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Existence of positive solutions for a system of nonlinear Caputo type fractional differential equations with two parameters

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Abstract

The main purpose of this paper is to prove the existence of positive solutions for a system of nonlinear Caputo-type fractional differential equations with two parameters. By using the Guo–Krasnosel'skii fixed point theorem, some existence theorems of positive solutions are obtained in terms of different values of parameters. Two examples are given to illustrate the main results.

MSC: 34A08; 34B27; 47H11

Keywords: Fixed point theorem; Fractional differential equation; Positive solution

1 Introduction

Fractional-order calculus, which is an important branch of mathematics, was introduced in 1695. Since fractional-order calculus can characterize many non-classical phenomena in natural sciences and engineering, it has been applied to various fields in recent years. At the same time, boundary value problems of fractional differential equations have appeared with applications of fractional-order calculus; so far, there have been many literature works about boundary value problems of fractional differential equations.

For some recent studies on fractional differential equations, we can refer to [1–27]. For example, in [10], the authors used the Guo–Krasnosel'skii fixed point theorem and the Leggett–Williams fixed point theorem to obtain the existence of positive solutions to the nonlinear Caputo fractional q-difference equation with integral boundary conditions. In [12], the authors investigated the following boundary value problem of Caputo-type fractional differential equation subject to Riemann–Stieltjes integral boundary conditions:

$$\begin{cases} {}^cD^\theta p(t) + \mu f(t,p(t)) = 0, \quad t \in [0,1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(t)\,dA(t), \end{cases}$$

where ${}^cD^\theta$ is the Caputo fractional derivative, $\theta \in (2,3)$, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, and $\mu > 0$ is a parameter. By using the Guo–Krasnosel'skii fixed point theorem, the authors obtained some new results about the existence and non-existence of positive solutions for the above equation.

In [18], the authors focused on the following boundary value problem:

$$\begin{cases} {}^{C}D^{q}u(t)+f(t,u(t))=0, \quad t\in[0,1], \\ u''(0)=0, \\ \alpha u(0)-\beta u'(0)=\int_0^1 h_1(s)u(s)ds, \\ \gamma u(1)+\delta({}^{C}D^{\sigma}_{0+}u)(1)=\int_0^1 h_2(s)u(s)\,ds, \end{cases}$$

where $2<q\leq 3$, $0<\sigma\leq 1$, $\alpha,\gamma,\delta\geq 0$ and $\beta>0$ satisfying

$$0<(\alpha+\beta)\gamma+\frac{\alpha\delta}{\Gamma(2-\sigma)}<\beta\left[\gamma+\frac{\delta\Gamma(q)}{\Gamma(q-\sigma)}\right].$$

The method they used is the Guo-Krasnoselskii fixed point theorem, and the existence theorems of positive solutions for the above equation were obtained.

In [23], the authors investigated a coupled system of Caputo fractional differential equations with coupled non-conjugate Riemann–Stieltjes type integro-multipoint boundary conditions. They obtained some new theorems by using the Leray–Schauder nonlinear alternative, the Krasnosel'skii fixed point theorem, and Banach's contraction mapping principle.

In [24], the authors studied the following nonlinear Caputo-type fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^{c}D^{\alpha}u(t)=f(t,u(t),v(t)), \quad t\in(0,1), \\ {}^{c}D^{\beta}v(t)=g(t,u(t),v(t)), \quad t\in(0,1), \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=u^{(n)}(0)=0, \qquad u(1)=\lambda\int_0^1 u(s)\,ds, \\ v(0)=v'(0)=\cdots=v^{(n-2)}(0)=v^{(n)}(0)=0, \qquad v(1)=\lambda\int_0^1 v(s)\,ds, \end{cases}$$

where $n<\alpha$, $\beta<n+1$, $n\geq 2$, $n\in N$, $0<\lambda<n$; $f,g\in C([0,1]\times R\times R,R)$. In this paper, by using Schauder's fixed point theorem and Banach's fixed point theorem, sufficient conditions were obtained for the existence and uniqueness of positive solutions of the above coupled system.

In [25], the authors considered the following fractional differential equations:

$$\begin{cases} -{}^{c}D^{\theta_1}x(t)=f_1(t,x(t),y(t)), \quad t\in[0,1], \\ -{}^{c}D^{\theta_2}y(t)=f_2(t,x(t),y(t)), \quad t\in[0,1], \\ x(0)=x''(0)=0, \qquad x(1)=\int_0^1 x(t)\,dA_1(t), \\ y(0)=y''(0)=0, \qquad y(1)=\int_0^1 y(t)\,dA_2(t), \end{cases}$$

where $f_i:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous; $\theta_i\in(2,3)$; A_i is a bounded variation function with positive measure $B_i=\int_0^1 t\,dA_i(t)<1$, $i=1,2$. By means of the fixed point index theory, the authors proved that the above system has at least two positive solutions.

In [26, 27], the authors used the Guo–Krasnosel'skii fixed point theorem to investigate the existence of positive solutions for systems of fractional differential equations nonlocal boundary value problems with two parameters, and the existence of positive solutions were obtained. In [26], the fractional derivative is the standard Riemann–Liouville derivative, and in [27], the fractional derivative is a conformable fractional derivative.

Inspired by [2–27], in this paper, we study the existence of positive solutions for the following system of fractional differential equations:

$$\begin{cases} -{}^cD^{\theta_1}u(t) = \lambda f_1(t, u(t), v(t)), & t \in [0,1], \\ -{}^cD^{\theta_2}v(t) = \mu f_2(t, u(t), v(t)), & t \in [0,1], \\ u(0) = u''(0) = 0, & u(1) = \int_0^1 u(t)\,dA_1(t), \\ v(0) = v''(0) = 0, & v(1) = \int_0^1 v(t)\,dA_2(t), \end{cases} \tag{1.1}$$

where ${}^cD^{\theta_i}$ is the Caputo fractional derivative; $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous; $\theta_i \in (2,3)$; A_i is a bounded variation function with positive measure $B_i = \int_0^1 t\,dA_i(t) < 1$, $i = 1,2$; λ and μ are positive parameters. By studying system (1.1), we improve and generalize paper [12]. Compared with literatures [26, 27], the definition of fractional derivative is different from those of [26, 27]. The main purpose of this paper is to demonstrate the existence of positive solutions about system (1.1). By the Guo–Krasnosel'skii fixed point theorem, we obtain some existence theorems of positive solutions under the conditions of various values of parameters. To illustrate the theoretical results, two examples are given in the last section of the paper.

2 Preliminaries

In the following, some concepts and lemmas of Caputo differential equations are presented, as well as some auxiliary results for proving the main theorems.

Definition 2.1 (see [1]) For a function $x \in C^n[0,+\infty)$, the Caputo fractional derivative of order $\theta > 0$ is defined as

$${}^cD^{\theta}x(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} x^{(n)}(s)\,ds, \quad n-1 < \theta < n.$$

Lemma 2.1 (see [1]) *Let $\theta > 0$. If we assume $x \in C(0,1) \bigcap L(0,1)$, then the fractional differential equation*

$${}^cD^{\theta}x(t) = 0$$

has the general solution $x(t) = C_0 + C_1 t + \cdots + C_{n-1}t^{n-1}$, $C_i \in R$, $i = 0,1,\ldots,n-1$.

Lemma 2.2 (see [1]) *Suppose that $x \in C(0,1) \bigcap L(0,1)$ with a fractional derivative of order θ that belongs to $C(0,1) \bigcap L(0,1)$. Then $I^{\theta\,c}D^{\theta}x(t) = x(t) + C_0 + C_1 t + \cdots + C_{n-1}t^{n-1}$, for $C_i \in R$, $i = 0,1,\ldots,n-1$.*

Lemma 2.3 (see [12]) *Let $x \in C[0,1]$ and $\theta_1, \theta_2 \in (2,3)$. Then p is a solution of the linear Caputo fractional differential equation*

$$\begin{cases} {}^cD^{\theta_i}p(t) + x(t) = 0, \quad t \in [0,1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(t)\,dA_i(t), \end{cases}$$

if and only if p is the solution of the integral equation

$$p(t) = \int_0^1 G_i(t,s)x(s)\,ds,$$

where

$$G_i(t,s) = \frac{1}{\Gamma(\theta_i)} \begin{cases} \frac{t}{1-B_i}[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1}\,dA_i(t)] - (t-s)^{\theta_i-1}, \\ \quad 0 \le s \le t \le 1, \\ \frac{t}{1-B_i}[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1}\,dA_i(t)], \\ \quad 0 \le t \le s \le 1, \end{cases} \tag{2.1}$$

and $B_i = \int_0^1 t\,dA_i(t) < 1$, $i = 1,2$.

Lemma 2.4 (see [12]) *Green's function $G_i(t,s)$ $(i = 1,2)$ defined by (2.1) has the following properties*:

(i) $\Gamma(\theta_i)G_i(t,s) \le \frac{1}{1-B_i}(1-s)^{\theta_i-1}$ *for* $t, s \in [0,1]$;

(ii) $\Gamma(\theta_i)G_i(t,s) \ge N_i(1-s)^{\theta_i-1}$ *for* $t \in [\frac{1}{4}, \frac{3}{4}]$, $s \in [0,1]$,

where

$$N_i = \min\left\{ \frac{1 - \int_0^1 t^{\theta_i-1}\,dA_i(t)}{4(1-B_i)}, \min_{t\in[\frac{1}{4},\frac{3}{4}]} t\bigl(1 - t^{\theta_i-2}\bigr) \right\}, \quad i = 1,2. \tag{2.2}$$

Lemma 2.5 (see [28]) *Let P be a cone of the Banach space X and Ω_1 and Ω_2 be two bounded open sets in X with $\theta \subset \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator. If one of the following two conditions holds*:

(1) $\|Ap\| \le \|p\|$ *for all* $p \in P \cap \partial\Omega_1$, $\|Ap\| \ge \|p\|$ *for all* $p \in P \cap \partial\Omega_2$;

(2) $\|Ap\| \ge \|p\|$ *for all* $p \in P \cap \partial\Omega_1$, $\|Ap\| \le \|p\|$ *for all* $p \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

Let $X = C[0,1] \times C[0,1]$. Define the norm $\|(x,y)\|_X = \|x\| + \|y\|$ on X,where $\|x\| = \max_{0\le t\le 1} |x(t)|$, then X is a Banach space.

We define the cone

$$P = \Bigl\{(u,v) \in X : u \ge 0, v \ge 0, \min_{\frac{1}{4}\le t\le\frac{3}{4}} \bigl(u(t) + v(t)\bigr) \ge K\bigl\|(u,v)\bigr\|_X\Bigr\},$$

where

$$K = \min\bigl\{N_1(1-B_1), N_2(1-B_2)\bigr\} < 1, \tag{3.1}$$

N_1, N_2 are defined by (2.2).

We define the operators L_1, L_2, and L as follows:

$$
\begin{aligned}
&L_1(u,v)(t)=\lambda\int_0^1 G_1(t,s)f_1\bigl(s,u(s),v(s)\bigr)\,ds,\quad t\in[0,1],\\
&L_2(u,v)(t)=\mu\int_0^1 G_2(t,s)f_2\bigl(s,u(s),v(s)\bigr)\,ds,\quad t\in[0,1],\\
&L(u,v)=\bigl(L_1(u,v),L_2(u,v)\bigr),\quad \forall(u,v)\in X,
\end{aligned}
$$

where $G_i(t,s)(i=1,2)$ is defined by (2.1).

Obviously, fixed points of the operator L in P are positive solutions of system (1.1).

Lemma 3.1 *$L:P\to P$ is completely continuous.*

Proof We easily know that $L_1(u,v)(t)\geq 0$, $L_2(u,v)(t)\geq 0$ for $(u,v)\in P$, $t\in[0,1]$.

Obviously, by Lemma 2.4, for $(u,v)\in P$, when $t\in[\frac{1}{4},\frac{3}{4}]$, we have

$$
\begin{aligned}
L_1(u,v)(t)&=\lambda\int_0^1 G_1(t,s)f_1\bigl(s,u(s),v(s)\bigr)\,ds\\
&\geq\frac{\lambda N_1}{\Gamma(\theta_1)}\int_0^1(1-s)^{\theta_1-1}f_1\bigl(s,u(s),v(s)\bigr)\,ds\\
&=\frac{\lambda N_1(1-B_1)}{\Gamma(\theta_1)}\int_0^1\frac{(1-s)^{\theta_1-1}}{(1-B_1)}f_1\bigl(s,u(s),v(s)\bigr)\,ds\\
&\geq\lambda N_1(1-B_1)\max_{t\in[0,1]}\int_0^1 G_1(t,s)f_1\bigl(s,u(s),v(s)\bigr)\,ds\\
&=N_1(1-B_1)\bigl\|L_1(u,v)\bigr\|.
\end{aligned}\tag{3.2}
$$

Similarly, we get

$$
L_2(u,v)(t)\geq N_2(1-B_2)\bigl\|L_2(u,v)\bigr\|,\quad (u,v)\in P,t\in\left[\frac{1}{4},\frac{3}{4}\right].\tag{3.3}
$$

From (3.2) and (3.3), we have

$$
\begin{aligned}
&\min_{t\in[\frac{1}{4},\frac{3}{4}]}\bigl(L_1(u,v)(t)+L_2(u,v)(t)\bigr)\\
&\quad\geq N_1(1-B_1)\bigl\|L_1(u,v)\bigr\|+N_2(1-B_2)\bigl\|L_2(u,v)\bigr\|\\
&\quad\geq K\bigl\|L(u,v)\bigr\|_X.
\end{aligned}\tag{3.4}
$$

By (3.4), we get $L(P)\subset P$. From the paper [12], we know that L_1, L_2 are completely continuous. So L is completely continuous. The proof is completed. □

For convenience, we first list the following denotations:

$$
z_0=\lim_{(u,v)\to(0^+,0^+)}\sup_{t\in[0,1]}\frac{f_1(t,u,v)}{u+v},\qquad z_0^*=\lim_{(u,v)\to(0^+,0^+)}\sup_{t\in[0,1]}\frac{f_2(t,u,v)}{u+v},\tag{3.5}
$$

$$z_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_1(t,u,v)}{u+v}, \qquad z_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_2(t,u,v)}{u+v}, \tag{3.6}$$

$$P_1 = \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1}\,ds, \qquad P_2 = \frac{1}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1}\,ds, \tag{3.7}$$

$$P_3 = \frac{N_1K}{\Gamma(\theta_1)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_1-1}\,ds, \qquad P_4 = \frac{N_2K}{\Gamma(\theta_2)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_2-1}\,ds. \tag{3.8}$$

Theorem 3.1 *Let $z_0, z_0^*, z_\infty, z_\infty^* \in (0,+\infty)$, $Q_1 < Q_2$, $Q_3 < Q_4$. Then when $\lambda \in (Q_1,Q_2)$ and $\mu \in (Q_3,Q_4)$ hold, we get that system (1.1) has at least one positive solution, where*

$$Q_1 = \frac{1}{2z_\infty P_3}, \qquad Q_2 = \frac{1}{2z_0 P_1}, \qquad Q_3 = \frac{1}{2z_\infty^* P_4}, \qquad Q_4 = \frac{1}{2z_0^* P_2}.$$

Proof It is easy to see that there exists $\varepsilon > 0$ such that, for $\lambda \in (Q_1,Q_2)$ and $\mu \in (Q_3,Q_4)$, we have

$$\frac{1}{2(z_\infty - \varepsilon)P_3} \le \lambda \le \frac{1}{2(z_0+\varepsilon)P_1}, \qquad \frac{1}{2(z_\infty^* - \varepsilon)P_4} \le \mu \le \frac{1}{2(z_0^*+\varepsilon)P_2}.$$

By (3.5), for the above $\varepsilon > 0$, there exists a constant $R_1 > 0$ such that

$$\begin{aligned} &f_1(t,u,v) \le (z_0+\varepsilon)(u+v), \quad 0 \le u+v \le R_1, t\in[0,1], \\ &f_2(t,u,v) \le \bigl(z_0^*+\varepsilon\bigr)(u+v), \quad 0 \le u+v \le R_1, t\in[0,1]. \end{aligned}$$

Let $\Omega_1 = \{(u,v)\in X \mid \|(u,v)\|_X < R_1\}$. For any $(u,v) \in P \bigcap \partial\Omega_1$, by Lemma 2.4 and (3.1), we have

$$\begin{aligned} L_1(u,v)(t) &= \lambda \int_0^1 G_1(t,s) f_1\bigl(s,u(s),v(s)\bigr)\,ds \\ &\le \frac{\lambda}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1}(z_0+\varepsilon)\bigl(u(s)+v(s)\bigr)\,ds \\ &\le \frac{\lambda(z_0+\varepsilon)}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1}\,ds\bigl(\|u\|+\|v\|\bigr) \\ &= \lambda(z_0+\varepsilon)P_1\bigl(\|u\|+\|v\|\bigr) \\ &\le \frac{1}{2}\bigl\|(u,v)\bigr\|_X. \end{aligned}$$

Also, we get

$$\begin{aligned} L_2(u,v)(t) &= \mu \int_0^1 G_2(t,s) f_2\bigl(s,u(s),v(s)\bigr)\,ds \\ &\le \frac{\mu}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1}\bigl(z_0^*+\varepsilon\bigr)\bigl(u(s)+v(s)\bigr)\,ds \\ &\le \frac{\mu(z_0^*+\varepsilon)}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1}\,ds\bigl(\|u\|+\|v\|\bigr) \\ &= \mu\bigl(z_0^*+\varepsilon\bigr)P_2\bigl(\|u\|+\|v\|\bigr) \\ &\le \frac{1}{2}\bigl\|(u,v))\bigr\|_X. \end{aligned}$$

So

$$\left\|L(u,v)\right\| = \left\|L_1(u,v)\right\| + \left\|L_2(u,v)\right\| \le \left\|(u,v)\right\|_X, \quad \forall (u,v) \in P \cap \partial\Omega_1. \tag{3.9}$$

From (3.6), we know that there exist $\varepsilon > 0$ and $\overline{R}_2 > 0$ such that

$$\begin{aligned}
&f_1(t,u,v) \ge (z_\infty - \varepsilon)(u+v), \quad u+v \ge \overline{R}_2, t \in \left[\frac{1}{4}, \frac{3}{4}\right],\\
&f_2(t,u,v) \ge \left(z^*_\infty - \varepsilon\right)(u+v), \quad u+v \ge \overline{R}_2, t \in \left[\frac{1}{4}, \frac{3}{4}\right].
\end{aligned}$$

Let $\Omega_2 = \{(u,v) \in X | \|(u,v)\|_X < R_2\}$, where $R_2 = \max\{2R_1, \frac{\overline{R}_2}{K}\}$. From (3.1) and Lemma 2.4, for any $(u,v) \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
L_1(u,v)\left(\frac{3}{4}\right) &= \lambda \int_0^1 G_1\left(\frac{3}{4}, s\right) f_1\bigl(s, u(s), v(s)\bigr)\,ds\\
&\ge \frac{\lambda N_1}{\Gamma(\theta_1)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_1 - 1}(z_\infty - \varepsilon)\bigl(u(s) + v(s)\bigr)\,ds\\
&\ge \frac{\lambda N_1 K(z_\infty - \varepsilon)}{\Gamma(\theta_1)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_1 - 1}\,ds\bigl(\|u\| + \|v\|\bigr)\\
&= \lambda(z_\infty - \varepsilon)P_3\bigl(\|u\| + \|v\|\bigr)\\
&\ge \frac{1}{2}\left\|(u,v)\right\|_X,
\end{aligned}$$

and

$$\begin{aligned}
L_2(u,v)\left(\frac{3}{4}\right) &= \mu \int_0^1 G_2\left(\frac{3}{4}, s\right) f_2\bigl(s, u(s), v(s)\bigr)\,ds\\
&\ge \frac{\mu N_2}{\Gamma(\theta_2)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_2 - 1}\left(z^*_\infty - \varepsilon\right)\bigl(u(s) + v(s)\bigr)\,ds\\
&\ge \frac{\mu N_2 K(z^*_\infty - \varepsilon)}{\Gamma(\theta_2)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_2 - 1}\,ds\bigl(\|u\| + \|v\|\bigr)\\
&= \mu\left(z^*_\infty - \varepsilon\right)P_4\bigl(\|u\| + \|v\|\bigr)\\
&\ge \frac{1}{2}\left\|(u,v)\right\|_X.
\end{aligned}$$

So

$$\left\|L(u,v)\right\| = \left\|L_1(u,v)\right\| + \left\|L_2(u,v)\right\| \ge \left\|(u,v)\right\|_X, \quad \forall (u,v) \in P \cap \partial\Omega_2. \tag{3.10}$$

By virtue of (3.9), (3.10), and Lemma 2.5, we know that L has at least a fixed point $(u,v) \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Therefore, (u,v) is one positive solution of system (1.1). □

Since the proofs of the following theorems are similar to Theorem 3.1, we only give the results as follows.

Theorem 3.2 *Let* $z_0 = 0$, $z_0^*, z_\infty, z_\infty^* \in (0,+\infty)$, $Q_3 < Q_4$. *Then when* $\lambda \in (Q_1,+\infty)$ *and* $\mu \in (Q_3, Q_4)$ *hold, we get that system* (1.1) *has at least one positive solution.*

Theorem 3.3 *Let* $z_0^* = 0$, $z_0, z_\infty, z_\infty^* \in (0,+\infty)$, $Q_1 < Q_2$. *Then when* $\lambda \in (Q_1, Q_2)$ *and* $\mu \in (Q_3,+\infty)$ *hold, we get that system* (1.1) *has at least one positive solution.*

Theorem 3.4 *Let* $z_0 = 0$, $z_0^* = 0$, $z_\infty, z_\infty^* \in (0,+\infty)$. *Then when* $\lambda \in (Q_1,+\infty)$ *and* $\mu \in (Q_3,+\infty)$ *hold, we get that system* (1.1) *has at least one positive solution.*

Theorem 3.5 *Let* $z_0, z_0^* \in (0,+\infty)$, $z_\infty = +\infty$, *and* $z_\infty^* = +\infty$. *Then when* $\lambda \in (0, Q_2)$, $\mu \in (0, Q_4)$ *hold, we get that system* (1.1) *has at least one positive solution.*

Theorem 3.6 *Let* $z_0 \in (0,+\infty)$, $z_0^* = 0$, $z_\infty^* = +\infty$, *and* $z_\infty = +\infty$. *Then when* $\lambda \in (0, Q_2)$, $\mu \in (0,+\infty)$ *hold, we get that system* (1.1) *has at least one positive solution.*

Theorem 3.7 *Let* $z_0 = 0$, $z_0^* \in (0,+\infty)$, $z_\infty^* = +\infty$, *and* $z_\infty = +\infty$. *Then when* $\lambda \in (0,+\infty)$, $\mu \in (0, Q_4)$ *hold, we get that system* (1.1) *has at least one positive solution.*

Theorem 3.8 *Let* $z_0 = z_0^* = 0$, $z_\infty = +\infty$, *and* $z_\infty^* = +\infty$. *Then when* $\lambda \in (0,+\infty)$ *and* $\mu \in (0,+\infty)$ *hold, we get that system* (1.1) *has at least one positive solution.*

For convenience, we give the other denotations as follows:

$$\overline{z}_0 = \lim_{(u,v)\to(0^+,0^+)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_1(t,u,v)}{u+v}, \qquad \overline{z}_0^* = \lim_{(u,v)\to(0^+,0^+)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_2(t,u,v)}{u+v}, \tag{3.11}$$

$$\overline{z}_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \sup_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v}, \qquad \overline{z}_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \sup_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v}. \tag{3.12}$$

Theorem 3.9 *Let* $\overline{z}_0, \overline{z}_0^*, \overline{z}_\infty, \overline{z}_\infty^* \in (0,+\infty)$, $\overline{Q}_1 < \overline{Q}_2$, $\overline{Q}_3 < \overline{Q}_4$. *Then when* $\lambda \in (\overline{Q}_1, \overline{Q}_2)$, $\mu \in (\overline{Q}_3, \overline{Q}_4)$ *hold, we have that system* (1.1) *has at least one positive solution, where*

$$\overline{Q}_1 = \frac{1}{2\overline{z}_0 P_3}, \qquad \overline{Q}_2 = \frac{1}{2\overline{z}_\infty P_1}, \qquad \overline{Q}_3 = \frac{1}{2\overline{z}_0^* P_4}, \qquad \overline{Q}_4 = \frac{1}{2\overline{z}_\infty^* P_2}.$$

Proof Since $\lambda \in (\overline{Q}_1, \overline{Q}_2)$, $\mu \in (\overline{Q}_3, \overline{Q}_4)$, so we can choose $\varepsilon > 0$ such that

$$\frac{1}{2(\overline{z}_0 - \varepsilon)P_3} \le \lambda \le \frac{1}{2(\overline{z}_\infty + \varepsilon)P_1}, \qquad \frac{1}{2(\overline{z}_0^* - \varepsilon)P_4} \le \mu \le \frac{1}{2(\overline{z}_\infty^* + \varepsilon)P_2}. \tag{3.13}$$

By (3.11)–(3.13), for the above $\varepsilon > 0$, there exists a constant $R_3 > 0$ such that

$$f_1(t,u,v) \ge (\overline{z}_0 - \varepsilon)(u+v), \quad 0 < u+v \le R_3, t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

$$f_2(t,u,v) \ge \left(\overline{z}_0^* - \varepsilon\right)(u+v), \quad 0 < u+v \le R_3, t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Set $\Omega_3 = \{(u,v) \in X | \ \|(u,v)\|_X < R_3\}$. From Lemma 2.4 and (3.1), for any $(u,v) \in P \bigcap \partial\Omega_3$, we have

$$\begin{aligned} L_1(u,v)\left(\frac{1}{4}\right) &= \lambda \int_0^1 G_1\left(\frac{1}{4},s\right) f_1\bigl(s,u(s),v(s)\bigr)\,ds \\ &\geq \frac{\lambda N_1}{\Gamma(\theta_1)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_1-1}(\overline{z}_0-\epsilon)\bigl(u(s)+v(s)\bigr)\,ds \\ &\geq \frac{\lambda N_1 K(\overline{z}_0-\epsilon)}{\Gamma(\theta_1)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_1-1}\,ds\bigl(\|u\|+\|v\|\bigr) \\ &= \lambda(\overline{z}_0-\epsilon)P_3\bigl(\|u\|+\|v\|\bigr) \\ &\geq \frac{1}{2}\bigl\|(u,v)\bigr\|_X, \end{aligned}$$

and

$$\begin{aligned} L_2(u,v)\left(\frac{1}{4}\right) &= \mu \int_0^1 G_2\left(\frac{1}{4},s\right) f_2\bigl(s,u(s),v(s)\bigr)\,ds \\ &\geq \frac{\mu N_2}{\Gamma(\theta_2)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_2-1}\bigl(\overline{z}_0^*-\epsilon\bigr)\bigl(u(s)+v(s)\bigr)\,ds \\ &\geq \frac{\mu N_2 K(\overline{z}_0^*-\epsilon)}{\Gamma(\theta_2)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\theta_2-1}\,ds\bigl(\|u\|+\|v\|\bigr) \\ &= \mu\bigl(\overline{z}_0^*-\epsilon\bigr)P_4\bigl(\|u\|+\|v\|\bigr) \\ &\geq \frac{1}{2}\bigl\|(u,v)\bigr\|_X. \end{aligned}$$

Then we have

$$\bigl\|L(u,v)\bigr\| = \bigl\|L_1(u,v)\bigr\| + \bigl\|L_2(u,v)\bigr\| \geq \bigl\|(u,v)\bigr\|_X, \quad \forall (u,v) \in P \cap \partial\Omega_3. \tag{3.14}$$

Let $\widetilde{f}_1(t,w) = \max_{0\leq u+v\leq w} f_1(t,u,v)$, $\widetilde{f}_2(t,w) = \max_{0\leq u+v\leq w} f_2(t,u,v)$. Obviously, $\widetilde{f}_1,\widetilde{f}_2 : [0,1] \times [0,+\infty) \to [0,+\infty)$, $f_1(t,u,v) \leq \widetilde{f}_1(t,w)$, $f_2(t,u,v) \leq \widetilde{f}_2(t,w)$, $u \geq 0$, $v \geq 0$, $u+v \leq w$, $t \in [0,1]$; $\widetilde{f}_1(t,w)$ and $\widetilde{f}_2(t,w)$ are nondecreasing on w, and

$$\limsup_{w\to+\infty} \max_{t\in[0,1]} \frac{\widetilde{f}_1(t,w)}{w} \leq \overline{z}_\infty, \tag{3.15}$$

$$\limsup_{w\to+\infty} \max_{t\in[0,1]} \frac{\widetilde{f}_2(t,w)}{w} \leq \overline{z}_\infty^*. \tag{3.16}$$

By (3.15) and (3.16), there exist $\varepsilon > 0$ and $\overline{R}_4 > 0$ such that

$$\widetilde{f}_1(t,w) \leq (\overline{z}_\infty+\epsilon)w, \qquad \widetilde{f}_2(t,w) \leq \bigl(\overline{z}_\infty^*+\epsilon\bigr)w, \quad t \in [0,1], w \geq \overline{R}_4. \tag{3.17}$$

Let $\Omega_4 = \{(u,v) \in X | \ \|(u,v)\|_X < R_4\}$, where $R_4 = \max\{2R_3, 3\overline{R}_4\}$. For any $(u,v) \in P \cap \partial\Omega_4$, we have $f_1(t,u,v) \leq \widetilde{f}_1(t,\|(u,v)\|_X)$, $f_2(t,u,v) \leq \widetilde{f}_2(t,\|(u,v)\|_X)$. So, by (3.13) and (3.17), we get that

$$\begin{aligned} L_1(u,v)(t) &= \lambda \int_0^1 G_1(t,s) f_1\big(s,u(s),v(s)\big)\,ds \\ &\le \frac{\lambda}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1}(\overline{z}_\infty+\epsilon)\big(\|(u,v)\|_X\big)\,ds \\ &= \lambda(\overline{z}_\infty+\epsilon)P_1\big(\|(u,v)\|_X\big) \\ &\le \frac{1}{2}\|(u,v)\|_X, \end{aligned}$$

and

$$\begin{aligned} L_2(u,v)(t) &= \mu \int_0^1 G_2(t,s) f_2\big(s,u(s),v(s)\big)\,ds \\ &\le \frac{\mu}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1}\big(\overline{z}^*_\infty+\epsilon\big)\big(\|(u,v)\|_X\big)\,ds \\ &= \mu\big(\overline{z}^*_\infty+\epsilon\big)P_2\big(\|(u,v)\|_X\big) \\ &\le \frac{1}{2}\|(u,v)\|_X. \end{aligned}$$

Then we get

$$\|L(u,v)\| = \|L_1(u,v)\| + \|L_2(u,v)\| \le \|(u,v)\|_X, \quad \forall (u,v) \in P \cap \partial\Omega_4. \tag{3.18}$$

By virtue of (3.14)(3.18) and Lemma 2.5, we know that L has at least a fixed point $(u,v) \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$. Therefore, (u,v) is one positive solution of system (1.1). □

Since the proofs of the following theorems are similar to Theorem 3.9, we only give the results as follows.

Theorem 3.10 *Let $\overline{z}_0, \overline{z}^*_0, \overline{z}_\infty \in (0,+\infty)$, $\overline{z}^*_\infty = 0$, $\overline{Q}_1 < \overline{Q}_2$. Then when $\lambda \in (\overline{Q}_1, \overline{Q}_2)$, $\mu \in (\overline{Q}_3, +\infty)$ hold, we get that system (1.1) has at least one positive solution.*

Theorem 3.11 *Let $\overline{z}_0, \overline{z}^*_0, \overline{z}^*_\infty \in (0,+\infty)$, $\overline{z}_\infty = 0, \overline{Q}_3 < \overline{Q}_4$. Then when $\lambda \in (\overline{Q}_1, +\infty)$, $\mu \in (\overline{Q}_3, \overline{Q}_4)$ hold, we get that system (1.1) has at least one positive solution.*

Theorem 3.12 *Let $\overline{z}_0, \overline{z}^*_0 \in (0,+\infty)$, $\overline{z}_\infty = \overline{z}^*_\infty = 0$. Then when $\lambda \in (\overline{Q}_1, +\infty)$, $\mu \in (\overline{Q}_3, +\infty)$ hold, we get that system (1.1) has at least one positive solution.*

Theorem 3.13 *Let $\overline{z}_\infty, \overline{z}^*_\infty \in (0,+\infty)$, $\overline{z}_0 = +\infty$, and $\overline{z}^*_0 = +\infty$. Then when $\lambda \in (0, \overline{Q}_2)$, $\mu \in (0, \overline{Q}_4)$ hold, we get that system (1.1) has at least one positive solution.*

Theorem 3.14 *Let $\overline{z}_0 = +\infty$, $\overline{z}_\infty \in (0,+\infty), \overline{z}^*_0 = +\infty$, and $\overline{z}^*_\infty = 0$. Then when $\lambda \in (0, \overline{Q}_2)$, $\mu \in (0,+\infty)$ hold, we get that system (1.1) has at least one positive solution.*

Theorem 3.15 *Let $\overline{z}^*_\infty \in (0,+\infty)$, $\overline{z}_\infty = 0$, $\overline{z}_0 = +\infty$, and $\overline{z}^*_0 = +\infty$. Then when $\lambda \in (0,+\infty)$, $\mu \in (0, \overline{Q}_4)$ hold, we get that system (1.1) has at least one positive solution.*

Theorem 3.16 *Let $\overline{z}_\infty = \overline{z}_\infty^* = 0$, $\overline{z}_0 = +\infty$, and $\overline{z}_0^* = +\infty$. Then when $\lambda \in (0,+\infty)$, $\mu \in (0,+\infty)$ hold, we get that system (1.1) has at least one positive solution.*

4 Applications

Example 4.1 Consider the following Caputo-type fractional system:

$$\begin{cases} -{}^cD^{\frac{5}{2}}u(t) = \lambda f_1(t,u(t),v(t)), & t \in [0,1], \\ -{}^cD^{\frac{5}{2}}u(t) = \mu f_2(t,u(t),v(t)), & t \in [0,1], \\ u(0) = u''(0) = 0, & u(1) = \frac{1}{2}\int_0^1 u(t)\,dt, \\ v(0) = v''(0) = 0, & v(1) = \frac{1}{2}\int_0^1 v(t)\,dt. \end{cases} \tag{4.1}$$

Take

$$f_1 = t(u+v)^3 + (u+v)e^{(u+v)} + (u+v),$$
$$f_2 = t(u+v)^3,$$

where $\theta_1 = \theta_2 = \frac{5}{2}$, $A_1(t) = A_2(t) = \frac{1}{2}t$, $B_1 = B_2 = \frac{1}{4}$.

We can get $P_1 = P_2 = \frac{32}{45\sqrt{\pi}}$, $P_3 = P_4 = \frac{9(9\sqrt{3}-1)}{327680\sqrt{\pi}}$. Obviously, we can infer that

$$z_0 = \lim_{(u,v)\to(0^+,0^+)} \sup_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v} = \lim_{(u,v)\to(0^+,0^+)} (u+v)^2 + e^{(u+v)} + 1 = 2,$$

$$z_0^* = \lim_{(u,v)\to(0^+,0^+)} \sup_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v} = \lim_{(u,v)\to(0^+,0^+)} (u+v)^2 = 0,$$

$$z_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_1(t,u,v)}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{1}{4}(u+v)^2 + e^{(u+v)} + 1 = +\infty,$$

$$z_\infty^* = \lim_{(u,v)\to(+\infty,+\infty)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_2(t,u,v)}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{1}{4}(u+v)^2 = +\infty.$$

Then, for each $\lambda \in (0, \frac{45\sqrt{\pi}}{8})$ and $\mu \in (0,+\infty)$, from Theorem 3.6, system (4.1) has at least a positive solution.

Example 4.2 Consider the following Caputo-type fractional system:

$$\begin{cases} -{}^cD^{\frac{5}{2}}u(t) = \lambda f_1(t,u(t),v(t)), & t \in [0,1], \\ -{}^cD^{\frac{5}{2}}u(t) = \mu f_2(t,u(t),v(t)), & t \in [0,1], \\ u(0) = u''(0) = 0, & u(1) = \frac{1}{2}\int_0^1 u(t)\,dt, \\ v(0) = v''(0) = 0, & v(1) = \frac{1}{2}\int_0^1 v(t)\,dt. \end{cases} \tag{4.2}$$

Take

$$f_1(t,u,v) = \frac{t}{u+v},$$
$$f_2(t,u,v) = \frac{t}{u+v} + 6,$$

where $\theta_1 = \theta_2 = \frac{5}{2}$, $A_1(t) = A_2(t) = \frac{1}{2}t$, $B_1 = B_2 = \frac{1}{4}$.

We can get $P_1 = P_2 = \frac{32}{45\sqrt{\pi}}$, $P_3 = P_4 = \frac{9(9\sqrt{3}-1)}{327680\sqrt{\pi}}$, and

$$\begin{aligned}
\overline{z}_0 &= \lim_{(u,v)\to(0^+,0^+)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_1(t,u,v)}{u+v} = \lim_{(u,v)\to(0^+,0^+)} \frac{1}{4(u+v)^2} = +\infty, \\
\overline{z}_0^* &= \lim_{(u,v)\to(0^+,0^+)} \inf_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f_2(t,u,v)}{u+v} = \lim_{(u,v)\to(0^+,0^+)} \frac{1}{4(u+v)^2} + \frac{6}{(u+v)} = +\infty, \\
\overline{z}_\infty &= \lim_{(u,v)\to(+\infty,+\infty)} \sup_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{1}{(u+v)^2} = 0, \\
\overline{z}_\infty^* &= \lim_{(u,v)\to(+\infty,+\infty)} \sup_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{1}{(u+v)^2} + \frac{1}{(u+v)} = 0.
\end{aligned}$$

Then, for each $\lambda \in (0,+\infty)$ and $\mu \in (0,+\infty)$, from Theorem 3.16, system (4.2) has at least a positive solution.

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