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Differential inequalities for spirallike and strongly starlike functions

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Abstract

In this paper, by using a technique of the first-order differential subordination, we find several sufficient conditions for an analytic function p such that $p(0) = 1$ to satisfy $\mathrm{Re}\{e^{i\beta}p(z)\} > \gamma$ or $|\arg\{p(z) - \gamma\}| < \delta$ for all $z \in \mathbb{D}$, where $\beta \in (-\pi/2, \pi/2)$, $\gamma \in [0, \cos\beta)$, $\delta \in (0, 1]$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The results obtained here will be applied to find some conditions for spirallike functions and strongly starlike functions in $\mathbb{D}$.

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1 Introduction and definitions

For real numbers β, γ, and δ satisfying $-\pi/2 < \beta < \pi/2$, $0 \le \gamma < \cos\beta$, and $0 < \delta \le 1$, define two domains $\Omega_\gamma(\beta)$ and $\Lambda_\gamma(\delta)$ in $\mathbb{C}$ by

$$\Omega_\gamma(\beta) = \left\{w \in \mathbb{C} : \mathrm{Re}\left(e^{-i\beta} w\right) > \gamma\right\}$$

and

$$\Lambda_\gamma(\delta) = \left\{w \in \mathbb{C} : \left|\arg(w - \gamma)\right| < \frac{\pi}{2}\delta\right\},$$

respectively. Then it clearly holds that

$$\Omega_\gamma(\beta) \cap \Omega_\gamma(-\beta) \subset \Lambda_\gamma\left(1 - \frac{2}{\pi}\beta\right). \tag{1.1}$$

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D}$, and let $\mathcal{H}_1$ be the class of functions $p \in \mathcal{H}$ with $p(0) = 1$. We introduce two subfamilies $\mathcal{P}_\gamma(\beta)$ and $\mathcal{Q}_\gamma(\delta)$ of $\mathcal{H}_1$ defined as follows:

$$\mathcal{P}_\gamma(\beta) = \left\{p \in \mathcal{H}_1 : p(z) \in \Omega_\gamma(\beta) \text{ for all } z \in \mathbb{D}\right\}$$

and

$$\mathcal{Q}_\gamma(\delta) = \{p \in \mathcal{H}_1 : p(z) \in \Lambda_\gamma(\delta) \text{ for all } z \in \mathbb{D}\}.$$

A function p in $\mathcal{P}_\gamma(0)$ is said to be a Carathéodory function of order γ in $\mathbb{D}$. In particular, $\mathcal{P}_0(0) \equiv \mathcal{P}$ is the well-known class of Carathéodory functions. Also, a function p in $\mathcal{P}_0(\beta)$ is said to be a tilted Carathéodory function by angle β [27]. We note that

$$\mathcal{P}_\gamma(\beta) \cap \mathcal{P}_\gamma(-\beta) \subset \mathcal{Q}_\gamma\left(1 - \frac{2}{\pi}\beta\right)$$

holds, by (1.1).

Let $\mathcal{A}$ denote the class of functions f in $\mathcal{H}$ normalized by $f(0) = 0 = f'(0) - 1$. And let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions. Further we denote by $\mathcal{S}^*_\gamma(\beta)$ and $\mathcal{SS}^*_\gamma(\delta)$ the subclass of $\mathcal{A}$ consisting of β-spirallike functions of order γ [8, II, p. 89] (see also [16, 24]) and strongly starlike functions of order δ and type γ [9]. That is, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\gamma(\beta)$ if f satisfies

$$\operatorname{Re}\left\{e^{-i\beta}\frac{zf'(z)}{f(z)}\right\} > \gamma, \quad z \in \mathbb{D},$$

and belongs to the class $\mathcal{SS}^*_\gamma(\delta)$ when f satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta, \quad z \in \mathbb{D}.$$

Thus we have

$$f \in \mathcal{S}^*_\gamma(\beta) \quad \Longleftrightarrow \quad J_f \in \mathcal{P}_\gamma(\beta)$$

and

$$f \in \mathcal{SS}^*_\gamma(\delta) \quad \Longleftrightarrow \quad J_f \in \mathcal{Q}_\gamma(\delta),$$

where $J_f(z) := zf'(z)/f(z)$, $z \in \mathbb{D}$. Note that $\mathcal{S}^*_\gamma(0) \equiv \mathcal{S}^*(\gamma)$ is the class of starlike functions of order γ, and $\mathcal{S}^*_0(\beta) \equiv \mathcal{SP}(\beta)$ is the class of β-spirallike functions. It is well known [24] (or [8, Vol. I, p. 149]) that $\mathcal{S}^*(\gamma)$ and $\mathcal{SP}(\beta)$ are the subclasses of $\mathcal{S}$. See [7, 12, 28] for sufficient conditions for spirallike functions. We also note that $\mathcal{SS}^*_\gamma(\delta) \subset \mathcal{S}^*_\gamma(0) \subset \mathcal{S}$. Especially, $\mathcal{SS}^*_0(\delta) \equiv \mathcal{SS}^*(\delta)$ which is the class of strongly starlike functions of order δ [4, 25]. Refer to [5, 6, 11, 13, 14, 17–20, 23, 26] for various sufficient conditions for strongly starlike functions.

In the present paper we investigate new sufficient conditions for functions in $\mathcal{P}_\gamma(\beta)$ or $\mathcal{Q}_\gamma(\delta)$. As direct consequences of these results, we will obtain several sufficient conditions for spirallike functions or strongly starlike functions in $\mathbb{D}$.

For analytic functions f and g, we say that f is subordinate to g, denoted by $f \prec g$, if there is an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $|\omega(z)| \le |z|$ such that $f(z) = g(\omega(z))$. Further, if g is univalent, then the definition of subordination $f \prec g$ simplifies to the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (see [21, p. 36]).

Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ be the closure and boundary of $\mathbb{D}$, respectively. We denote by $\mathcal{R}$ the class of functions q that are analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) = \left\{\zeta : \zeta \in \partial\mathbb{D} \text{ and } \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that

$$q'(\zeta) \neq 0 \quad \big(\zeta \in \partial\mathbb{D} \setminus E(q)\big).$$

Furthermore, let the subclass of $\mathcal{R}$ for which $q(0) = a$ be denoted by $\mathcal{R}(a)$. We recall the following lemma which will be used for our results.

Lemma 1.1 ([15, p. 24]) *Let $q \in \mathcal{R}(a)$ and let*

$$p(z) = a + a_n z^n + \cdots \quad (n \geq 1)$$

be an analytic function in $\mathbb{D}$ with $p(0) = a$. If p is not subordinate to q, then there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus E(q)$ for which

(i) $p(z_0) = q(\zeta_0)$;

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ $(m \geq n \geq 1)$.

2 Main results

Throughout this section, let β and γ be real numbers such that $-\pi/2 < \beta < \pi/2$ and $0 \leq \gamma < \cos\beta$ unless we mention it. We define a function $\varphi_{\beta,\gamma} : \mathbb{D} \to \mathbb{C}$ by

$$\varphi_{\beta,\gamma}(z) = \frac{1 + (e^{i\beta} - 2\gamma)e^{i\beta} z}{1 - z}. \tag{2.1}$$

Then it is easy to check that the bilinear function $\varphi_{\beta,\gamma}$ maps the unit disk $\mathbb{D}$ onto the half-plane $\Omega_\gamma(\beta)$. By using the function $\varphi_{\beta,\gamma}$ we obtain the following results.

Theorem 2.1 *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$. If $p \in \mathcal{H}_1$ satisfies*

$$\left|p(z) + \alpha\frac{zp'(z)}{p(z)} - 1\right| < (\cos\beta - \gamma)\left(1 + \frac{1}{2}\mathrm{Re}(\alpha)\right)\big|p(z)\big|, \quad z \in \mathbb{D}, \tag{2.2}$$

then $1/p \in \mathcal{P}_\gamma(-\beta)$. That is, $\mathrm{Re}\{e^{i\beta}/p(z)\} > \gamma$ for all $z \in \mathbb{D}$.

Proof Let us define functions q and $h : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{e^{i\beta}}{p(z)} \tag{2.3}$$

and

$$h(z) = e^{i\beta}\varphi_{-\beta,\gamma}(z) = \frac{e^{i\beta} + (e^{-i\beta} - 2\gamma)z}{1 - z}, \tag{2.4}$$

where $\varphi_{\beta,\gamma}$ is the function defined by (2.1). Then the functions q and h are analytic in $\mathbb{D}$ with

$$q(0) = h(0) = \mathrm{e}^{\mathrm{i}\beta} \in \mathbb{C} \quad \text{and} \quad h(\mathbb{D}) = \bigl\{w \in \mathbb{C} : \mathrm{Re}\{w\} > \gamma\bigr\}.$$

Suppose now that q is not subordinate to h. Then, by Lemma 1.1, there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus \{1\}$ such that

$$q(z_0) = h(\zeta_0) = \gamma + \mathrm{i}\rho \quad (\rho \in \mathbb{R}) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 h'(\zeta_0) = m\sigma \quad (m \geq 1), \tag{2.5}$$

where

$$\sigma = \frac{-\rho^2 + 2\rho\sin\beta + 2\gamma\cos\beta - 1 - \gamma^2}{2(\cos\beta - \gamma)}. \tag{2.6}$$

Since $\gamma < \cos\beta$, we get $\sigma < 0$. Indeed, we have

$$2(\cos\beta - \gamma)\sigma = -(\rho - \sin\beta)^2 - (\cos\beta - \gamma)^2 \leq -(\cos\beta - \gamma)^2,$$

which implies that

$$\sigma \leq -\frac{1}{2}(\cos\beta - \gamma) < 0.$$

Using (2.3) and (2.5), we have

$$\begin{aligned}\left|\frac{p(z_0) + \alpha\frac{z_0 p'(z_0)}{p(z_0)} - 1}{p(z_0)}\right| &= \bigl|\alpha z_0 q'(z_0) + q(z_0) - \mathrm{e}^{\mathrm{i}\beta}\bigr| \\ &= \bigl|\alpha m\sigma + \gamma + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\beta}\bigr|. \end{aligned}\tag{2.7}$$

Let $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ with $\alpha_1 \geq 0$ and $\alpha_2 \in \mathbb{R}$. Then we have

$$\begin{aligned}&\bigl|\alpha m\sigma + \gamma + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\beta}\bigr|^2 \\ &\quad = \bigl(\alpha m\sigma + \gamma + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\beta}\bigr)\bigl(\overline{\alpha} m\sigma + \gamma - \mathrm{i}\rho - \mathrm{e}^{-\mathrm{i}\beta}\bigr) \\ &\quad = |\alpha|^2 m^2\sigma^2 + (\gamma - \cos\beta)^2 + 2\alpha_1 m\sigma(\gamma - \cos\beta) + \kappa, \end{aligned}\tag{2.8}$$

where

$$\kappa = (\rho - \sin\beta)^2 + 2\alpha_2 m\sigma(\rho - \sin\beta).$$

Furthermore it is easy to see that

$$\kappa = (\rho - \sin\beta + m\alpha_2\sigma)^2 - (m\alpha_2\sigma)^2 \geq -(m\alpha_2\sigma)^2.$$

Since $m \geq 1$, from (2.8), we have

$$\begin{aligned}
&\left|\alpha m\sigma + \gamma + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\beta}\right|^2 \\
&\quad \geq |\alpha|^2 m^2\sigma^2 + (\gamma - \cos\beta)^2 + 2\alpha_1 m\sigma(\gamma - \cos\beta) - \alpha_2^2 m^2\sigma^2 \\
&\quad = \alpha_1^2 m^2\sigma^2 + (\gamma - \cos\beta)^2 + 2\alpha_1 m\sigma(\gamma - \cos\beta) \\
&\quad \geq \alpha_1^2\sigma^2 + (\gamma - \cos\beta)^2 + 2\alpha_1\sigma(\gamma - \cos\beta) \\
&\quad = [\alpha_1\sigma + \gamma - \cos\beta]^2.
\end{aligned} \tag{2.9}$$

Since $\sigma < 0$, $\alpha_1 \geq 0$, and $\cos\beta > \gamma$, inequality (2.9) implies

$$\left|\alpha m\sigma + \gamma + \mathrm{i}\rho - \mathrm{e}^{\mathrm{i}\beta}\right| \geq -\alpha_1\sigma + \cos\beta - \gamma. \tag{2.10}$$

Furthermore, since $\sigma \leq -(\cos\beta - \gamma)/2$, we have

$$-\alpha_1\sigma + \cos\beta - \gamma \geq (\cos\beta - \gamma)\left(1 + \frac{1}{2}\alpha_1\right). \tag{2.11}$$

Finally, from (2.7), (2.10), and (2.11), we obtain

$$\left|p(z_0) + \alpha\frac{z_0 p'(z_0)}{p(z_0)} - 1\right| \geq (\cos\beta - \gamma)\left(1 + \frac{1}{2}\alpha_1\right)\left|p(z_0)\right|.$$

This inequality contradicts hypothesis (2.2). Therefore, we obtain $q \prec h$ in $\mathbb{D}$ and the inequality $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\beta}/p(z)\} > \gamma$ holds for all $z \in \mathbb{D}$. □

We remark that the hypothesis in Theorem 2.1 implies also $1/p \in \mathcal{P}_\gamma(\beta)$. And we also remark that Theorem 2.1 reduces the result [13] when $\alpha = 1$.

By the above remark, taking $\gamma = 1/2$ in Theorem 2.1 gives the following corollary.

Corollary 2.1 *Let α and $\beta \in \mathbb{R}$ with $\alpha \geq 0$ and $\beta \in [0, \pi/3)$. If $p \in \mathcal{H}_1$ satisfies (2.2) with $\gamma = 1/2$, then $p(\mathbb{D}) \subset \Xi_\beta$, where*

$$\Xi_\beta = \left\{w \in \mathbb{C} : \left|\mathrm{e}^{-\mathrm{i}\beta} w - 1\right| < 1 \text{ and } \left|\mathrm{e}^{\mathrm{i}\beta} w - 1\right| < 1\right\},$$

and we have $\operatorname{Re}\{p(z)\} > 0$ for all $z \in \mathbb{D}$. Furthermore, if $\beta \neq 0$, then $|\arg\{p(z)\}| < \cot\beta$ for all $z \in \mathbb{D}$.

Taking $p(z) = zf'(z)/f(z)$, $f \in \mathcal{A}$, in Corollary 2.1 gives the following result.

Corollary 2.2 *Let $\alpha \in \mathbb{R}$ with $\alpha \geq 0$. If $\beta \in (0, \pi/3)$ and $f \in \mathcal{A}$ satisfies*

$$\left|(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < \sqrt{\alpha + 1}\left(\cos\beta - \frac{1}{2}\right)\left|\frac{zf'(z)}{f(z)}\right|, \quad z \in \mathbb{D}, \tag{2.12}$$

then $f \in \mathcal{SS}_0^*(\cot\beta)$, i.e., f is strongly starlike of order $2(\cot\beta)/\pi$ in $\mathbb{D}$. If $f \in \mathcal{A}$ satisfies (2.12) with $\beta = 0$, then $f \in \mathcal{S}_0^*(0)$, i.e., f is a starlike function in $\mathbb{D}$.

Example 2.1 Let $a \in \mathbb{C}$ be given, and let $f_a(z) = z/(1-az)$, $z \in \mathbb{D}$. Then a computation shows that

$$\frac{1}{2}\left|\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - 1\right| = \frac{3|a||z|}{2|1-az|} < \frac{3|a|}{2|1-az|} = \frac{3|a|}{2}\left|\frac{zf'(z)}{f(z)}\right|, \quad z \in \mathbb{D}.$$

Hence if

$$|a| < \frac{\sqrt{6}}{3}\left(\cos\beta - \frac{1}{2}\right), \tag{2.13}$$

then inequality (2.12) with $\alpha = 1/2$ holds. Thus, by Corollary 2.2 with $\alpha = 1/2$, we conclude that f_a is strongly starlike of order $2(\cot\beta)/\pi$ in $\mathbb{D}$ provided inequality (2.13) holds.

Example 2.2 Let $g_a(z) = z/(1-az)^2$, $z \in \mathbb{D}$, with $a \in \mathbb{C}$. Then a similar computation with Example 2.1 and Corollary 2.2 gives that if $a \in \mathbb{C}$ satisfies

$$\frac{2|a|(3+2|a|)}{(1-|a|)^2} \le \frac{\sqrt{6}}{2}\left(\cos\beta - \frac{1}{2}\right),$$

then g_a is strongly starlike of order $2(\cot\beta)/\pi$ in $\mathbb{D}$.

Theorem 2.2 *Let $\alpha \in \mathbb{R}$ with $\alpha \ge 0$. Assume that*

$$(2\lambda + \gamma)|\sin\beta| < 2\sqrt{\Delta}, \tag{2.14}$$

where

$$\Delta = \lambda(\lambda + \cos\beta)\bigl(-2\gamma\cos\beta + 1 + \gamma^2\bigr), \tag{2.15}$$

with

$$\lambda = \frac{\alpha}{2(\cos\beta - \gamma)} \ge 0.$$

Let $p \in \mathcal{H}_1$ with $\gamma e^{-i\beta} \notin p(\mathbb{D})$. If

$$\left|\operatorname{Im}\left\{p(z) + \frac{\alpha z p'(z)}{p(z) - \gamma e^{-i\beta}} + i(2\lambda + \gamma)\sin\beta\right\}\right| < 2\sqrt{\Delta}, \quad z \in \mathbb{D}, \tag{2.16}$$

then $p \in \mathcal{P}_\gamma(-\beta)$. That is, $\operatorname{Re}\{e^{i\beta}p(z)\} > \gamma$ for all $z \in \mathbb{D}$.

Proof We first note that, since $p(0) = 1$, (2.14) implies that inequality (2.16) is well-defined. Next we define functions q and h by

$$q(z) = e^{i\beta}p(z) \tag{2.17}$$

and (2.4), respectively. If q is not subordinate to h, then there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}\setminus\{1\}$ satisfying (2.5) with $\rho \in \mathbb{R}$. We note that $\rho \neq 0$. Indeed, if $\rho = 0$, then $e^{i\beta}p(z_0) = q(z_0) = \gamma$. Therefore we have $p(z_0) = \gamma e^{-i\beta}$, which contradicts the condition $\gamma e^{-i\beta} \notin p(\mathbb{D})$.

Simple computations give

$$\begin{aligned}
&p(z_0) + \frac{\alpha z_0 p'(z_0)}{p(z_0) - \gamma \mathrm{e}^{-\mathrm{i}\beta}} \\
&\quad = \mathrm{e}^{-\mathrm{i}\beta} q(z_0) + \frac{\alpha z_0 q'(z_0)}{q(z_0) - \gamma} \\
&\quad = \gamma \cos\beta + \rho \sin\beta + \mathrm{i}\left(\rho\cos\beta - \gamma \sin\beta - \frac{\alpha m \sigma}{\rho}\right),
\end{aligned}$$

where σ is given by (2.6). Therefore we get

$$\begin{aligned}
&\operatorname{Im}\left\{p(z_0) + \frac{\alpha z_0 p'(z_0)}{p(z_0) - \gamma \mathrm{e}^{-\mathrm{i}\beta}} + \mathrm{i}(2\lambda + \gamma)\sin\beta\right\} \\
&\quad = \rho\cos\beta - \frac{\alpha m \sigma}{\rho} + 2\lambda \sin\beta \\
&\quad = \rho\cos\beta + m\lambda\left[\rho - 2\sin\beta + \frac{-2\gamma\cos\beta + 1 + \gamma^2}{\rho}\right] + 2\lambda\sin\beta. \qquad (2.18)
\end{aligned}$$

Assume that $\rho > 0$, and put

$$\tilde{\lambda} = \rho - 2\sin\beta + \frac{-2\gamma\cos\beta + 1 + \gamma^2}{\rho}.$$

Note that

$$-2\gamma\cos\beta + 1 + \gamma^2 = (\gamma - \cos\beta)^2 + \sin^2\beta \ge 0$$

and

$$-2\gamma\cos\beta + 1 + \gamma^2 - \sin^2\beta = (\cos\beta - \gamma)^2 \ge 0.$$

Since $\rho > 0$, these inequalities yield that

$$\tilde{\lambda} \ge 2\sqrt{-2\gamma\cos\beta + 1 + \gamma^2} - 2\sin\beta \ge 0.$$

Therefore, since $m \ge 1$ and $\lambda \ge 0$, from (2.18), we obtain

$$\begin{aligned}
&\operatorname{Im}\left\{p(z_0) + \frac{\alpha z_0 p'(z_0)}{p(z_0) - \gamma \mathrm{e}^{-\mathrm{i}\beta}} + \mathrm{i}(2\lambda + \gamma)\sin\beta\right\} \\
&\quad \ge \rho\cos\beta + \lambda\tilde{\lambda} + 2\lambda\sin\beta \\
&\quad = (\cos\beta + \lambda)\rho + \frac{\lambda(-2\gamma\cos\beta + 1 + \gamma^2)}{\rho} \\
&\quad \ge 2\sqrt{\Delta}, \qquad (2.19)
\end{aligned}$$

where Δ is given by (2.15). This contradicts condition (2.16).

Now assume that $\rho < 0$. From (2.18), we have

$$\begin{aligned}
&\operatorname{Im}\left\{p(z_0) + \frac{\alpha z_0 p'(z_0)}{p(z_0) - \gamma \mathrm{e}^{-\mathrm{i}\beta}} + \mathrm{i}(2\lambda + \gamma)\sin\beta\right\} \\
&\quad = -\left[\tilde{\rho}\cos\beta + m\lambda\left(\tilde{\rho} + 2\sin\beta + \frac{-2\gamma\cos\beta + 1 + \gamma^2}{\tilde{\rho}}\right)\right] + 2\lambda\sin\beta,
\end{aligned}$$

where $\tilde{\rho} = -\rho > 0$. A similar calculation with (2.19) gives us to get

$$\operatorname{Im}\left\{p(z_0) + \frac{\alpha z_0 p'(z_0)}{p(z_0) - \gamma \mathrm{e}^{-\mathrm{i}\beta}} + \mathrm{i}(2\lambda + \gamma)\sin\beta\right\} \leq -2\sqrt{\Delta},$$

where Δ is given by (2.15). This also contradicts condition (2.16). Therefore we get $q \prec h$ in $\mathbb{D}$, and the inequality $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\beta} p(z)\} > \gamma$, $z \in \mathbb{D}$, follows. □

We remark that Theorem 2.2 reduces the result [13] when $\alpha = 1$.

Theorem 2.3 *Let* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) \geq 0$. *Assume that* $\Psi(\alpha, \beta, \gamma) < \cos\beta$, *where*

$$\Psi(\alpha, \beta, \gamma) = \frac{\sin^2\beta(\gamma + s)^2}{\cos\beta + s} + \gamma^2\cos\beta + s\left(2\gamma\cos\beta - 1 - \gamma^2\right), \tag{2.20}$$

with

$$s = \frac{\operatorname{Re}(\alpha)}{2(\cos\beta - \gamma)}.$$

If $p \in \mathcal{H}_1$ *satisfies*

$$\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\beta}\left[\left(p(z)\right)^2 + \alpha z p'(z)\right]\right\} > \Psi(\alpha, \beta, \gamma), \quad z \in \mathbb{D}, \tag{2.21}$$

then $\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\beta} p(z)\} > \gamma$ *for all* $z \in \mathbb{D}$.

Proof We first note that, since $p(0) = 1$, the hypothesis $\Psi(\alpha, \beta, \gamma) < \cos\beta$ implies that inequality (2.21) is well defined. Now we define the functions q and h by (2.17) and (2.4), respectively. If q is not subordinate to h, then there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus \{1\}$ satisfying (2.5) with $\rho \in \mathbb{R}$.

Put $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ with $\alpha_1 \geq 0$ and $\alpha_2 \in \mathbb{R}$. By (2.17) and (2.5), we obtain

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}\beta}\left[\left(p(z_0)\right)^2 + \alpha z_0 p'(z_0)\right] \\
&\quad = \mathrm{e}^{\mathrm{i}\beta}\left[\mathrm{e}^{-2\mathrm{i}\beta}\left(q(z_0)\right)^2 + \alpha \mathrm{e}^{-\mathrm{i}\beta} z_0 q'(z_0)\right] \\
&\quad = \mathrm{e}^{-\mathrm{i}\beta}(\gamma + \mathrm{i}\rho)^2 + (\alpha_1 + \mathrm{i}\alpha_2)m\sigma \\
&\quad = \left(\gamma^2 - \rho^2\right)\cos\beta + 2\gamma\rho\sin\beta + m\sigma\alpha_1 \\
&\qquad + \mathrm{i}\left[2\gamma\rho\cos\beta - \left(\gamma^2 - \rho^2\right)\sin\beta + m\sigma\alpha_2\right].
\end{aligned}$$

Hence taking real parts in the above, and from $\sigma\alpha_1 \leq 0$ and $m \geq 1$, we have

$$\begin{aligned}
\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\beta}\left[\left(p(z_0)\right)^2 + \alpha z_0 p'(z_0)\right]\right\} &= \left(\gamma^2 - \rho^2\right)\cos\beta + 2\gamma\rho\sin\beta + m\sigma\alpha_1 \\
&\leq \left(\gamma^2 - \rho^2\right)\cos\beta + 2\gamma\rho\sin\beta + \sigma\alpha_1.
\end{aligned} \tag{2.22}$$

Now equation (2.6) gives

$$\left(\gamma^2-\rho^2\right)\cos\beta+2\gamma\rho\sin\beta+\sigma\alpha_1=-a_2\rho^2+a_1\rho+a_0, \tag{2.23}$$

where

$$a_2=\cos\beta+\frac{\alpha_1}{2\mu},\qquad a_1=\left(2\gamma+\frac{\alpha_1}{\mu}\right)\sin\beta$$

and

$$a_0=\gamma^2\cos\beta+\frac{\alpha_1(2\gamma\cos\beta-1-\gamma^2)}{2\mu},$$

with $\mu=\cos\beta-\gamma$.

Clearly, $a_2>0$. Thus we have

$$-a_2\rho^2+a_1\rho+a_0\le\frac{a_1^2}{4a_2}+a_0,\quad \rho\in\mathbb{R}. \tag{2.24}$$

Consequently, by (2.22), (2.23), and (2.24), we obtain

$$\operatorname{Re}\left\{e^{i\beta}\left[\left(p(z_0)\right)^2+\alpha z_0p'(z_0)\right]\right\}\le\frac{a_1^2}{4a_2}+a_0=\Psi(\alpha,\beta,\gamma).$$

This contradicts (2.21). Therefore we obtain $q\prec h$ in $\mathbb{D}$, and it follows that the inequality $\operatorname{Re}\{e^{i\beta}p(z)\}>\gamma$ holds for all $z\in\mathbb{D}$. □

Since the condition

$$\left|\arg(w-a\sec\beta)\right|<\frac{\pi}{2}-\beta$$

implies

$$\operatorname{Re}\left\{e^{i\beta}w\right\}>a\quad\text{and}\quad\operatorname{Re}\left\{e^{-i\beta}w\right\}>a,$$

for $w\in\mathbb{C}$, $a\in\mathbb{R}$ and $\beta\in(-\pi/2,\pi/2)$, by noting that $\Psi(\alpha,\beta,\gamma)=\Psi(\alpha,-\beta,\gamma)$, the following result can be obtained from Theorem 2.3.

Theorem 2.4 *Let* $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}(\alpha)\ge0$. *Assume that* $\Psi(\alpha,\beta,\gamma)<\cos\beta$, *where* Ψ *is given by* (2.20). *If* $p\in\mathcal{H}_1$ *satisfies*

$$\left|\arg\left\{\left(p(z)\right)^2+\alpha zp'(z)-\Psi(\alpha,\beta,\gamma)\sec\beta\right\}\right|<\frac{\pi}{2}-\beta,\quad z\in\mathbb{D},$$

then

$$\left|\arg\left(p(z)-\gamma\right)\right|<\frac{\pi}{2}-\beta,\quad z\in\mathbb{D}.$$

Taking $\alpha=1$ and $p(z)=zf'(z)/f(z)$, $f\in\mathcal{A}$, in Theorems 2.3 and 2.4 we have the following corollary.

Corollary 2.3 *Assume that $\Psi(1,\beta,\gamma) < \cos\beta$, where Ψ is given by (2.20). If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left\{e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \Psi(1,\beta,\gamma), \quad z \in \mathbb{D},$$

then f is a β-spirallike function of order γ in $\mathbb{D}$. If $f \in \mathcal{A}$ satisfies

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) - \Psi(1,\beta,\gamma)\sec\beta\right\}\right| < \frac{\pi}{2} - \beta, \quad z \in \mathbb{D},$$

then f is strongly starlike of order $1-(2/\pi)\beta$ and type γ in $\mathbb{D}$.

Example 2.3 Let $a \in \mathbb{C}$, and define a function $f_a : \mathbb{D} \to \mathbb{C}$ by $f_a(z) = z/(1-az)$. Then, since $|z| < 1$, we have

$$\begin{aligned}&\mathrm{Re}\left\{e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) - \cos\beta\right\}\\&\quad = \mathrm{Re}\left\{e^{i\beta}\frac{az(3-az)}{(1-az)^2}\right\} \geq -\frac{|a||z||3-az|}{|1-az|^2} > -\frac{|a|(3+|a|)}{(1-|a|)^2},\end{aligned}$$

or, equivalently,

$$\mathrm{Re}\left\{e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \cos\beta - \frac{|a|(3+|a|)}{(1-|a|)^2}.$$

By Corollary 2.3, f_a is a β-spirallike function of order γ provided

$$\frac{|a|(3+|a|)}{(1-|a|)^2} \leq \cos\beta - \Psi(1,\beta,\gamma),$$

where Ψ is given by (2.20). In particular, if

$$|a| \leq \frac{1}{76}(-239 + 3\sqrt{6769}) =: \tau = 0.1029\cdots,$$

then f_a is $(\pi/3)$-spirallike function of order $1/3$. Indeed, when $\beta = \pi/3$ and $\gamma = 1/3$, we have $\cos\beta - \Psi(1,\beta,\gamma) = 25/63$. Solving the inequality $|a|(3+|a|)/(1-|a|)^2 \leq 25/63$ gives us to get $|a| \leq \tau$.

Example 2.4 Let $a \in \mathbb{C}$ be given, and let $g_a(z) = z/(1-az)^2$, $z \in \mathbb{D}$. Then, from a similar computation with Example 2.3 and Corollary 2.3, we have that g_a is a β-spirallike function of order γ, if

$$\frac{1+4|a|+|a|^2}{(1-|a|)^2} \leq \cos\beta - \Psi(1,\beta,\gamma).$$

3 Concluding remarks and observations

In the present investigation, we have found several conditions for Carathéodory functions by using a technique of the first-order differential subordination. In particular, one can obtain conditions for Carathéodory functions of order γ $(0 < \gamma \leq 1)$ and for tilted

Carathéodory functions by angle β ($-\pi/2 < \beta < \pi/2$). We have applied these results to obtain new criteria for geometric properties such as spirallikeness and strongly starlikeness, and several examples were given here.

We conclude this paper by remarking that the results here reduce the earlier conditions [13] for Carathéodory functions. Also, as the examples in this paper show, the first-order differential subordination with the conformal mapping $\varphi_{\beta,\gamma}$ defined by (2.1) gives some nice criteria for spirallike functions and strongly starlike functions. This observation will indeed apply to any attempt to produce the conditions for other geometric properties such as convexity, q-starlikeness, etc. [1–3, 10, 22, 29, 30].

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