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Framed slant helices in Euclidean 3-space

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Abstract

In this paper, we define framed slant helices and give a necessary and sufficient condition for them in three-dimensional Euclidean space. Then, we introduce the spherical images of a framed curve. Also, we examine the relations between a framed slant helix and its spherical images. Moreover, we give an example of a framed slant helix and its spherical images with figures.

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1 Introduction

Let γ be a regular curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ in three-dimensional Euclidean space \mathbb{R}^3 . We know that the curve γ is a general helix if the tangent vector of γ makes a constant angle with a fixed straight line. In 1802, a classical characterization of general helices was given by M.A. Lanceret and was proved first by B. de Saint Venant in 1845: “The curve γ is a general helix if and only if its curvatures ratio is constant.” (see [1, 2]). On the other hand, the curve γ is a slant helix if its normal vector makes a constant angle with a fixed straight line. It is well known that slant helices have the following characterization: “The curve γ is a slant helix in \mathbb{R}^3 if and only if the geodesic curvature of the principal normal indicatrix (N) of the curve is a constant function.” That is

$$\sigma_N(s) = \left(\frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \right) \left(\frac{\tau}{\kappa} \right)'$$

is a constant function (see [3]). To date, general helices and slant helices have been introduced in different spaces and characterizations obtained of these curves by many researchers (see [4–9]).

Recently, Shun’ichi Honda and Masatomo Takahashi investigated framed curves in Euclidean space (see [10]). Moreover, Yongqiao Wang et al. defined framed helices and gave a characterization for a framed curve to be a framed helix (see [11]). Tuncer et al. introduced pedal and contrapedal curves of fronts by using the Legendrian Frenet frame in the Euclidean plane and also examined singularities of pedal and contrapedal curves of fronts in [12]. Yazıcı et al. investigated framed rectifying curves via the dilation of framed curves on S^2 in \mathbb{R}^3 , [13]. Yıldız studied the evolution of framed curves in \mathbb{R}^3 , [14].

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In this paper, we define framed slant helices and give a characterization to be a framed slant helix of a framed curve in three-dimensional Euclidean space. We then define framed spherical images of a framed curve. Also, we obtain some relations between a framed slant helix and its spherical images. Finally, we present an illustrated example to support the theory.

2 Basic materials

In this section, we outline the definitions of framed curves and framed helices in three-dimensional Euclidean space (see for details [10, 11]).

Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve with singular points. The set Δ_2 , which is defined by

$$\Delta_2 = \{ \mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \mu_i \cdot \mu_j = \delta_{ij}, i, j = 1, 2 \},$$

is a three-dimensional smooth manifold. For a given $\mu = (\mu_1, \mu_2) \in \Delta_2$, we can define a unit vector of \mathbb{R}^3 , as follows:

$$v = \mu_1 \times \mu_2.$$

This means that v is orthogonal to μ_1 and μ_2 .

Definition 2.1 We say that $(\gamma, \mu) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ is a framed curve if $\langle \gamma'(s), \mu_i(s) \rangle = 0$ for all $s \in I$ and $i = 1, 2$. We also say that $\gamma : I \rightarrow \mathbb{R}^3$ is a framed base curve if there exists $\mu : I \rightarrow \Delta_2$ such that (γ, μ) is a framed curve.

Let $(\gamma, \mu) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve, then the Frenet–Serret-type formula of the curve γ is as follows:

$$\begin{aligned} \gamma'(s) &= -m(s)\mu_1(s) - n(s)\mu_2(s) \\ \mu_1'(s) &= l(s)\mu_2(s) + m(s)v(s), \\ \mu_2'(s) &= -l(s)\mu_1(s) + n(s)v(s), \end{aligned}$$

where $l(s) = \langle \mu_1'(s), \mu_2(s) \rangle$, $m(s) = \langle \mu_1'(s), v(s) \rangle$ and $n(s) = \langle \mu_2'(s), v(s) \rangle$. Also, there exists a smooth mapping $\alpha : I \rightarrow \mathbb{R}$ such that:

$$\gamma'(s) = \alpha(s)v(s).$$

The functions $l(s)$, $m(s)$, $n(s)$ and $\alpha(s)$ are called the curvature functions of γ . Clearly, $\alpha(s_0) = 0$ if and only if s_0 is a singular point of the curve γ . The curvature of the framed curve is quite useful to analyze the framed curves and singularities (see [10, 11]).

Theorem 2.1 Let $(l, m, n, \alpha) : I \rightarrow \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma, \mu) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ whose associated curvature of the framed curve is (l, m, n, α) (see [11]).

Theorem 2.2 Let (γ, μ) and $(\bar{\gamma}, \bar{\mu}) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be framed curves whose curvatures of the framed curves (l, m, n, α) and $(\bar{l}, \bar{m}, \bar{n}, \bar{\alpha})$ coincide. Then, (γ, μ) and $(\bar{\gamma}, \bar{\mu})$ are congruent as framed curves (see [11]).

In [11], the authors defined an adapted frame with $\{v(s), \bar{\mu}_1(s), \bar{\mu}_2(s)\}$ along the γ and gave Frenet formulas for the adapted frame as follows:

$$\begin{pmatrix} v'(s) \\ \bar{\mu}'_1(s) \\ \bar{\mu}'_2(s) \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & q(s) \\ 0 & -q(s) & 0 \end{pmatrix} \begin{pmatrix} v(s) \\ \bar{\mu}_1(s) \\ \bar{\mu}_2(s) \end{pmatrix}, \tag{1}$$

where $(\bar{\mu}_1, \bar{\mu}_2) \in \Delta_2$ is defined by

$$\begin{pmatrix} \bar{\mu}_1(s) \\ \bar{\mu}_2(s) \end{pmatrix} = \begin{pmatrix} \cos \theta(s) & -\sin \theta(s) \\ \sin \theta(s) & \cos \theta(s) \end{pmatrix} \begin{pmatrix} \mu_1(s) \\ \mu_2(s) \end{pmatrix}.$$

Here, $\theta(s)$ is a smooth function. Clearly, $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ is also a framed curve, so we have:

$$\bar{v}(s) = \mu_1(s) \times \mu_2(s) = \bar{\mu}_1(s) \times \bar{\mu}_2(s) = v(s).$$

The vectors $v(s), \bar{\mu}_1(s), \bar{\mu}_2(s)$ are called the generalized tangent vector, the generalized principal normal vector and the generalized binormal vector of the framed curve γ , respectively. The functions $(p(s), q(s), \alpha(s))$ are referred to as the framed curvatures of the framed curve γ , where $p(s) = |v'(s)| > 0$ and $q(s) = l(s) - \theta'(s)$.

Definition 2.2 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve. If the framed base curve γ is a curve on S^2 , the γ is a framed spherical curve (see [11]).

Definition 2.3 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve with $p(s) > 0$. The γ is called a framed helix if its generalized tangent vector v makes a constant angle with a fixed unit vector ζ . That is

$$\langle v(s), \zeta \rangle = \cos \phi,$$

where ϕ is a constant angle (see [11]).

Theorem 2.3 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve with the adapted frame apparatus $\{v(s), \bar{\mu}_1(s), \bar{\mu}_2(s), p(s), q(s)\}$. Then, γ is a framed helix if and only if the following equation holds:

$$\frac{q(s)}{p(s)} = \mp \cot \phi(s),$$

where ϕ is a constant angle (see [11]).

3 Framed slant helices in \mathbb{R}^3

In this section we define a framed slant helix and its axis in three-dimensional Euclidean space \mathbb{R}^3 . Also, we give a characterization of a framed slant helix.

Definition 3.1 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve. Then, γ is called a framed slant helix if its generalized principal normal vector $\bar{\mu}_1$ makes a constant angle with a fixed unit vector ζ . That is

$$\langle \bar{\mu}_1(s), \zeta \rangle = \cos \varphi \quad \text{for all } s \in I,$$

where $\varphi \neq \frac{\pi}{2}$ is a constant angle between ζ and $\bar{\mu}_1(s)$.

Definition 3.2 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve with the adapted frame apparatus $\{\nu(s), \bar{\mu}_1(s), \bar{\mu}_2(s), p(s), q(s)\}$. Then, the framed harmonic curvature of the framed curve (γ, μ) is defined by

$$H(s) = \frac{q(s)}{p(s)}.$$

Proposition 3.1 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\{\nu(s), \bar{\mu}_1(s), \bar{\mu}_2(s)\}$ denote the adapted frame of γ . If the curve γ is a framed slant helix, then the axis of γ is

$$\zeta = \left(\frac{pH(1 + H^2)}{H'} \nu + \bar{\mu}_1 + \frac{p(1 + H^2)}{H'} \bar{\mu}_2 \right) \cos \varphi,$$

where H is the framed harmonic curvature function of the curve γ and $\varphi \neq \frac{\pi}{2}$ is a constant angle.

Proof If the axis of a framed slant helix γ is ζ , then we can write

$$\zeta = \lambda_1(s)\nu(s) + \lambda_2(s)\bar{\mu}_1(s) + \lambda_3(s)\bar{\mu}_2(s),$$

where $\lambda_1(s) = \langle \nu, \zeta \rangle$, $\lambda_2(s) = \langle \bar{\mu}_1, \zeta \rangle$ and $\lambda_3(s) = \langle \bar{\mu}_2, \zeta \rangle$.

Also, we know from Definition 3.1 that

$$\langle \bar{\mu}_1, \zeta \rangle = \cos \varphi. \tag{2}$$

By differentiating equation (2), we obtain

$$\langle \bar{\mu}'_1, \zeta \rangle + \langle \bar{\mu}_1, \zeta' \rangle = 0$$

and using the Frenet formulas for the adapted frame of the curve γ given in equation (1), we have

$$\langle \nu, \zeta \rangle = H \langle \bar{\mu}_2, \zeta \rangle. \tag{3}$$

Again, differentiating equation (3) and using equation (1), we obtain

$$\langle \bar{\mu}_2, \zeta \rangle = \frac{p(1 + H^2)}{H'} \langle \bar{\mu}_1, \zeta \rangle. \tag{4}$$

Then, if we substitute equation (4) into equation (3), we obtain

$$\langle \nu, \zeta \rangle = \frac{pH(1 + H^2)}{H'} \langle \bar{\mu}_1, \zeta \rangle. \tag{5}$$

Consequently, using equations (2), (4) and (5) the axis of the framed slant helix γ is given by

$$\zeta = \left(\frac{pH(1+H^2)}{H'}v + \bar{\mu}_1 + \frac{p(1+H^2)}{H'}\bar{\mu}_2 \right) \cos \varphi,$$

which completes the proof. □

Theorem 3.1 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve with the adapted frame apparatus $\{v, \bar{\mu}_1, \bar{\mu}_2, p, q\}$. Then, γ is a framed slant helix if and only if*

$$\sigma = \frac{H'}{p(1+H^2)^{\frac{3}{2}}}$$

is a constant function, where H is the framed harmonic curvature function of the curve γ .

Proof If the axis of the framed slant helix γ is ζ , we have from Proposition 3.1:

$$\zeta = \left(\frac{pH(1+H^2)}{H'}v + \bar{\mu}_1 + \frac{p(1+H^2)}{H'}\bar{\mu}_2 \right) \cos \varphi.$$

As ζ is a unit vector we can readily see that

$$\sigma = \frac{H'}{p(1+H^2)^{\frac{3}{2}}} = \pm \cot \varphi,$$

where φ is the constant angle between ζ and $\bar{\mu}_1$.

Conversely, if σ is a constant function then the result is obvious. This completes the proof. □

Corollary 3.1 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\{v(s), \bar{\mu}_1(s), \bar{\mu}_2(s)\}$ denote the adapted frame of γ . If the curve γ is a framed slant helix, then the axis of γ is*

$$\zeta = \cos \psi(s) \sin \theta v(s) + \cos \theta \bar{\mu}_1(s) + \sin \psi(s) \sin \theta \bar{\mu}_2(s),$$

where $\psi(s) = \arccos(\frac{H}{1+H^2})$ is the angle between v and ζ .

4 Spherical images of framed slant helices in \mathbb{R}^3

In this section, first we define the spherical indicatrices of a framed curve and we investigate the relations between framed slant helices and their spherical indicatrices.

4.1 v -Indicatrices of framed slant helices

Definition 4.1 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve. Its v -indicatrix is the framed curve $(\beta, \bar{\mu}_{1\beta}, \bar{\mu}_{2\beta}) : I \rightarrow S^2 \times \Delta_2$ defined by

$$\beta(s) = v(s) \quad \text{for all } s \in I.$$

The adapted frame apparatus of β is given by the notation $\{v_\beta, \bar{\mu}_{1\beta}, \bar{\mu}_{2\beta}\}$. Clearly, there exists a smooth mapping $\alpha_\beta : I \rightarrow \mathbb{R}$ such that:

$$\beta'(s) = \alpha_\beta(s)v_\beta(s).$$

Theorem 4.1 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve in \mathbb{R}^3 and $(\beta, \bar{\mu}_{1\beta}, \bar{\mu}_{2\beta})$ be the framed v -indicatrix of γ . Then, γ is a framed slant helix if and only if β is a framed helix.*

Proof We assume that γ is a framed slant helix in \mathbb{R}^3 and β is the framed v -indicatrix of γ . From Definition 4.1, we have

$$\beta(s) = v(s),$$

then, differentiating the last equation according to parameter s and using equation (1), we obtain

$$\beta'(s) = v'(s),$$

that is

$$\alpha_\beta(s)v_\beta(s) = p(s)\bar{\mu}_1(s).$$

From the norm of the last equation, we obtain

$$\alpha_\beta(s) = p(s).$$

Hence, we obtain the following equation:

$$v_\beta(s) = \bar{\mu}_1(s). \tag{6}$$

If we differentiate the last equation and use equation (1), we obtain

$$p_\beta(s)\bar{\mu}_{1\beta}(s) = -p(s)v(s) + q(s)\bar{\mu}_2(s).$$

From the norm of the above equation, we obtain

$$p_\beta(s) = p(s)\sqrt{1 + H^2(s)}. \tag{7}$$

Hence, we obtain the following equation

$$\bar{\mu}_{1\beta}(s) = -\frac{1}{\sqrt{1 + H^2(s)}}v(s) + \frac{H(s)}{\sqrt{1 + H^2(s)}}\bar{\mu}_2(s). \tag{8}$$

Then, using equations (6) and (8), we obtain

$$\bar{\mu}_{2\beta}(s) = v_\beta(s) \times \bar{\mu}_{1\beta}(s) = \frac{H(s)}{\sqrt{1 + H^2(s)}}v(s) + \frac{1}{\sqrt{1 + H^2(s)}}\bar{\mu}_2(s). \tag{9}$$

From the norm of the derivative of the last equation, we obtain the following equation:

$$q_\beta(s) = \frac{H'(s)}{1 + H^2(s)}. \tag{10}$$

Then, we can readily see that $\frac{q_\beta(s)}{p_\beta(s)} = \frac{H'(s)}{p(s)(1+H^2(s))^{\frac{3}{2}}} = \sigma$ is a constant function since γ is a framed slant helix. In other words, using Theorem 2.3 we can readily see that β is a framed helix.

Conversely, if we assume that β is a framed helix then it is clear that γ is a framed slant helix. This completes the proof. □

Corollary 4.1 *The v -indicatrix curve β of a framed curve γ is a regular framed curve.*

Proof It is obvious from the equation $\alpha_\beta(s) = p(s) > 0$ in the proof of Theorem 4.1. □

Corollary 4.2 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve with the adapted frame $\{v(s), \bar{\mu}_1(s), \bar{\mu}_2(s)\}$ in \mathbb{R}^3 and $(\beta, \bar{\mu}_{1\beta}, \bar{\mu}_{2\beta})$ be the framed v -indicatrix of γ with the adapted frame $\{v_\beta(s), \bar{\mu}_{1\beta}(s), \bar{\mu}_{2\beta}(s)\}$. Then, we have the following relations between the adapted frames of γ and β*

$$\begin{aligned} v_\beta(s) &= \bar{\mu}_1(s), \\ \bar{\mu}_{1\beta}(s) &= -\frac{1}{\sqrt{1+H^2(s)}}v(s) + \frac{H(s)}{\sqrt{1+H^2(s)}}\bar{\mu}_2(s), \\ \bar{\mu}_{2\beta}(s) &= \frac{H(s)}{\sqrt{1+H^2(s)}}v(s) + \frac{1}{\sqrt{1+H^2(s)}}\bar{\mu}_2(s), \end{aligned}$$

where H is the framed harmonic curvature function of the curve γ .

Proof It is obvious from equations (6), (8) and (9). □

Corollary 4.3 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve with the framed curvatures $p(s), q(s)$ in \mathbb{R}^3 and $(\beta, \bar{\mu}_{1\beta}, \bar{\mu}_{2\beta})$ be the framed v -indicatrix of γ with the framed curvatures $p_\beta(s), q_\beta(s)$. Then, the relations between these framed curvatures functions are*

$$\begin{aligned} p_\beta(s) &= p(s)\sqrt{1+H^2(s)}, \\ q_\beta(s) &= \frac{H'(s)}{1+H^2(s)}, \end{aligned}$$

where H is the framed harmonic curvature function of the curve γ .

Proof It is obvious from equations (7) and (10). □

4.2 $\bar{\mu}_1$ -Indicatrices of framed slant helices

Definition 4.2 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve. Its $\bar{\mu}_1$ -indicatrix is the framed curve $(\eta, \bar{\mu}_{1\eta}, \bar{\mu}_{2\eta}) : I \rightarrow S^2 \times \Delta_2$ defined by

$$\eta(s) = \bar{\mu}_1(s) \quad \text{for all } s \in I.$$

The adapted frame apparatus of η is given by the notation $\{v_\eta, \bar{\mu}_{1\eta}, \bar{\mu}_{2\eta}\}$. Clearly, there exists a smooth mapping $\alpha_\eta : I \rightarrow \mathbb{R}$ such that:

$$\eta'(s) = \alpha_\eta(s)v_\eta(s).$$

Theorem 4.2 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed slant helix in \mathbb{R}^3 and $(\eta, \bar{\mu}_{1\eta}, \bar{\mu}_{2\eta})$ be the framed $\bar{\mu}_1$ -indicatrix of γ . Then, the curve η is a plane curve on S^2 .*

Proof Let γ be a framed slant helix in \mathbb{R}^3 and η a framed $\bar{\mu}_1$ -indicatrix of γ . From Definition 4.2, we have

$$\eta(s) = \bar{\mu}_1(s). \tag{11}$$

Then, differentiating equation (11) according to the parameter s and using equation (1), we obtain

$$\eta'(s) = \bar{\mu}'_1(s)$$

or

$$\alpha_\eta(s)v_\eta(s) = -p(s)v(s) + q(s)\bar{\mu}_2(s).$$

From the norm of the last equality, we obtain

$$\alpha_\eta(s) = p(s)\sqrt{1 + H^2(s)}.$$

Hence, we obtain the following equation:

$$v_\eta(s) = -\frac{1}{\sqrt{1 + H^2(s)}}v(s) + \frac{H(s)}{\sqrt{1 + H^2(s)}}\bar{\mu}_2(s). \tag{12}$$

If we differentiate the last equation and use equation (1), we obtain

$$p_\eta(s)\bar{\mu}_{1\eta}(s) = \frac{H(s)H'(s)}{(1 + H^2(s))^{\frac{3}{2}}}v(s) - p(s)\sqrt{1 + H^2(s)}\bar{\mu}'_1(s) + \frac{H'(s)}{(1 + H^2(s))^{\frac{3}{2}}}\bar{\mu}_2(s).$$

Since γ is a framed slant helix $\sigma(s)$ is a constant function. Hence, we can obtain

$$p_\eta(s)\bar{\mu}_1(s) = p(s)\sigma H(s)v(s) - p(s)\sqrt{1 + H^2(s)}\bar{\mu}'_1(s) + p(s)\sigma\bar{\mu}_2(s). \tag{13}$$

Then, the norm of the last equation gives us

$$p_\eta(s) = p(s)\sqrt{(1 + H^2(s))(1 + \sigma^2)}$$

and so

$$\bar{\mu}_{1\eta}(s) = \frac{\sigma H(s)}{\sqrt{(1 + H^2(s))(1 + \sigma^2)}}v(s) - \frac{1}{\sqrt{1 + \sigma^2}}\bar{\mu}'_1(s) + \frac{\sigma}{\sqrt{(1 + H^2(s))(1 + \sigma^2)}}\bar{\mu}_2(s). \tag{14}$$

Then, using equations (12) and (14), we obtain

$$\begin{aligned} \bar{\mu}_{2,\eta}(s) &= v_\eta(s) \times \bar{\mu}_{1,\eta}(s) \\ &= \frac{H(s)}{\sqrt{(1+H^2(s))(1+\sigma^2)}} v(s) + \frac{\sigma}{\sqrt{1+\sigma^2}} \bar{\mu}_1(s) \\ &\quad + \frac{1}{\sqrt{(1+H^2(s))(1+\sigma^2)}} \bar{\mu}_2(s). \end{aligned} \tag{15}$$

From the norm of the derivative of $\bar{\mu}_2(s_\eta)$, we obtain

$$q_\eta(s) = 0.$$

Hence, γ is a plane curve. This completes the proof. □

Corollary 4.4 *The $\bar{\mu}_1$ -indicatrix curve η of a framed curve γ is a regular framed curve.*

Proof It is obvious from the equation $\alpha_\eta(s) = p(s)\sqrt{1+H^2} > 0$ in the proof of Theorem 4.2. □

Corollary 4.5 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve with the adapted frame $\{v(s), \bar{\mu}_1(s), \bar{\mu}_2(s)\}$ in \mathbb{R}^3 and $(\eta, \bar{\mu}_{1,\eta}, \bar{\mu}_{2,\eta})$ be the framed $\bar{\mu}_1$ -indicatrix of γ with the adapted frame $\{v_\eta(s), \bar{\mu}_{1,\eta}(s), \bar{\mu}_{2,\eta}(s)\}$. Then, we have the following relations between the adapted frames of γ and η*

$$\begin{aligned} v_\eta(s) &= -\frac{1}{\sqrt{1+H^2(s)}} v(s) + \frac{H(s)}{\sqrt{1+H^2(s)}} \bar{\mu}_2(s), \\ \bar{\mu}_{1,\eta}(s) &= \frac{1}{\sqrt{1+\sigma^2}} \left(\frac{\sigma H(s)}{\sqrt{1+H^2(s)}} v(s) - \bar{\mu}_1(s) + \frac{\sigma}{\sqrt{1+H^2(s)}} \bar{\mu}_2(s) \right), \\ \bar{\mu}_{2,\eta}(s) &= \frac{1}{\sqrt{1+\sigma^2}} \left(\frac{H(s)}{\sqrt{1+H^2(s)}} v(s) + \sigma \bar{\mu}_1(s) + \frac{1}{\sqrt{1+H^2(s)}} \bar{\mu}_2(s) \right), \end{aligned}$$

where H is the framed harmonic curvature function of the curve γ .

Proof It is obvious from equations (12), (14) and (15). □

4.3 $\bar{\mu}_2$ -Indicatrices of framed slant helices

Definition 4.3 Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve. Its $\bar{\mu}_2$ -indicatrix is the framed curve $(\delta, \bar{\mu}_{1,\delta}, \bar{\mu}_{2,\delta}) : I \rightarrow S^2 \times \Delta_2$ defined by

$$\delta(s) = \bar{\mu}_2(s) \quad \text{for all } s \in I.$$

The adapted frame apparatus of δ is given by the notation $\{v_\delta, \bar{\mu}_{1,\delta}, \bar{\mu}_{2,\delta}\}$. Clearly, there exists a smooth mapping $\alpha_\delta : I \rightarrow \mathbb{R}$ such that:

$$\delta'(s) = \alpha_\delta(s)v_\delta(s_\delta).$$

Theorem 4.3 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve in \mathbb{R}^3 and $(\delta, \bar{\mu}_{1,\delta}, \bar{\mu}_{2,\delta})$ be the framed μ_2 -indicatrix of γ . Then, γ is a framed slant helix if and only if δ is a framed helix.*

Proof We assume that γ is a framed slant helix in \mathbb{R}^3 and δ is the framed $\bar{\mu}_2$ -indicatrix of γ . From Definition 4.3, we have

$$\delta(s) = \bar{\mu}_2(s).$$

Then, differentiating the last equation according to the parameter s and using equation (1), we obtain

$$\delta'(s) = \bar{\mu}'_2(s)$$

that is

$$\alpha_\delta(s)v_\delta(s) = -p(s)H(s)\bar{\mu}_1(s),$$

where $H(s)$ is the framed harmonic curvature function of γ . From the norm of the above equation, assuming that $\epsilon = 1$ if $H > 0$ or $\epsilon = -1$ if $H < 0$, we obtain

$$\alpha_\delta(s) = \epsilon p(s)H(s).$$

Hence, we obtain the following equation:

$$v_\delta(s) = -\epsilon \bar{\mu}_1(s). \tag{16}$$

If we differentiate the last equation and use equation (1), we obtain

$$p_\delta(s)\bar{\mu}_{1_\delta}(s) = -\epsilon(-p(s)v(s) + q(s)\bar{\mu}_2(s)).$$

From the norm of the last equation, we obtain

$$p_\delta(s) = p(s)\sqrt{1 + H^2(s)}. \tag{17}$$

Hence, we have

$$\bar{\mu}_{1_\delta}(s) = \frac{\epsilon}{\sqrt{1 + H^2(s)}}v(s) - \frac{\epsilon H(s)}{\sqrt{1 + H^2(s)}}\bar{\mu}_2(s). \tag{18}$$

Then, using equations (16) and (18), we have

$$\bar{\mu}_{2_\delta}(s) = v_\delta(s) \times \bar{\mu}_{1_\delta}(s) = \frac{H(s)}{\sqrt{1 + H^2(s)}}v(s) + \frac{1}{\sqrt{1 + H^2(s)}}\bar{\mu}_2(s). \tag{19}$$

From the norm of the derivative of the last equation, we obtain the following equation:

$$q_\delta(s) = \frac{H'(s)}{1 + H^2(s)}. \tag{20}$$

Then, we can readily see that $\frac{q_\delta(s)}{p_\delta(s)} = \frac{H'(s)}{p(s)(1+H^2(s))^{\frac{3}{2}}} = \sigma$ is a constant function since γ is a framed slant helix. In other words, using Theorem 2.3 we can see readily that δ is a framed helix.

Conversely, we assume that δ is a framed helix then it is clear that γ is a framed slant helix. This completes the proof. □

Corollary 4.6 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve with the adapted frame $\{v(s), \bar{\mu}_1(s), \bar{\mu}_2(s)\}$ in \mathbb{R}^3 and $(\delta, \bar{\mu}_{1\delta}, \bar{\mu}_{2\delta})$ be the framed $\bar{\mu}_2$ -indicatrix of γ with the adapted frame $\{v_\delta(s), \bar{\mu}_{1\delta}(s), \bar{\mu}_{2\delta}(s)\}$. Then, we have the following relations between the adapted frames of γ and δ*

$$\begin{aligned} v_\delta(s) &= -\epsilon \bar{\mu}_1(s), \\ \bar{\mu}_{1\delta}(s) &= \frac{\epsilon}{\sqrt{1+H^2(s)}} v(s) - \frac{\epsilon H(s)}{\sqrt{1+H^2(s)}} \bar{\mu}_2(s), \\ \bar{\mu}_{2\delta}(s) &= \frac{H(s)}{\sqrt{1+H^2(s)}} v(s) + \frac{1}{\sqrt{1+H^2(s)}} \bar{\mu}_2(s), \end{aligned}$$

where $\epsilon = 1$ if $H > 0$ or $\epsilon = -1$ if $H < 0$, H is the framed harmonic curvature function of the curve γ .

Proof It is obvious from equations (16), (18) and (19). □

Corollary 4.7 *Let $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ be a framed curve with the framed curvatures $p(s), q(s)$ in \mathbb{R}^3 and $(\delta, \bar{\mu}_{1\delta}, \bar{\mu}_{2\delta})$ be the framed $\bar{\mu}_2$ -indicatrix of γ with the framed curvatures $p_\delta(s), q_\delta(s)$. Then, the relations between these framed curvatures functions are*

$$\begin{aligned} p_\delta(s) &= p(s)\sqrt{1+H^2(s)}, \\ q_\delta(s) &= \frac{H'(s)}{1+H^2(s)}, \end{aligned}$$

where H is the framed harmonic curvature function of the curve γ .

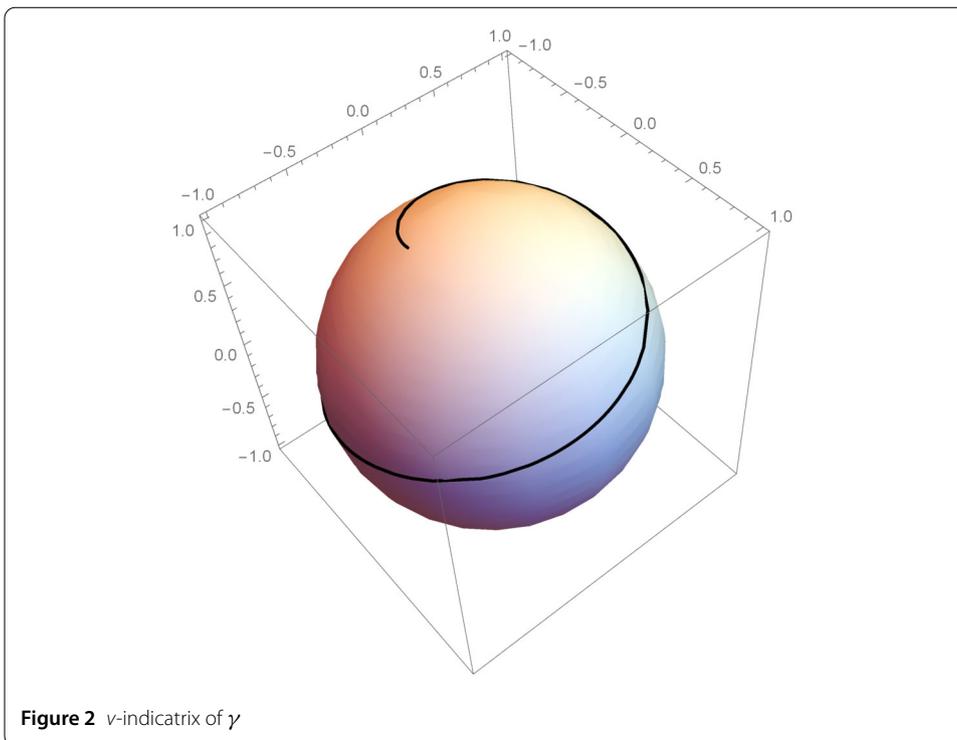
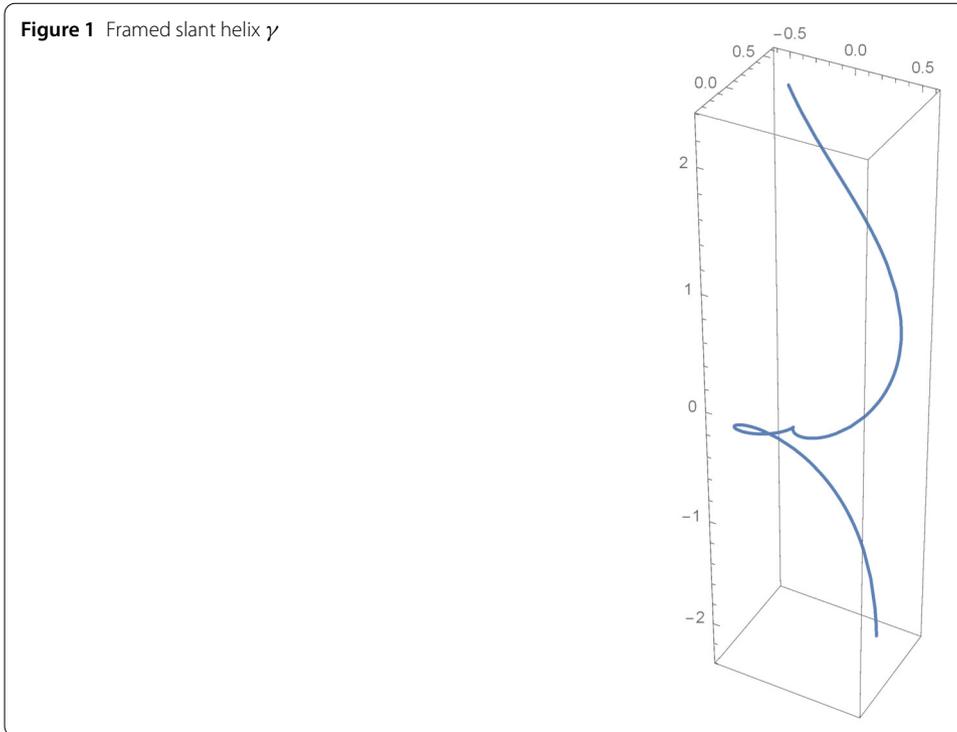
Proof It is obvious from equations (17) and (20). □

Example 4.1 Let $\gamma : (-2\pi, 2\pi) \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve defined by

$$\begin{aligned} \gamma(t) &= \frac{\sqrt{6}}{5} \left(\sin\left(\frac{3t}{5}\right) - \frac{2}{7} \sin\left(\frac{7t}{5}\right) - \frac{\sin(t)}{5}, -\cos\left(\frac{3t}{5}\right) + \frac{2}{7} \cos\left(\frac{7t}{5}\right) + \frac{\cos(t)}{5}, \right. \\ &\quad \left. \frac{2\sqrt{6}t}{5} - \sqrt{6} \sin\left(\frac{2t}{5}\right) \right). \end{aligned}$$

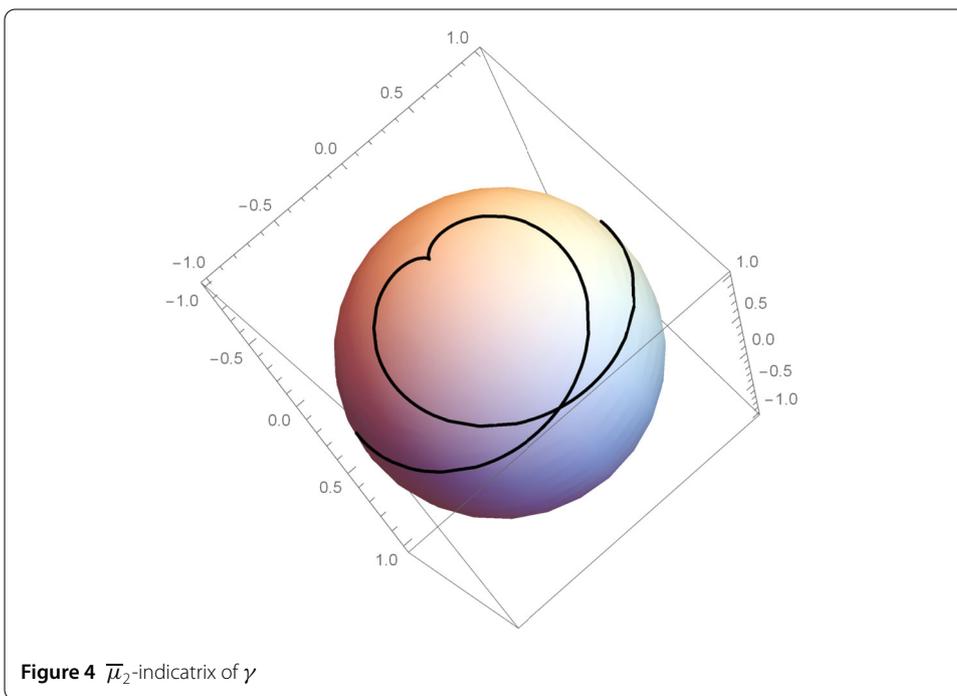
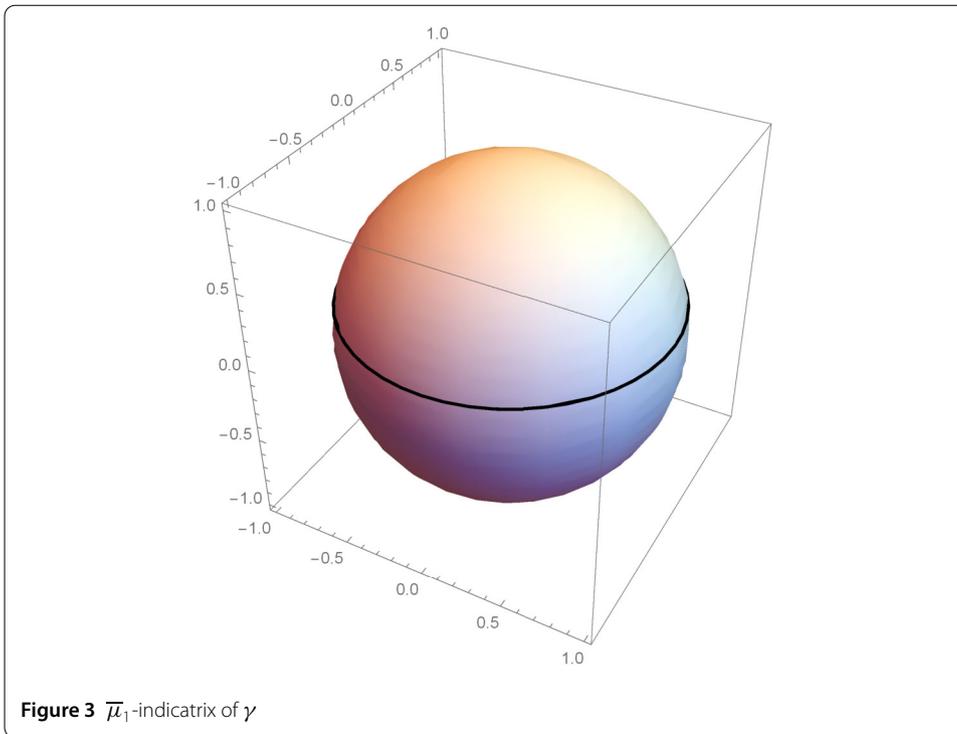
The curve γ has a singular point at $t = 0$, so that it is not a Frenet curve. On the other hand, the curve γ is a framed curve with the mapping $(\gamma, \bar{\mu}_1, \bar{\mu}_2) : (-2\pi, 2\pi) \subset \mathbb{R} \rightarrow \mathbb{R}^3 \times \Delta_2$. The adapted frame vectors of the framed curve $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ are given by

$$\begin{aligned} v(t) &= \frac{1}{5} \left(3 \sin\left(\frac{4t}{5}\right) + 2 \sin\left(\frac{6t}{5}\right), -3 \cos\left(\frac{4t}{5}\right) - 2 \cos\left(\frac{6t}{5}\right), \frac{2\sqrt{6}}{5} \sin\left(\frac{t}{5}\right) \right), \\ \bar{\mu}_1 &= \left(\frac{2}{5} \sqrt{6} \cos(t), \frac{2}{5} \sqrt{6} \sin(t), \frac{1}{5} \right), \\ \bar{\mu}_2 &= \frac{1}{5} \left(2 \cos\left(\frac{6t}{5}\right) - 3 \cos\left(\frac{4t}{5}\right), 2 \sin\left(\frac{6t}{5}\right) - 3 \sin\left(\frac{4t}{5}\right), \frac{2\sqrt{6}}{5} \cos\left(\frac{t}{5}\right) \right). \end{aligned}$$



Also, the framed curvatures of the framed curve $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ are as follows:

$$p(t) = \frac{2}{5}\sqrt{6}\cos\left(\frac{t}{5}\right) \quad \text{and} \quad q(t) = \frac{2}{5}\sqrt{6}\sin\left(\frac{t}{5}\right).$$



Moreover, we can readily see that the $\sigma = \frac{1}{2\sqrt{6}}$ for the framed curve $(\gamma, \bar{\mu}_1, \bar{\mu}_2)$ with the help of Theorem 3.1, so it is a framed slant helix. Finally, we show Figs. 1–4, which are the framed slant helix, the ν -indicatrix of γ , the $\bar{\mu}_1$ -indicatrix of γ , and the $\bar{\mu}_2$ -indicatrix of γ , respectively.

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Authors' contributions

All the authors contributed equally to this work. All the authors read and approved the manuscript.

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