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RESEARCH

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Asymptotic equivalence relations for rapidly varying solutions of sublinear differential equations of Emden–Fowler type

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Abstract

We discuss sublinear differential equations of the Emden–Fowler type $x'' = q(t)x^{\gamma}$ under the assumption that the coefficient q is a rapidly varying function. We show that all of their strongly decreasing and strongly increasing solutions are rapidly varying functions and are in the asymptotic equivalence relation with a precisely defined function determined by the coefficient q.

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1 Introduction

This paper is concerned with positive solutions of differential equations of the Emden–Fowler type of the form

$$x''(t) = q(t)x(t)^{\gamma}, \quad t \geq a > 0, \tag{E}$$

where $\gamma \neq 1$ is a positive constant, and q is positive, continuous function on $[a,\infty)$.

Equation (E) is called *sublinear* or *superlinear* according to $\gamma < 1$ or $\gamma > 1$. We consider the sublinear case, i.e., when $0 < \gamma < 1$.

Any positive solution x of (E), continuable at infinity and eventually different from zero, is either increasing or decreasing. A positive decreasing solution of (E) is said to be

- *strongly decreasing* if $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} x'(t) = 0$,
- *asymptotically constant* if $\lim_{t\to\infty} x(t) = \text{const} > 0$, $\lim_{t\to\infty} x'(t) = 0$.

A positive increasing solution of (E) is said to be

- *asymptotically linear* if $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0$,
- *strongly increasing* if $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} x'(t) = \infty$.

The existence of the above four types has been studied in [2, 25]. In the sublinear case, the existence of strongly increasing solutions is completely characterized, while for the existence of strongly decreasing solutions, only the sufficient condition is known, as it is stated in the following propositions.

Proposition 1.1 ([25, Theorem 3.8]) *Sublinear equation* (E) *has a strongly increasing solution if and only if*

$$\int_a^\infty t^\gamma q(t)\,dt = \infty. \tag{1.1}$$

Proposition 1.2 ([25, Theorem 3.2]) *Sublinear equation* (E) *has a strongly decreasing solution if*

$$\int_a^\infty tq(t)\,dt < \infty. \tag{1.2}$$

The existence and asymptotic behavior of regularly varying solutions of nonlinear differential equations were extensively studied in [8, 9, 11, 13–16, 18–22, 24]. Unlike regularly varying solutions, rapidly varying solutions of linear and nonlinear equations are much less studied. The study of second-order linear differential equation in the framework of rapid variation was initiated by Marić [23]. Half-linear differential equations in the framework of the Karamata theory and the de Haan theory were treated in [26–28]. Also, the existence of regularly and rapidly varying solutions of third-order nonlinear differential equations was studied in [17], while in [10, 12] the conditions for the existence and asymptotic representations of solutions are given assuming that the coefficient of the equation belongs to the subclass of rapidly varying functions. Although the results in [10, 12] can be applied to (E), the problem of determining the conditions for all solutions to be rapidly varying functions is not considered in these papers. Therefore, our goal in this paper is to prove that all strongly decreasing and strongly increasing solutions are rapidly varying functions under the assumption that the coefficient q is rapidly varying and to examine the properties of these solutions in more detail. In addition, the existence conditions and asymptotic representations of solutions are given in [10, 12] under the assumption that the coefficient of the equation belongs to the subclass of rapidly varying functions. The solutions considered in these papers also belong to the subclass of rapidly varying functions. Therefore, the results obtained in this paper improve the results in [10, 12], since we consider the equation with rapidly varying coefficient and its rapidly varying solutions.

This paper is organized as follows: In Sect. 2, we give the basic definitions and properties of the regularly and rapidly varying functions. We also present asymptotic equivalence relations in the class of rapidly varying functions of index ∞, which are defined in [1, 5], and some of their basic properties that are useful for our research. In addition, we introduce analogous relations in the class of rapidly varying functions of index $-\infty$ and examine their properties. The main results are stated in Sect. 3. In Sect. 4, we prove some important lemmas that significantly shorten the proof of the main results. Section 5 contains the proofs of the main results. Some illustrative examples are presented in Sect. 6.

2 Preliminaries

In this section, first, we recall basic information on the Karamata theory of regularly varying functions and the de Haan theory of rapidly varying functions.

Definition 2.1 A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be regularly varying of index $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0.$$

The set of all regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$.

Definition 2.2 A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be rapidly varying of index ∞ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \infty \quad \text{for all } \lambda > 1.$$

A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be rapidly varying of index $-\infty$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = 0 \quad \text{for all } \lambda > 1.$$

The set of rapidly varying functions of index ∞ (or $-\infty$) is denoted by $\mathrm{RPV}(\infty)$ (or $\mathrm{RPV}(-\infty)$). For more information on regular and rapid variation, the reader is referred to the monograph by Bingaham, Goldie, and Teugels [1]. For more recent contribution of the theory of rapid variation, see [6, 7].

Example 2.1

1. It is easy to see that function $f(t) = a^t$, $a > 1$ is a typical representative of the class $\mathrm{RPV}(\infty)$, while the function $f(t) = a^t$, $a \in (0, 1)$ is a typical representative of the class $\mathrm{RPV}(-\infty)$.
2. The function $f(t) = g(t)a^{h(t)}$, $g \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$, $h \in \mathrm{RV}(m)$, $m > 0$ belongs to the $\mathrm{RPV}(\infty)$, when $a > 1$ and $\mathrm{RPV}(-\infty)$, when $a \in (0, 1)$.

Next, we give some properties of rapidly varying functions.

Proposition 2.1
(1) $f \in \mathrm{RPV}(\infty)$ *if and only if* $1/f \in \mathrm{RPV}(-\infty)$.
(2) *If* $f, g \in \mathrm{RPV}(\infty)$ *and* $h \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$, *then*
(i) $f^p \in \mathrm{RPV}(\infty)$ *for any* $p > 0$.
(ii) $f \cdot h \in \mathrm{RPV}(\infty)$.
(iii) $f \cdot g \in \mathrm{RPV}(\infty)$.

Proof (1) This part of the proposition is shown in [29] on time scales.
(2) Since $\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \infty$, $\lim_{t\to\infty} \frac{g(\lambda t)}{g(t)} = \infty$ and $\lim_{t\to\infty} \frac{h(\lambda t)}{h(t)} = \lambda^{\rho}$ for all $\lambda > 1$, we have
(i) $\lim_{t\to\infty} \frac{(f(\lambda t))^p}{(f(t))^p} = (\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)})^p = \infty$, for any $p > 0$,
(ii) $\lim_{t\to\infty} \frac{f(\lambda t)\cdot h(\lambda t)}{f(t)\cdot h(t)} = \lambda^{\rho} \cdot \lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \infty$
(iii) $\lim_{t\to\infty} \frac{f(\lambda t)\cdot g(\lambda t)}{f(t)\cdot g(t)} = \infty$. □

Next, we consider some useful equivalence relations on the classes $\mathrm{RPV}(\infty)$ and $\mathrm{RPV}(-\infty)$. The following relation is introduced in [1] and further considered in [3, 4].

Definition 2.3 Let f and g be positive functions in $[a,\infty)$. These two functions are called mutually inversely asymptotic at ∞, denoted by $f(t) \overset{\star}{\sim} g(t), t \to \infty$, if for every $\lambda > 1$, there exists $t_0 = t_0(\lambda)$ such that

$$f\left(\frac{t}{\lambda}\right) \le g(t) \le f(\lambda t), \quad \text{for all } t \ge t_0.$$

Definition of the next relation and its properties are given in [5].

Definition 2.4 Let f and g be positive functions in $[a,\infty)$. These two functions are called mutually rapidly equivalent at ∞, denoted by $f(t) \overset{r}{\sim} g(t), t \to \infty$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{g(t)} = \lim_{t\to\infty} \frac{g(\lambda t)}{f(t)} = \infty, \quad \text{for all } \lambda > 1.$$

Proposition 2.2 *Let f and g be positive functions in $[a,\infty)$. Then, the following assertions hold*:

(a) *if f and g are measurable functions such that $f(t) \overset{r}{\sim} g(t)$ for $t \to \infty$, then f and g both belong to* $\mathrm{RPV}(\infty)$;
(b) *the relation $\overset{r}{\sim}$ is an equivalence relation in the class* $\mathrm{RPV}(\infty)$.

Proposition 2.3 *Let $f \in \mathrm{RPV}(\infty)$ be a locally bounded function on $[a,\infty)$. Also, let $1/f$ be a locally bounded function on $[a,\infty)$. Then the following assertions are true*:

(a) $f(t) \overset{r}{\sim} \frac{1}{t}\int_a^t f(s)\,ds,\ t \to \infty$;
(b) $f(t) \overset{r}{\sim} \frac{1}{t\int_t^\infty \frac{ds}{s^2 f(s)}},\ t \to \infty$;
(c) $F \in \mathrm{RPV}(\infty)$, *where* $F(t) = \int_a^t f(s)\,ds,\ t > a$;
(d) $\varphi \in \mathrm{RPV}(\infty)$, *where* $\varphi(t) = \frac{1}{\int_t^\infty \frac{ds}{f(s)}},\ t > a$.

Remark 2.1 It is easy to prove that if $f(t) \overset{r}{\sim} g(t),\ t \to \infty$, then

(a) $f(t)^p \overset{r}{\sim} g(t)^p,\ t \to \infty$ for all $p > 0$,
(b) $h(t) \cdot f(t) \overset{r}{\sim} h(t) \cdot g(t),\ t \to \infty$ for $h \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$ or $h \in \mathrm{RPV}(\infty)$.

Here, we introduce two new relations on $\mathrm{RPV}(-\infty)$.

Definition 2.5 Let f and g be positive functions in $[a,\infty)$. These two functions are called mutually inversely asymptotic at $-\infty$, denoted by $f(t) \underset{\star}{\sim} g(t), t \to \infty$, if for every $\lambda > 1$, there exists $t_0 = t_0(\lambda)$ such that

$$f(\lambda t) \le g(t) \le f\left(\frac{t}{\lambda}\right), \quad \text{for all } t \ge t_0.$$

Definition 2.6 Let f and g be positive functions in $[a,\infty)$. These two functions are called mutually rapidly equivalent at $-\infty$, denoted by $f(t) \underset{r}{\sim} g(t), t \to \infty$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{g(t)} = \lim_{t\to\infty} \frac{g(\lambda t)}{f(t)} = 0, \quad \text{for all } \lambda > 1.$$

The next proposition establishes a connection between relations $\overset{r}{\sim}$ and $\underset{r}{\sim}$.

Proposition 2.4 *Let f and g be positive functions in $[a,\infty)$. Then*

$$f(t) \overset{r}{\sim} g(t), \quad t\to\infty \quad \text{if and only if} \quad \frac{1}{f(t)} \underset{r}{\sim} \frac{1}{g(t)}, \quad t\to\infty.$$

Proof The proposition directly follows from the equalities

$$\lim_{t\to\infty}\frac{f(\lambda t)}{g(t)} = \left[\lim_{t\to\infty}\frac{\frac{1}{f(\lambda t)}}{\frac{1}{g(t)}}\right]^{-1} \quad \text{and} \quad \lim_{t\to\infty}\frac{g(\lambda t)}{f(t)} = \left[\lim_{t\to\infty}\frac{\frac{1}{g(\lambda t)}}{\frac{1}{f(t)}}\right]^{-1}.$$ □

The next proposition directly follows from Proposition 2.4, Proposition 2.2, and Proposition 2.1.

Proposition 2.5 *Let f and g be positive functions in $[a,\infty)$. Then the following assertions hold*:

(a) *if f and g are measurable functions such that $f(t) \underset{r}{\sim} g(t)$ for $t\to\infty$, then f and g both belong to* $\mathrm{RPV}(-\infty)$;

(b) *the relation $\underset{r}{\sim}$ is an equivalence relation in the class* $\mathrm{RPV}(-\infty)$.

Remark 2.2 Proposition 2.3(b) will be easier to use if we rewrite it in a different form. Denote $g(t) = \frac{1}{t^2 f(t)}$. Hence, due to Remark 2.1, we have

$$t^2 f(t) \overset{r}{\sim} \frac{t}{\int_t^\infty g(s)\,ds}, \quad t\to\infty$$

yielding

$$\frac{1}{t^2 f(t)} \underset{r}{\sim} \frac{1}{t}\int_t^\infty g(s)\,ds, \quad t\to\infty$$

by using the Proposition 2.4. Since $f \in \mathrm{RPV}(\infty)$, from Proposition 2.1, we conclude that $g \in \mathrm{RPV}(-\infty)$. Also, since $1/f$ is a locally bounded function on $[a,\infty)$, so is g.

Therefore, we have the following proposition.

Proposition 2.6 *Let $g \in \mathrm{RPV}(-\infty)$ be a locally bounded function on $[a,\infty)$. Then*

$$g(t) \underset{r}{\sim} \frac{1}{t}\int_t^\infty g(s)\,ds, \quad t\to\infty. \tag{2.1}$$

3 Main results

Theorem 3.1 *Suppose that $q \in \mathrm{RPV}(\infty)$ satisfies the condition (1.1). Every strongly increasing solution of (E) is rapidly varying of index ∞. Moreover, any such solution x satisfies the asymptotic relation*

$$x(t) \overset{\star}{\sim} X(t), \quad t\to\infty, \tag{3.1}$$

where the function X is given by

$$X(t) = \left(t^2 q(t)\right)^{\frac{1}{1-\gamma}}. \tag{3.2}$$

Theorem 3.2 *Suppose that $q \in \mathrm{RPV}(-\infty)$ satisfies the condition (1.2). Every strongly decreasing solution of (E) is rapidly varying of index $-\infty$. Moreover, any such solution x satisfies the asymptotic relation*

$$x(t) \underset{\star}{\sim} X(t), \quad t \to \infty. \tag{3.3}$$

4 Auxiliary lemmas

Let us denote

$$X_1(t) = \left(t \int_a^t q(s)\,ds \right)^{\frac{1}{1-\gamma}}, \tag{4.1}$$

$$X_2(t) = \left(\int_a^t \left(\int_a^s q(r)^{\frac{2}{3+\gamma}}\,dr \right)^{\frac{3+\gamma}{1-\gamma}} ds \right)^{\frac{1}{2}}. \tag{4.2}$$

First, we show that functions X, X_1, and X_2 are in the relation $\overset{r}{\sim}$ under the assumption that q is a rapidly varying function of index ∞.

Lemma 4.1 *Suppose that $q \in \mathrm{RPV}(\infty)$. Then*

$$X(t) \overset{r}{\sim} X_1(t), \quad t \to \infty, \tag{4.3}$$

where the functions X and X_1 are given by (3.2) and (4.1), respectively.

Proof Using Proposition 2.3(a), we have

$$q(t) \overset{r}{\sim} \frac{1}{t} \int_a^t q(s)\,ds, \quad t \to \infty. \tag{4.4}$$

Multiplying (4.4) by t^2, in the view of Remark 2.1, we get

$$t^2 q(t) \overset{r}{\sim} t \int_a^t q(s)\,ds, \quad t \to \infty,$$

implying

$$\left(t^2 q(t)\right)^{\frac{1}{1-\gamma}} \overset{r}{\sim} \left(t \int_a^t q(s)\,ds \right)^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

since $\frac{1}{1-\gamma} > 0$. This completes the proof of Lemma 4.1. □

Lemma 4.2 *Suppose that $q \in \mathrm{RPV}(\infty)$. Then*

$$X(t) \overset{r}{\sim} X_2(t), \quad t \to \infty, \tag{4.5}$$

where the functions X and X_2 are given by (3.2) and (4.2), respectively.

Proof Applying Proposition 2.3(a), we conclude that

$$\int_a^t q(s)^{\frac{2}{3+\gamma}}\,ds \overset{r}{\sim} t\cdot q(t)^{\frac{2}{3+\gamma}}, \quad t\to\infty.$$

Since $\frac{3+\gamma}{1-\gamma}>0$, due to Remark 2.1, we get

$$\left(\int_a^t q(s)^{\frac{2}{3+\gamma}}\,ds\right)^{\frac{3+\gamma}{1-\gamma}} \overset{r}{\sim} t^{\frac{3+\gamma}{1-\gamma}} q(t)^{\frac{2}{1-\gamma}}, \quad t\to\infty. \tag{4.6}$$

On the other hand, another application of Proposition 2.3(a) gives us

$$\int_a^t\left(\int_a^s q(r)^{\frac{2}{3+\gamma}}\,dr\right)^{\frac{3+\gamma}{1-\gamma}} ds \overset{r}{\sim} t\left(\int_a^t q(s)^{\frac{2}{3+\gamma}}\,ds\right)^{\frac{3+\gamma}{1-\gamma}}, \quad t\to\infty. \tag{4.7}$$

By combining (4.6) and (4.7), we have

$$\int_a^t\left(\int_a^s q(r)^{\frac{2}{3+\gamma}}\,dr\right)^{\frac{3+\gamma}{1-\gamma}} ds \overset{r}{\sim} t^{\frac{4}{1-\gamma}} q(t)^{\frac{2}{1-\gamma}}, \quad t\to\infty,$$

implying

$$\left(\int_a^t\left(\int_a^s q(r)^{\frac{2}{3+\gamma}}\,dr\right)^{\frac{3+\gamma}{1-\gamma}} ds\right)^{\frac{1}{2}} \overset{r}{\sim} \left(t^2 q(t)\right)^{\frac{1}{1-\gamma}}, \quad t\to\infty.$$

This completes the proof of Lemma 4.2. □

Denote by

$$Y_1(t)=\left(\int_t^\infty\int_s^\infty q(r)\,dr\,ds\right)^{\frac{1}{1-\gamma}}, \tag{4.8}$$

$$Y_2(t)=\left(\int_t^\infty\left(\int_s^\infty q(r)^{\frac{2}{3+\gamma}}\,dr\right)^{\frac{3+\gamma}{1-\gamma}} ds\right)^{\frac{1}{2}}. \tag{4.9}$$

Next, we show that functions X, Y_1, and Y_2 are in the relation $\underset{r}{\sim}$ under the assumption that q is a rapidly varying function of index $-\infty$.

Lemma 4.3 *Suppose that* $q\in\mathrm{RPV}(-\infty)$. *Then*

$$X(t)\underset{r}{\sim} Y_1(t), \quad t\to\infty, \tag{4.10}$$

where the functions X *and* Y_1 *are given by* (3.2) *and* (4.8), *respectively.*

Proof Using Proposition 2.6, we get

$$\int_t^\infty q(s)\,ds \underset{r}{\sim} tq(t), \quad t\to\infty. \tag{4.11}$$

On the other hand, another application of Proposition 2.6 gives us

$$\int_t^\infty \int_s^\infty q(r)\,dr\,ds \underset{r}{\sim} t\int_t^\infty q(s)\,ds, \quad t\to\infty. \tag{4.12}$$

From (4.11) and (4.12), we conclude

$$\int_t^\infty \int_s^\infty q(r)\,dr\,ds \underset{r}{\sim} t^2 q(t), \quad t\to\infty$$

implying (4.10). □

Lemma 4.4 *Suppose that* $q\in \mathrm{RPV}(-\infty)$. *Then*

$$X(t) \underset{r}{\sim} Y_2(t), \quad t\to\infty, \tag{4.13}$$

where the functions X *and* Y_2 *are given by* (3.2) *and* (4.9), *respectively.*

Proof Applying Proposition 2.6, we conclude that

$$\int_t^\infty q(s)^{\frac{2}{3+\gamma}}\,ds \underset{r}{\sim} t\cdot q(t)^{\frac{2}{3+\gamma}}, \quad t\to\infty,$$

implying

$$\left(\int_t^\infty q(s)^{\frac{2}{3+\gamma}}\,ds\right)^{\frac{3+\gamma}{1-\gamma}} \underset{r}{\sim} t^{\frac{3+\gamma}{1-\gamma}} q(t)^{\frac{2}{1-\gamma}}, \quad t\to\infty, \tag{4.14}$$

since $\frac{3+\gamma}{1-\gamma}>0$. On the other hand, another use of Proposition 2.6 gives us

$$\int_t^\infty \left(\int_s^\infty q(r)^{\frac{2}{3+\gamma}}\,dr\right)^{\frac{3+\gamma}{1-\gamma}} ds \underset{r}{\sim} t\left(\int_t^\infty q(s)^{\frac{2}{3+\gamma}}\,ds\right)^{\frac{3+\gamma}{1-\gamma}}, \quad t\to\infty. \tag{4.15}$$

By combining (4.14) and (4.15), we get

$$\int_t^\infty \left(\int_s^\infty q(r)^{\frac{2}{3+\gamma}}\,dr\right)^{\frac{3+\gamma}{1-\gamma}} ds \underset{r}{\sim} t^{\frac{4}{1-\gamma}} q(t)^{\frac{2}{1-\gamma}}, \quad t\to\infty$$

yielding (4.13). □

5 Proofs of main results

Proof of Theorem 3.1 Since q satisfies the condition (1.1), we obtain that the equation (E) has a strongly increasing solution.

Let x be arbitrary strongly increasing solution of (E) defined on $[T,\infty)$, $T\ge a$. First, we show that there exist positive constants m, M such that

$$mX_2(t)\le x(t)\le MX_1(t), \quad \text{for large } t, \tag{5.1}$$

where X_1 and X_2 are given by (4.1) and (4.2), respectively. Integrating x' on $[T,t]$, we get

$$x(t) = x(T) + \int_T^t x'(s)\,ds \le x(T) + x'(t)(t-T), \quad t \ge T,$$

because x' is increasing. Hence, we find $K_1 > 0$ such that

$$x(t) \le K_1 t x'(t), \quad t \ge T. \tag{5.2}$$

Since x is increasing, integration of (E) from T to t gives

$$x'(t) = x'(T) + \int_T^t q(s)x(s)^\gamma\,ds \le x'(T) + x(t)^\gamma \int_T^t q(s)\,ds, \quad t \ge T,$$

implying, due to the fact $\int_T^t q(s)\,ds \to \infty$ as $t \to \infty$, that we find $K_2 > 0$ such that

$$x'(t) \le K_2 x(t)^\gamma \int_T^t q(s)\,ds, \quad t \ge T. \tag{5.3}$$

By combining (5.2) and (5.3), we have

$$x(t) \le K_1 K_2 t x(t)^\gamma \int_T^t q(s)\,ds, \quad t \ge T.$$

Thus, there exists $M > 0$ such that

$$x(t) \le M \left(t \int_T^t q(s)\,ds \right)^{\frac{1}{1-\gamma}}, \quad t \ge T.$$

The right-hand side of the inequality (5.1) is proved.

Next, we prove the left-hand side of the inequality (5.1). Set $w(t) = x(t)x'(t)$ and

$$\nu = \frac{\gamma+1}{\gamma+3}, \qquad \mu = \frac{2}{\gamma+3}, \qquad \kappa = \frac{1-\gamma}{\gamma+3}. \tag{5.4}$$

An application of Young's inequality gives

$$\begin{aligned} w'(t) &= w(t)\left(\frac{q(t)x(t)^\gamma}{x'(t)} + \frac{x'(t)}{x(t)} \right) \ge \frac{w(t)}{\mu^\mu \nu^\nu} \left(\frac{q(t)x(t)^\gamma}{x'(t)} \right)^\mu \left(\frac{x'(t)}{x(t)} \right)^\nu \\ &= \frac{w(t)}{\mu^\mu \nu^\nu} x(t)^{\gamma\mu-\nu} x'(t)^{\nu-\mu} q(t)^\mu. \end{aligned}$$

Since, $\gamma\mu - \nu = \nu - \mu = -\kappa$, we get

$$w'(t) \ge \frac{1}{\mu^\mu \nu^\nu} w(t)^{1-\kappa} q(t)^\mu. \tag{5.5}$$

After dividing (5.5) by $w(t)^{1-\kappa}$ and integrating the obtained inequality on $[T,t]$, we get that there is $k_1 > 0$ such that

$$w(t)^\kappa \ge k_1 \int_T^t q(s)^\mu\,ds, \quad t \ge T$$

or

$$x(t)x'(t) \geq k_1^{\frac{1}{\kappa}} \left(\int_T^t q(s)^{\mu} \, ds \right)^{\frac{1}{\kappa}}, \quad t \geq T. \tag{5.6}$$

Integrating (5.6) from T to t, we find $k_2 > 0$ and $T^* \geq T$ sufficiently large such that

$$\frac{x(t)^2}{2} \geq k_2 \int_a^t \left(\int_a^s q(r)^{\frac{2}{\gamma+3}} \, dr \right)^{\frac{\gamma+3}{1-\gamma}} ds, \quad t \geq T^*. \tag{5.7}$$

From (5.7), we obtain that there exists $m > 0$ such that the left-hand side of the inequality (5.1) is satisfied.

Next, we prove that x is a rapidly varying function of index ∞. Fix arbitrary $\lambda > 1$. Indeed, from (5.1) for sufficiently large t, we have

$$mX_2(\lambda t) \leq x(\lambda t) \leq MX_1(\lambda t), \tag{5.8}$$

and

$$\frac{1}{MX_1(t)} \leq \frac{1}{x(t)} \leq \frac{1}{mX_2(t)}. \tag{5.9}$$

From (5.8) and (5.9), we obtain

$$\frac{m}{M} \frac{X_2(\lambda t)}{X_1(t)} \leq \frac{x(\lambda t)}{x(t)} \leq \frac{M}{m} \frac{X_1(\lambda t)}{X_2(t)} \tag{5.10}$$

for sufficiently large t. By Lemma 4.1 and Lemma 4.2, we have $X_1(t) \overset{r}{\sim} X_2(t)$, $t \to \infty$, which means

$$\lim_{t\to\infty} \frac{X_2(\lambda t)}{X_1(t)} = \lim_{t\to\infty} \frac{X_1(\lambda t)}{X_2(t)} = \infty. \tag{5.11}$$

Since λ was arbitrary, combining (5.10) and (5.11) gives us $\lim_{t\to\infty} \frac{x(\lambda t)}{x(t)} = \infty$ for all $\lambda > 1$, that is, $x \in \mathrm{RPV}(\infty)$.

It remains to prove that the solution x satisfies the asymptotic relation (3.1). Fix arbitrary $\lambda > 1$. Let m and M be positive numbers, satisfying (5.1) for $t \geq T_1 \geq T$. By Lemma 4.1 and Lemma 4.2, we have (4.3) and (4.5), so there exists $T_2 = T_2(\lambda) \geq T_1$ such that

$$MX_1(t) \leq X(\lambda t) \wedge X\left(\frac{t}{\lambda}\right) \leq mX_2(t), \quad t \geq T_2.$$

Therefore, from (5.1), we conclude that

$$X\left(\frac{t}{\lambda}\right) \leq x(t) \leq X(\lambda t), \quad t \geq T_2, \tag{5.12}$$

implying $x(t) \overset{\star}{\sim} X(t)$, $t \to \infty$. This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2 Assumption (1.2) ensures the existence of strongly decreasing solution of (E). Assume that x is the arbitrary strongly decreasing solution of (E) defined on $[T,\infty)$, $T \geq a$. First, we show that there exist positive constants m and M such that

$$mY_2(t) \leq x(t) \leq MY_1(t), \quad \text{for large } t, \tag{5.13}$$

where Y_1 and Y_2 are given by (4.8) and (4.9), respectively. Since $x'(t) \to 0$, $t \to \infty$, and x is decreasing, integrating (E) from t to ∞, we get

$$-x'(t) = \int_t^\infty q(s)x(s)^\gamma\,ds \leq x(t)^\gamma \int_t^\infty q(s)\,ds, \quad t \geq T. \tag{5.14}$$

Dividing (5.14) by $x(t)^\gamma$ and then integrating from t to ∞, since $x(t) \to 0$, $t \to \infty$, we have

$$\frac{1}{1-\gamma}x(t)^{1-\gamma} \leq \int_t^\infty \int_s^\infty q(r)\,dr\,ds, \quad t \geq T$$

implying that there exists $M > 0$ such that the right-hand side of the inequality (5.13) is satisfied.

Next, we prove the left-hand side of the inequality (5.13). Setting $w(t) = x(t)|x'(t)|$ and ν, μ, κ as in (5.4), application of Young's inequality gives

$$\begin{aligned} -w'(t) &= w(t)\left(\frac{q(t)x(t)^\gamma}{|x'(t)|} + \frac{|x'(t)|}{x(t)}\right) \geq \frac{w(t)}{\mu^\mu \nu^\nu}\left(\frac{q(t)x(t)^\gamma}{|x'(t)|}\right)^\mu \left(\frac{|x'(t)|}{x(t)}\right)^\nu \\ &= \frac{w(t)}{\mu^\mu \nu^\nu}x(t)^{\gamma\mu-\nu}\left|x'(t)\right|^{\nu-\mu}q(t)^\mu = \frac{1}{\mu^\mu \nu^\nu}w(t)^{1-\kappa}q(t)^\mu, \end{aligned}$$

afterwards multiplying by $w(t)^{\kappa-1}$ and integrating from t to ∞, we find $k_1 > 0$ such that

$$w(t)^\kappa \geq k_1 \int_t^\infty q(s)^\mu\,ds, \quad t \geq T,$$

or

$$-x(t)x'(t) \geq k_1^{\frac{1}{\kappa}}\left(\int_t^\infty q(s)^\mu\,ds\right)^{\frac{1}{\kappa}}, \quad t \geq T. \tag{5.15}$$

Since $x(t) \to 0$, $t \to \infty$, integrating (5.15) from t to ∞ yields that there is $k_2 > 0$ such that

$$\frac{x(t)^2}{2} \geq k_2 \int_t^\infty \left(\int_s^\infty q(r)^{\frac{2}{\gamma+3}}\,dr\right)^{\frac{\gamma+3}{1-\gamma}} ds. \tag{5.16}$$

From (5.16), we obtain that there exists $m > 0$ such that the left-hand side of the inequality (5.13) is satisfied.

That $x \in \mathrm{RPV}(-\infty)$ and satisfies the asymptotic relation (3.3) can be proved in the same way as in the proof of Theorem 3.1, using Lemma 4.3 and Lemma 4.4. □

6 Examples

Now, we present two examples that illustrate main results stated by Theorem 3.1 and Theorem 3.2.

Example 6.1 Consider the equation

$$x''(t) = q_1(t)x^{\gamma}(t), \quad t > 0, 0 < \gamma < 1, \tag{6.1}$$

where $q_1(t) = e^{t+(1-\gamma)e^t}(1 + e^t)$. Since

$$q_1 \in \mathrm{RPV}(\infty) \wedge \int_a^{\infty} t^{\gamma} q_1(t)\, dt = \infty,$$

by Theorem 3.1 follows that every strongly increasing solution of (6.1) is rapidly varying of index ∞, and any such solution x satisfies the asymptotic relation

$$x(t) \overset{\star}{\sim} Q_1(t), \quad t \to \infty, \tag{6.2}$$

where $Q_1(t) = (t^2 q_1(t))^{\frac{1}{1-\gamma}}$. It is easy to check that $x_1(t) = e^{e^t}$ is such a solution of (6.1), since $x_1 \in \mathrm{RPV}(\infty)$ and

$$\lim_{t\to\infty} \frac{x_1(\lambda t)}{Q_1(t)} = \lim_{t\to\infty} \frac{Q_1(\lambda t)}{x_1(t)} = \infty,$$

implying that x_1 satisfies the asymptotic relation (6.2).

Example 6.2 Consider the equation

$$x''(t) = q_2(t)x^{\gamma}(t), \quad t > 0, 0 < \gamma < 1, \tag{6.3}$$

where $q_2(t) = k^2 e^{k(\gamma-1)t}$, $k > 0$. Since

$$q_2 \in \mathrm{RPV}(-\infty) \wedge \int_a^{\infty} t q_2(t)\, dt < \infty,$$

by Theorem 3.2 follows that every strongly decreasing solution of (6.3) is rapidly varying of index $-\infty$, and any such solution x satisfies the asymptotic relation

$$x(t) \underset{\star}{\sim} Q_2(t), \quad t \to \infty, \tag{6.4}$$

where $Q_2(t) = (t^2 q_2(t))^{\frac{1}{1-\gamma}}$. It is easy to check that $x_2(t) = e^{-kt}$, $k > 0$ is such a solution of (6.3), since $x_2 \in \mathrm{RPV}(-\infty)$ and

$$\lim_{t\to\infty} \frac{x_2(\lambda t)}{Q_2(t)} = \lim_{t\to\infty} \frac{Q_2(\lambda t)}{x_2(t)} = 0,$$

implying that x_2 satisfies the asymptotic relation (6.4).

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