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Half-linear differential equations of fourth order: oscillation criteria of solutions

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Abstract

In this paper, we are concerned with the oscillation of solutions to a class of fourth-order delay differential equations with p-Laplacian like operators $(r(t)|x'''(t)|^{p_1-2}x'''(t))' + q(t)|x(\tau(t))|^{p_2-2}x(\tau(t)) = 0$ and $(r(t)|x'''(t)|^{p_1-2}x'''(t))' + \sigma(t)|x'''(t)|^{p_1-2}x'''(t) + q(t)|x(\tau(t))|^{p_2-2}x(\tau(t)) = 0$. New oscillation criteria are presented by the comparison technique and employing the Riccati transformation. Moreover, our results are an extension and complement to previous results in the literature. Two examples are shown to illustrate the conclusions.

Keywords: Oscillation; Fourth-order; Delay differential equations; p-Laplacian like operators

1 Introduction

In this work, we investigate the oscillation of solutions to a class of fourth-order half-linear differential equations with p-Laplacian like operators

$$\left(r(t)\left|x'''(t)\right|^{p_1-2}x'''(t)\right)' + q(t)\left|x\left(\tau(t)\right)\right|^{p_2-2}x\left(\tau(t)\right) = 0 \tag{1}$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{1/p_1-1}(s)}\,\mathrm{d}s = \infty. \tag{2}$$

Also, we establish new criteria for the oscillatory behavior of fourth-order differential equations with middle term

$$\left(r(t)\left|x'''(t)\right|^{p_1-2}x'''(t)\right)' + \sigma(t)\left|x'''(t)\right|^{p_1-2}x'''(t) + q(t)\left|x\left(\tau(t)\right)\right|^{p_2-2}x\left(\tau(t)\right) = 0 \tag{3}$$

under the condition

$$\int_{t_0}^{\infty} \left[\frac{1}{r(s)} \exp\left(-\int_{t_0}^{s} \frac{\sigma(\eta)}{r(\eta)}\,\mathrm{d}\eta\right)\right]^{1/p_1-1} \mathrm{d}s = \infty. \tag{4}$$

Throughout this paper, we assume that $p_i > 1$, $i = 1, 2$, are real numbers and $j \geq 1$, $r, \sigma, q \in C([t_0, \infty), [0, \infty))$, $r(t) > 0$, $q(t) > 0$, $r'(t) + \sigma(t) \geq 0$, $\tau(t) \in C([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$.

Definition 1.1 A nontrivial solution x of (1) and (3) is termed oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros.

Definition 1.2 Equations (1) and (3) are called oscillatory if all their solutions are oscillatory.

Half-linear delay differential equations arise in a variety of phenomena including mixing liquids, economics problems, biology, medicine, physics, engineering and automatic control problems, as well as vibrational motion in flight, and human self-balancing, see [1–6]. In particular, differential equations with p-Laplacian like operators, as the classical half-linear or Emden–Fowler differential equations, have numerous applications in the study of non-Newtonian fluid theory, porous medium problems, chemotaxis models, etc.; see [7–10]. We can also refer to [11–13]for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms.

In what follows, we state some background details that motivate the analysis of (1) and (3). In recent years, numerous significant results for the oscillation of delay differential equations have been shown in [14–26].

Chiu and Li [4] considered the oscillatory behavior of a class of scalar advanced and delayed differential equations with piecewise constant generalized arguments, which extended the theory of functional differential equations. The authors in [27–34] studied the asymptotic properties of different orders of some differential equations. For more details on this theory, we refer the reader to the papers [35–43].

In 2014, Li et al. [44] presented some open problems for the study of qualitative properties of solutions to differential equations, and the authors used the Riccati technique to find oscillation conditions for the studied equations.

Zhang et al. [45] investigated a higher-order half-linear/Emden–Fowler delay equation with p-Laplacian like operators

$$\left(r(t)\left(x^{(\kappa-1)}(t)\right)^{p-1}\right)' + \sigma(t)\left|x^{(\kappa-1)}(t)\right|^{p_1-2}x^{(\kappa-1)}(t) + q(t)f\left(x\left(\tau(t)\right)\right) = 0.$$

In particular, the authors in [46] used the integral average technique and obtained several oscillation criteria of the delay equation

$$\left(r(t)\left|x^{(\kappa-1)}(t)\right|^{p-2}x^{(\kappa-1)}(t)\right)' + q(t)g\left(x\left(\tau(t)\right)\right) = 0,$$

where κ is even and under the condition

$$\int_{\upsilon_0}^{\infty} \frac{1}{r^{1/(p-1)}(s)}\,\mathrm{d}s = \infty.$$

The motivation for this article is to continue the previous works [23, 31].

On the basis of the above discussion, we will establish criteria for the oscillation of (1) and (3) by Riccati and comparison techniques under (2) and (4). Finally, two examples are presented to show the significance of the conclusions.

2 Auxiliary results

To establish oscillation criteria for (1) and (3), we give the following lemmas in this section.

Lemma 2.1 ([33]) *Let $h \in C^n([t_0,\infty),(0,\infty))$. Suppose that $h^{(n)}(t)$ is of a fixed sign on $[t_0,\infty)$, $h^{(n)}(t)$ not identically zero and that there exists $t_1 \geq t_0$ such that, for all $t \geq t_1$,*

$$h^{(n-1)}(t)h^{(n)}(t) \leq 0.$$

If we have $\lim_{t\to\infty} h(t) \neq 0$, then there exists $t_\lambda \geq t_0$ such that

$$h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| h^{(n-1)}(t) \right|$$

for every $\lambda \in (0,1)$ and $t \geq t_\lambda$.

Lemma 2.2 ([32]) *If the function x satisfies $x^{(i)}(t) > 0$, $i = 0,1,\ldots,n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 2.3 ([34]) *Let $V > 0$. Then*

$$Uu - Vu^{(\kappa+1)/\kappa} \leq \frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}} U^{\kappa+1} V^{-\kappa}. \tag{5}$$

Lemma 2.4 *Let (2) hold. If x is an eventually positive solution of (1), then $x' > 0$ and $x''' > 0$.*

Proof The proof is obvious and therefore is omitted. □

Lemma 2.5 *If*

$$\int_{t_0}^{\infty} \left(M^{p_2-p_1} \beta(s) q(s) \frac{\tau^{3(p_2-1)}(s)}{s^{3\kappa}} - \frac{2^{p_1-1}}{p_1^{p_1}} \frac{r(s)(\beta'(s))^{p_1}}{\mu^{p_1-1} s^{2(p_1-1)} \beta^{p_1-1}(s)} \right) \mathrm{d}s = \infty \tag{6}$$

for some $\mu \in (0,1)$, then $x'' < 0$.

Proof Let $x''(t) > 0$. From Lemmas 2.2 and 2.1, we find

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\tau^3(t)}{t^3} \tag{7}$$

and

$$x'(t) \geq \frac{\mu}{2} t^2 x'''(t). \tag{8}$$

Let

$$\zeta(t) := \beta(t) \frac{r(t)(x'''(t))^{p_1-1}}{x^{p_1-1}(t)} > 0. \tag{9}$$

From (7), (8), and (9), we find

$$\begin{aligned}\zeta'(t) &\leq \frac{\beta'(t)}{\beta(t)}\phi(t) - \beta(t)q(t)\frac{\tau^{3(p_1-1)}(t)}{t^{3(p_1-1)}}x^{p_2-p_1}\bigl(\tau(t)\bigr)\\ &\quad - \frac{(p_1-1)\mu}{2}\frac{t^2}{\beta^{1/p_1-1}(t)r^{1/p_1-1}(t)}\zeta^{1+(1/(p_1-1))}(t).\end{aligned} \tag{10}$$

Since $x'(t) > 0$, there exist $t_2 \geq t_1$ and a constant $M > 0$ such that $x(t) > M$ for all $t \geq t_2$. Using inequality (5) with $U = \beta'/\beta$, $V = \kappa\mu t^2/(2r^{1/\kappa}(t)\beta^{1/\kappa}(t))$ and $u = \zeta$, we get

$$\zeta'(t) \leq -M^{p_2-p_1}\beta(t)q(t)\frac{\tau^{3(p_1-1)}(t)}{t^{3(p_1-1)}} + \frac{2^{p_1-1}}{p_1^{p_1}}\frac{r(t)(\beta'(t))^{p_1}}{\mu^{p_1-1}t^{2(p_1-1)}\beta^{p_1-1}(t)}.$$

This implies that

$$\int_{t_1}^{t}\left(M^{p_2-p_1}\beta(s)q(s)\frac{\tau^{3(p_2-1)}(s)}{s^{3\kappa}} - \frac{2^{p_1-1}}{{p_1}^{p_1}}\frac{r(s)(\beta'(s))^{p_1}}{\mu^{p_1-1}s^{2(p_1-1)}\beta^{p_1-1}(s)}\right)\mathrm{d}s \leq \zeta(t_1),$$

which contradicts (6). The proof is complete. □

For convenience, we denote

$$\begin{aligned}R(t) &:= \int_t^\infty\left(\frac{1}{r(\eta)}\int_\eta^\infty q(s)\,\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\eta,\\ \widetilde{R}(t) &:= \mu_2^{(p_2-1)/(p_1-1)}\int_t^\infty\left(\frac{1}{r(\eta)}\int_\eta^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{p_2-1}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\eta,\\ \vartheta_{t_0}(t) &:= \exp\left(\int_{t_0}^t\frac{\sigma(\eta)}{r(\eta)}\,\mathrm{d}\eta\right),\end{aligned}$$

and

$$\widehat{R}(t) := \mu_2^{(p_2-1)/(p_1-1)}\int_t^\infty\left(\frac{1}{r(\eta)\vartheta_{t_0}(t)}\int_\eta^\infty \vartheta_{t_0}(t)q(s)\left(\frac{\tau(s)}{s}\right)^{p_2-1}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\eta,$$

where $\mu_2 \in (0,1)$.

We shall establish oscillation conditions for (3) by converting into the form (1). It is not difficult to see that

$$\begin{aligned}\frac{1}{\vartheta_{t_0}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\mu(t)r(t)\bigl(x'''(t)\bigr)^{p_1-1}\bigr) &= \frac{1}{\vartheta_{t_0}(t)}\bigl[\vartheta_{t_0}(t)\bigl(r(t)\bigl(x'''(t)\bigr)^{p_1-1}\bigr)' + \vartheta'_{t_0}(t)r(t)\bigl(x'''(t)\bigr)^{p_1-1}\bigr]\\ &= \bigl(r(t)\bigl(x'''(t)\bigr)^{p_1-1}\bigr)' + \frac{\vartheta'_{t_0}(t)}{\vartheta_{t_0}(t)}r(t)\bigl(x'''(t)\bigr)^{p_1-1},\\ &= \bigl(r(t)\bigl(x'''(t)\bigr)^{p_1-1}\bigr)' + \sigma(t)\bigl(x'''(t)\bigr)^{p_1-1},\end{aligned}$$

which with (3) gives

$$\bigl(\vartheta_{t_0}(t)r(t)\bigl(x'''(t)\bigr)^{p_1-1}\bigr)' + \vartheta_{t_0}(t)q(t)x^{p_2-1}\bigl(\tau(t)\bigr) = 0.$$

3 Main results

In this section, we establish oscillation criteria for (1) and (3) by the Riccati transformation and comparison technique.

Theorem 3.1 *If the equation*

$$\eta'(t) + \frac{\lambda^{p_2-1}}{6^{p_2-1}} \frac{q_i(t)\tau^{3(p_2-1)}(t)}{r^{(p_2-1)/(p_1-1)}(\tau(t))} \eta^{(p_2-1)/(p_1-1)}\bigl(\tau(t)\bigr) = 0 \tag{11}$$

is oscillatory, then (1) *is oscillatory.*

Proof Let (1) have a nonoscillatory solution in $[t_0, \infty)$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$. Let

$$\eta(t) := r(t)\bigl(x'''(t)\bigr)^{p_1-1} > 0 \quad \text{[from Lemma 2.4]},$$

which with (1) gives

$$\eta'(t) + q(t)x^{p_2-1}\bigl(\tau(t)\bigr) = 0. \tag{12}$$

Since x is positive and increasing, we see $\lim_{t\to\infty} x(t) \neq 0$. So, using Lemma 2.1, we find

$$x^{p_2-1}\bigl(\tau(t)\bigr) \geq \frac{\lambda^{p_2-1}}{6^{p_2-1}} \tau^{3(p_2-1)}(t)\bigl(x'''\bigl(\tau(t)\bigr)\bigr)^{p_2-1} \tag{13}$$

for all $\lambda \in (0, 1)$. By (12) and (13), we see that

$$\eta'(t) + \frac{\lambda^{p_2-1}}{6^{p_2-1}} q_i(t)\tau^{3(p_2-1)}(t)\bigl(x'''\bigl(\tau(t)\bigr)\bigr)^{p_2-1} \leq 0.$$

So, η is a positive solution of the inequality

$$\eta'(t) + \frac{\lambda^{p_2-1}}{6^{p_2-1}} \frac{q(t)\tau^{3(p_2-1)}(t)}{r^{(p_2-1)/(p_1-1)}(\tau(t))} \eta^{(p_2-1)/(p_1-1)}\bigl(\tau(t)\bigr) \leq 0.$$

By using [40, Theorem 1], we find that (11) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 3.2 *Let $p_2 = p_1$ and* (2) *hold. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{\lambda^{p_2-1}}{6^{p_2-1}} \frac{q(s)\tau^{3(p_2-1)}(s)}{r^{(p_2-1)/(p_1-1)}(\tau(s))} \,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{14}$$

then (1) *is oscillatory.*

Theorem 3.3 *Let $p_2 \geq p_1$ and* (6) *hold for some $\mu \in (0, 1)$. If*

$$u''(t) + M^{p_2-p_1}\widetilde{R}(t)u(t) = 0 \tag{15}$$

is oscillatory, then (1) *is oscillatory.*

Proof Assume to the contrary that (1) has a nonoscillatory solution in $[t_0,\infty)$. Without loss of generality, we only need to be concerned with positive solutions of equation (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$. From Lemmas 2.2 and 2.4, we have that

$$x'(t) > 0, \qquad x''(t) < 0 \quad \text{and} \quad x'''(t) > 0 \tag{16}$$

for $t \geq t_2$, where t_2 is sufficiently large. Now, integrating (1) from t to l, we have

$$r(l)\big(x'''(l)\big)^{p_1-1} = r(t)\big(x'''(t)\big)^{p_1-1} - \int_t^l q(s)x^{p_2-1}\big(\tau(s)\big)\,\mathrm{d}s. \tag{17}$$

Using Lemma 3 in [34] with (16), we get

$$\frac{x(\tau(t))}{x(t)} \geq \lambda\frac{\tau(t)}{t},$$

which with (17) gives

$$r(l)\big(x'''(l)\big)^{p_1-1} - r(t)\big(x'''(t)\big)^{p_1-1} + \lambda^{p_2-1}\int_t^l q_i(s)\left(\frac{\tau(s)}{s}\right)^{p_2-1} x^{p_1-1}(s)\,\mathrm{d}s \leq 0.$$

It follows, by $x' > 0$, that

$$r(l)\big(x'''(l)\big)^{p_1-1} - r(t)\big(x'''(t)\big)^{p_1-1} + \lambda^{p_2-1}x^{p_1-1}(t)\int_t^l q(s)\left(\frac{\tau(s)}{s}\right)^{p_2-1}\mathrm{d}s \leq 0. \tag{18}$$

Taking $l \to \infty$, we have

$$-r(t)\big(x'''(t)\big)^{p_1-1} + \lambda^{p_2-1}x^{p_1-1}(t)\int_t^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{p_2-1}\mathrm{d}s \leq 0,$$

that is,

$$x'''(t) \geq \frac{\lambda^{(p_2-1)/(p_1-1)}}{r^{1/(p_1-1)}(t)}x^{(p_2-1)/(p_1-1)}(t)\left(\int_t^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{p_2-1}\mathrm{d}s\right)^{1/(p_1-1)}.$$

Integrating the above inequality from t to ∞, we obtain

$$-x''(t) \geq \lambda^{(p_2-1)/(p_1-1)}x^{(p_2-1)/(p_1-1)}(t)\int_t^\infty\left(\frac{1}{r(\eta)}\int_\eta^\infty q(s)\left(\frac{\tau_i(s)}{s}\right)^{p_2-1}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\eta,$$

hence

$$x''(t) \leq -\widetilde{R}(t)x^{(p_2-1)/(p_1-1)}(t). \tag{19}$$

Letting

$$\phi(t) = \frac{x'(t)}{x(t)},$$

then $\phi(t) > 0$ for $t \ge t_1$ and

$$\phi'(t) = \frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2.$$

By using (19) and the definition of $\phi(t)$, we see that

$$\phi'(t) \le -\widetilde{R}(t)\frac{x^{(p_2-1)/(p_1-1)}(t)}{x(t)} - \phi^2(t). \tag{20}$$

Since $x'(t) > 0$, there exists a constant $M > 0$ such that $x(t) \ge M$ for all $t \ge t_2$. Then (20) becomes

$$\phi'(t) + \phi^2(t) + M^{p_2-p_1}\widetilde{R}(t) \le 0. \tag{21}$$

From [39], we obtain that (15) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that (21) holds, which is a contradiction. Theorem is proved. □

Theorem 3.4 *Let $p_2 \ge p_1$, $\tau_i'(t) > 1$ and (6) hold for some $\mu \in (0,1)$. If*

$$\left(\frac{1}{\tau'(t)}u'(t)\right)' + M^{(p_2-1)/(p_1-2)}R(t)u(t) = 0 \tag{22}$$

is oscillatory, then (1) is oscillatory.

Proof From the proof of Theorem 3.3, we find that (17) holds. So, it follows from $\tau_i'(t) \ge 0$ and $x'(t) \ge 0$ that

$$r(l)\big(x'''(l)\big)^{p_1-1} - r(t)\big(x'''(t)\big)^{p_1-1} + x^{p_2-1}\big(\tau(t)\big)\int_t^l q(s)\,\mathrm{d}s \le 0. \tag{23}$$

Thus, (16) becomes

$$x''(t) \le -R(t)x^{(p_2-1)/(p_1-1)}\big(\tau_i(t)\big). \tag{24}$$

Letting

$$\delta(t) = \frac{x'(t)}{x(\tau(t))}, \tag{25}$$

then $\delta(t) > 0$ for $t \ge t_1$, and

$$\begin{aligned} \delta'(t) &= \frac{x''(t)}{x(\tau(t))} - \frac{x'(t)}{x^2(\tau(t))}x'\big(\tau(t)\big)\tau'(t) \\ &\le \frac{x''(t)}{x(\tau(t))} - \tau'(t)\left(\frac{x'(t)}{x(\tau(t))}\right)^2. \end{aligned}$$

From (24) and (25), we find that

$$\delta'(t) + M^{(p_2-1)/(p_1-2)}R(t) + \tau'(t)\delta^2(t) \le 0. \tag{26}$$

From [39], we find that (22) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that (26) holds, which is a contradiction. Theorem is proved. □

Corollary 3.5 *Let $p_2 = p_1$ and (6) hold. If*

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\widetilde{R}(s) - \frac{1}{4}h^2(t,s) \right) \mathrm{d}s = \infty$$

or

$$\liminf_{t\to\infty} t \int_{t}^{\infty} \widetilde{R}(s)\,\mathrm{d}s > \frac{1}{4}, \tag{27}$$

then (1) is oscillatory.

Corollary 3.6 *Let $p_2 = p_1$ and (6) hold. If $\varepsilon \in (0, 1/4]$ such that*

$$t^2\widetilde{R}(s) \geq \varepsilon$$

and

$$\limsup_{t\to\infty} \left(t^{\varepsilon-1} \int_{t_0}^{t} s^{2-\varepsilon}\widetilde{R}(s)\,\mathrm{d}s + t^{1-\widetilde{\varepsilon}} \int_{t}^{\infty} s^{\widetilde{\varepsilon}}\widetilde{R}(s)\,\mathrm{d}s \right) > 1,$$

where $\widetilde{\varepsilon} = \frac{1}{2}(1 - \sqrt{1-4\varepsilon})$, then (1) is oscillatory.

Corollary 3.7 *Let $p_1 = p_2$ and (4) hold. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{\lambda^{p_2-1}}{6^{p_2-1}} \frac{\vartheta_{t_0}(s)q(s)\tau_i^{3(p_2-1)}(s)}{\vartheta_{t_0}^{(p_2-1)/(p_1-1)}(\tau(s))r^{(p_2-1)/(p_1-1)}(\tau(s))}\,\mathrm{d}s > \frac{1}{\mathrm{e}},$$

then (3) is oscillatory.

Corollary 3.8 *Let $p_1 = p_2$, (4), and*

$$\int_{t_0}^{\infty} \left(M^{p_2-p_1}\beta(s)\vartheta_{t_0}(s)q(s)\frac{\tau^{3\kappa}(s)}{s^{3\kappa}} - \frac{2^{p_1-1}}{p_1{}^{p_1}} \frac{r(s)(\beta'(s))^{p_1}}{\mu^{p_1-1}s^{2(p_1-1)}\beta^{p_1-1}(s)} \right) \mathrm{d}s = \infty, \tag{28}$$

hold for some $\mu \in (0,1)$. If

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\widehat{R}(s) - \frac{1}{4}h^2(t,s) \right) \mathrm{d}s = \infty$$

or

$$\liminf_{t\to\infty} \int_{t}^{\infty} \widehat{R}(s)\,\mathrm{d}s > \frac{1}{4},$$

then (3) is oscillatory.

Corollary 3.9 *Let $p_1 = p_2$ and (28) hold. If $\varepsilon \in (0, 1/4]$ such that*

$$t^2\widehat{R}(s) \geq \varepsilon$$

and

$$\limsup_{t\to\infty}\left(t^{\varepsilon-1}\int_{t_0}^{t} s^{2-\varepsilon}\widehat{R}(s)\,\mathrm{d}s + t^{1-\widetilde{\varepsilon}}\int_t^\infty s^{\widetilde{\varepsilon}}\widehat{R}(s)\,\mathrm{d}s\right) > 1,$$

where $\widetilde{\varepsilon}$ is defined as in Corollary 3.6, then (3) is oscillatory.

4 Examples and discussion

Two examples are presented to show the applications of our results. The first example is given to demonstrate Corollaries 3.2 and 3.5.

Example 4.1 For $t \geq 1$, consider the equation

$$\left(t^3\left(x'''(t)\right)^3\right)' + \frac{q_0}{t^7}x^3(\gamma t) = 0, \tag{29}$$

we see that $p_1 = p_2 = 4$, $r(t) = t^3$, $\tau(t) = \gamma t$ and $q(t) = q_0/t^7$, $\gamma \in (0, 1]$ and $q_0 > 0$. So, we obtain

$$\widetilde{R}(t) = \lambda\left(\frac{q_0}{6}\right)^{1/3}\gamma\frac{1}{2t^2}.$$

By Corollary 3.2 and Corollary 3.5, equation (29) is oscillatory if

$$q_0 > \frac{6^3}{e(\ln\frac{1}{\gamma})\gamma^6},$$

$$q_0 > \left(\frac{3^4}{2}\right)\frac{1}{\gamma^9},$$

and

$$q_0 > 6\left(\frac{1}{4\gamma}\right)^3,$$

respectively. Thus, equation (29) is oscillatory if

$$q_0 > \max\left\{\left(\frac{3^4}{2}\right)\frac{1}{\gamma^9}, 6\left(\frac{1}{4\gamma}\right)^3\right\} = \left(\frac{3^4}{2}\right)\frac{1}{\gamma^9}. \tag{30}$$

Now, we give the second example to demonstrate Corollary 3.8.

Example 4.2 Consider the equation

$$\left(t^3\left(x'''(t)\right)^3\right)' + \left(x'''(t)\right)^3 + \frac{q_0}{t^5}x^3(t/2) = 0, \quad t \geq 1, q_0 > 0. \tag{31}$$

Let $p_1 = p_2 = 4$, $r(t) = t^3$, $\sigma(t) = 1$, $\tau(t) = t/2$, and $q(t) = q_0/t^5$. Thus, it is easy to verify that

$$\begin{aligned} &\int_{t_0}^{\infty} \left[\frac{1}{r(s)} \exp\left(- \int_{t_0}^{s} \frac{\sigma(\eta)}{r(\eta)} \,\mathrm{d}x \right) \right]^{1/p_1 - 1} \mathrm{d}s \\ &\quad = \int_{t_0}^{\infty} \left[\frac{1}{s^3} \exp\left(- \int_{t_0}^{s} \frac{1}{s^3} \,\mathrm{d}x \right) \right]^{1/3} \mathrm{d}s = \infty. \end{aligned}$$

Using Corollary 3.8, equation (31) is oscillatory.

5 Conclusion

The oscillation conditions of the fourth-order differential equations with p-Laplacian like operators are obtained in this study. In order to improve and simplify prior results in the literature, we expanded the results in [23, 31] to fourth-order equations and used the Riccati transformation and comparison techniques. It is interesting to extend our results to even-order damped differential equations with p-Laplacian like operators

$$\left(r(t) \left(x^{(\kappa-1)}(t) \right)^{p-1} \right)' + \sigma(t) \left| x'''(t) \right|^{p_1 - 2} x'''(t) + q(t) f\left(x\left(\tau(t) \right) \right) = 0$$

under the condition

$$\int_{t_0}^{\infty} \left[\frac{1}{r(s)} \exp\left(- \int_{t_0}^{s} \frac{\sigma(\eta)}{r(\eta)} \,\mathrm{d}x \right) \right]^{1/p_1 - 1} \mathrm{d}s < \infty.$$

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The authors declare that they have no competing interests.

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