(T-t,z)\big) \\
&\quad + q_1 z \int_t^T e^{-q_1(\eta-t)} \mathcal{N}\left(d_1\left(\eta-t, \frac{z}{\xi_{\mathrm{up}}(\eta)}\right)\right) d\eta \\
&\quad + q_2 \int_t^T e^{-q_2(\eta-t)} \mathcal{N}\left(-d_2\left(\eta-t, \frac{z}{\xi_{\mathrm{low}}(\eta)}\right)\right) d\eta,
\end{aligned}
$$

where

$$
d_1(t,z) = \frac{\log z + (q_2 - q_1 + \frac{1}{2}\sigma_z^2)t}{\sigma_z\sqrt{t}}, \qquad d_2(t,z) = \frac{\log z + (q_2 - q_1 - \frac{1}{2}\sigma_z^2)t}{\sigma_z\sqrt{t}}.
$$

Moreover, the *smooth-pasting* condition (8) allows us to state the next corollary.

Corollary 1 *Two free boundaries* $\xi_{\mathrm{up}}(t)$ *and* $\xi_{\mathrm{low}}(t)$ *satisfy the following coupled integral equations*:

$$
\begin{aligned}
\xi_{\mathrm{up}}(t) &= \xi_{\mathrm{up}}(t) e^{-q_1(T-t)} \mathcal{N}\big(d_1\big(T-t, \xi_{\mathrm{up}}(t)\big)\big) + e^{-q_2(T-t)} \mathcal{N}\big(-d_2\big(T-t, \xi_{\mathrm{up}}(t)\big)\big) \\
&\quad + q_1 \xi_{\mathrm{up}}(t) \int_t^T e^{-q_1(\eta-t)} \mathcal{N}\left(d_1\left(\eta-t, \frac{\xi_{\mathrm{up}}(t)}{\xi_{\mathrm{up}}(\eta)}\right)\right) d\eta \\
&\quad + q_2 \int_t^T e^{-q_2(\eta-t)} \mathcal{N}\left(-d_2\left(\eta-t, \frac{\xi_{\mathrm{up}}(t)}{\xi_{\mathrm{low}}(\eta)}\right)\right) d\eta
\end{aligned}
$$

and

$$
\begin{aligned}
1 &= \xi_{\mathrm{low}}(t) e^{-q_1(T-t)} \mathcal{N}\big(d_1\big(T-t, \xi_{\mathrm{low}}(t)\big)\big) + e^{-q_2(T-t)} \mathcal{N}\big(-d_2\big(T-t, \xi_{\mathrm{low}}(t)\big)\big) \\
&\quad + q_1 \xi_{\mathrm{low}}(t) \int_t^T e^{-q_1(\eta-t)} \mathcal{N}\left(d_1\left(\eta-t, \frac{\xi_{\mathrm{low}}(t)}{\xi_{\mathrm{up}}(\eta)}\right)\right) d\eta \\
&\quad + q_2 \int_t^T e^{-q_2(\eta-t)} \mathcal{N}\left(-d_2\left(\eta-t, \frac{\xi_{\mathrm{low}}(t)}{\xi_{\mathrm{low}}(\eta)}\right)\right) d\eta.
\end{aligned}
$$

From substitution (4), we finally have the integral equation representation for $V(t,s_1,s_2)$, which is the price of American better-of option on two assets, in the following theorem.

Theorem 1 *The price of American better-of option on two assets $V(t,s_1,s_2)$ in (2) is presented as the following formula*:

$$\begin{aligned} V(t,s_1,s_2) &= s_1 e^{-q_1(T-t)}\mathcal{N}\left(d_1\left(T-t,\frac{s_1}{s_2}\right)\right) + s_2 e^{-q_2(T-t)}\mathcal{N}\left(-d_2\left(T-t,\frac{s_1}{s_2}\right)\right) \\ &\quad + q_1 s_1 \int_t^T e^{-q_1(\eta-t)}\mathcal{N}\left(d_1\left(\eta-t,\xi_{\mathrm{up}}(\eta)\frac{s_1}{s_2}\right)\right)d\eta \\ &\quad + q_2 s_2 \int_t^T e^{-q_2(\eta-t)}\mathcal{N}\left(-d_2\left(\eta-t,\xi_{\mathrm{low}}(\eta)\frac{s_1}{s_2}\right)\right)d\eta. \end{aligned}$$

4 Numerical results

Since the explicit analytic formula of $V(t,s_1,s_2)$ in Theorem 1 is expressed by two free boundaries ξ_{low} and ξ_{up}, we need to solve the coupled integral equations of two boundaries in Corollary 1. Although the coupled integral equations are rather complicated, we can solve the equations by using the numerical scheme combined with Chiarella and Ziogas [43] and Huang, Subrahmanyam, and Yu [44]. We briefly summarize our numerical scheme in the next subsection.

4.1 Numerical implementation

We can rewrite the couple integral equations in Corollary 1 as

$$\begin{aligned} 0 &= \Psi_1\big(t,\xi_{\mathrm{up}}(t)\big) + \int_t^T \Phi\big(\eta-t,\eta,\xi_{\mathrm{up}}(t),\xi_{\mathrm{up}}(\eta),\xi_{\mathrm{low}}(\eta)\big)\,d\eta, \\ 0 &= \Psi_2\big(t,\xi_{\mathrm{low}}(t)\big) + \int_t^T \Phi\big(\eta-t,\eta,\xi_{\mathrm{low}}(t),\xi_{\mathrm{up}}(\eta),\xi_{\mathrm{low}}(\eta)\big)\,d\eta, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Psi_1(t,x) &= xe^{-q_1(T-t)}\mathcal{N}\big(d_1(T-t,x)\big) + e^{-q_2(T-t)}\mathcal{N}\big(-d_2(T-t,x)\big) - x, \\ \Psi_2(t,x) &= xe^{-q_1(T-t)}\mathcal{N}\big(d_1(T-t,x)\big) + e^{-q_2(T-t)}\mathcal{N}\big(-d_2(T-t,x)\big) - 1, \\ \Phi\big(t,\eta,x,\xi_{\mathrm{up}}(\eta),\xi_{\mathrm{low}}(\eta)\big) &= xq_1 e^{-q_1(\eta-t)}\mathcal{N}\left(d_1\left(\eta-t,\frac{x}{\xi_{\mathrm{up}}(\eta)}\right)\right) \\ &\quad + q_2 e^{-q_2(\eta-t)}\mathcal{N}\left(-d_2\left(\eta-t,\frac{x}{\xi_{\mathrm{low}}(\eta)}\right)\right). \end{aligned}$$

We now present how to solve numerically the coupled integral equations of ξ_{up} and ξ_{low} in (16). First, we partition the time-interval $[0,T]$ into $(N+1)$ time-steps with end points

$$t_i = (N-i)\Delta t \quad i = 0,1,2,\dots,N \text{ with } \Delta t = \frac{T}{N}.$$

Let us denote ξ_{up}^i and ξ_{low}^i by the numerical approximated value of $\xi_{\mathrm{up}}(t_i)$ and $\xi_{\mathrm{low}}(t_i)$, respectively.

It follows from $\xi_{\mathrm{up}}(T) = \xi_{\mathrm{low}}(T) = 1$ that $\xi_{\mathrm{up}}^0 = \xi_{\mathrm{low}}^0 = 1$.

For $t = t_1$, we can approximate the coupled integral equations (16) by utilizing the trapezoidal rule as follows:

$$\begin{aligned}
0 &= \Psi_1\big(t_1, \xi_{\mathrm{up}}^1\big) + \frac{\Delta t}{2}\big[\Phi\big(t_1, t_0, \xi_{\mathrm{up}}^1, \xi_{\mathrm{up}}^0, \xi_{\mathrm{low}}^0\big) + \Phi\big(t_1, t_1, \xi_{\mathrm{up}}^1, \xi_{\mathrm{up}}^1, \xi_{\mathrm{low}}^1\big)\big], \\
0 &= \Psi_2\big(t_1, \xi_{\mathrm{low}}^1\big) + \frac{\Delta t}{2}\big[\Phi\big(t_1, t_0, \xi_{\mathrm{low}}^1, \xi_{\mathrm{up}}^0, \xi_{\mathrm{low}}^0\big) + \Phi\big(t_1, t_1, \xi_{\mathrm{up}}^1, \xi_{\mathrm{up}}^1, \xi_{\mathrm{low}}^1\big)\big].
\end{aligned} \tag{17}$$

Since $\xi_{\mathrm{up}}(t)$ and $\xi_{\mathrm{low}}(t)$ are decreasing and increasing functions for $t \in [0, T]$, respectively, it follows from $\xi_{\mathrm{up}}(T) = \xi_{\mathrm{low}}(T) = 1$ that

$$\xi_{\mathrm{up}}^i > \xi_{\mathrm{low}}^i \quad \text{for all } i = 0, 1, \dots, N.$$

This implies that

$$\lim_{\eta \to t+} \mathcal{N}\left(-d_2\left(\eta - t, \frac{\xi_{\mathrm{up}}^i}{\xi_{\mathrm{low}}^i}\right)\right) = 0 \quad \text{and} \quad \lim_{\eta \to t+} \mathcal{N}\left(d_1\left(\eta - t, \frac{\xi_{\mathrm{low}}^i}{\xi_{\mathrm{up}}^i}\right)\right) = 0. \tag{18}$$

Hence, we deduce that, for $i = 1, 2, \dots N$,

$$\Phi_1\big(t_i, t_i, \xi_{\mathrm{up}}^i, \xi_{\mathrm{up}}^i, \xi_{\mathrm{low}}^i\big) = \lim_{\eta \to t_i+} \xi_{\mathrm{up}}^i q_1 \mathcal{N}\left(d_1\left(\eta - t_i, \frac{\xi_{\mathrm{up}}^i}{\xi_{\mathrm{up}}^i}\right)\right) = \frac{1}{2} q_1 \xi_{\mathrm{up}}^i \tag{19}$$

and

$$\Phi_2\big(t_i, t_i, \xi_{\mathrm{low}}^i, \xi_{\mathrm{up}}^i, \xi_{\mathrm{low}}^i\big) \lim_{\eta \to t_i+} \Phi_2\big(t_i, \eta - t_i, \xi_{\mathrm{low}}^i, \xi_{\mathrm{up}}^i, \xi_{\mathrm{low}}^i\big) = \frac{1}{2} q_2. \tag{20}$$

Thus, we can rewrite the coupled equations in (17) as

$$0 = \Psi_1\big(t_1, \xi_{\mathrm{up}}^1\big) + \frac{\Delta t}{2}\left[\frac{1}{2} q_1 \xi_{\mathrm{up}}^1 + \Phi_1\big(t_1, t_0, \xi_{\mathrm{up}}^1, \xi_{\mathrm{up}}^0, \xi_{\mathrm{low}}^0\big)\right], \tag{21}$$

$$0 = \Psi_2\big(\tau_1, \xi_{\mathrm{low}}^1\big) + \frac{\Delta t}{2}\left[\frac{1}{2} q_2 + \Phi_2\big(t_1, t_0, \xi_{\mathrm{low}}^1, \xi_{\mathrm{up}}^0, \xi_{\mathrm{low}}^0\big)\right]. \tag{22}$$

Note that the only unknowns in (21) and (22) are ξ_{up}^1 and ξ_{low}^1, respectively. By applying the bisection method to (21) and (22), we can find ξ_{up}^1 and ξ_{low}^1.

Recursively, we find ξ_{up}^i and ξ_{low}^i for $i = 2, 3, \dots, N$ by solving the following coupled equations:

$$\begin{aligned}
0 &= \Psi_1\big(t_i, \xi_{\mathrm{up}}^i\big) \\
&\quad + \frac{\Delta \tau}{2}\left[\frac{1}{2} q_1 \xi_{\mathrm{up}}^i + 2 \sum_{j=1}^{i-1} \Phi_1\big(t_i, t_{i-j}, \xi_{\mathrm{up}}^i, \xi_{\mathrm{up}}^j, \xi_{\mathrm{low}}^j\big) + \Phi_1\big(t_i, t_0, \xi_{\mathrm{up}}^i, \xi_{\mathrm{up}}^0, \xi_{\mathrm{low}}^0\big)\right], \\
0 &= \Psi_2\big(t_i, \xi_{\mathrm{low}}^i\big) \\
&\quad + \frac{\Delta \tau}{2}\left[\frac{1}{2} q_2 + 2 \sum_{j=1}^{i-1} \Phi_2\big(t_i, \tau_{i-j}, \xi_{\mathrm{low}}^i, \xi_{\mathrm{up}}^j, \xi_{\mathrm{low}}^j\big) + \Phi_2\big(t_i, t_0, \xi_{\mathrm{low}}^i, \xi_{\mathrm{up}}^0, \xi_{\mathrm{low}}^0\big)\right].
\end{aligned} \tag{23}$$

Using the values $\{\xi_{\mathrm{up}}^i\}_{i=0}^N$ and $\{\xi_{\mathrm{low}}^i\}_{i=0}^N$, we can approximate the value function $\mathcal{P}(t,z)$ as

$$\mathcal{P}_n(0,z) \equiv \Psi(0,z) + \frac{\Delta t}{2}\Bigg[\Phi\big(0,T,z,\xi_{\mathrm{up}}^N,\xi_{\mathrm{low}}^N\big) + 2\sum_{j=1}^{N-1}\Phi\big(t_j,t_{n-j},z,\xi_{\mathrm{up}}^j,\xi_{\mathrm{low}}^j\big) + \Phi\big(0,T,z,\xi_{\mathrm{up}}^0,\xi_{\mathrm{low}}^0\big)\Bigg], \tag{24}$$

where

$$\Psi(t,z) = ze^{-q_1(T-t)}\mathcal{N}\big(d_1(T-t,z)\big) + e^{-q_2(T-t)}\mathcal{N}\big(-d_2(T-t,z)\big).$$

For a sufficiently large number of sub-intervals N, ξ_{up}^N, ξ_{low}^N, and $\mathcal{P}_n(t,z)$ converge to $\xi_{\mathrm{up}}(0)$, $\xi_{\mathrm{low}}(0)$, and $\mathcal{P}(t,z)$, respectively (see Huang, Subrahmanyam, and Yu [44]). To accelerate the convergence speed, we can apply a three-point Richardson extrapolation scheme developed by Geske and Johnson [8] as follows:

$$\mathcal{P}(t,z) \approx \frac{9\mathcal{P}_3 - 8\mathcal{P}_2 + \mathcal{P}_1}{2}.$$

4.2 Numerical experiments

In this subsection, we present the results of numerical experiments. Specifically, using the numerical scheme proposed in Sect. 4.1 and the formula in Theorem 1, we demonstrate the accuracy and efficiency of our approach and examine the significant movements of the boundaries and prices with respect to some parameters. For the experiments, the baseline parameters are based on the works of [29, 42].

In Table 1, we present a comparison between our explicit pricing formula and the binomial tree method (BTM) [42]. The values obtained by BTM with 20,000 time steps are considered as the benchmark values, and 'R-err' in Table 1 denotes a relative error defined by

$$\text{R-err} := \left|\frac{\text{'Our approach' - 'BTM '}}{\text{'Pricing formula'}}\right|.$$

Comparing the values obtained by our formula with the values obtained by the BTM, we can find that 'R-err' is very small in Table 1. Additionally, to calculate each option price,

Table 1 Price of American better-of option. All experiments are conducted using the MATLAB on a PC with Intel(R) Core(TM) i7 3.4 GHz CPU. The parameter values used in the experiments are $T = 1$, $r = 0.04$, $q_1 = 0.02$, $\sigma_1 = 0.3$, $\rho = 0.3$, $s_1 = s_2 = 100$

σ_2	q_2	BTM (20000)	Time (s)	Our approach	Time (s)	R-err
0.2	0.02	101.969	37.876	102.008	0.079	3.76×10^{-4}
	0.03	101.728	38.066	101.832	0.082	1.09×10^{-3}
	0.04	101.522	37.815	101.738	0.083	2.13×10^{-3}
0.3	0.02	103.108	37.925	103.098	0.081	9.64×10^{-5}
	0.03	102.805	38.123	102.822	0.074	1.71×10^{-4}
	0.04	102.546	38.075	102.616	0.095	6.87×10^{-4}
0.4	0.02	105.041	37.919	105.010	0.081	2.87×10^{-4}
	0.03	104.689	38.034	104.659	0.079	2.88×10^{-4}
	0.04	104.374	37.926	104.358	0.092	1.53×10^{-4}

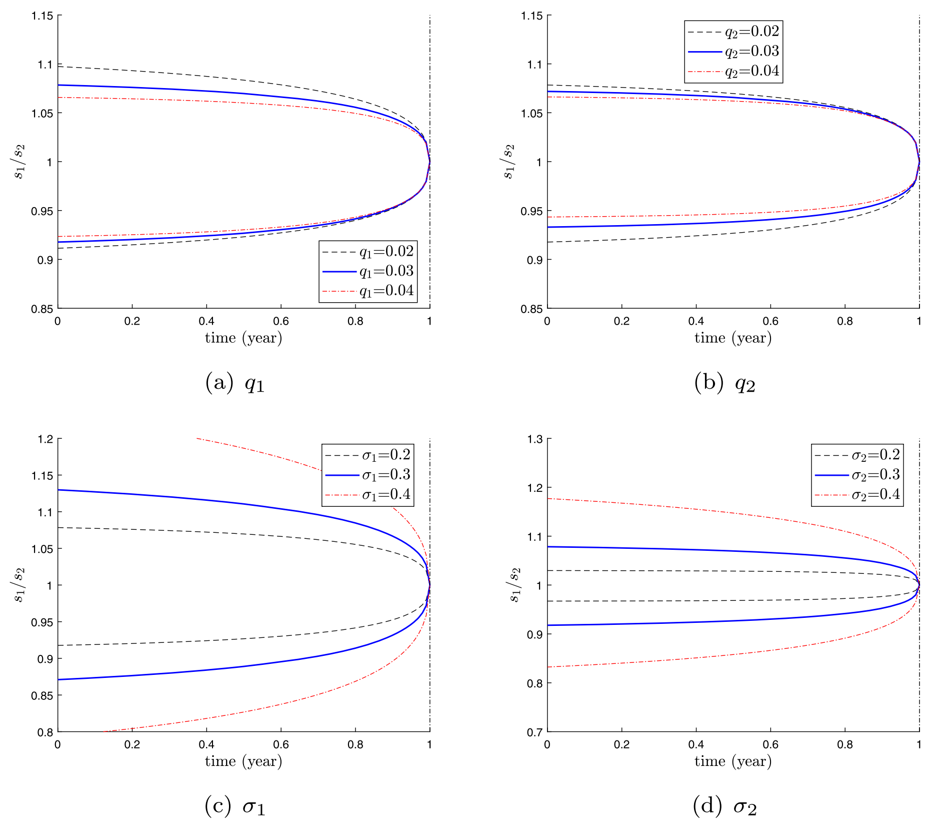

Figure 2 Two free boundaries $\xi_{up}(t)$ and $\xi_{low}(t)$ varying with parameters. The parameter values used in the experiments are $T = 1$, $r = 0.04$, $q_1 = 0.02$, $q_2 = 0.03$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $\rho = 0.3$, $s_1 = s_2 = 100$

our approach takes less than 0.01 seconds. On the other hand, the BTM approach takes more than 37 seconds. That is, we conclude that the approach based on our explicit pricing formula is accurate and efficient.

Figure 2 illustrates the behavior of two free boundaries (optimal stopping boundaries) of American better-of option with respect to two dividends (q_1, q_2) and two volatilities of underlying assets (σ_1, σ_2). Figure 2(a) and Fig. 2(b) show that the areas of continuation region become narrower as q_1 and q_2 increase, respectively. In Fig. 2(a), we find that the upper free boundary is more sensitive to variable q_1 than variable q_2. On the other hand, in Fig. 2(b), we can see that the lower free boundary moves more sensitively with respect to variable q_2. Figure 2(c) and Fig. 2(d) show that the area of stopping region becomes wider when the volatilities of two underlying assets increase. From Fig. 2(c) and Fig. 2(d), we can find that the stopping region is more affected by the volatility σ_1 of underlying asset $S_{1,t}$ than by the volatility σ_2 of underlying asset $S_{2,t}$. We note that the boundaries rarely change as time to maturity $(T - t)$ increase if the volatility σ_2 is very small. Figure 3 illustrates how the prices of option change when the initial value of $S_{1,t}$ increases. As shown in Fig. 3, there exist significant differences between prices near at-the-money. Figure 3(a) and Fig. 3(b) show the effects of dividends on the option price. As expected, we can see that the option with high dividend is cheaper than the option with low dividend. Figure 3(c) and Fig. 3(d) present the movements of the option prices for different volatilities. We observe that the

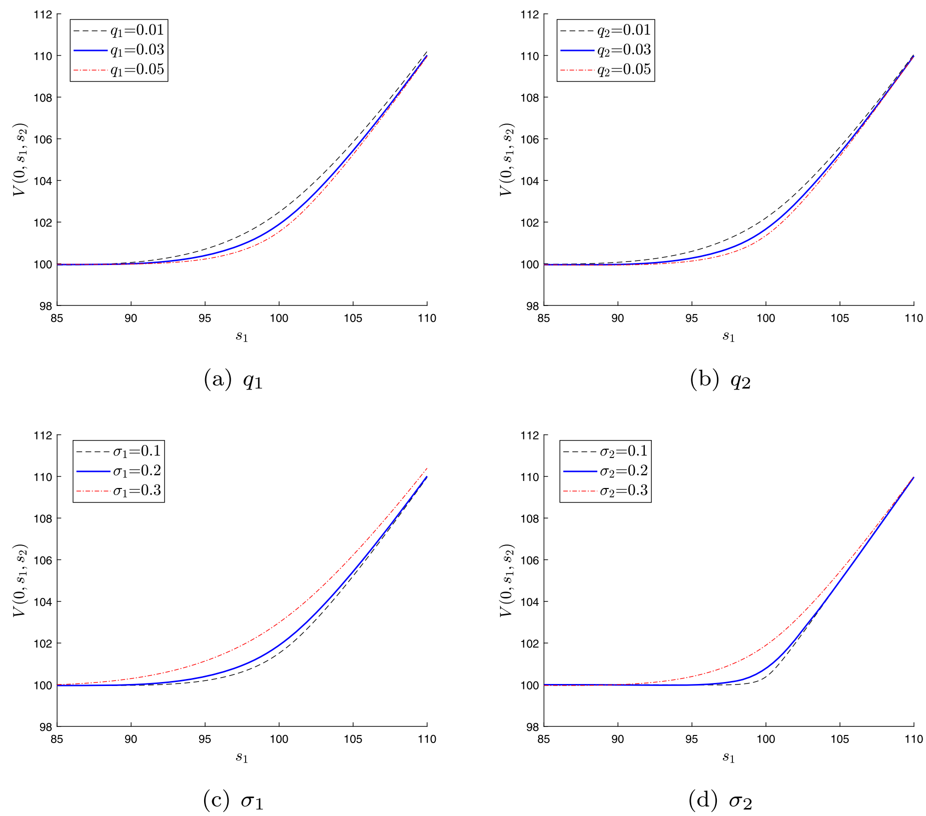

Figure 3 Value $V(t, s_1, s_2)$ varying with parameters. The parameter values used in the experiments are $T = 1$, $r = 0.04$, $q_1 = 0.02$, $q_2 = 0.03$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $\rho = 0.3$, $s_1 = s_2 = 100$

option price has a high value as the volatility increases. We also find that the option prices are more sensitive to σ_1 than σ_2.

5 Concluding remarks

In this paper, we proposed a new approach for pricing of American better-of option based on the PDE approach. We represented the option pricing problem as a free boundary problem and considered the Mellin transforms to solve the PDE. From these approaches, we derived an explicit pricing formula of American better-of option with two free boundaries, which satisfy the coupled integral equations. Hence, the pricing formula was provided as the integral equation representation.

The derived integral equation involves simple integrals. Thus, the prices and the boundaries for American better-of options can be computed more efficiently. To show the efficiency and accuracy of our approach, we performed some numerical experiments with the binomial tree method for the simulations and compared the values of American better-of options by the formula with the simulation results. The results show that the pricing formula is computationally efficient and accurate. Moreover, we presented several graphs to analyze the behaviors or sensitivities of the prices and free boundaries. From the graphs, we found the significant movements of option prices and free boundaries with respect to the selected parameters.

We also note that our approach can be extended to the valuation of other types of options with two free boundaries such as American strangle options, American Eagle options, British strangle options, etc. These topics will be considered as future works.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JJ and GK designed the model; JJ contributed analysis of the mathematical model; JJ proved the theorems in the paper; GK carried out the numerical experiments; JJ and GK wrote the paper. All authors read and approved the final manuscript.

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