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Fractional versions of Minkowski-type integral inequalities via unified Mittag-Leffler function

Shuang-Shuang Zhou[1], Ghulam Farid[2] and Ayyaz Ahmad[2*]

*Correspondence:
ayyaz3007@gmail.com
[2]Department of Mathematics,
COMSATS University Islamabad,
Attock Campus, Pakistan
Full list of author information is
available at the end of the article

Abstract

We present unified versions of Minkowski-type fractional integral inequalities with the help of fractional integral operator based on a unified Mittag-Leffler function. These inequalities provide new as well as already known fractional versions of Minkowski-type inequalities.

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1 Introduction

A Swedish mathematician Gosta Mittag-Leffler introduced a function in the form of a power series [1]

$$E_\alpha(t) = \sum_{l=0}^{\infty} \frac{t^l}{\Gamma(\alpha l + 1)},$$

where $t, \alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$. This function is called the Mittag-Leffler function. It plays a vital role in the representation of solutions of fractional differential equations. Many researchers have given its various generalizations and extensions, which are used to formulate solutions of real-world problems in different fields of science and engineering [2, 3]. The Mittag-Leffler function is also used to introduce new generalized fractional integral operators. These integral operators are frequently used for extensions and generalizations of well-known classical integral inequalities. For a detailed study of the Mittag-Leffler function, we refer the readers to [4–8].

In [9] a generalization of the Mittag-Leffler function is given in the form of Q-function. In [6] the extended generalized Mittag-Leffler function and its related fractional integral operator are described along with their applications to generalizing classical Opial-type inequalities. In [10] a unified form of the Mittag-Leffler function, which generates a generalized Q-function and the extended generalized Mittag-Leffler function, is studied; also, a

fractional integral operator containing a unified Mittag-Leffler function is introduced, and its boundedness is proved. For some recent related work, we refer the readers to [11–14].

In this paper, we present Minkowski-type inequalities by using the fractional integral operator corresponding to the unified Mittag-Leffler function. The findings of this paper are implicitly related with several Minkowski-type inequalities already studied for different kinds of known fractional integral operators. Some particular cases of the main results are explicitly given in the form of corollaries. First, we give the definition of the unified Mittag-Leffler function and the associated fractional integral operator (see [10]).

Definition 1 Let $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, where $a_i, b_i, c_i \in \mathbb{C}$, $i = 1, \ldots, n$. Also, let α, β, γ, δ, μ, ν, λ, ρ, $\theta, t \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda), \Re(\theta)\} > 0$, and $k \in (0,1) \cup \mathbb{N}$. Let $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$ with $\mathrm{Im}(\rho) = \mathrm{Im}(\delta + \nu + \alpha)$. Then the unified Mittag-Leffler function is defined by

$$M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p) = \sum_{l=0}^{\infty} \frac{\Pi_{i=1}^{n} B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\Pi_{i=1}^{n} B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}} \frac{t^l}{\Gamma(\alpha l + \beta)}. \tag{1.1}$$

If we put $n = 1$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$ in (1.1), then we get the extended generalized Mittag-Leffler function defined in [6]. Also, by substituting $a_i = l$, $p = 0$, and $\Re(\rho) > 0$ into (1.1) we get the generalized Q-function defined in [9].

Definition 2 Let $f \in L_1[a,b]$. Then for $\xi \in [a,b]$, the fractional integral operators corresponding to (1.1) are defined by

$$I_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n} f(\xi;\underline{a},\underline{b},\underline{c},p) = \int_a^{\xi} (\xi - t)^{\beta-1} M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(\omega(\xi - t)^{\alpha};\underline{a},\underline{b},\underline{c},p) f(t)dt, \tag{1.2}$$

$$I_{b^-,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n} f(\xi;\underline{a},\underline{b},\underline{c},p) = \int_{\xi}^{b} (t - \xi)^{\beta-1} M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(\omega(t - \xi)^{\alpha};\underline{a},\underline{b},\underline{c},p) f(t)dt. \tag{1.3}$$

By fixing $n = 1$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$ we get the fractional integral operator given in [6]. By taking $a_i = l$, $p = 0$, and $\Re(\rho) > 0$ in (1.2) and (1.3) we define the fractional integral operators corresponding to the generalized Q-function as follows:

$$_Q I_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n} f(\xi;\underline{a},\underline{b}) = \int_a^{\xi} (\xi - t)^{\beta-1} Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(\omega(\xi - t)^{\alpha};\underline{a},\underline{b}) f(t)dt, \tag{1.4}$$

$$_Q I_{b^-,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n} f(\xi;\underline{a},\underline{b}) = \int_{\xi}^{b} (t - \xi)^{\beta-1} Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(\omega(t - \xi)^{\alpha};\underline{a},\underline{b}) f(t)dt. \tag{1.5}$$

From (1.1) various generalized Mittag-Leffler functions given by Wiman [15], Prabhakar [8], Shukla and Parajapati [4], Salim and Faraj [7], Rahman et al. [5], Andrić et al. [6], and Bhatnagar and Pandey [9] can be easily deduced. Also, the fractional integral operators associated with these Mittag-Leffler functions can be obtained. Before moving toward our main results, we give the following Minkowski-type inequalities. First, we state the integral version of the classical Minkowski inequality.

Theorem 1 *Let $\phi, \psi \in L_r[d,e]$. Then for $r \geq 1$,*

$$\left(\int_d^e \left(\phi(\xi) + \psi(\xi)\right)^r d\xi\right)^{\frac{1}{r}} \leq \left(\int_d^e \phi^r(\xi)d\xi\right)^{\frac{1}{r}} + \left(\int_d^e \psi^r(\xi)d\xi\right)^{\frac{1}{r}}. \tag{1.6}$$

A reversed Minkowski-type inequality is given as follows.

Theorem 2 ([16]) *Let $\phi, \psi \in L_r[d,e]$ be such that $\phi, \psi \in \Re^+$ and $0 \leq k_1 \leq \frac{\phi(\xi)}{\psi(\xi)} \leq k_2$ for all $\xi \in [d,e]$. Then for $r \geq 1$, we have*

$$\left(\int_d^e \phi^r(\xi)d\xi\right)^{\frac{1}{r}} + \left(\int_d^e \psi^r(\xi)d\xi\right)^{\frac{1}{r}} \leq \frac{k_2(k_1 + 1) + (k_2 + 1)}{(k_1 + 1)(k_2 + 1)} \tag{1.7}$$

$$\times \left(\int_d^e \left(\phi(\xi) + \psi(\xi)\right)^r d\xi\right)^{\frac{1}{r}}.$$

Another reversed Minkowski-type inequality is given as follows.

Theorem 3 ([17]) *Under the assumptions of Theorem 2, we have the inequality*

$$\left(\int_d^e \phi^r(\xi)d\xi\right)^{\frac{2}{r}} + \left(\int_d^e \psi^r(\xi)d\xi\right)^{\frac{2}{r}} \tag{1.8}$$

$$\geq \left(\frac{(k_1 + 1)(k_2 + 1)}{k_2} - 2\right)\left(\int_d^e \phi^r(\xi)d\xi\right)^{\frac{2}{r}}\left(\int_d^e \psi^r(\xi)d\xi\right)^{\frac{2}{r}}.$$

In the next section, we give some generalized versions of Minkowski-type integral inequalities using fractional integral operators containing the unified Mittag-Leffler function (1.1) defined in Definition 2. Also, the reversed Minkowski-type integral inequalities for these fractional integral operators are proved.

We will use the following notations to make a smart representation of results of this paper: $M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n} = \mathbf{M}$, $I_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n} = \mathbf{I}$, $_QI_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n} = _Q\mathbf{I}$.

2 Generalized versions of Minkowski-type inequalities

In this section, we give generalized Minkowski-type integral inequalities for fractional operators defined in (1.2) and (1.3).

Theorem 4 *Let $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, \ldots, n$. Also, let $\mu, \nu, \rho, t \in \mathbb{R}$, $\alpha, \beta, \gamma, \delta, \lambda, \theta > 0$, and $k \in (0,1) \cup \mathbb{N}$. Let $k + \rho < \delta + \nu + \alpha$. Let $r > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$, and let ϕ and ψ be positive and rth-power integrable functions on $[d,e]$. If $\frac{\phi}{\psi}$ is bounded above by k_2 and bounded below by k_1 with $k_1 > 0$, then we have*

$$\left[(\mathbf{I}\phi)(\xi;\nu)\right]^{\frac{1}{r}}\left[(\mathbf{I}\psi)(\xi;\nu)\right]^{\frac{1}{s}} \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{rs}}\left[(\mathbf{I}\phi^{\frac{1}{r}}\psi^{\frac{1}{s}})(\xi;\nu)\right]. \tag{2.1}$$

Proof Under the conditions on $\frac{\phi}{\psi}$, we have the inequalities

$$\psi(t) \leq \frac{1}{k_1^{\frac{1}{r}}}\phi^{\frac{1}{r}}(t)\psi^{\frac{1}{s}}(t),$$

$$\phi(t) \le k_2^{\frac{1}{s}} \phi^{\frac{1}{r}}(t) \psi^{\frac{1}{s}}(t).$$

Multiplying both sides of these inequalities by $(\xi - t)^{\beta-1}\mathbf{M}(\omega(\xi - t)^{\alpha}; \nu)$ and integrating over $[d, \xi]$, we get

$$(\mathbf{I}\psi)(\xi; \nu) \le \frac{1}{k_1^{\frac{1}{r}}} \big(\mathbf{I}\phi^{\frac{1}{r}}\psi^{\frac{1}{s}}\big)(\xi; \nu),$$

$$(\mathbf{I}\phi)(\xi; \nu) \le k_2^{\frac{1}{s}} \big(\mathbf{I}\phi^{\frac{1}{r}}\psi^{\frac{1}{s}}\big)(\xi; \nu).$$

These two inequalities further take the following forms

$$\big[(\mathbf{I}\psi)(\xi; \nu)\big]^{\frac{1}{s}} \le \frac{1}{k_1^{\frac{1}{rs}}} \big[\big(\mathbf{I}\phi^{\frac{1}{r}}\psi^{\frac{1}{s}}\big)(\xi; \nu)\big]^{\frac{1}{s}}, \qquad (2.2)$$

$$\big[(\mathbf{I}\phi)(\xi; \nu)\big]^{\frac{1}{r}} \le k_2^{\frac{1}{rs}} \big[\big(\mathbf{I}\phi^{\frac{1}{r}}\psi^{\frac{1}{s}}\big)(\xi; \nu)\big]^{\frac{1}{r}}. \qquad (2.3)$$

Now multiplying inequalities (2.2) and (2.3), we get required result. □

Corollary 1 *Under the assumptions of Theorem 4, together with $a_i = l$, $p = 0$, and $\rho > 0$ in (2.1), we get*

$$\big[(\mathbf{Q}\mathbf{I}\phi)(\xi; \nu)\big]^{\frac{1}{r}} \big[(\mathbf{Q}\mathbf{I}\psi)(\xi; \nu)\big]^{\frac{1}{s}} \le \left(\frac{k_2}{k_1}\right)^{\frac{1}{rs}} \big[\big(\mathbf{Q}\mathbf{I}\phi^{\frac{1}{r}}\psi^{\frac{1}{s}}\big)(\xi; \nu)\big].$$

Remark 1 The Minkowski-type inequality containing the extended Mittag-Leffler function introduced by Andrić et al. [18] can be deduced from the theorem by setting $n = 1$, $\alpha, \beta, \gamma, \delta, \mu, \lambda, \theta, k > 0$, $\theta > \lambda$, $0 < k \le \delta + \alpha$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, and $\rho = 0 = \nu$.

Before moving toward the proof of our next result, we state a particular case of the GM-AM inequality for $x, y \ge 0$ with $r, s > 1$ satisfying $r^{-1} + s^{-1} = 1$,

$$xy \le r^{-1}x^r + s^{-1}y^s,$$

and also the elementary inequality

$$(x + y)^r \le 2^{r-1}\big(x^r + y^r\big), \quad x, y \ge 0, r > 1. \qquad (2.4)$$

Theorem 5 *Let $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, \ldots, n$. Also, let α, β, γ, δ, μ, ν, λ, ρ, θ, $t \in \mathbb{R}$, $k \in (0, 1) \cup \mathbb{N}$, and $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$. Let $k + \rho < \delta + \nu + \alpha$. Let $r, s \ge 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$, and let ϕ and ψ be positive rth-power integrable functions on $[d, e]$ such that their ratio is bounded above by k_2 and bounded below by k_1 with $k_1, k_2 > 0$. Then we have the following inequality:*

$$\big(\mathbf{I}(\phi\psi)\big)(\xi, \nu) \le r^{-1}2^{r-1}\left(\frac{k_2}{k_2 + 1}\right)^r \big(\mathbf{I}(\phi^r + \psi^r)\big)(\xi, \nu) \qquad (2.5)$$

$$+ s^{-1}2^{s-1}\left(\frac{1}{k_1 + 1}\right)^s \big(\mathbf{I}(\phi^s + \psi^s)\big)(\xi, \nu).$$

Proof Under the conditions on $\frac{\phi}{\psi}$ given in the theorem, we have the following inequalities:

$$(k_1 + 1)^s \psi^s(t) \leq \big(\phi(t) + \psi(t)\big)^s,$$

$$(k_2 + 1)^r \phi^r(t) \leq k_2^r \big(\psi(t) + \phi(t)\big)^r.$$

Multiplying both sides of the above inequalities by $(\xi - t)^{\beta-1} \mathbf{M}(\omega(\xi - t)^\alpha, \nu)$ and integrating on $[d, \xi]$, these inequalities take the following form

$$s^{-1}\big(\mathbf{I}\psi^s\big)(\xi, \nu) \leq \frac{1}{(k_1 + 1)^s} s^{-1}\big(\mathbf{I}(\psi + \phi)^s\big)(\xi, \nu). \tag{2.6}$$

$$r^{-1}\big(\mathbf{I}\phi^r\big)(\xi, \nu) \leq r^{-1}\frac{k_2^r}{(k_2 + 1)^r}\big(\mathbf{I}(\psi + \phi)^r\big)(\xi, \nu). \tag{2.7}$$

By a particular case of the GM-AM inequality we have

$$\phi(t)\psi(t) \leq r^{-1}\phi^r(t) + s^{-1}\psi^r(t).$$

Multiplying both sides of this inequality by $(\xi - t)^{\beta-1} \mathbf{M}(\omega(\xi - t)^\alpha; \nu)$ and integrating on $[d, \xi]$, we have

$$\big(\mathbf{I}(\phi\psi)\big)(\xi; \nu) \leq r^{-1}\big(\mathbf{I}\phi^r\big)(\xi; \nu) + s^{-1}\big(\mathbf{I}\psi^s\big)(\xi; \nu). \tag{2.8}$$

Applying (2.8) to the sum of (2.6) and (2.7), we get the inequality

$$\big(\mathbf{I}(\phi\psi)\big)(\xi; \nu) \leq r^{-1}\left(\frac{k_2}{k_2 + 1}\right)^r \big(\mathbf{I}\big(\psi(t) + \phi(t)\big)^r\big)(\xi; \nu) \tag{2.9}$$

$$+ s^{-1}\left(\frac{1}{k_1 + 1}\right)^s \big(\mathbf{I}\big(\psi(t) + \phi(t)\big)^s\big)(\xi; \nu).$$

Applying (2.4) in (2.9), we obtain (2.5). $\qquad\square$

Corollary 2 *Under the assumptions of Theorem 5 with $a_i = l$, $p = 0$, and $\rho > 0$ in (2.5), we get the inequality*

$$\big({}_{\mathbf{Q}}\mathbf{I}(\phi\psi)\big)(\xi, \nu) \leq r^{-1}2^{r-1}\left(\frac{k_2}{k_2 + 1}\right)^r \big({}_{\mathbf{Q}}\mathbf{I}\big(\phi^r + \psi^r\big)\big)(\xi, \nu)$$

$$+ s^{-1}2^{s-1}\left(\frac{1}{k_1 + 1}\right)^s \big({}_{\mathbf{Q}}\mathbf{I}\big(\phi^s + \psi^s\big)\big)(\xi, \nu).$$

Remark 2 For $n = 1$, $\alpha, \beta, \gamma, \delta, \mu, \lambda, \theta, k > 0$, $\theta > \lambda$, $0 < k \leq \delta + \alpha$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$ and $\rho = 0 = \nu$, (2.5) produces the result presented by Andrić et al. [18].

Theorem 6 *Let $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, \ldots, n$. Also, let α, β, γ, δ, μ, ν, λ, ρ, θ, $t \in \mathbb{R}$, $k \in (0,1) \cup \mathbb{N}$, and $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$. Let $k + \rho < \delta + \nu + \alpha$. Let $r \geq 1$, and let ϕ and ψ be positive rth-power*

integrable functions on $[d,e]$ such that their ratio is bounded above by k_2 and bounded below by k_1 with $k_1, k_2 > 0$. Then we have the inequalities

$$k_2^{-1}\big(\mathbf{I}(\phi\psi)\big)(\xi;\nu) \le (k_1+1)^{-1}(k_2+1)^{-1}\big(\mathbf{I}(\phi+\psi)^2\big)(\xi;\nu) \tag{2.10}$$
$$\le k_1^{-1}\big(\mathbf{I}(\phi\psi)\big)(\xi;\nu).$$

Proof From the boundedness of the ratio of the functions ϕ and ψ we have

$$0 \le k_1 \le \frac{\phi(t)}{\psi(t)} \le k_2, \quad t \in [d,e].$$

Using the upper bound $\frac{\phi(t)}{\psi(t)} \le k_2$, we have

$$\phi + \psi \le (k_2+1)\psi, \tag{2.11}$$
$$k_2^{-1}(k_2+1)\phi \le \phi + \psi. \tag{2.12}$$

Using the lower bound $k_1 \le \frac{\phi(t)}{\psi(t)}$, we have

$$\phi + \psi \ge (k_1+1)\psi, \tag{2.13}$$
$$k_1^{-1}(k_1+1)\phi \ge \phi + \psi. \tag{2.14}$$

From inequalities (2.11) and (2.13) we obtain the inequality

$$(k_1+1)\psi \le \phi + \psi \le (k_2+1)\psi. \tag{2.15}$$

From inequalities (2.12) and (2.14) we obtain the inequality

$$k_2^{-1}(k_2+1)\phi \le \phi + \psi \le k_1^{-1}(k_1+1)\phi. \tag{2.16}$$

Now multiplying $(\xi-t)^{\beta-1}\mathbf{M}(\omega(\xi-t)^{\alpha};\nu)$ by the product of (2.15) and (2.16) and integrating on $[d,\xi]$, we get (2.10). $\qquad\square$

Corollary 3 *Under the assumptions of Theorem* 6 *along with* $a_i = l$, $p = 0$, *and* $\rho > 0$ *in* (2.10), *we have the inequality*

$$k_2^{-1}\big(_{\mathbf{Q}}\mathbf{I}(\phi\psi)\big)(\xi;\nu) \le (k_1+1)^{-1}(k_2+1)^{-1}\big(_{\mathbf{Q}}\mathbf{I}(\phi+\psi)^2\big)(\xi;\nu)$$
$$\le k_1^{-1}\big(_{\mathbf{Q}}\mathbf{I}(\phi\psi)\big)(\xi;\nu).$$

Remark 3 Inequality (2.10) is the generalization of the inequalities proved by Andrić et al. [18]. By setting $n = 1$, $\alpha, \beta, \gamma, \delta, \mu, \lambda, \theta, k > 0$, $\theta > \lambda$, $0 < k \le \delta + \alpha$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, and $\rho = 0 = \nu$ the result of [18] can be deduced.

Theorem 7 *Let* $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, *where* $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, \ldots, n$. *Also, let* α, β, γ, δ, μ, ν, λ, ρ, θ, $t \in \mathbb{R}$, $k \in (0,1) \cup \mathbb{N}$, *and* $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$. *Let* $k + \rho < \delta + \nu + \alpha$. *Let* $r \ge 1$, *and let* ϕ *and* ψ *be positive and*

rth-power integrable functions on $[d,e]$ such that their ratio is bounded above by k_2 and bounded below by k_1 with $k_1, k_2 > 0$ and $0 < \eta < k_1$. Then we have the following inequalities:

$$\frac{k_2 + 1}{k_2 - \eta} \left[\left(\mathbf{I}(\phi - \eta \psi)^{\frac{1}{r}} \right)(\xi; \nu) \right] \leq \left[\left(\mathbf{I}\phi^r \right)(\xi; \nu) \right]^{\frac{1}{r}} + \left[\left(\mathbf{I}\psi^r \right)(\xi; \nu) \right]^{\frac{1}{r}} \tag{2.17}$$

$$\leq \frac{k_1 + 1}{k_1 - \eta} \left[\left(\mathbf{I}(\phi - \eta \psi)^{\frac{1}{r}} \right)(\xi; \nu) \right].$$

Proof By the conditions on ϕ and ψ in the statement we have

$$0 < \eta < k_1 \leq \frac{\phi(t)}{\psi(t)} \leq k_2, \quad t \in [d, e], \tag{2.18}$$

and the above inequality takes the form

$$\frac{(\phi - \eta \psi)^r}{(k_2 - \eta)^r} \leq \psi^r \leq \frac{(\phi - \eta \psi)^r}{(k_1 - \eta)^r}.$$

Multiplying the last inequality by $(\xi - t)^{\beta - 1} \mathbf{M}(\omega(\xi - t)^{\alpha}; \nu)$ and integrating on $[d, \xi]$, we obtain

$$\frac{[(\mathbf{I}(\phi - \eta \psi)^r)(\xi; \nu)]^{\frac{1}{r}}}{k_2 - \eta} \leq \left[\left(\mathbf{I}\psi^r \right)(\xi; \nu) \right]^{\frac{1}{r}} \leq \frac{[(\mathbf{I}(\phi - \eta \psi)^r)(\xi; \nu)]^{\frac{1}{r}}}{k_1 - \eta}. \tag{2.19}$$

Inequality (2.18) can also be written as

$$\frac{1}{k_2} \leq \frac{\psi(t)}{\phi(t)} \leq \frac{1}{k_1},$$

and after certain steps, the above inequality takes the form

$$\frac{k_2(\phi - \eta \psi)}{k_2 - \eta \psi} \leq \phi \leq \frac{k_1(\phi - \eta \psi)}{k_1 - \eta}.$$

By multiplying with $(\xi - t)^{\beta - 1} \mathbf{M}(\omega(\xi - t)^{\alpha}; \nu)$ and integrating on $[d, \xi]$ the above inequality becomes

$$\left(\frac{k_2}{k_2 - \eta} \right)^r \left(\mathbf{I}(\phi - \eta \psi)^r \right)(\xi; \nu) \leq \left(\mathbf{I}\phi^r \right)(\xi; \nu) \leq \left(\frac{k_1}{k_1 - \eta} \right)^r \left(\mathbf{I}(\phi - \eta \psi)^r \right)(\xi; \nu),$$

$$\left(\frac{k_2}{k_2 - \eta} \right) \left[\left(\mathbf{I}(\phi - \eta \psi)^r \right)(\xi; \nu) \right]^{\frac{1}{r}} \tag{2.20}$$

$$\leq \left[\left(\mathbf{I}\phi^r \right)(\xi; \nu) \right]^{\frac{1}{r}} \leq \left(\frac{k_1}{k_1 - \eta} \right) \left[\left(\mathbf{I}(\phi - \eta \psi)^r \right)(\xi; \nu) \right]^{\frac{1}{r}}.$$

Adding (2.19) and (2.20), we obtain (2.17).   □

Corollary 4 *Under the assumptions of Theorem* 7 *with* $a_i = l$, $p = 0$, *and* $\rho > 0$ *in* (2.17), *we have the following inequality*:

$$\frac{k_2 + 1}{k_2 - \eta} \big[\big({}_\mathbf{Q}\mathbf{I}(\phi - \eta\psi)^{\frac{1}{r}} \big)(\xi; \nu) \big] \leq \big[\big({}_\mathbf{Q}\mathbf{I}\phi^r \big)(\xi; \nu) \big]^{\frac{1}{r}} + \big[\big({}_\mathbf{Q}\mathbf{I}\psi^r \big)(\xi; \nu) \big]^{\frac{1}{r}}$$

$$\leq \frac{k_1 + 1}{k_1 - \eta} \big[\big({}_\mathbf{Q}\mathbf{I}(\phi - \eta\psi)^{\frac{1}{r}} \big)(\xi; \nu) \big].$$

Remark 4 The above theorem reproduces the Minkowski-type inequality involving the extended Mittag-Leffler function presented by Andrić et al. [18] by setting $n = 1$, $\alpha, \beta, \gamma, \delta, \mu, \lambda, \theta, k > 0$, $\theta > \lambda$, $0 < k \leq \delta + \alpha$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, and $\rho = 0 = \nu$.

3 Fractional integral inequalities of reverse Minkowski type

In this section, we state and prove some reverse versions of Minkowski-type inequalities, which are generalizations of (1.6) and (1.8).

Theorem 8 *Let* $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, *where* $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, \ldots, n$. *Also, let* α, β, γ, δ, μ, ν, λ, ρ, $\theta, t \in \mathbb{R}$, $k \in (0, 1) \cup \mathbb{N}$, *and* $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$. *Let* $k + \rho < \delta + \nu + \alpha$. *Let* $r \geq 1$, *and let* ϕ *and* ψ *be positive and rth-power integrable functions on* $[d, e]$ *such that their ratio is bounded above by* k_2 *and bounded below by* k_1 *with* $k_1, k_2 > 0$. *Then*

$$\big[(\mathbf{I}\phi^r)(\xi; \nu) \big]^{\frac{1}{r}} + \big[(\mathbf{I}\psi^r)(\xi; \nu) \big]^{\frac{1}{r}} \leq m \big[(\mathbf{I}(\phi + \psi)^r)(\xi; \nu) \big]^{\frac{1}{r}}, \tag{3.1}$$

where $m = \frac{k_2(k_1 + 1) + (k_2 + 1)}{(k_1 + 1)(k_2 + 1)}$.

Proof Under the conditions of the theorem on $\frac{\phi}{\psi}$, we have the following inequalities:

$$(k_1 + 1)^r \psi^r(t) \leq \big(\phi(t) + \psi(t) \big)^r,$$

$$(k_2 + 1)^r \phi^r(t) \leq k_2^r \big(\psi(t) + \phi(t) \big)^r.$$

By multiplying both sides of inequalities by $(\xi - t)^{\beta-1} \mathbf{M}(\omega(\xi - t)^\alpha; \nu)$ and integrating on $[d, \xi]$ the above inequalities become

$$(k_1 + 1)^r \big(\mathbf{I}\psi^r \big)(\xi; \nu) \leq \big(\mathbf{I}(\psi + \phi)^r \big)(\xi; \nu), \tag{3.2}$$

$$\big(\mathbf{I}\phi^r \big)(\xi; \nu) \leq \frac{k_2^r}{(k_2 + 1)^r} \big(\mathbf{I}(\psi + \phi)^r \big)(\xi; \nu). \tag{3.3}$$

The above inequalities further produce the following inequalities:

$$\big[\big(\mathbf{I}\psi^r \big)(\xi; \nu) \big]^{\frac{1}{r}} \leq \frac{1}{k_1 + 1} \big[\big(\mathbf{I}(\psi + \phi)^r \big)(\xi; \nu) \big]^{\frac{1}{r}}, \tag{3.4}$$

$$\big[\big(\mathbf{I}\phi^r \big)(\xi; \nu) \big]^{\frac{1}{r}} \leq \frac{k_2}{k_2 + 1} \big[\big(\mathbf{I}(\psi + \phi)^r \big)(\xi; \nu) \big]^{\frac{1}{r}}. \tag{3.5}$$

The sum of (3.4) and (3.5) gives (3.1). □

Corollary 5 *Under the assumptions of Theorem* 8 *with* $a_i = l$, $p = 0$, *and* $\rho > 0$ *in* (3.1), *we obtain the following inequality*:

$$\left[\left(\mathbf{Q}\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{1}{r}} + \left[\left(\mathbf{Q}\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{1}{r}} \leq m\left[\left(\mathbf{Q}\mathbf{I}(\phi + \psi)^r\right)(\xi;\nu)\right]^{\frac{1}{r}},$$

where $m = \frac{k_2(k_1+1)+(k_2+1)}{(k_1+1)(k_2+1)}$.

Remark 5 The Minkowski-type fractional integral inequality containing the extended Mittag-Leffler function introduced by Andrić et al. [18] turns out to be a particular case of (3.1) by setting $n = 1$, $\alpha, \beta, \gamma, \delta, \mu, \lambda, \theta, k > 0$, $\theta > \lambda$, $0 < k \leq \delta + \alpha$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, and $\rho = 0 = \nu$.

Theorem 9 *Let* $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, *where* $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, \ldots, n$. *Also let* α, β, γ, δ, μ, ν, λ, ρ, θ, $t \in \mathbb{R}$, $k \in (0,1) \cup \mathbb{N}$ *and* $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$. *Let* $k + \rho < \delta + \nu + \alpha$. *Let* $r \geq 1$, *and let* ϕ *and* ψ *be positive and rth-power integrable functions on* $[d, e]$ *such that their ratio is bounded above by* k_2 *and bounded below by* k_1 *with* $k_1, k_2 > 0$. *Then we have the following inequality*:

$$\left[\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{2}{r}} + \left[\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{2}{r}} \geq M\left[\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{1}{r}}\left[\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{1}{r}}, \tag{3.6}$$

where $M = k_2^{-1}(k_2 + 1)(k_1 + 1) - 2$.

Proof From the preceding theorem we have the following integral inequalities:

$$\left[\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{1}{r}} \leq \frac{1}{k_1 + 1}\left[\left(\mathbf{I}\left(\psi(t) + \phi(t)\right)^r\right)(\xi;\nu)\right]^{\frac{1}{r}},$$

$$\left[\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{1}{r}} \leq \frac{k_2}{k_2 + 1}\left[\left(\mathbf{I}\left(\psi(t) + \phi(t)\right)^r\right)(\xi;\nu)\right]^{\frac{1}{r}}.$$

Taking the product of these inequalities and using Minkowski's inequality, we get

$$\left[\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{1}{r}}\left[\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{1}{r}} \leq \frac{k_2}{(k_2 + 1)(k_1 + 1)}$$
$$\times \left[\left(\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right)^{\frac{1}{r}} + \left(\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right)^{\frac{1}{r}}\right]^2.$$

This inequality takes the following form by simplification:

$$M\left[\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{1}{r}}\left[\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{1}{r}} \leq \left[\left(\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{2}{r}} + \left[\left(\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{2}{r}}. \qquad \square$$

Corollary 6 *Under the assumptions of Theorem* 9 *and setting* $a_i = l$, $p = 0$, *and* $\rho > 0$ *in* (3.6), *we have the following inequality*:

$$\left[\left(\mathbf{Q}\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{2}{r}} + \left[\left(\mathbf{Q}\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{2}{r}} \geq M\left[\left(\mathbf{Q}\mathbf{I}\phi^r\right)(\xi;\nu)\right]^{\frac{1}{r}}\left[\left(\mathbf{Q}\mathbf{I}\psi^r\right)(\xi;\nu)\right]^{\frac{1}{r}},$$

where $M = k_2^{-1}(k_2 + 1)(k_1 + 1) - 2$.

Remark 6 Putting $n = 1$, $\alpha, \beta, \gamma, \delta, \mu, \lambda, \theta, k > 0$, $\theta > \lambda$, $0 < k \leq \delta + \alpha$, $b_1 = c_1 + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$ and $\rho = 0 = \nu$, the above theorem reproduces the Minkowski-type inequality involving the extended Mittag-Leffler function presented by Andrić et al. [18].

4 Conclusions

We have proved some generalized Minkowski-type integral inequalities using fractional integral operators associated with unified Mittag-Leffler function. A number of such inequalities already studied for various types of known fractional integral operators can be deduced from the results of this paper. The unified Mittag-Leffler function and associated integral operators can be applied to extend and generalize the classical concepts.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contribution in this paper. All authors read and approved the final manuscript.

Author details
[1]College of Science, Hunan City University, Yiyang 413000, P.R. China. [2]Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan.

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