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On some accelerated optimization algorithms based on fixed point and linesearch techniques for convex minimization problems with applications

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Abstract

In this paper, we introduce and study a new accelerated algorithm based on forward–backward and SP-algorithm for solving a convex minimization problem of the sum of two convex and lower semicontinuous functions in a Hilbert space. Under some suitable control conditions, a weak convergence theorem of the proposed algorithm based on a fixed point is established. Moreover, we choose the stepsize of our algorithm which is independent on the Lipschitz constant of the gradient of the objective function by using a linesearch technique, and then a weak convergence result of the proposed algorithm is analyzed. As applications, we apply the proposed algorithm for solving the image restoration problems and compare its convergence behavior with other well-known algorithms in the literature. By our experiment, the algorithms have a higher efficiency than the others.

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Keywords: Convex minimization problems; Fixed points; Forward–backward algorithms; Image restoration problems

1 Introduction

Throughout this paper, let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real numbers and the set of positive integers, respectively. Let I denote the identity operator on $\mathcal{H}$. The symbols $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively.

In this work, we are interested in solving the convex minimization problems of the following form:

$$\underset{x\in\mathcal{H}}{\text{minimize}}\ \psi_1(x)+\psi_2(x), \tag{1}$$

where $\psi_1:\mathcal{H}\to\mathbb{R}$ is a convex and differentiable function with a L-Lipschitz continuous gradient of ψ_1 and $\psi_2:\mathcal{H}\to\mathbb{R}\cup\{\infty\}$ is a proper lower semi-continuous and convex function. If x is a solution of problem (1), then x is characterized by the fixed point equation of the forward–backward operator

$$x = \underbrace{\mathrm{prox}_{\alpha\psi_2}}_{\text{backward step}} \underbrace{\big(x - \alpha\nabla\psi_1(x)\big)}_{\text{forward step}}, \tag{2}$$

where $\alpha > 0$, prox_{ψ_2} is the proximity operator of ψ_2, and $\nabla\psi_1$ stands for the gradient of ψ_1.

In the recent years, various iterative algorithms for solving a convex minimization problem of the sum of two convex functions were introduced and studied by many mathematicians, see [1, 4, 7–10, 14–16, 18, 21, 25] for instance.

One of the popular iterative algorithms, called *forward–backward splitting* (FBS) algorithm [8, 16], is defined by the following: let $x_1 \in \mathcal{H}$ and set

$$x_{n+1} = \mathrm{prox}_{c_n\psi_2}\big(x_n - c_n\nabla\psi_1(x_n)\big), \quad \forall n \in \mathbb{N}, \tag{3}$$

where $0 < c_n < 2/L$.

In 2005, Combettes and Wajs [8] introduced the following *relaxed forward–backward splitting* (R-FBS) algorithm, which is defined by the following: let $\varepsilon \in (0, \min(1, \frac{1}{L}))$, $x_1 \in \mathbb{R}^N$ and set

$$y_n = x_n - c_n\nabla\psi_1(x_n), \qquad x_{n+1} = x_n + \beta_n\big(\mathrm{prox}_{c_n\psi_2}(y_n) - x_n\big), \quad \forall n \in \mathbb{N}, \tag{4}$$

where $c_n \in [\varepsilon, \frac{2}{L} - \varepsilon]$ and $\beta_n \in [\varepsilon, 1]$.

To accelerate the forward–backward splitting algorithm, an inertial technique is employed. So, various inertial algorithms were introduced and studied in order to accelerate convergence behavior of the algorithms, see [3, 6, 11, 26] for example. Recently, Beck and Teboulle [3] introduced a *fast iterative shrinkage-thresholding algorithm* (FISTA) for solving problem (1). FISTA is defined by the following: let $x_1 = y_0 \in \mathbb{R}^N$, $t_1 = 1$ and set

$$\begin{cases} t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}, \qquad \alpha_n = \frac{t_n - 1}{t_{n+1}}, \\ y_n = \mathrm{prox}_{\frac{1}{L}\psi_2}(x_n - \frac{1}{L}\nabla\psi_1(x_n)), \\ x_{n+1} = y_n + \alpha_n(y_n - y_{n-1}), \quad n \in \mathbb{N}. \end{cases} \tag{5}$$

Note that α_n is called an *inertial parameter* which controls the momentum $y_n - y_{n-1}$.

It is observed that both FBS and FISTA algorithms need to assume the Lipschitz continuity condition on the gradient of ψ_1, and the stepsize depends on the Lipschitz constant L, which is not an easy task to find in general practice.

In 2016, Cruz and Nghia [9] proposed a linesearch technique for selecting the stepsize which is independent of the Lipschitz constant L. Their linesearch technique is given by the following process:

Linesearch. Fix $x \in \mathcal{H}$, $\sigma > 0$, $\delta > 0$, and $\theta \in (0,1)$.
Input $\alpha = \sigma$.
While

$$\alpha \left\| \nabla\psi_1\bigl(\mathrm{prox}_{\alpha\psi_2}(I - \alpha\nabla\psi_1)(x)\bigr) - \nabla\psi_1(x) \right\| > \delta \left\| \mathrm{prox}_{\alpha\psi_2}(I - \alpha\nabla\psi_1)(x) - x \right\|,$$

do

$$\alpha = \theta\alpha.$$

End
Output α.

The forward–backward splitting algorithm where the stepsize c_n is generated by above linesearch was introduced by Cruz and Nghia [9] and defined by the following:

(FBSL). Let $x_1 \in \mathcal{H}$, $\sigma > 0$, $\delta \in (0,1/2)$, and $\theta \in (0,1)$. For $n \geq 1$, let

$$x_{n+1} = \mathrm{prox}_{c_n\psi_2}\bigl(x_n - c_n\nabla\psi_1(x_n)\bigr),$$

where $c_n :=$ Linesearch$(x_n, \sigma, \theta, \delta)$.

Moreover, they also proposed an accelerated algorithm with an inertial technical term as follows.

(FISTAL). Let $x_0 = x_1 \in \mathcal{H}$, $\alpha_0 = \sigma > 0$, $\delta \in (0,1/2)$, $\theta \in (0,1)$, and $t_1 = 1$. For $n \geq 1$, let

$$\begin{aligned}
t_{n+1} &= \frac{1 + \sqrt{1 + 4t_n^2}}{2}, \qquad \alpha_n = \frac{t_n - 1}{t_{n+1}}, \\
y_n &= x_n + \alpha_n(x_n - x_{n-1}), \\
x_{n+1} &= \mathrm{prox}_{c_n\psi_2}\bigl(y_n - c_n\nabla\psi_1(y_n)\bigr),
\end{aligned}$$

where $c_n :=$ Linesearch$(y_n, c_{n-1}, \theta, \delta)$.

For the past decade, various fixed point algorithms for nonexpansive operators were introduced and studied for solving convex minimization problems, problem (1), see [11, 13, 17, 23]. In 2011, Phuengrattana and Suantai [23] introduced a new fixed point algorithm known as SP-iteration and showed that this algorithm has a convergence rate better than that of Ishikawa [13] and Mann [17] iterations. The SP-iteration for nonexpansive operator S was defined as follows:

$$\begin{aligned}
v_n &= (1 - \beta_n)x_n + \beta_n Sx_n, \\
y_n &= (1 - \gamma_n)v_n + \gamma_n Sv_n, \\
x_{n+1} &= (1 - \theta_n)y_n + \theta_n Sy_n, \quad n \in \mathbb{N},
\end{aligned}$$

where $x_1 \in \mathcal{H}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\theta_n\}$ are sequences in $(0,1)$.

Motivated by these works, we combine the idea of SP-iteration, FBS algorithm, and a linesearch technique to propose a new accelerated algorithm for a convex minimization problem which can be applied to solve the image restoration problems. We obtain weak convergence theorems in Hilbert spaces under some suitable conditions.

2 Preliminaries

In this section, we give some definitions and basic properties for proving our results in the next sections.

Let $\psi : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semi-continuous, and convex function. The *proximity* (or *proximal*) operator [2, 19] of ψ, denoted by $\operatorname{prox}_{\psi}$, is defined for each $x \in \mathcal{H}$, $\operatorname{prox}_{\psi} x$ is the unique solution of the minimization problem

$$\underset{y\in\mathcal{H}}{\operatorname{minimize}}\, \psi(y) + \frac{1}{2}\|x-y\|^2. \tag{6}$$

The proximity operator can be formulated in the equivalent form

$$\operatorname{prox}_{\psi} = (I + \partial\psi)^{-1} : \mathcal{H} \to \mathcal{H}, \tag{7}$$

where $\partial\psi$ is the subdifferential of ψ defined by

$$\partial\psi(x) := \bigl\{u \in \mathcal{H} : \psi(x) + \langle u, y-x\rangle \le \psi(y), \forall y \in \mathcal{H}\bigr\}, \quad \forall x \in \mathcal{H}.$$

Moreover, we have the following useful fact:

$$\frac{x - \operatorname{prox}_{\alpha\psi}(x)}{\alpha} \in \partial\psi\bigl(\operatorname{prox}_{\alpha\psi}(x)\bigr), \quad \forall x \in \mathcal{H}, \alpha > 0. \tag{8}$$

Note that the subdifferential operator $\partial\psi$ is maximal monotone (see [5] for more details) and the solution of (1) is a fixed point of the following operator:

$$x \in \operatorname{Argmin}(\psi_1 + \psi_2) \quad \Longleftrightarrow \quad x = \operatorname{prox}_{c\psi_2}(I - c\nabla\psi_1)(x),$$

where $c > 0$. If $0 < c < \frac{2}{L}$, we know that $\operatorname{prox}_{c\psi_2}(I - c\nabla\psi_1)$ is a nonexpansive operator.

An operator $S : \mathcal{H} \to \mathcal{H}$ is said to be *Lipschitz continuous* if there exists $L > 0$ such that

$$\|Sx - Sy\| \le L\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

If S is 1-Lipschitz continuous, then S is called a *nonexpansive operator*. A point $x \in \mathcal{H}$ is called a *fixed point* of S if $x = Sx$. The set of all fixed points of S is denoted by $\operatorname{Fix}(S)$.

The operator $I - S$ is called *demiclosed at zero* if for any sequence $\{x_n\}$ in $\mathcal{H}$ which converges weakly to x and the sequence $\{x_n - Sx_n\}$ converges strongly to 0, then $x \in \operatorname{Fix}(S)$. It is known [22] that if S is a nonexpansive operator, then $I - S$ is demiclosed at zero. Let $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator and $\{S_n : \mathcal{H} \to \mathcal{H}\}$ be a sequence of nonexpansive operators such that $\emptyset \neq \operatorname{Fix}(S) \subset \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n)$. Then $\{S_n\}$ is said to satisfy *NST-condition (I)* with S [20] if for each bounded sequence $\{x_n\}$ in $\mathcal{H}$,

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Let $x, y \in \mathcal{H}$ and $t \in [0,1]$. The following inequalities hold on $\mathcal{H}$:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{9}$$

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2. \tag{10}$$

The following lemmas are crucial for our main results.

Lemma 2.1 ([6]) *Let $\psi_1 : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function with an L-Lipschitz continuous gradient of ψ_1, and let $\psi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a proper lower semi-continuous and convex function. Let $S_n := \operatorname{prox}_{c_n\psi_2}(I - c_n\nabla\psi_1)$ and $S := \operatorname{prox}_{c\psi_2}(I - c\nabla\psi_1)$, where $c_n, c \in (0, 2/L)$ with $c_n \to c$ as $n \to \infty$. Then $\{S_n\}$ satisfies NST-condition (I) with S.*

Lemma 2.2 ([24]) *If $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semi-continuous, and convex function, then the graph of ∂f defined by $\operatorname{Gph}(\partial f) := \{(x,y) \in \mathcal{H} \times \mathcal{H} : y \in \partial f(x)\}$ is demiclosed, i.e., if the sequence $\{(x_k, y_k)\}$ in $\operatorname{Gph}(\partial f)$ satisfies $x_k \rightharpoonup x$ and $y_k \to y$, then $(x,y) \in \operatorname{Gph}(\partial f)$.*

Lemma 2.3 ([12]) *Let $\psi_1, \psi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be two proper, lower semi-continuous, and convex functions. Then, for any $x \in \mathcal{H}$ and $c_2 \geq c_1 > 0$, we have*

$$\begin{aligned} \frac{c_2}{c_1}\|x - \operatorname{prox}_{c_1\psi_2}(x - c_1\nabla\psi_1(x))\| &\geq \|x - \operatorname{prox}_{c_2\psi_2}(x - c_2\nabla\psi_1(x))\| \\ &\geq \|x - \operatorname{prox}_{c_1\psi_2}(x - c_1\nabla\psi_1(x))\|. \end{aligned}$$

Lemma 2.4 ([11]) *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+t_n)a_n + t_n a_{n-1}, \quad \forall n \in \mathbb{N}.$$

Then $a_{n+1} \leq M \cdot \prod_{j=1}^{n}(1+2t_j)$, where$M = \max\{a_1, a_2\}$. Moreover, if $\sum_{n=1}^{\infty} t_n < \infty$, then $\{a_n\}$ is bounded.

Lemma 2.5 ([27]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.6 ([22]) *Let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that there exists a nonempty set $\Omega \subset \mathcal{H}$ satisfying*:

(i) *For every $p \in \Omega$, $\lim_{n\to\infty} \|x_n - p\|$ exists;*

(ii) *$\omega_w(x_n) \subset \Omega$,*

where $\omega_w(x_n)$ is the set of all weak-cluster points of $\{x_n\}$. Then $\{x_n\}$ converges weakly to a point in Ω.

3 The SP-forward–backward splitting based on a fixed point algorithm

In this section, we introduce a new accelerated algorithm by using FBS and SP-iteration with the inertial technique to solve a convex minimization problem of the sum of two convex functions ψ_1 and ψ_2, where

- $\psi_1 : \mathcal{H} \to \mathbb{R}$ is a convex and differentiable function with an L-Lipschitz continuous gradient of ψ_1;

Algorithm 1 SP-forward–backward splitting (SP-FBS)

Take $x_0, x_1 \in \mathcal{H}$ arbitrarily and calculate x_{n+1} as follows:

$$\begin{aligned} u_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ v_n &= u_n + \beta_n\big(\operatorname{prox}_{c_n\psi_2}\big(u_n - c_n\nabla\psi_1(u_n)\big) - u_n\big), \\ y_n &= v_n + \gamma_n\big(\operatorname{prox}_{c_n\psi_2}\big(v_n - c_n\nabla\psi_1(v_n)\big) - v_n\big), \\ x_{n+1} &= y_n + \theta_n\big(\operatorname{prox}_{c_n\psi_2}\big(y_n - c_n\nabla\psi_1(y_n)\big) - y_n\big), \quad \forall n \geq 1. \end{aligned}$$

- $\psi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a proper lower semi-continuous and convex function;
- $\Omega := \operatorname{Argmin}(\psi_1 + \psi_2) \neq \emptyset$.

Now, we are ready to prove the convergence theorem of Algorithm 1 (SP-FBS).

Theorem 3.1 *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\theta_n\}$, and $\{c_n\}$ satisfy the following conditions*:

(C1) $\gamma_n, \theta_n \in [0,1]$, $\beta_n \in [a,b] \subset (0,1)$;
(C2) $\alpha_n \geq 0$, $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(C3) $0 < c_n, c < 2/L$ *such that* $\lim_{n\to\infty} c_n = c$.

Then the following statements hold:

(i) $\|x_{n+1} - p^*\| \leq M \cdot \prod_{j=1}^{n}(1 + 2\alpha_j)$, *where* $M = \max\{\|x_1 - p^*\|, \|x_2 - p^*\|\}$ *and* $p^* \in \Omega$.
(ii) $\{x_n\}$ *converges weakly to a point in* Ω.

Proof For each $n \in \mathbb{N}$, set $S_n := \operatorname{prox}_{c_n\psi_2}(I - c_n\nabla\psi_1)andS := \operatorname{prox}_{c\psi_2}(I - c\nabla\psi_1)$. Then the sequence $\{x_n\}$ generated by Algorithm 1 is the same as that generated by the following inertial SP-iteration:

$$\begin{aligned} u_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ v_n &= (1-\beta_n)u_n + \beta_n S_n u_n, \\ y_n &= (1-\gamma_n)v_n + \gamma_n S_n v_n, \\ x_{n+1} &= (1-\theta_n)y_n + \theta_n S_n y_n. \end{aligned} \tag{11}$$

By condition (C3), we know that S_n and S are nonexpansive operators with $\bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) = \operatorname{Fix}(S) = \operatorname{Argmin}(\psi_1 + \psi_2) := \Omega$. By Lemma 2.1, we obtain that $\{S_n\}$ satisfies NST-condition (I) with S.

(i) Let $p^* \in \Omega$. By (11), we have

$$\|u_n - p^*\| \leq \|x_n - p^*\| + \alpha_n\|x_n - x_{n-1}\| \tag{12}$$

and

$$\|v_n - p^*\| \leq (1-\beta_n)\|u_n - p^*\| + \beta_n\|S_n u_n - p^*\| \leq \|u_n - p^*\|. \tag{13}$$

Similarly, we get that

$$\|y_n - p^*\| \leq \|v_n - p^*\| \quad \text{and} \quad \|x_{n+1} - p^*\| \leq \|y_n - p^*\|. \tag{14}$$

From (12), (13), and (14), we get

$$\begin{aligned}\|x_{n+1}-p^*\| &\le \|y_n-p^*\|\\ &\le \|v_n-p^*\|\\ &\le \|u_n-p^*\|\\ &\le \|x_n-p^*\|+\alpha_n\|x_n-x_{n-1}\|.\end{aligned}\tag{15}$$

This implies that

$$\|x_{n+1}-p^*\| \le (1+\alpha_n)\|x_n-p^*\|+\alpha_n\|x_{n-1}-p^*\|.\tag{16}$$

Apply Lemma 2.4, we get $\|x_{n+1}-p^*\| \le M\cdot\prod_{j=1}^{n}(1+2\alpha_j)$, where $M=\max\{\|x_1-p^*\|,\|x_2-p^*\|\}$.

(ii) It follows from (i) that $\{x_n\}$ is bounded. This implies $\sum_{n=1}^{\infty}\alpha_n\|x_n-x_{n-1}\|<\infty$. By (15) and Lemma 2.5, we obtain that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. By (10), we have

$$\|u_n-x^*\|^2 \le \|x_n-p^*\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2+2\alpha_n\|x_n-p^*\|\|x_n-x_{n-1}\|.\tag{17}$$

From (9), we also have

$$\begin{aligned}\|v_n-p^*\|^2 &= (1-\beta_n)\|u_n-p^*\|^2+\beta_n\|S_nu_n-p^*\|^2\\ &\quad-\beta_n(1-\beta_n)\|u_n-S_nu_n\|^2\\ &\le \|u_n-p^*\|^2-\beta_n(1-\beta_n)\|u_n-S_nu_n\|^2.\end{aligned}\tag{18}$$

By (14), (17), and (18), we obtain

$$\begin{aligned}\|x_{n+1}-p^*\|^2 &\le \|y_n-p^*\|^2\\ &\le \|v_n-p^*\|^2\\ &\le \|u_n-p^*\|^2-\beta_n(1-\beta_n)\|u_n-S_nu_n\|^2\\ &\le \|x_n-p^*\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2+2\alpha_n\|x_n-p^*\|\|x_n-x_{n-1}\|\\ &\quad-\beta_n(1-\beta_n)\|u_n-S_nu_n\|^2.\end{aligned}\tag{19}$$

Since $0<a\le\beta_n\le b<1$, $\sum_{n=1}^{\infty}\alpha_n\|x_n-x_{n-1}\|<\infty$ and $\lim_{n\to\infty}\|x_n-p^*\|$ exists, the above inequality implies $\lim_{n\to\infty}\|u_n-S_nu_n\|=0$. Since $\{u_n\}$ is bounded and $\{S_n\}$ satisfies NST-condition (I) with S, we have $\lim_{n\to\infty}\|u_n-Su_n\|=0$. By the demiclosedness of $I-S$, we have $\omega_w(u_n)\subset\mathrm{Fix}(S)=\Omega$. Since $\lim_{n\to\infty}\|u_n-x_n\|=0$, we have $\omega_w(x_n)\subset\omega_w(u_n)\subset\mathrm{Fix}(S)=\Omega$. By Lemma 2.6, we can conclude that $\{x_n\}$ converges weakly to a point in Ω. This completes the proof. □

Remark 3.2 If we set $\alpha_n=0$, $S_n=S$ for all $n\in\mathbb{N}$, then Algorithm 1 is reduced to the SP-algorithm [23]:

$$v_n=(1-\beta_n)x_n+\beta_nSx_n,$$

$$y_n = (1-\gamma_n)v_n + \gamma_n S v_n,$$
$$x_{n+1} = (1-\theta_n)y_n + \theta_n S y_n,$$

where $\beta_n, \gamma_n, \theta_n \in (0,1)$.

Remark 3.3 If we set $\alpha_n = \gamma_n = \theta_n = 0$ for all $n \in \mathbb{N}$, then Algorithm 1 is reduced to the Krasnosel'skii–Mann algorithm [8]:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n S_n x_n, \quad n \geq 1,$$

where $\beta_n \in (0,1)$.

4 The SP-forward–backward splitting algorithm with linesearch technique

In this section, we introduce a new accelerated algorithm by using the inertial and linesearch technique to solve a convex minimization problem of the sum of two convex functions ψ_1 and ψ_2, where

(B1) $\psi_1 : \mathcal{H} \to \mathbb{R}$ and $\psi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ are two proper, lower semi-continuous, and convex functions and $\Omega := \operatorname{Argmin}(\Psi := \psi_1 + \psi_2) \neq \emptyset$;

(B2) ψ_1 is differentiable on $\mathcal{H}$. The gradient $\nabla\psi_1$ is uniformly continuous on $\mathcal{H}$.

We note that assumption (B2) is a weaker than the Lipschitz continuity assumption on $\nabla\psi_1$.

Lemma 4.1 ([9]) *If* $\{x_n\}$ *is a sequence generated by the following algorithm*:

$$x_{n+1} = \operatorname{prox}_{c_n\psi_2}\big(x_n - c_n\nabla\psi_1(x_n)\big),$$

where $c_n := \text{Linesearch}(x_n, \sigma, \theta, \delta)$. *Then, for each* $n \geq 1$ *and* $p \in \mathcal{H}$,

$$\begin{aligned}\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \geq{}& 2c_n\big[(\psi_1+\psi_2)(x_{n+1}) - (\psi_1+\psi_2)(p)\big] \\ &+ (1-2\delta)\|x_{n+1} - x_n\|^2.\end{aligned}$$

Now, we are ready to prove the convergence theorem of Algorithm 2 (SP-FBSL).

Algorithm 2 SP-forward–backward splitting with linesearch (SP-FBSL)

Take $x_0, x_1 \in \mathcal{H}$ arbitrarily and calculate x_{n+1} as follows:

$$u_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$v_n = u_n + \beta_n\big(\operatorname{prox}_{c_n^1\psi_2}\big(u_n - c_n^1\nabla\psi_1(u_n)\big) - u_n\big),$$
$$y_n = v_n + \gamma_n\big(\operatorname{prox}_{c_n^2\psi_2}\big(v_n - c_n^2\nabla\psi_1(v_n)\big) - v_n\big),$$
$$x_{n+1} = y_n + \theta_n\big(\operatorname{prox}_{c_n^3\psi_2}\big(y_n - c_n^3\nabla\psi_1(y_n)\big) - y_n\big), \quad \forall n \geq 1,$$

where $c_n^1 := \text{Linesearch}(u_n, \sigma, \theta, \delta)$, $c_n^2 := \text{Linesearch}(v_n, \sigma, \theta, \delta)$ and $c_n^3 := \text{Linesearch}(y_n, \sigma, \theta, \delta)$.

Theorem 4.2 *Let $\{x_n\}$ be the sequence generated by Algorithm 2. If $\{\gamma_n\}, \{\theta_n\} \subset [0,1]$, $\beta_n \in [a,b] \subset (0,1)$, $\alpha_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\{x_n\}$ converges weakly to a point in Ω.*

Proof We denote

$$\bar{u_n} := \operatorname{prox}_{c_n^1 \psi_2}\bigl(u_n - c_n^1 \nabla \psi_1(u_n)\bigr), \qquad \bar{v_n} := \operatorname{prox}_{c_n^2 \psi_2}\bigl(v_n - c_n^2 \nabla \psi_1(v_n)\bigr), \quad \text{and}$$
$$\bar{y_n} := \operatorname{prox}_{c_n^3 \psi_2}\bigl(y_n - c_n^3 \nabla \psi_1(y_n)\bigr).$$

Let $p^* \in \Omega$. Apply Lemma 4.1, we have for any $n \in \mathbb{N}$ and $p \in \mathcal{H}$

$$\|u_n - p\|^2 - \|\bar{u_n} - p\|^2 \geq 2c_n^1\bigl[\Psi(\bar{u_n}) - \Psi(p)\bigr] + (1 - 2\delta)\|\bar{u_n} - u_n\|^2, \tag{20}$$
$$\|v_n - p\|^2 - \|\bar{v_n} - p\|^2 \geq 2c_n^2\bigl[\Psi(\bar{v_n}) - \Psi(p)\bigr] + (1 - 2\delta)\|\bar{v_n} - v_n\|^2, \tag{21}$$
$$\|y_n - p\|^2 - \|\bar{y_n} - p\|^2 \geq 2c_n^3\bigl[\Psi(\bar{y_n}) - \Psi(p)\bigr] + (1 - 2\delta)\|\bar{y_n} - y_n\|^2. \tag{22}$$

Putting $p = p^*$ in (20)–(22), we have

$$\bigl\|\bar{u_n} - p^*\bigr\| \leq \bigl\|u_n - p^*\bigr\|, \qquad \bigl\|\bar{v_n} - p^*\bigr\| \leq \bigl\|v_n - p^*\bigr\| \quad \text{and} \quad \bigl\|\bar{y_n} - p^*\bigr\| \leq \bigl\|y_n - p^*\bigr\|.$$

So, we obtain

$$\begin{aligned} \bigl\|x_{n+1} - p^*\bigr\| &= \bigl\|(1 - \theta_n)\bigl(y_n - p^*\bigr) + \theta_n\bigl(\bar{y_n} - p^*\bigr)\bigr\| \\ &\leq (1 - \theta_n)\bigl\|y_n - p^*\bigr\| + \theta_n\bigl\|\bar{y_n} - p^*\bigr\| \\ &\leq \bigl\|y_n - p^*\bigr\|. \end{aligned} \tag{23}$$

Similarly, we get

$$\bigl\|y_n - p^*\bigr\| \leq \bigl\|v_n - p^*\bigr\| \quad \text{and} \quad \bigl\|v_n - p^*\bigr\| \leq \bigl\|u_n - p^*\bigr\|. \tag{24}$$

From (23) and (24), we obtain

$$\begin{aligned} \bigl\|x_{n+1} - p^*\bigr\| &\leq \bigl\|u_n - p^*\bigr\| \\ &= \bigl\|x_k + \alpha_n(x_n - x_{n-1}) - p^*\bigr\| \\ &\leq \bigl\|x_n - p^*\bigr\| + \alpha_n\|x_n - x_{n-1}\| \\ &\leq (1 + \alpha_n)\bigl\|x_n - p^*\bigr\| + \alpha_n\bigl\|x_{n-1} - p^*\bigr\|. \end{aligned} \tag{25}$$

This implies by Lemma 2.4 that $\{x_n\}$ is bounded, and hence $\sum_{n=1}^{\infty} \alpha_n\|x_n - x_{n-1}\| < \infty$. It follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{26}$$

By (25) and Lemma 2.5, $\lim_{n\to\infty} \|x_n - p^*\|$ exists and $\lim_{n\to\infty} \|x_n - p^*\| = \lim_{n\to\infty} \|u_n - p^*\|$.

Next, we show that $\omega_w(x_n) \subset \Omega$. Let $x \in \omega_w(x_n)$, i.e., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. By (26), we have $u_{n_k} \rightharpoonup x$.

From (23), (24), and (9), we have

$$\begin{aligned}
\|x_{n+1} - p^*\|^2 &\leq \|v_n - p^*\|^2 \\
&= (1-\beta_n)\|u_n - p^*\|^2 + \beta_n\|\bar{u_n} - p^*\|^2 - \beta_n(1-\beta_n)\|u_n - \bar{u_n}\|^2 \\
&\leq \|u_n - p^*\|^2 - \beta_n(1-\beta_n)\|u_n - \bar{u_n}\|^2 \\
&= \|x_k + \alpha_n(x_n - x_{n-1}) - p^*\|^2 - \beta_n(1-\beta_n)\|u_n - \bar{u_n}\|^2 \\
&\leq \|x_n - p^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - p^*\|\|x_n - x_{n-1}\| \\
&\quad - \beta_n(1-\beta_n)\|u_n - \bar{u_n}\|^2.
\end{aligned} \tag{27}$$

Since $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty}\|x_n - p^*\|$ exists, and $\sum_{n=1}^{\infty}\alpha_n\|x_n - x_{n-1}\| < \infty$, the above inequality implies

$$\lim_{n\to\infty}\|u_n - \bar{u_n}\| = 0. \quad \text{Hence } \bar{u_{n_k}} \rightharpoonup x. \tag{28}$$

Now, let us split our further analysis into two cases.

Case 1. Suppose that the sequence $\{c^1_{n_k}\}$ does not converge to 0. Without loss of generality, there exists $c > 0$ such that $c^1_{n_k} \geq c > 0$. By (B2), we have

$$\lim_{n\to\infty}\|\nabla\psi_1(u_n) - \nabla\psi_1(\bar{u_n})\| = 0. \tag{29}$$

From (8), we get

$$\frac{u_{n_k} - \bar{u_{n_k}}}{c^1_{n_k}} + \nabla\psi_1(\bar{u_{n_k}}) - \nabla\psi_1(u_{k_i}) \in \partial\psi_2(\bar{u_{n_k}}) + \nabla\psi_1(\bar{u_{n_k}}) = \partial\Psi(\bar{u_{n_k}}). \tag{30}$$

By (28)–(30), it follows from Lemma 2.2 that $0 \in \partial\Psi(x)$, that is, $x \in \Omega$.

Case 2. Suppose that the sequence $\{c^1_{n_k}\}$ converges to 0. Define $\widehat{c^1_{n_k}} = \frac{c^1_{n_k}}{\theta} > c^1_{n_k} > 0$ and

$$\widehat{u_{n_k}} := \operatorname{prox}_{\widehat{c^1_{n_k}}\psi_2}\big(u_{n_k} - \widehat{c^1_{n_k}}\nabla\psi_1(u_{n_k})\big).$$

By Lemma 2.3, we have

$$\|u_{n_k} - \widehat{u_{n_k}}\| \leq \frac{\widehat{c^1_{n_k}}}{c^1_{n_k}}\|u_{n_k} - \bar{u_{n_k}}\| = \frac{1}{\theta}\|u_{n_k} - \bar{u_{n_k}}\|. \tag{31}$$

Since $\|u_{n_k} - \bar{u_{n_k}}\| \to 0$, we have $\|u_{n_k} - \widehat{u_{n_k}}\| \to 0$. By (B2), we have

$$\lim_{k\to\infty}\|\nabla\psi_1(u_{n_k}) - \nabla\psi_1(\widehat{u_{n_k}})\| = 0. \tag{32}$$

It follows from the definition of Linesearch that

$$\widehat{c^1_{n_k}}\|\nabla\psi_1(u_{n_k}) - \nabla\psi_1(\widehat{u_{n_k}})\| > \delta\|u_{n_k} - \widehat{u_{n_k}}\|. \tag{33}$$

By (32) and (33), we get

$$\lim_{k\to\infty} \frac{\|u_{n_k} - \widehat{u_{n_k}}\|}{\widehat{c^1_{n_k}}} = 0. \tag{34}$$

From (8), we get

$$\frac{u_{n_k} - \widehat{u_{n_k}}}{\widehat{c^1_{n_k}}} + \nabla\psi_1(\widehat{u_{n_k}}) - \nabla\psi_1(u_{n_k}) \in \partial\psi_2(\widehat{u_{n_k}}) + \nabla\psi_1(\widehat{u_{n_k}}) = \partial\Psi(\widehat{u_{n_k}}). \tag{35}$$

Since $u_{n_k} \rightharpoonup x$ and $\|u_{n_k} - \widehat{u_{n_k}}\| \to 0$, we have $\widehat{u_{n_k}} \rightharpoonup x$. By (34) and (35), it follows from Lemma 2.2 that $0 \in \partial\Psi(x)$, that is, $x \in \Omega$. Therefore, $\omega_w(x_n) \subset \Omega$. Using Lemma 2.6, we obtain that $x_n \rightharpoonup \bar{x}$ for some $\bar{x} \in \Omega$. This completes the proof. □

5 Application in image restoration problems

In this section, we apply the convex minimization problem (1) to image restoration problems. We analyze and compare efficiency of SP-FBS and SP-FBSL algorithms with FBS algorithm, R-FBS algorithm, FISTA algorithm, FBSL algorithm, and FISTAL algorithm. All experiments and visualizations are performed on a laptop computer (Intel Core-i5/4.00 GB RAM/Windows 8/64-bit) with MATLAB.

The image restoration problem is a basic linear inverse problem of the form

$$Ax = y + \varepsilon, \tag{36}$$

where $A \in \mathbb{R}^{M\times N}$ and $y \in \mathbb{R}^M$ are known, ε is an unknown noise, and $x \in \mathbb{R}^N$ is the true image to be estimated. To approximate the original image in (36), we need to minimize the value of ε by using the LASSO model [28]:

$$\min_{x\in\mathbb{R}^N} \left\{ \frac{1}{2}\|Ax - y\|_2^2 + \lambda\|x\|_1 \right\}, \tag{37}$$

where λ is a positive parameter, $\|\cdot\|_1$ is the l_1-norm, and $\|\cdot\|_2$ is the Euclidean norm. It is noted that problem (1) can be applied to LASSO model (37) by setting

$$\psi_1(x) = \frac{1}{2}\|y - Ax\|_2^2 \quad \text{and} \quad \psi_2(x) = \lambda\|x\|_1,$$

where y represents the observed image and $A = RW$, where R is the kernel matrix and W is 2-D fast Fourier transform.

We take two RGB test images (Wat Chedi Luang and antique kitchen with size of 256×256 and 512×512, respectively) and use the *peak signal-to-noise ratio* (PSNR) in decibel (dB) [28] as the image quality measures, which is formulated as follows:

$$\mathrm{PSNR}(x_k) = 10\log_{10}\left(\frac{M\cdot 255^2}{\|x_k - x\|_2^2}\right),$$

where M is the number of image samples, and x is the original image.

Table 1 Details of blurring processes

Scenarios	Kernel matrix
I	Gaussian blur of filter size 9×9 with standard deviation $\hat{\sigma} = 10$
II	Out-of-focus blur (disk) with radius $r = 6$
III	Motion blur specifying with motion length of 21 pixels and motion orientation 11^o

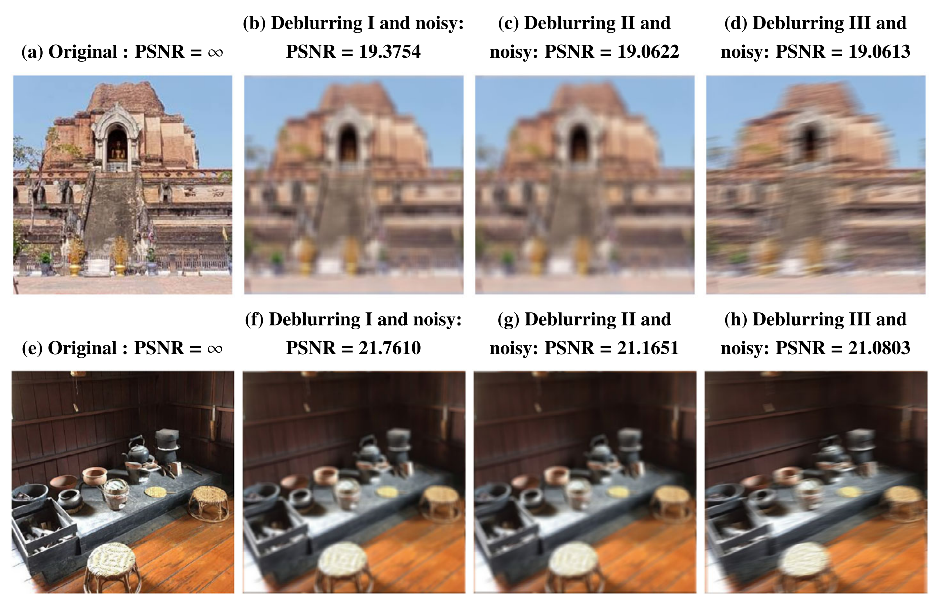

Figure 1 Deblurring of the Wat Chedi Luang and Antique kitchen

Next, we will present three scenarios of blurring processes and noise 10^{-4} in Table 1 and see the original images and the blurred images in Fig. 1.

Next, we test the image recovery performance of the studied algorithms for recovering the images (Wat Chedi Luang and antique kitchen) by setting the parameters as in (38) and by choosing the blurred images as the starting points. The maximum iteration number for all methods is fixed at 200. In LASSO model (37), the regularization parameter is taken by $\lambda = 10^{-4}$. Details of parameters for the studied algorithms are chosen as follows:

$$
\begin{aligned}
&c_n = \frac{1}{L}, \qquad \sigma = 10, \qquad \delta = 0.1, \qquad \theta = 0.9, \qquad \beta_n = \gamma_n = \theta_n = \frac{0.99n}{n+1}, \\
&\alpha_n = \begin{cases} \frac{n}{n+1} & \text{if } 1 \le n \le \mathcal{M}, \\ \frac{1}{2^n} & \text{otherwise}, \end{cases}
\end{aligned}
\tag{38}
$$

where $\mathcal{M}$ is a large positive number which depends on the number of iterations.

The obtained results for deblurring test images (scenarios I–III) are presented in Figs. 2–7. We observe from Figs. 2–8 that if the iteration number is fixed at 200, the PSNR of SP-FBSL algorithm and SP-FBS algorithm are slightly higher than that of the others.

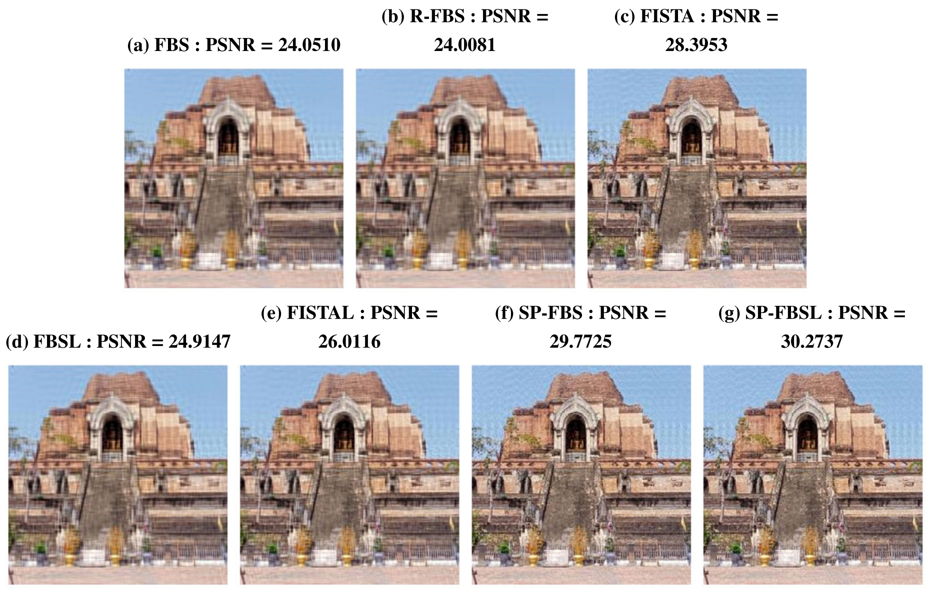

Figure 2 PSNR at the 200th number of iteration of FBS, R-FBS, FISTA, FBSL, FISTAL, SP-FBS and SP-FBSL algorithms for deblurring (scenario I) of the Wat Chedi Luang

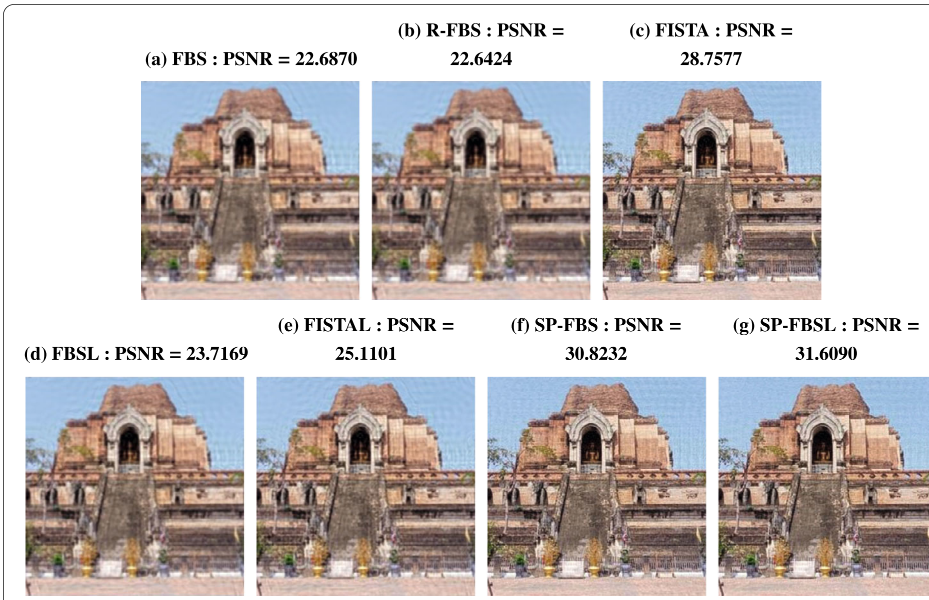

Figure 3 PSNR at the 200th number of iteration of FBS, R-FBS, FISTA, FBSL, FISTAL, SP-FBS and SP-FBSL algorithms for deblurring (scenario II) of the Wat Chedi Luang

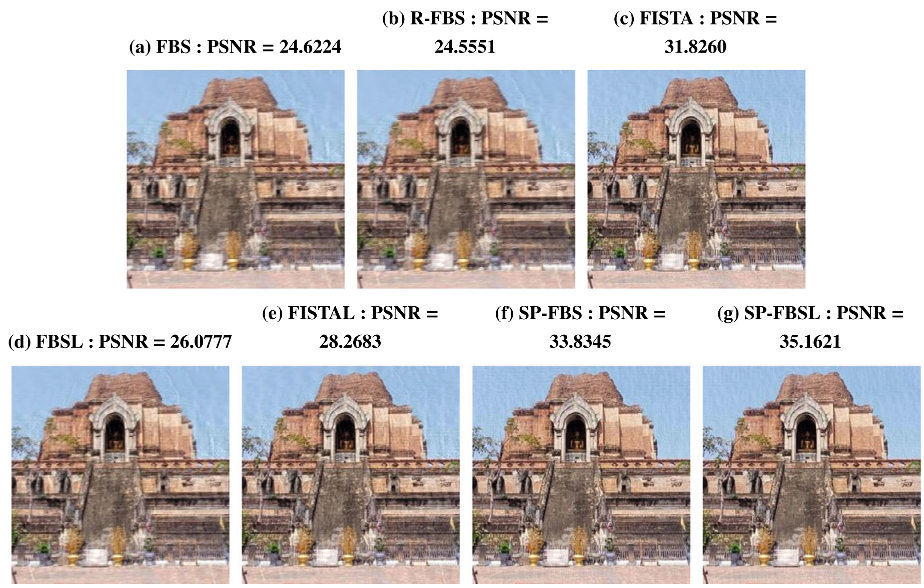

Figure 4 PSNR at the 200th number of iteration of FBS, R-FBS, FISTA, FBSL, FISTAL, SP-FBS and SP-FBSL algorithms for deblurring (scenario III) of the Wat Chedi Luang

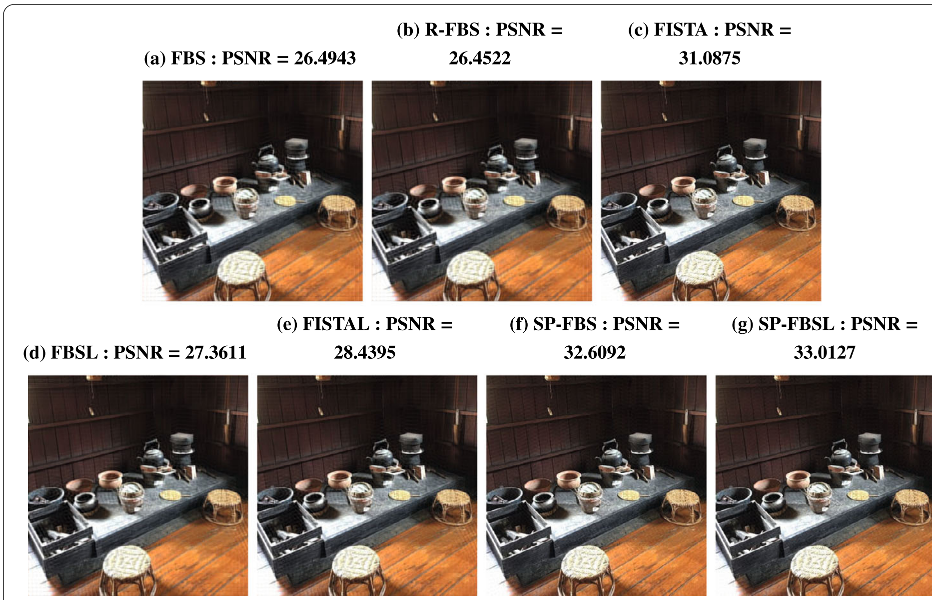

Figure 5 PSNR at the 200th number of iteration of FBS, R-FBS, FISTA, FBSL, FISTAL, SP-FBS and SP-FBSL algorithms for deblurring (scenario I) of the Antique kitchen

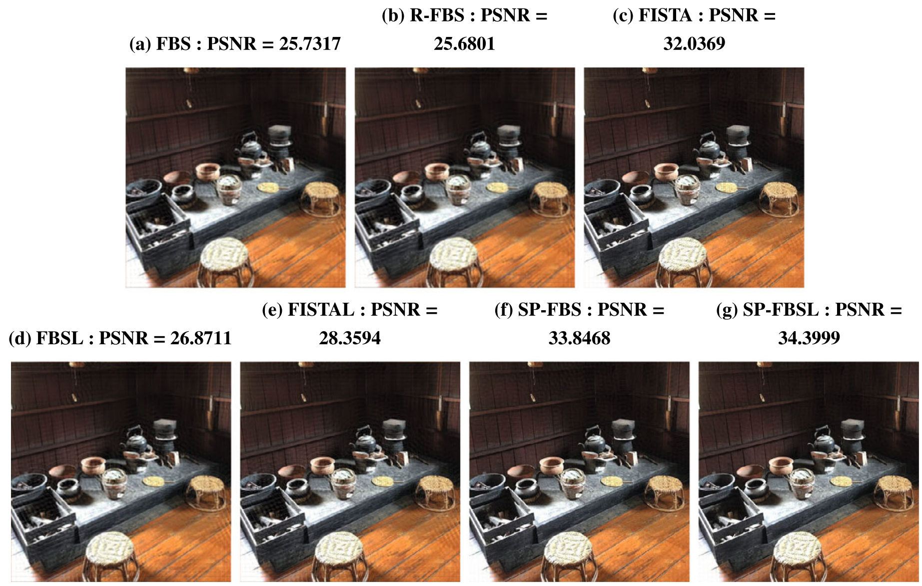

Figure 6 PSNR at the 200th number of iteration of FBS, R-FBS, FISTA, FBSL, FISTAL, SP-FBS and SP-FBSL algorithms for deblurring (scenario II) of the Antique kitchen

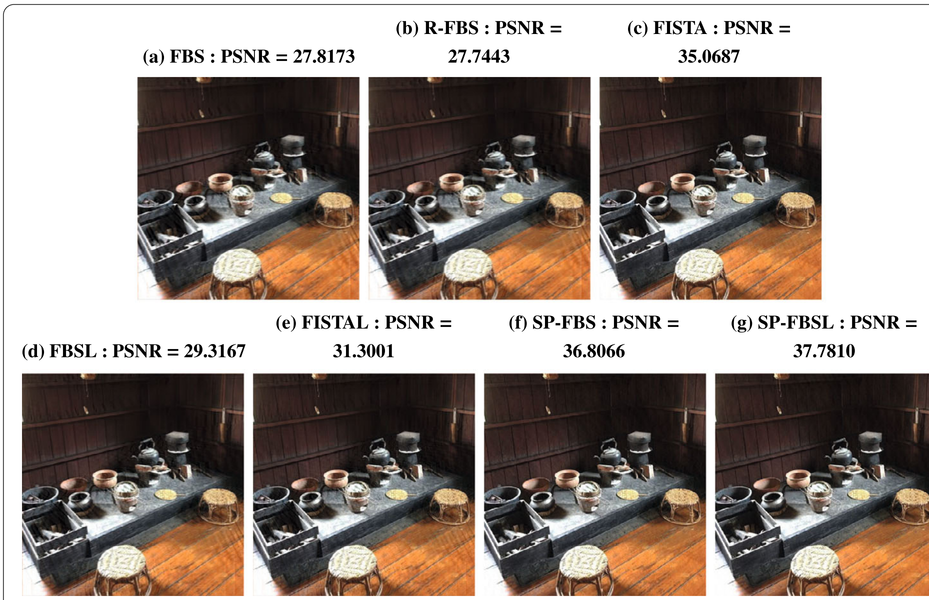

Figure 7 PSNR at the 200th number of iteration of FBS, R-FBS, FISTA, FBSL, FISTAL, SP-FBS and SP-FBSL algorithms for deblurring “scenario (III)” of the Antique kitchen

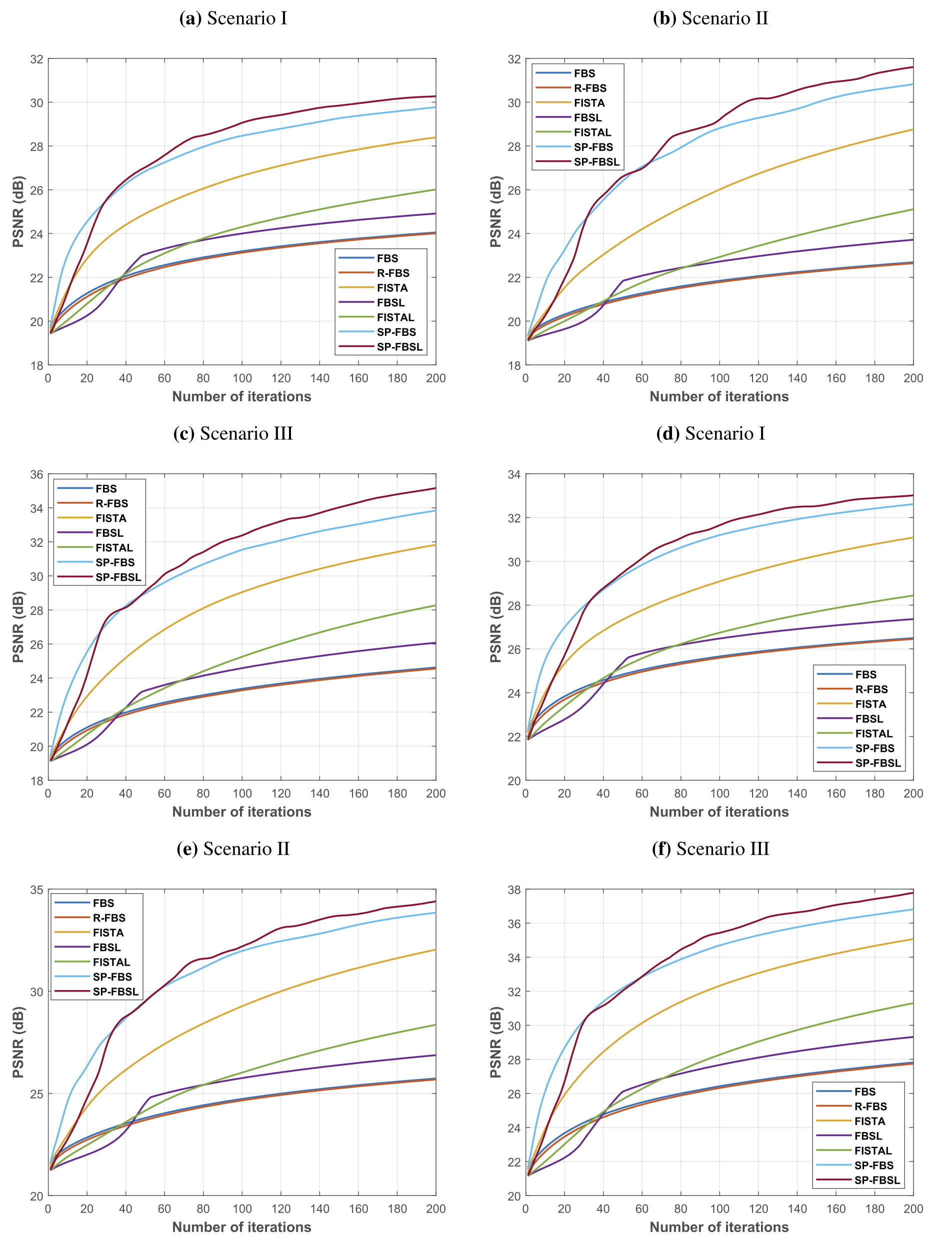

Figure 8 The graphs of PSNR of the algorithms: (**a**)–(**c**) for "Wat Chedi Luang" image and (**d**)–(**f**) for "Antique kitchen" image

6 Conclusions

In this work, we propose an inertial SP-forward–backward splitting (SP-FBS) algorithm for solving convex minimization problems. We prove that a sequence generated by SP-FBS algorithm converges weakly to a solution of problem (1) under the assumption of the Lipschitz continuity of the gradient of the objective function and the stepsize of the algorithm depends on the Lipschitz constant of the gradient of the objective function. Moreover, we remove the Lipschitz continuity assumption on the gradient of the objective function by using the linesearch technique of Cruz and Nghia [9] and propose an inertial SP-forward–backward splitting algorithm with linesearch (SP-FBSL) to solve a convex minimization problem. We also prove that a sequence generated by SP-FBSL converges weakly to a minimizer of the sum of those two convex functions under suitable control conditions. Finally, we present numerical experiments of the studied algorithms for solving image restoration problems. From our experiments, we see that our algorithms have a higher efficiency than the well-known algorithms in [3, 8, 9, 16].

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Availability of data and materials
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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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