Some parameterized Simpson's type inequalities for differentiable convex functions involving generalized fractional integrals

Xuexiao You[1], Muhammad Aamir Ali[2], Hüseyin Budak[3], Hasan Kara[3] and Dafang Zhao[1,4*]

*Correspondence:
dafangzhao@163.com
[1]School of Mathematics and
Statistics, Hubei Normal University,
Huangshi, P.R. China
[4]Northwest Institute of
Eco-Environment and Resources,
Chinese Academy of Sciences,
Lanzhou, P.R. China
Full list of author information is
available at the end of the article

Abstract

In this paper, we establish some new inequalities of Simpson's type for differentiable convex functions involving some parameters and generalized fractional integrals. The results given in this study are a generalization of results proved in (Du, Li and Yang in Appl. Math. Comput. 293:358–369, 2017).

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1 Introduction

Simpson's rules are well-known ways for numerical integration and numerical estimation of definite integrals. This method is known as developed by Simpson (1710–1761). However, Kepler used the same approximation about 100 years ago, so that this method is also known as Kepler's rule. Simpson's rule includes the three-point Newton–Cotes quadrature rule, so estimation based on three-step quadratic kernel is sometimes called a Newton-type result.

1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\, d\varkappa \approx \frac{\kappa_2 - \kappa_1}{6} \left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \mathcal{F}(\kappa_2) \right].$$

2) Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\, d\varkappa \approx \frac{\kappa_2 - \kappa_1}{8} \left[\mathcal{F}(\kappa_1) + 3\mathcal{F}\left(\frac{2\kappa_1 + \kappa_2}{3}\right) + 3\mathcal{F}\left(\frac{\kappa_1 + 2\kappa_2}{3}\right) + \mathcal{F}(\kappa_2) \right].$$

There are a large number of estimations related to these quadrature rules in the literature; one of them is the following estimation known as Simpson's inequality.

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Theorem 1 *Suppose that* $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is a four times continuously differentiable mapping on* (κ_1, κ_2), *and let* $\|\mathcal{F}^{(4)}\|_\infty = \sup_{\varkappa \in (\kappa_1, \kappa_2)} |\mathcal{F}^{(4)}(\varkappa)| < \infty$. *Then we have the inequality*

$$\left| \frac{1}{3} \left[\frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} + 2\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) \right] - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \, d\varkappa \right|$$
$$\leq \frac{1}{2880} \|\mathcal{F}^{(4)}\|_\infty (\kappa_2 - \kappa_1)^4.$$

Integral-type inequalities have numerous applications in the study of qualitative theory of different classes of differential equations and partial differential equations; see, for instance, [3, 4, 8, 11, 14, 15, 20–22] for more detail. In recent years, many authors have focused on Simpson's type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson's and Newton's type results for convex mappings, because convexity theory is an effective and powerful method for solving a large number of problems that arise within different branches of pure and applied mathematics. For example, Dragomir et al. [9] presented new Simpson's type results and their applications to quadrature formulas in numerical integration. Moreover, some inequalities of Simpson's type for *s*-convex functions are deduced by Alomari et al. [2]. Afterwards, Sarikaya et al. [30] observed variants of Simpson's type inequalities based on convexity. In [25] and [26] the authors provided some Newton's type inequalities for harmonic convex and *p*-harmonic convex functions. Additionally, new Newton's type inequalities for functions whose local fractional derivatives are generalized convex are given by Iftikhar et al. [17]. For more recent developments, the reader can consult [1, 5, 12, 27, 31].

2 Generalized fractional integrals

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral [29].

Let a function $\varphi : [0, \infty) \to \{0, \infty)$ satisfy the following condition:

$$\int_0^1 \frac{\varphi(\tau)}{\tau} \, d\tau < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{\kappa_1^+} I_\varphi \mathcal{F}(\varkappa) = \int_{\kappa_1}^{\varkappa} \frac{\varphi(\varkappa - \tau)}{\varkappa - \tau} \mathcal{F}(\tau) \, d\tau, \quad \varkappa > \kappa_1, \tag{2.1}$$

$$_{\kappa_2^-} I_\varphi \mathcal{F}(\varkappa) = \int_{\varkappa}^{\kappa_2} \frac{\varphi(\tau - \varkappa)}{\tau - \varkappa} \mathcal{F}(\tau) \, d\tau, \quad \varkappa < \kappa_2. \tag{2.2}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as the Riemann–Liouville fractional integral, *k*-Riemann–Liouville fractional integral, Katugampola fractional integral, conformable fractional integral, Hadamard fractional integral, etc. These important particular cases of the integral operators (2.1) and (2.2) are mentioned below.

i) If we take $\varphi(\tau) = \tau$, then operators (2.1) and (2.2) reduce to the Riemann integral:

$$I_{\kappa_1^+} \mathcal{F}(\varkappa) = \int_{\kappa_1}^{\varkappa} \mathcal{F}(\tau) \, d\tau, \quad \varkappa > \kappa_1,$$

$$I_{\kappa_2^-}\mathcal{F}(\varkappa) = \int_\varkappa^{\kappa_2} \mathcal{F}(\tau)\,d\tau, \quad \varkappa < \kappa_2.$$

ii) If we take $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$, then operators (2.1) and (2.2) reduce to the Riemann–Liouville fractional integral:

$$I_{\kappa_1^+}^\alpha \mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^\varkappa (\varkappa - \tau)^{\alpha-1} \mathcal{F}(\tau)\,d\tau, \quad \varkappa > \kappa_1,$$

$$I_{\kappa_2^-}^\alpha \mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_\varkappa^{\kappa_2} (\tau - \varkappa)^{\alpha-1} \mathcal{F}(\tau)\,d\tau, \quad \varkappa < \kappa_2.$$

iii) If we take $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)} \tau^{\frac{\alpha}{k}}$, then operators (2.1) and (2.2) reduce to the k-Riemann–Liouville fractional integral:

$$I_{\kappa_1^+,k}^\alpha \mathcal{F}(\varkappa) = \frac{1}{k\Gamma_k(\alpha)} \int_{\kappa_1}^\varkappa (\varkappa - \tau)^{\frac{\alpha}{k}-1} \mathcal{F}(\tau)\,d\tau, \quad \varkappa > \kappa_1,$$

$$I_{\kappa_2^-,k}^\alpha \mathcal{F}(\varkappa) = \frac{1}{k\Gamma_k(\alpha)} \int_\varkappa^{\kappa_2} (\tau - \varkappa)^{\frac{\alpha}{k}-1} \mathcal{F}(\tau)\,d\tau, \quad \varkappa < \kappa_2,$$

where

$$\Gamma_k(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\frac{\tau^k}{k}}\,d\tau, \quad \mathcal{R}(\alpha) > 0,$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0,\ k > 0,$$

are given by Mubeen and Habibullah [24].

Sarikaya and Ertuğral also established the following Hermite–Hadamard inequality for the generalized fractional integral operators.

Theorem 2 ([29]) *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$. Then we have the following inequalities for fractional integral operators:*

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \le \frac{1}{2\Delta(1)}\left[{}_{\kappa_1^+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\kappa_2^-}I_\varphi \mathcal{F}(\kappa_1)\right] \le \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}, \tag{2.3}$$

where the mapping $\Delta : [0, 1] \to \mathbb{R}$ is defined by

$$\Delta(\tau) = \int_0^\tau \frac{\varphi((\kappa_2 - \kappa_1)u)}{u}\,du.$$

In the literature, there are several papers on inequalities for generalized fractional integrals. We refer the reader to [6, 7, 13, 16, 18, 19, 23, 28, 32].

3 A lemma

In this section, we propose a parameterized identity involving the ordinary first derivative via generalized fractional integrals.

Lemma 1 *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable function on (κ_1, κ_2). If $\mathcal{F}'$ is continuous on $[\kappa_1, \kappa_2]$, then for $\lambda, \mu \geq 0$, we have the identity*

$$
\begin{aligned}
&\Delta(1)\lambda\mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2) \\
&\quad - \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\right] \\
&= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}} \left(\Delta(\tau) - \Delta(1)\lambda\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right) d\tau \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left(\Delta(\tau) - \Delta(1)\mu\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right) d\tau\right].
\end{aligned}
\tag{3.1}
$$

Proof Applying fundamental rules of integration, we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \left(\Delta(\tau) - \Delta(1)\lambda\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right) d\tau \\
&= \frac{1}{\kappa_2 - \kappa_1}\left[\Delta\left(\frac{1}{2}\right) - \Delta(1)\lambda\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \lambda\Delta(1)\mathcal{F}(\kappa_1) - {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\right]
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(\Delta(\tau) - \Delta(1)\mu\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right) d\tau \\
&= \frac{1}{\kappa_2 - \kappa_1}\left[\Delta(1)(1 - \mu)\mathcal{F}(\kappa_2) - \Delta\left(\frac{1}{2}\right)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \right. \\
&\quad \left. + \Delta(1)\mu\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) - {}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2)\right].
\end{aligned}
\tag{3.3}
$$

By adding (3.1) and (3.3) we obtain the required equality (3.1).   □

Remark 1 If $\varphi(\tau) = \tau$ in Lemma 1, then Lemma 1 becomes [10, Lemma 2.1 for $m = 1$].

Corollary 1 *In Lemma 1, if we set $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$, then we obtain the following new Riemann–Liouville fractional integral identity*:

$$
\begin{aligned}
&\lambda\mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + (1 - \mu)\mathcal{F}(\kappa_2) \\
&\quad - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J_{\frac{\kappa_1+\kappa_2}{2}+}^\alpha \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1+\kappa_2}{2}-}^\alpha \mathcal{F}(\kappa_1)\right] \\
&= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}} \left(\tau^\alpha - \lambda\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right) d\tau \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left(\tau^\alpha - \mu\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right) d\tau\right].
\end{aligned}
$$

Corollary 2 *In Lemma* 1, *if we set* $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then we obtain the following new k-Riemann–Liouville fractional integral identity*:

$$\lambda\mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + (1 - \mu)\mathcal{F}(\kappa_2)$$

$$- \frac{\Gamma_k(\alpha + 1)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathcal{F}(\kappa_2) + J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathcal{F}(\kappa_1)\right]$$

$$= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}}\left(\tau^{\frac{\alpha}{k}} - \lambda\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right)d\tau\right.$$

$$\left.+ \int_{\frac{1}{2}}^1\left(\tau^{\frac{\alpha}{k}} - \mu\right)\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right)d\tau\right].$$

4 Simpson's inequalities for generalized fractional integrals

In this section, we establish some new Simpson's type inequalities for differentiable convex functions via generalized fractional integrals.

Theorem 3 *We assume that the conditions of Lemma* 1 *hold. If the mapping* $|\mathcal{F}'|$ *is convex on* $[\kappa_1, \kappa_2]$, *then we have the following inequality for generalized fractional integrals*:

$$\left|\Delta(1)\lambda\mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2)\right.$$

$$\left.- \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\right]\right|$$

$$\leq (\kappa_2 - \kappa_1)\left[\left|\mathcal{F}'(\kappa_2)\right|\left\{\Pi_1^\varphi(\lambda) + \Pi_3^\varphi(\mu)\right\} + \left|\mathcal{F}'(\kappa_1)\right|\left\{\Pi_2^\varphi(\lambda) + \Pi_4^\varphi(\mu)\right\}\right], \tag{4.1}$$

where

$$\Pi_1^\varphi(\lambda) = \int_0^{\frac{1}{2}}\tau\left|\Delta(\tau) - \Delta(1)\lambda\right|d\tau, \qquad \Pi_2^\varphi(\lambda) = \int_0^{\frac{1}{2}}(1 - \tau)\left|\Delta(\tau) - \Delta(1)\lambda\right|d\tau,$$

$$\Pi_3^\varphi(\mu) = \int_{\frac{1}{2}}^1\tau\left|\Delta(\tau) - \Delta(1)\mu\right|d\tau, \qquad \Pi_4^\varphi(\mu) = \int_{\frac{1}{2}}^1(1 - \tau)\left|\Delta(\tau) - \Delta(1)\mu\right|d\tau.$$

Proof By taking the modulus in Lemma 1 and using the properties of the modulus, we obtain that

$$\left|\Delta(1)\lambda\mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2)\right.$$

$$\left.- \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\right]\right|$$

$$\leq (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}}\left|\Delta(\tau) - \Delta(1)\lambda\right|\left|\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right)\right|d\tau\right.$$

$$\left.+ \int_{\frac{1}{2}}^1\left|\Delta(\tau) - \Delta(1)\mu\right|\left|\mathcal{F}'\left(\tau\kappa_2 + (1 - \tau)\kappa_1\right)\right|d\tau\right]. \tag{4.2}$$

Since the mapping $|\mathcal{F}'|$ is convex on $[\kappa_1, \kappa_2]$, we have

$$
\left| \Delta(1)\lambda\mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2) \right.
$$

$$
\left. - \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1)\left[|\mathcal{F}'(\kappa_2)|\left(\int_0^{\frac{1}{2}} \tau\,|\Delta(\tau) - \Delta(1)\lambda|\,d\tau + \int_{\frac{1}{2}}^1 \tau\,|\Delta(\tau) - \Delta(1)\mu|\,d\tau \right) \right.
$$

$$
\left. + |\mathcal{F}'(\kappa_1)|\left(\int_0^{\frac{1}{2}} (1 - \tau)\,|\Delta(\tau) - \Delta(1)\lambda|\,d\tau + \int_{\frac{1}{2}}^1 (1 - \tau)\,|\Delta(\tau) - \Delta(1)\mu|\,d\tau \right) \right]
$$

$$
= (\kappa_2 - \kappa_1)\left[|\mathcal{F}'(\kappa_2)|\left\{ \Pi_1^\varphi(\lambda) + \Pi_3^\varphi(\mu) \right\} + |\mathcal{F}'(\kappa_1)|\left\{ \Pi_2^\varphi(\lambda) + \Pi_4^\varphi(\mu) \right\} \right],
$$

which ends the proof.   □

Remark 2 In Theorem 3, if we take $\varphi(\tau) = \tau$, then Theorem 3 reduces to [10, Theorem 2.1 for $s = m = 1$].

Corollary 3 *In Theorem* 3, *if we use* $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\kappa_1)}$, *then we obtain the following parameterized Simpson's type inequality for Riemann–Liouville fractional integrals*:

$$
\left| \lambda\mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + (1 - \mu)\mathcal{F}(\kappa_2) \right.
$$

$$
\left. - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J_{\frac{\kappa_1+\kappa_2}{2}+}^\alpha \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1+\kappa_2}{2}-}^\alpha \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1)\left[|\mathcal{F}'(\kappa_2)|\left\{ \Pi_1^\alpha(\lambda) + \Pi_3^\alpha(\mu) \right\} + |\mathcal{F}'(\kappa_1)|\left\{ \Pi_2^\alpha(\lambda) + \Pi_4^\alpha(\mu) \right\} \right], \tag{4.3}
$$

where

$$
\Pi_1^\alpha(\lambda) = \frac{\alpha}{\alpha + 2}\lambda^{\frac{\alpha+2}{\alpha}} - \frac{\lambda}{8} + \frac{1}{2^{\alpha+2}(\alpha + 2)},
$$

$$
\Pi_2^\alpha(\lambda) = \frac{2\alpha}{\alpha + 1}\lambda^{\frac{\alpha+1}{\alpha}} - \frac{\lambda}{2} + \frac{1}{2^{\alpha+1}(\alpha + 1)} - \Pi_1^\alpha(\lambda),
$$

$$
\Pi_3^\alpha(\mu) = \frac{\alpha}{\alpha + 2}\mu^{\frac{\alpha+2}{\alpha}} - \frac{5}{8}\mu + \frac{2^{\alpha+2} + 1}{2^{\alpha+2}(\alpha + 2)},
$$

and

$$
\Pi_4^\alpha(\mu) = \frac{2\alpha}{\alpha + 1}\mu^{\frac{\alpha+1}{\alpha}} - \frac{3}{2}\mu + \frac{2^{\alpha+1} + 1}{2^{\alpha+1}(\alpha + 1)} - \Pi_3^\alpha(\mu).
$$

Corollary 4 *In Theorem* 3, *if we use* $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\kappa_1)}$, *then we obtain the following parameterized Simpson's type inequality for k-Riemann–Liouville fractional integrals*:

$$
\left| \lambda\mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + (1 - \mu)\mathcal{F}(\kappa_2) \right.
$$

$$
\left. - \frac{\Gamma_k(\alpha + 1)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J_{\frac{\kappa_1+\kappa_2}{2}+,k}^\alpha \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1+\kappa_2}{2}-,k}^\alpha \mathcal{F}(\kappa_1) \right] \right|
$$

$$\leq (\kappa_2 - \kappa_1)\Big[\big|\mathcal{F}'(\kappa_2)\big|\big\{\Pi_1^{\frac{\alpha}{k}}(\lambda) + \Pi_3^{\frac{\alpha}{k}}(\mu)\big\} + \big|\mathcal{F}'(\kappa_1)\big|\big\{\Pi_2^{\frac{\alpha}{k}}(\lambda) + \Pi_4^{\frac{\alpha}{k}}(\mu)\big\}\Big], \tag{4.4}$$

where

$$\Pi_1^{\frac{\alpha}{k}}(\lambda) = \frac{\alpha}{\alpha + 2k}\lambda^{\frac{\alpha+2k}{\alpha}} - \frac{\lambda}{8} + \frac{1}{2^{\frac{\alpha+2k}{k}}(\frac{\alpha+2k}{k})},$$

$$\Pi_2^{\frac{\alpha}{k}}(\lambda) = \frac{2\alpha}{\alpha + k}\lambda^{\frac{\alpha+k}{\alpha}} - \frac{\lambda}{2} + \frac{1}{2^{\frac{\alpha+k}{k}}(\frac{\alpha+k}{k})} - \Pi_1^{\frac{\alpha}{k}},$$

$$\Pi_3^{\frac{\alpha}{k}}(\mu) = \frac{\alpha}{\alpha + 2k}\mu^{\frac{\alpha+2k}{\alpha}} - \frac{5}{8}\mu + \frac{2^{\frac{\alpha+2k}{k}} + 1}{2^{\frac{\alpha+2k}{k}}(\frac{\alpha+2k}{k})},$$

and

$$\Pi_4^{\frac{\alpha}{k}}(\mu) = \frac{2\alpha}{\alpha + k}\mu^{\frac{\alpha+1}{\alpha}} - \frac{3}{2}\mu + \frac{2^{\frac{\alpha+k}{k}} + 1}{2^{\frac{\alpha+k}{k}}(\frac{\alpha+k}{k})} - \Pi_3^{\frac{\alpha}{k}}.$$

Theorem 4 *We assume that the conditions of Lemma* 1 *hold. If the mapping* $|\mathcal{F}|^{p_1}$, $p_1 \geq 1$, *is convex on* $[\kappa_1, \kappa_2]$, *then we have the following inequality of Simpson's type for generalized fractional integrals:*

$$\left|\Delta(1)\lambda\mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2)\right.$$
$$\left. - \Big[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\Big]\right|$$
$$\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}}\big|\Delta(\tau) - \Delta(1)\lambda\big|\,d\tau\right)^{1-\frac{1}{p_1}}\left(\Pi_1^{\varphi}(\lambda)\big|\mathcal{F}'(\kappa_2)\big|^{p_1} + \Pi_2^{\varphi}(\lambda)\big|\mathcal{F}'(\kappa_1)\big|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1\big|\Delta(\tau) - \Delta(1)\mu\big|\,d\tau\right)^{1-\frac{1}{p_1}}\left(\Pi_3^{\varphi}(\mu)\big|\mathcal{F}'(\kappa_2)\big|^{p_1} + \Pi_4^{\varphi}(\mu)\big|\mathcal{F}'(\kappa_1)\big|^{p_1}\right)^{\frac{1}{p_1}}\right], \tag{4.5}$$

where $\Pi_1^{\varphi}(\lambda)$, $\Pi_2^{\varphi}(\lambda)$, $\Pi_3^{\varphi}(\mu)$, *and* $\Pi_4^{\varphi}(\mu)$ *are defined in Theorem* 3.

Proof Reusing inequality (4.2), by the power mean inequality we have

$$\left|\Delta(1)\lambda\mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2)\right.$$
$$\left. - \Big[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\Big]\right|$$
$$\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}}\big|\Delta(\tau) - \Delta(1)\lambda\big|\,d\tau\right)^{1-\frac{1}{p_1}}\right.$$
$$\times \left(\int_0^{\frac{1}{2}}\big|\Delta(\tau) - \Delta(1)\lambda\big|\big|\mathcal{F}'(\tau\kappa_2 + (1 - \tau)\kappa_1)\big|^{p_1}\,d\tau\right)^{\frac{1}{p_1}}$$
$$+ \left(\int_{\frac{1}{2}}^1\big|\Delta(\tau) - \Delta(1)\mu\big|\,d\tau\right)^{1-\frac{1}{p_1}}$$

$$\times \left(\int_{\frac{1}{2}}^{1} \left| \Delta(\tau) - \Delta(1)\mu \right| \left| \mathcal{F}'\big(\tau\kappa_2 + (1-\tau)\kappa_1\big) \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \right].$$

Using the convexity of $|\mathcal{F}'|^{p_1}$, we have

$$\left| \Delta(1)\lambda \mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \Delta(1)(1-\mu)\mathcal{F}(\kappa_2) \right.$$

$$\left. - \left[{}_{\frac{\kappa_1+\kappa_2}{2}+} I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-} I_\varphi \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \Delta(\tau) - \Delta(1)\lambda \right| d\tau \right)^{1-\frac{1}{p_1}} \right.$$

$$\times \left(\left| \mathcal{F}'(\kappa_2) \right|^{p_1} \int_0^{\frac{1}{2}} \tau \left| \Delta(\tau) - \Delta(1)\lambda \right| d\tau \right.$$

$$+ \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \int_0^{\frac{1}{2}} (1-\tau) \left| \Delta(\tau) - \Delta(1)\lambda \right| d\tau \right)^{\frac{1}{p_1}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \left| \Delta(\tau) - \Delta(1)\mu \right| d\tau \right)^{1-\frac{1}{p_1}}$$

$$\times \left(\left| \mathcal{F}'(\kappa_2) \right|^{p_1} \int_{\frac{1}{2}}^{1} \tau \left| \Delta(\tau) - \Delta(1)\mu \right| d\tau \right.$$

$$+ \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \int_{\frac{1}{2}}^{1} (1-\tau) \left| \Delta(\tau) - \Delta(1)\mu \right| d\tau \right)^{\frac{1}{p_1}} \right]$$

$$= (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \Delta(\tau) - \Delta(1)\lambda \right| d\tau \right)^{1-\frac{1}{p_1}} \left(\Pi_1^\varphi(\lambda) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^\varphi(\lambda) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right.$$

$$+ \left(\int_{\frac{1}{2}}^{1} \left| \Delta(\tau) - \Delta(1)\mu \right| d\tau \right)^{1-\frac{1}{p_1}} \left(\Pi_3^\varphi(\mu) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^\varphi(\mu) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right],$$

which finishes the proof. $\square$

Remark 3 In Theorem 4, if we assume that $\varphi(\tau) = \tau$, then Theorem 4 reduces to [10, Theorem 2.3 for $s = m = 1$].

Corollary 5 *If we assume that $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 4, then we have the following parameterized Simpson's type inequality for Riemann–Liouville fractional integrals:*

$$\left| \lambda \mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + (1-\mu)\mathcal{F}(\kappa_2) \right.$$

$$\left. - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[\left(\Pi_1^\alpha(\lambda) + \Pi_2^\alpha(\lambda) \right)^{1-\frac{1}{p_1}} \left(\Pi_1^\alpha(\lambda) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^\alpha(\lambda) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right.$$

$$+ \left(\Pi_3^\alpha(\mu) + \Pi_4^\alpha(\mu) \right)^{1-\frac{1}{p_1}} \left(\Pi_3^\alpha(\mu) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^\alpha(\mu) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right], \tag{4.6}$$

where $\Pi_1^\alpha(\lambda)$, $\Pi_2^\alpha(\lambda)$, $\Pi_3^\alpha(\mu)$, and $\Pi_4^\alpha(\mu)$ are defined in Corollary 3.

Corollary 6 *If we assume that* $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem* 4, *then we have the following parameterized Simpson's type inequality for* k*-Riemann–Liouville fractional integrals:*

$$
\begin{aligned}
&\left| \lambda \mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + (1 - \mu)\mathcal{F}(\kappa_2) \right. \\
&\quad \left. - \frac{\Gamma_k(\alpha + 1)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{\frac{\kappa_1 + \kappa_2}{2}+,k}\mathcal{F}(\kappa_2) + J^{\alpha}_{\frac{\kappa_1 + \kappa_2}{2}-,k}\mathcal{F}(\kappa_1) \right] \right| \\
&\leq (\kappa_2 - \kappa_1)\left[\left(\Pi_1^{\frac{\alpha}{k}}(\lambda) + \Pi_2^{\frac{\alpha}{k}}(\lambda)\right)^{1 - \frac{1}{p_1}} \left(\Pi_1^{\frac{\alpha}{k}}(\lambda)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_2^{\frac{\alpha}{k}}(\lambda)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \right. \\
&\quad \left. + \left(\Pi_3^{\frac{\alpha}{k}}(\mu) + \Pi_4^{\frac{\alpha}{k}}(\mu)\right)^{1 - \frac{1}{p_1}} \left(\Pi_3^{\frac{\alpha}{k}}(\mu)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_4^{\frac{\alpha}{k}}(\mu)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \right],
\end{aligned} \tag{4.7}
$$

where $\Pi_1^{\frac{\alpha}{k}}(\lambda)$, $\Pi_2^{\frac{\alpha}{k}}(\lambda)$, $\Pi_3^{\frac{\alpha}{k}}(\mu)$, *and* $\Pi_4^{\frac{\alpha}{k}}(\mu)$ *are described in Corollary* 4.

Theorem 5 *We assume that the conditions of Lemma* 1 *hold. If the mapping* $|\mathcal{F}'|^{r_1}$, $r_1 > 1$ *is convex on* $[\kappa_1, \kappa_2]$, *then we have the following inequality of Simpson's type for generalized fractional integrals:*

$$
\begin{aligned}
&\left| \Delta(1)\lambda \mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2) \right. \\
&\quad \left. - \left[{}_{\frac{\kappa_1 + \kappa_2}{2}+}I_{\varphi}\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1 + \kappa_2}{2}-}I_{\varphi}\mathcal{F}(\kappa_1) \right] \right| \\
&\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}} \left|\Delta(\tau) - \Delta(1)\lambda\right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \left|\Delta(\tau) - \Delta(1)\mu\right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right],
\end{aligned} \tag{4.8}
$$

where $\frac{1}{p_1} + \frac{1}{r_1} = 1$.

Proof Reusing inequality (4.2), by the well-known Hölder inequality we have

$$
\begin{aligned}
&\left| \Delta(1)\lambda \mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2) \right. \\
&\quad \left. - \left[{}_{\frac{\kappa_1 + \kappa_2}{2}+}I_{\varphi}\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1 + \kappa_2}{2}-}I_{\varphi}\mathcal{F}(\kappa_1) \right] \right| \\
&\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}} \left|\Delta(\tau) - \Delta(1)\lambda\right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\int_0^{\frac{1}{2}} \left|\mathcal{F}'\bigl(\tau\kappa_2 + (1 - \tau)\kappa_1\bigr)\right|^{r_1} d\tau \right)^{\frac{1}{r_1}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \left|\Delta(\tau) - \Delta(1)\mu\right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\int_{\frac{1}{2}}^1 \left|\mathcal{F}'\bigl(\tau\kappa_2 + (1 - \tau)\kappa_1\bigr)\right|^{r_1} d\tau \right)^{\frac{1}{r_1}} \right].
\end{aligned}
$$

Since $|\mathcal{F}'|^{r_1}$ is convex, we have

$$
\left| \Delta(1)\lambda \mathcal{F}(\kappa_1) + \Delta(1)(\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)(1 - \mu)\mathcal{F}(\kappa_2) \right.
$$
$$
\left. - \left[{}_{\frac{\kappa_1 + \kappa_2}{2}+}I_{\varphi}\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1 + \kappa_2}{2}-}I_{\varphi}\mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \Bigg[\left(\int_0^{\frac{1}{2}} \left| \Delta(\tau) - \Delta(1)\lambda \right|^{p_1} d\tau \right)^{\frac{1}{p_1}}
$$

$$
\times \left(\left| \mathcal{F}'(\kappa_2) \right|^{r_1} \int_0^{\frac{1}{2}} \tau \, d\tau + \left| \mathcal{F}'(\kappa_1) \right|^{r_1} \int_0^{\frac{1}{2}} (1 - \tau) \, d\tau \right)^{\frac{1}{r_1}}
$$

$$
+ \left(\int_{\frac{1}{2}}^1 \left| \Delta(\tau) - \Delta(1)\mu \right|^{p_1} d\tau \right)^{\frac{1}{p_1}}
$$

$$
\times \left(\left| \mathcal{F}'(\kappa_2) \right|^{r_1} \int_{\frac{1}{2}}^1 \tau \, d\tau + \left| \mathcal{F}'(\kappa_1) \right|^{r_1} \int_{\frac{1}{2}}^1 (1 - \tau) \, d\tau \right)^{\frac{1}{r_1}} \Bigg]
$$

$$
= (\kappa_2 - \kappa_1) \Bigg[\left(\int_0^{\frac{1}{2}} \left| \Delta(\tau) - \Delta(1)\lambda \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + 3\left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}}
$$

$$
+ \left(\int_{\frac{1}{2}}^1 \left| \Delta(\tau) - \Delta(1)\mu \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + \left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}} \Bigg],
$$

which completes the proof. □

Remark 4 In Theorem 5, if we set $\varphi(\tau) = \tau$, then Theorem 5 reduces to [10, Theorem 2.2 for $s = m = 1$].

Corollary 7 *In Theorem 5, if we set $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$, then we obtain the following parameterized Simpson's type inequality for Riemann-Liouville fractional integrals:*

$$
\left| \lambda \mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + (1 - \mu)\mathcal{F}(\kappa_2) \right.
$$

$$
\left. - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \Bigg[\left(\int_0^{\frac{1}{2}} \left| \tau^\alpha - \lambda \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + 3\left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}}
$$

$$
+ \left(\int_{\frac{1}{2}}^1 \left| \tau^\alpha - \mu \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + \left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}} \Bigg].
$$

Corollary 8 *In Theorem 5, if we set $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we obtain the following parameterized Simpson's type inequality for k-Riemann–Liouville fractional integrals:*

$$
\left| \lambda \mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + (1 - \mu)\mathcal{F}(\kappa_2) \right.
$$

$$
\left. - \frac{\Gamma_k(\alpha + 1)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \Bigg[\left(\int_0^{\frac{1}{2}} \left| \tau^{\frac{\alpha}{k}} - \lambda \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + 3\left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}}
$$

$$
+ \left(\int_{\frac{1}{2}}^1 \left| \tau^{\frac{\alpha}{k}} - \mu \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + \left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}} \Bigg].
$$

5 Particular cases

In this section, we give some particular cases of our main results.

Remark 5 From Lemma 1 we get the following identities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the new identity:

$$
\frac{1}{6}\left[\Delta(1)\mathcal{F}(\kappa_1) + \Delta(1)4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \Delta(1)\mathcal{F}(\kappa_2)\right]
$$
$$
- \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi \mathcal{F}(\kappa_1)\right]
$$
$$
= (\kappa_2 - \kappa_1) \times \left[\int_0^{\frac{1}{2}}\left(\Delta(\tau) - \frac{\Delta(1)}{6}\right)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau\right.
$$
$$
\left. + \int_{\frac{1}{2}}^1\left(\Delta(\tau) - \frac{5\Delta(1)}{6}\right)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau\right].
$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the identity

$$
\frac{1}{\Delta(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi \mathcal{F}(\kappa_1)\right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}
$$
$$
= \frac{(\kappa_2 - \kappa_1)}{2\Delta(1)}\int_0^1\big(\Delta(1) - 2\Delta(\tau)\big)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau.
$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the identity

$$
\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{1}{\Delta(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi \mathcal{F}(\kappa_1)\right]
$$
$$
= \frac{(\kappa_2 - \kappa_1)}{\Delta(1)}\left[\int_0^{\frac{1}{2}}\Delta(\tau)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau\right.
$$
$$
\left. + \int_{\frac{1}{2}}^1\big(\Delta(\tau) - \Delta(1)\big)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau\right].
$$

Remark 6 From Corollary 1 we have the following identities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the identity

$$
\frac{1}{6}\left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \mathcal{F}(\kappa_2)\right]
$$
$$
- \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathcal{F}(\kappa_1)\right]
$$
$$
= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}}\left(\tau^\alpha - \frac{1}{6}\right)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau\right.
$$
$$
\left. + \int_{\frac{1}{2}}^1\left(\tau^\alpha - \frac{5}{6}\right)\mathcal{F}'\big(\tau\kappa_2 + (1 - \tau)\kappa_1\big)\,d\tau\right].
$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the identity

$$\frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathcal{F}(\kappa_1)\right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}$$

$$= \frac{(\kappa_2 - \kappa_1)}{2}\int_0^1 \left(1 - 2\tau^\alpha\right)\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau.$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the identity

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathcal{F}(\kappa_1)\right]$$

$$= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}}\left(\tau^\alpha - 1\right)\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \tau^\alpha \mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau\right].$$

Remark 7 From Corollary 2 we have the following identities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the identity

$$\frac{1}{6}\left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \mathcal{F}(\kappa_2)\right]$$

$$- \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathcal{F}(\kappa_1)\right]$$

$$= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}}\left(\tau^{\frac{\alpha}{k}} - \frac{1}{6}\right)\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \left(\tau^{\frac{\alpha}{k}} - \frac{5}{6}\right)\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau\right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the identity

$$\frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathcal{F}(\kappa_1)\right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}$$

$$= \frac{(\kappa_2 - \kappa_1)}{2}\int_0^1 \left(1 - 2\tau^{\frac{\alpha}{k}}\right)\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau.$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the identity

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathcal{F}(\kappa_1)\right]$$

$$= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{2}}\left(\tau^{\frac{\alpha}{k}} - 1\right)\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \tau^{\frac{\alpha}{k}}\mathcal{F}'\left(\tau\kappa_2 + (1-\tau)\kappa_1\right)d\tau\right].$$

Remark 8 From Theorem 3 we have the following new inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for generalized fractional integrals:

$$\left| \frac{1}{6} \left[\Delta(1)\mathcal{F}(\kappa_1) + \Delta(1)4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \Delta(1)\mathcal{F}(\kappa_2) \right] \right.$$

$$- \left. \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[|\mathcal{F}'(\kappa_2)| \left\{ \Pi_1^\varphi\left(\frac{1}{6} \right) + \Pi_3^\varphi\left(\frac{5}{6} \right) \right\} + |\mathcal{F}'(\kappa_1)| \left\{ \Pi_2^\varphi\left(\frac{1}{6} \right) + \Pi_4^\varphi\left(\frac{5}{6} \right) \right\} \right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for generalized fractional integrals:

$$\left| \frac{1}{\Delta(1)} \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[|\mathcal{F}'(\kappa_2)| \left\{ \Pi_1^\varphi\left(\frac{1}{2} \right) + \Pi_3^\varphi\left(\frac{1}{2} \right) \right\} + |\mathcal{F}'(\kappa_1)| \left\{ \Pi_2^\varphi\left(\frac{1}{2} \right) + \Pi_4^\varphi\left(\frac{1}{2} \right) \right\} \right].$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\left| \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{1}{\Delta(1)} \left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[|\mathcal{F}'(\kappa_2)| \left\{ \Pi_1^\varphi(0) + \Pi_3^\varphi(1) \right\} + |\mathcal{F}'(\kappa_1)| \left\{ \Pi_2^\varphi(0) + \Pi_4^\varphi(1) \right\} \right].$$

Remark 9 From Corollary 3 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for Riemann–Liouville fractional integrals:

$$\left| \frac{1}{6} \left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \mathcal{F}(\kappa_2) \right] \right.$$

$$- \left. \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[|\mathcal{F}'(\kappa_2)| \left\{ \Pi_1^\alpha\left(\frac{1}{6} \right) + \Pi_3^\alpha\left(\frac{5}{6} \right) \right\} + |\mathcal{F}'(\kappa_1)| \left\{ \Pi_2^\alpha\left(\frac{1}{6} \right) + \Pi_4^\alpha\left(\frac{5}{6} \right) \right\} \right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for Riemann–Liouville fractional integrals:

$$\left| \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[|\mathcal{F}'(\kappa_2)| \left\{ \Pi_1^\alpha\left(\frac{1}{2} \right) + \Pi_3^\alpha\left(\frac{1}{2} \right) \right\} + |\mathcal{F}'(\kappa_1)| \left\{ \Pi_2^\alpha\left(\frac{1}{2} \right) + \Pi_4^\alpha\left(\frac{1}{2} \right) \right\} \right].$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\left| \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] \right|$$
$$\leq (\kappa_2 - \kappa_1) \left[\left| \mathcal{F}'(\kappa_2) \right| \left\{ \Pi^\alpha_1(0) + \Pi^\alpha_3(1) \right\} + \left| \mathcal{F}'(\kappa_1) \right| \left\{ \Pi^\alpha_2(0) + \Pi^\alpha_4(1) \right\} \right].$$

Remark 10 From Corollary 4 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for k-Riemann–Liouville fractional integrals:

$$\left| \frac{1}{6} \left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \mathcal{F}(\kappa_2) \right] \right.$$
$$\left. - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] \right|$$
$$\leq (\kappa_2 - \kappa_1) \left[\left| \mathcal{F}'(\kappa_2) \right| \left\{ \Pi^{\frac{\alpha}{k}}_1\left(\frac{1}{6} \right) + \Pi^{\frac{\alpha}{k}}_3\left(\frac{5}{6} \right) \right\} \right.$$
$$\left. + \left| \mathcal{F}'(\kappa_1) \right| \left\{ \Pi^{\frac{\alpha}{k}}_2\left(\frac{1}{6} \right) + \Pi^{\frac{\alpha}{k}}_4\left(\frac{5}{6} \right) \right\} \right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for k-Riemann–Liouville fractional integrals:

$$\left| \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|$$
$$\leq (\kappa_2 - \kappa_1) \left[\left| \mathcal{F}'(\kappa_2) \right| \left\{ \Pi^{\frac{\alpha}{k}}_1\left(\frac{1}{2} \right) + \Pi^{\frac{\alpha}{k}}_3\left(\frac{1}{2} \right) \right\} \right.$$
$$\left. + \left| \mathcal{F}'(\kappa_1) \right| \left\{ \Pi^{\frac{\alpha}{k}}_2\left(\frac{1}{2} \right) + \Pi^{\frac{\alpha}{k}}_4\left(\frac{1}{2} \right) \right\} \right].$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for k-Riemann–Liouville fractional integrals:

$$\left| \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] \right|$$
$$\leq (\kappa_2 - \kappa_1) \left[\left| \mathcal{F}'(\kappa_2) \right| \left\{ \Pi^{\frac{\alpha}{k}}_1(0) + \Pi^{\frac{\alpha}{k}}_3(1) \right\} + \left| \mathcal{F}'(\kappa_1) \right| \left\{ \Pi^{\frac{\alpha}{k}}_2(0) + \Pi^{\frac{\alpha}{k}}_4(1) \right\} \right].$$

Remark 11 From Theorem 4 we have the following new inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for generalized fractional integrals:

$$\left| \frac{1}{6} \left[\Delta(1)\mathcal{F}(\kappa_1) + \Delta(1)4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \Delta(1)\mathcal{F}(\kappa_2) \right] \right.$$
$$\left. - \left[{}_{\frac{\kappa_1+\kappa_2}{2}+} I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-} I_\varphi \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau) - \frac{\Delta(1)}{6}\right|d\tau\right)^{1-\frac{1}{p_1}}\right.$$

$$\times \left(\Pi_1^\varphi\left(\frac{1}{6}\right)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_2^\varphi\left(\frac{1}{6}\right)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

$$+ \left(\int_{\frac{1}{2}}^1\left|\Delta(\tau) - \frac{5\Delta(1)}{6}\right|d\tau\right)^{1-\frac{1}{p_1}}$$

$$\times \left.\left(\Pi_3^\varphi\left(\frac{5}{6}\right)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_4^\varphi\left(\frac{5}{6}\right)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for generalized fractional integrals:

$$\left|\frac{1}{\Delta(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}\right|$$

$$\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau) - \frac{\Delta(1)}{2}\right|d\tau\right)^{1-\frac{1}{p_1}}\right.$$

$$\times \left(\Pi_1^\varphi\left(\frac{1}{2}\right)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_2^\varphi\left(\frac{1}{2}\right)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

$$+ \left(\int_{\frac{1}{2}}^1\left|\Delta(\tau) - \frac{\Delta(1)}{2}\right|d\tau\right)^{1-\frac{1}{p_1}}$$

$$\times \left.\left(\Pi_3^\varphi\left(\frac{1}{2}\right)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_4^\varphi\left(\frac{1}{2}\right)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\left|\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{1}{\Delta(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathcal{F}(\kappa_1)\right]\right|$$

$$\leq (\kappa_2 - \kappa_1)\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)\right|d\tau\right)^{1-\frac{1}{p_1}}\left(\Pi_1^\varphi(0)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_2^\varphi(0)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$

$$+ \left.\left(\int_{\frac{1}{2}}^1\left|\Delta(\tau) - \Delta(1)\right|d\tau\right)^{1-\frac{1}{p_1}}\left(\Pi_3^\varphi(1)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_4^\varphi(1)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].$$

Remark 12 From Corollary 5 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for Riemann–Liouville fractional integrals:

$$\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \mathcal{F}(\kappa_2)\right] - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathcal{F}(\kappa_1)\right]\right|$$

$$\leq (\kappa_2 - \kappa_1)\left[\left(\Pi_1^\alpha\left(\frac{1}{6}\right) + \Pi_2^\alpha\left(\frac{1}{6}\right)\right)^{1-\frac{1}{p_1}}\right.$$

$$\times \left(\Pi_1^\alpha\left(\frac{1}{6}\right)\left|\mathcal{F}'(\kappa_2)\right|^{p_1} + \Pi_2^\alpha\left(\frac{1}{6}\right)\left|\mathcal{F}'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

$$
+ \left(\Pi_3^\alpha \left(\frac{5}{6} \right) + \Pi_4^\alpha \left(\frac{5}{6} \right) \right)^{1 - \frac{1}{p_1}}
$$

$$
\times \left(\Pi_3^\alpha \left(\frac{5}{6} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^\alpha \left(\frac{5}{6} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \Bigg].
$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for Riemann–Liouville fractional integrals:

$$
\left| \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J_{\frac{\kappa_1 + \kappa_2}{2}+}^\alpha \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1 + \kappa_2}{2}-}^\alpha \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \Bigg[\left(\Pi_1^\alpha \left(\frac{1}{2} \right) + \Pi_2^\alpha \left(\frac{1}{2} \right) \right)^{1 - \frac{1}{p_1}}
$$

$$
\times \left(\Pi_1^\alpha \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^\alpha \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}}
$$

$$
+ \left(\Pi_3^\alpha \left(\frac{1}{2} \right) + \Pi_4^\alpha \left(\frac{1}{2} \right) \right)^{1 - \frac{1}{p_1}}
$$

$$
\times \left(\Pi_3^\alpha \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^\alpha \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \Bigg].
$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$
\left| \mathcal{F} \left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J_{\frac{\kappa_1 + \kappa_2}{2}+}^\alpha \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1 + \kappa_2}{2}-}^\alpha \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \Big[\left(\Pi_1^\alpha(0) + \Pi_2^\alpha(0) \right)^{1 - \frac{1}{p_1}} \left(\Pi_1^\alpha(0) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^\alpha(0) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}}
$$

$$
+ \left(\Pi_3^\alpha(1) + \Pi_4^\alpha(1) \right)^{1 - \frac{1}{p_1}} \left(\Pi_3^\alpha(1) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^\alpha(1) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \Big].
$$

Remark 13 From Corollary 6 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for k-Riemann–Liouville fractional integrals:

$$
\left| \frac{1}{6} \left[\mathcal{F}(\kappa_1) + 4\mathcal{F} \left(\frac{\kappa_1 + \kappa_2}{2} \right) + \mathcal{F}(\kappa_2) \right] \right.
$$

$$
\left. - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J_{\frac{\kappa_1 + \kappa_2}{2}+,k}^\alpha \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1 + \kappa_2}{2}-,k}^\alpha \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \Bigg[\left(\Pi_1^{\frac{\alpha}{k}} \left(\frac{1}{6} \right) + \Pi_2^{\frac{\alpha}{k}} \left(\frac{1}{6} \right) \right)^{1 - \frac{1}{p_1}}
$$

$$
\times \left(\Pi_1^{\frac{\alpha}{k}} \left(\frac{1}{6} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^{\frac{\alpha}{k}} \left(\frac{1}{6} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}}
$$

$$+ \left(\Pi_3^{\frac{\alpha}{k}} \left(\frac{5}{6} \right) + \Pi_4^{\frac{\alpha}{k}} \left(\frac{5}{6} \right) \right)^{1 - \frac{1}{p_1}}$$

$$\times \left(\Pi_3^{\frac{\alpha}{k}} \left(\frac{5}{6} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^{\frac{\alpha}{k}} \left(\frac{5}{6} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for k-Riemann–Liouville fractional integrals:

$$\left| \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J_{\frac{\kappa_1 + \kappa_2}{2} +, k}^{\alpha} \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1 + \kappa_2}{2} -, k}^{\alpha} \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[\left(\Pi_1^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) + \Pi_2^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) \right)^{1 - \frac{1}{p_1}} \right.$$

$$\times \left(\Pi_1^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}}$$

$$+ \left(\Pi_3^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) + \Pi_4^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) \right)^{1 - \frac{1}{p_1}}$$

$$\times \left. \left(\Pi_3^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^{\frac{\alpha}{k}} \left(\frac{1}{2} \right) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right].$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for k-Riemann–Liouville fractional integrals:

$$\left| \mathcal{F} \left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J_{\frac{\kappa_1 + \kappa_2}{2} +, k}^{\alpha} \mathcal{F}(\kappa_2) + J_{\frac{\kappa_1 + \kappa_2}{2} -, k}^{\alpha} \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[\left(\Pi_1^{\frac{\alpha}{k}} (0) + \Pi_2^{\frac{\alpha}{k}} (0) \right)^{1 - \frac{1}{p_1}} \left(\Pi_1^{\frac{\alpha}{k}} (0) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_2^{\frac{\alpha}{k}} (0) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right.$$

$$+ \left. \left(\Pi_3^{\frac{\alpha}{k}} (1) + \Pi_4^{\frac{\alpha}{k}} (1) \right)^{1 - \frac{1}{p_1}} \left(\Pi_3^{\frac{\alpha}{k}} (1) \left| \mathcal{F}'(\kappa_2) \right|^{p_1} + \Pi_4^{\frac{\alpha}{k}} (1) \left| \mathcal{F}'(\kappa_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right].$$

Remark 14 From Theorem 5 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for generalized fractional integrals:

$$\left| \frac{1}{6} \left[\Delta(1) \mathcal{F}(\kappa_1) + \Delta(1) 4 \mathcal{F} \left(\frac{\kappa_1 + \kappa_2}{2} \right) + \Delta(1) \mathcal{F}(\kappa_2) \right] \right.$$

$$\left. - \left[{}_{\frac{\kappa_1 + \kappa_2}{2} +} I_{\varphi} \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1 + \kappa_2}{2} -} I_{\varphi} \mathcal{F}(\kappa_1) \right] \right|$$

$$\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \Delta(\tau) - \frac{\Delta(1)}{6} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{\left| \mathcal{F}'(\kappa_2) \right|^{r_1} + 3 \left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.$$

$$+ \left. \left(\int_{\frac{1}{2}}^1 \left| \Delta(\tau) - \frac{5\Delta(1)}{6} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3 \left| \mathcal{F}'(\kappa_2) \right|^{r_1} + \left| \mathcal{F}'(\kappa_1) \right|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for generalized fractional integrals:

$$
\left| \frac{1}{\Delta(1)} \left[{}_{\frac{\kappa_1+\kappa_2}{2}+} I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-} I_\varphi \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \Delta(\tau) - \frac{\Delta(1)}{2} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$

$$
\left. + \left(\int_{\frac{1}{2}}^1 \left| \Delta(\tau) - \frac{\Delta(1)}{2} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$
\left| \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{1}{\Delta(1)} \left[{}_{\frac{\kappa_1+\kappa_2}{2}+} I_\varphi \mathcal{F}(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-} I_\varphi \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} |\Delta(\tau)|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$

$$
\left. + \left(\int_{\frac{1}{2}}^1 |\Delta(\tau) - 1|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

Remark 15 From Corollary 7 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for Riemann–Liouville fractional integrals:

$$
\left| \frac{1}{6} \left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \mathcal{F}(\kappa_2) \right] - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \tau^\alpha - \frac{1}{6} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$

$$
\left. + \left(\int_{\frac{1}{2}}^1 \left| \tau^\alpha - \frac{5}{6} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for Riemann–Liouville fractional integrals:

$$
\left| \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|
$$

$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \tau^\alpha - \frac{1}{2} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$

$$
\left. + \left(\int_{\frac{1}{2}}^1 \left| \tau^\alpha - \frac{1}{2} \right|^{p_1} d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

(2) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$
\left| \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} \left[J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}+} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}-} \mathcal{F}(\kappa_1) \right] \right|
$$
$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} |\tau^\alpha|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$
$$
\left. + \left(\int_{\frac{1}{2}}^1 |\tau^\alpha - 1|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

Remark 16 From Corollary 8 we have the following inequalities.

(1) For $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$, we have the following Simpson's type inequality for k-Riemann–Liouville fractional integrals:

$$
\left| \frac{1}{6} \left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) + \mathcal{F}(\kappa_2) \right] \right.
$$
$$
\left. - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] \right|
$$
$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \tau^{\frac{\alpha}{k}} - \frac{1}{6} \right|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$
$$
\left. + \left(\int_{\frac{1}{2}}^1 \left| \tau^{\frac{\alpha}{k}} - \frac{5}{6} \right|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

(2) For $\lambda = \mu = \frac{1}{2}$, we have the following trapezoidal-type inequality for k-Riemann–Liouville fractional integrals:

$$
\left| \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] - \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} \right|
$$
$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} \left| \tau^{\frac{\alpha}{k}} - \frac{1}{2} \right|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$
$$
\left. + \left(\int_{\frac{1}{2}}^1 \left| \tau^{\frac{\alpha}{k}} - \frac{1}{2} \right|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

(3) For $\lambda = 0$ and $\mu = 1$, we have the following midpoint-type inequality for k-Riemann–Liouville fractional integrals:

$$
\left| \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2} \right) - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}+,k} \mathcal{F}(\kappa_2) + J^\alpha_{\frac{\kappa_1 + \kappa_2}{2}-,k} \mathcal{F}(\kappa_1) \right] \right|
$$
$$
\leq (\kappa_2 - \kappa_1) \left[\left(\int_0^{\frac{1}{2}} |\tau^{\frac{\alpha}{k}}|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{|\mathcal{F}'(\kappa_2)|^{r_1} + 3|\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right.
$$
$$
\left. + \left(\int_{\frac{1}{2}}^1 |\tau^{\frac{\alpha}{k}} - 1|^{p_1} \, d\tau \right)^{\frac{1}{p_1}} \left(\frac{3|\mathcal{F}'(\kappa_2)|^{r_1} + |\mathcal{F}'(\kappa_1)|^{r_1}}{8} \right)^{\frac{1}{r_1}} \right].
$$

6 Application to some particular means

Let us recall the following means for real numbers κ_1 and κ_2.

(1) The arithmetic mean of $\kappa_1, \kappa_2 \geq 0$ is

$$A = A(\kappa_1, \kappa_2) = \frac{\kappa_1 + \kappa_2}{2}.$$

(2) The generalized logarithmic mean of $\kappa_1, \kappa_2 > 0$ is

$$L_n = L_n(\kappa_1, \kappa_2) = \begin{cases} \kappa_1 & \text{if } \kappa_1 = \kappa_2, \\ \frac{\kappa_2^{n+1} - \kappa_1^{n+1}}{(n+1)(\kappa_2 - \kappa_1)} & \text{if } \kappa_1 \neq \kappa_2, \end{cases} \quad n \in R\backslash\{-1, 0\}.$$

(3) The logarithmic mean of $\kappa_1, \kappa_2 > 0$ is

$$L = L(\kappa_1, \kappa_2) = \begin{cases} \kappa_1 & \text{if } \kappa_1 = \kappa_2, \\ [\frac{\kappa_2 - \kappa_1}{\ln \kappa_2 - \ln \kappa_1}]^{\frac{1}{n}} & \text{if } \kappa_1 \neq \kappa_2. \end{cases}$$

We further give some applications to above given means using the new inequalities.

Proposition 1 *Let $\kappa_1, \kappa_2 \in \mathbb{R}$, $0 < \kappa_1 < \kappa_2$, and $p \in \mathbb{Z}$ with $|p| \geq 2$. Then we have the following inequalities:*

$$\left| L_p^p(\kappa_1, \kappa_2) - A\left(\kappa_1^p, \kappa_2^p\right) \right| \leq \frac{p(\kappa_2 - \kappa_1)}{4} A\left(\kappa_1^{p-1}, \kappa_2^{p-1}\right),$$

$$\left| A^p(\kappa_1, \kappa_2) - L_p^P(\kappa_1, \kappa_2) \right| \leq \frac{p(\kappa_2 - \kappa_1)}{4} A\left(\kappa_1^{p-1}, \kappa_2^{p-1}\right).$$

Proof Using Remark 9, parts (2) and (3), for $\alpha = 1$ and $\mathcal{F}(\varkappa) = \varkappa^p$, we immediately obtain the inequalities. □

Proposition 2 *Let $\kappa_1, \kappa_2 \in \mathbb{R}$, $0 < \kappa_1 < \kappa_2$, and $p \in \mathbb{Z}$ with $|p| \geq 2$. Then we have the following inequalities:*

$$\left| L_p^p(\kappa_1, \kappa_2) - A\left(\kappa_1^p, \kappa_2^p\right) \right| \leq \frac{p(\kappa_2 - \kappa_1)}{2} A\left(\kappa_1^{p-1}, \kappa_2^{p-1}\right),$$

$$\left| A^p(\kappa_1, \kappa_2) - L_p^P(\kappa_1, \kappa_2) \right| \leq \frac{p(\kappa_2 - \kappa_1)}{2} A\left(\kappa_1^{p-1}, \kappa_2^{p-1}\right).$$

Proof Using Remark 12, parts (2) and (3), for $\alpha = 1$ and $\mathcal{F}(\varkappa) = \varkappa^p$, we immediately obtain the inequalities. □

Proposition 3 *Let $\kappa_1, \kappa_2 \in \mathbb{R}$, $0 < \kappa_1 < \kappa_2$, $0 \notin [\kappa_1, \kappa_2]$. Then we have the inequality*

$$\left| A^{-1}(\kappa_1, \kappa_2) - L^{-1}(\kappa_1, \kappa_2) \right| \leq \frac{(\kappa_2 - \kappa_1)}{4} A\left(\kappa_1^{-2}, \kappa_2^{-2}\right).$$

Proof Using Remark 9, part (3), for $\alpha = 1$ and $\mathcal{F}(\varkappa) = \frac{1}{\varkappa}$, we immediately obtain the inequality. □

Proposition 4 *Let* $\kappa_1, \kappa_2 \in \mathbb{R}$, $0 < \kappa_1 < \kappa_2$, $0 \notin [\kappa_1, \kappa_2]$. *Then we have the inequality*

$$\left| A^{-1}(\kappa_1, \kappa_2) - L^{-1}(\kappa_1, \kappa_2) \right| \le \frac{(\kappa_2 - \kappa_1)}{2} A\left(\kappa_1^{-2}, \kappa_2^{-2}\right).$$

Proof Using Remark 12, part (3), for $\alpha = 1$ and $\mathcal{F}(\varkappa) = \frac{1}{\varkappa}$, we immediately obtain the inequality. □

7 Concluding remarks

In this study, we proved some new bounds for Simpson's inequalities for differentiable convex functions via generalized fractional integrals. We also show that the results proved here are a strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study to obtain similar inequalities for different kinds of convexity in their future work.

Availability of data and materials
Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Hubei Normal University, Huangshi, P.R. China. [2]Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, P.R. China. [3]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey. [4]Northwest Institute of Eco-Environment and Resources, Chinese Academy of Sciences, Lanzhou, P.R. China.

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