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Another class of nonterminating $_3F_2$-series with a free argument

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Abstract

By means of the linearization method, we evaluate another class of nonterminating $_3F_2$-series with a free argument x and two perturbing integer parameters m and n.

MSC: Primary 33C20; secondary 05A10

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1 Introduction and motivation

There has always been a strong interest in discovering novel summation formulae for (generalized) hypergeometric series due to their broad variety of applications in mathematics, physics, and computer science (see [5–7, 13, 14, 19–21, 23]). The purpose of this paper is to evaluate, in closed forms, the following class of nonterminating $_3F_2$-series with a free variable x (with $|x| < 1$ for convergence) and two perturbing integer parameters m and n:

$$\Omega_{m,n}(a,x) := {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}+m, & 3a+n \end{matrix}\middle| x^2\right], \tag{1}$$

where, according to Bailey [2, §2.1], the classical hypergeometric series reads as

$${}_{1+p}F_p\left[\begin{matrix} a_0, & a_1,\dots,a_p \\ & b_1,\dots,b_p \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_p)_k}{k!(b_1)_k\cdots(b_p)_k} z^k.$$

Denote by $\mathbb{Z}$ and $\mathbb{N}$, respectively, sets of integers and natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For indeterminate y and $n \in \mathbb{Z}$, the rising and falling factorials are defined by the following quotients of Euler's Γ-function:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad \langle x\rangle_n = \frac{\Gamma(1+x)}{\Gamma(1+x-n)},$$

where the multiparameter notation for the former one will be abbreviated to

$$\begin{bmatrix} A, B, \dots, C \\ \alpha, \beta, \dots, \gamma \end{bmatrix}_n = \frac{(A)_n(B)_n\cdots(C)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

Our work is motivated by Lambert's binomial series (see Riordan [22, §4.5] and [1, 8–10, 15, 20]) which is well known in classical analysis. Let u and v be the two variables related through the equation $u = v/(1+v)^{\beta}$. Then

$$\phi_\alpha(u) := (1+v)^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}u^k,$$

$$\psi_\alpha(u) := \frac{(1+v)^{\alpha+1}}{1+v-\beta v} = \sum_{k=0}^{\infty}\binom{\alpha+k\beta}{k}u^k.$$

By the bisection of series, we have further four generating functions

$$\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+2\beta k}\binom{\alpha+2\beta k}{2k}u^{2k} = \frac{\phi_\alpha(u)+\phi_\alpha(-u)}{2},$$

$$\sum_{k=0}^{\infty}\binom{\alpha+2\beta k}{2k}u^{2k} = \frac{\psi_\alpha(u)+\psi_\alpha(-u)}{2};$$

$$\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+\beta(2k+1)}\binom{\alpha+\beta(2k+1)}{2k+1}u^{2k+1} = \frac{\phi_\alpha(u)-\phi_\alpha(-u)}{2},$$

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta(2k+1)}{2k+1}u^{2k+1} = \frac{\psi_\alpha(u)-\psi_\alpha(-u)}{2}.$$

Specifying with $\beta = \frac{3}{2}$, making the replacements $u \to \frac{2x}{3\sqrt{3}}, v \to y$, and then letting

$$\alpha \to 3a-1, \alpha \to 3a-2, \alpha \to 3a-\frac{5}{2}, \alpha \to 3a-\frac{7}{2},$$

respectively, in the above four equations, we get four hypergeometric formulae:

$${}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3} & a+\frac{1}{3} \\ & \frac{1}{2}, & 3a \end{matrix}\middle| x^2\right] = \frac{1}{2}\left\{(1+y_+)^{3a-1} + (1+y_-)^{3a-1}\right\}, \qquad [\Omega_{0,0}]$$

$${}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{1}{2}, & 3a-1 \end{matrix}\middle| x^2\right] = \frac{(1+y_+)^{3a-1}}{2-y_+} + \frac{(1+y_-)^{3a-1}}{2-y_-}; \qquad [\Omega_{0,-1}]$$

$${}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{3}{2}, & 3a-1 \end{matrix}\middle| x^2\right] = \frac{(1+y_+)^{3a-1}}{(6a-5)y_+} + \frac{(1+y_-)^{3a-1}}{(6a-5)y_-}, \qquad [\Omega_{1,-1}]$$

$${}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{3}{2}, & 3a-2 \end{matrix}\middle| x^2\right] = \frac{(1+y_+)^{3a-1}}{(3a-2)(2-y_+)y_+} + \frac{(1+y_-)^{3a-1}}{(3a-2)(2-y_-)y_-}. \qquad [\Omega_{1,-2}]$$

Here and forth, x and y are two variables related via equations

$$\pm x = \frac{3\sqrt{3}y_\pm}{2\sqrt{(1+y_\pm)^3}}, \tag{2}$$

where $y_\pm$ are computed from x through the fundamental algebraic relationship

$$\frac{2x}{3\sqrt{3}} = \frac{y}{(1+y)^{3/2}} \quad \text{or equivalently} \quad \left(\frac{2x}{y}\right)^2 = \left(\frac{3}{1+y}\right)^3.$$

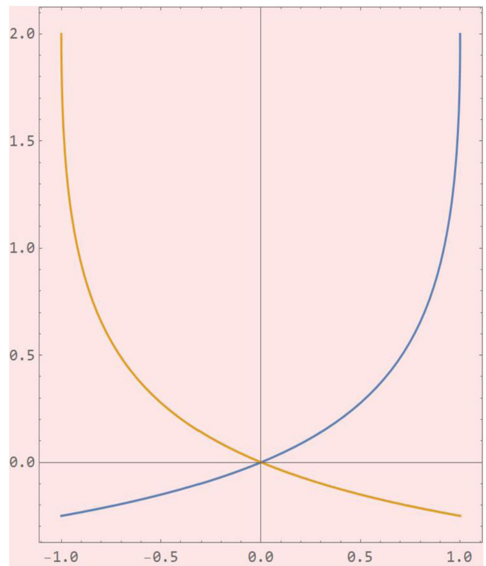

Figure 1 The "$x-y$" curve

Recall that the hypergeometric ${}_3F_2(x^2)$-series converge (generically) only if their argument is less than 1 in magnitude. Therefore x is restricted to $(-1,1)$. There are exactly two solutions y_+ and y_- of the above equation in the region $(-1/4,2)$ whenever x satisfies $-1<x<1$. By equating both members of the last equation to t^6, we can parameterize the algebraic "x–y curve" by rational functions:

$$x=\frac{t}{2}\left(3-t^2\right) \quad \text{and} \quad y=\frac{3-t^2}{t^2}.$$

The portions of the curve with $t\in(-2,-1)$ and $t\in(1,2)$ lie, in the "x–y plane", in the abovementioned region. For any x, the corresponding $y_\pm$ are the y-coordinates of the points (x,y) that lie on these two branches that are illustrated in the Fig. 1.

The four identities of ${}_3F_2$-series highlighted in the last page are not isolated examples. As we shall show, there exists a large number of closed formulae for the series $\Omega_{m,n}$. By means of the linearization method (cf. [3, 4, 11, 12, 16–18]), we shall reduce in the next section, for $m,n\in\mathbb{Z}$, the series $\Omega_{m,n}$ to $\Omega_{m',0}$ with $m'<0$. Then this last series will be evaluated in Sect. 3 via differential operators. The conclusive theorem affirms that, for all the $m,n\in\mathbb{Z}$, the nonterminating $\Omega_{m,n}$-series can be always evaluated explicitly in terms of a finite number of algebraic functions in $y_\pm$. Finally, by making use of *Mathematica* commands, 26 closed formulae are presented as exemplification.

2 Linearization method

By means of the linearization method, we shall establish, in this section, three reduction formulae that express ultimately the series $\Omega_{m,n}$ with $m,n\in\mathbb{Z}$ in terms of the series $\Omega_{m',0}$, but with $m'<0$.

2.1 $m>0$

By employing the Chu–Vandermonde formula on binomial convolutions, it is routine to prove the following linear representation lemma.

Lemma 1 (Linear representation) *For a natural number m and a variable y, the following linear relation holds*:

$$\langle y\rangle_m=\sum_{i=0}^m(-1)^i\binom{m}{i}\langle A+y\rangle_{m-i}(A)_i.$$

Now specifying in this lemma the parameters

$$y=k\quad\text{and}\quad A=3a-m+n-1,$$

we get the equality

$$\langle k\rangle_m=\sum_{i=0}^m(-1)^i\binom{m}{i}\langle 3a-m+n-1+k\rangle_{m-i}(3a-m+n-1)_i.$$

By inserting this relation in the $\Omega_{m,n}$-series, we have

$$\begin{aligned}\Omega_{m,n}(a,x)&=\sum_{k=0}^\infty\frac{(a)_k(a-\frac{1}{3})_k(a+\frac{1}{3})_k}{k!(\frac{1}{2}+m)_k(3a+n)_k}x^{2k}\\&=\sum_{k=m}^\infty\frac{(a)_{k-m}(a-\frac{1}{3})_{k-m}(a+\frac{1}{3})_{k-m}}{(k-m)!(\frac{1}{2}+m)_{k-m}(3a+n)_{k-m}}x^{2k-2m}\\&\quad\times\sum_{i=0}^m(-1)^i\binom{m}{i}\frac{\langle 3a-m+n-1+k\rangle_{m-i}(3a-m+n-1)_i}{\langle k\rangle_m}.\end{aligned}$$

Observing that

$$\begin{aligned}\frac{\langle 3a-m+n-1+k\rangle_{m-i}}{(3a+n)_{k-m}}&=\frac{(1-3a-n)_{2m}}{(3a-2m+n)_{k+i}},\\\frac{(a)_{k-m}(a-\frac{1}{3})_{k-m}(a+\frac{1}{3})_{k-m}}{(k-m)!\langle k\rangle_m}&=(-27)^m\frac{(a-m)_k(a-m-\frac{1}{3})_k(a-m+\frac{1}{3})_k}{k!(2-3a)_{3m}};\end{aligned}$$

we can reformulate the double sum

$$\begin{aligned}\Omega_{m,n}(a,x)={}&\left(-\frac{27}{x^2}\right)^m\frac{(\frac{1}{2})_m(1-3a-n)_{2m}}{(2-3a)_{3m}}\sum_{i=0}^m(-1)^i\binom{m}{i}\frac{(3a-m+n-1)_i}{(3a-2m+n)_i}\\&\times\sum_{k=m}^\infty\left[\begin{matrix}a-m, & a-m-\frac{1}{3}, & a-m+\frac{1}{3}\\ 1, & \frac{1}{2}, & 3a-2m+n+i\end{matrix}\right]_k x^{2k}.\end{aligned}$$

Expressing the last sum with respect to k in terms of $\Omega_{0,m+n+i}(a-m,x)$, we derive the first reduction formula.

Proposition 2 ($m\in\mathbb{N}_0$ and $n\in\mathbb{Z}$)

$$\Omega_{m,n}(a,x)=\left(-\frac{27}{x^2}\right)^m\frac{(\frac{1}{2})_m(1-3a-n)_{2m}}{(2-3a)_{3m}}\sum_{i=0}^m(-1)^i\binom{m}{i}\frac{(3a-m+n-1)_i}{(3a-2m+n)_i}$$

$$\times \left\{ \Omega_{0,m+n+i}(a-m,x) - \sum_{k=0}^{m-1} \frac{(3a-3m-1)_{3k}}{(2k)!(3a-2m+n+i)_k} \left(\frac{4x^2}{27}\right)^k \right\}.$$

2.2 $n < 0$

Analogously, we can also prove, without difficulty, another linear representation lemma.

Lemma 3 (Linear representation) *For a negative integer n and a variable y, the following linear relation holds*:

$$(A+y)_{-n} = \sum_{i=0}^{-n} \binom{-n}{i} \langle B+y\rangle_i (A-B+i)_{-n-i}.$$

Under the parameter specification

$$y = k, \qquad A = 3a+n, \qquad B = m - \frac{1}{2},$$

the equality in Lemma 3 can be restated as

$$(3a+n+k)_{-n} = \sum_{i=0}^{-n} \binom{-n}{i} \left\langle m+k-\frac{1}{2}\right\rangle_i \left(3a+n-m+\frac{1}{2}+i\right)_{-n-i}.$$

By putting this relation inside the $\Omega_{m,n}$-series, we can manipulate the double sum

$$\begin{aligned}
\Omega_{m,n}(a,x) &= \sum_{k=0}^{\infty} \frac{(a)_k(a-\frac{1}{3})_k(a+\frac{1}{3})_k}{k!(\frac{1}{2}+m)_k(3a+n)_{k-n}} x^{2k} \\
&\quad \times \sum_{i=0}^{-n} \binom{-n}{i} \left\langle m+k-\frac{1}{2}\right\rangle_i \left(3a+n-m+\frac{1}{2}+i\right)_{-n-i} \\
&= \sum_{i=0}^{-n} \binom{-n}{i} \left(3a+n-m+\frac{1}{2}+i\right)_{-n-i} \\
&\quad \times \sum_{k=0}^{\infty} \left\langle m+k-\frac{1}{2}\right\rangle_i \frac{(a)_k(a-\frac{1}{3})_k(a+\frac{1}{3})_k}{k!(\frac{1}{2}+m)_k(3a+n)_{k-n}} x^{2k} \\
&= \sum_{i=0}^{-n} \binom{-n}{i} \frac{(\frac{1}{2}+m-i)_i(3a)_n}{(3a-m+\frac{1}{2})_{n+i}} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(a)_k(a-\frac{1}{3})_k(a+\frac{1}{3})_k}{k!(\frac{1}{2}+m-i)_k(3a)_k} x^{2k}.
\end{aligned}$$

Writing the last sum by $\Omega_{m-i,0}(a,x)$, we get the second reduction formula.

Proposition 4 ($m,n \in \mathbb{Z}$ with $n<0$)

$$\Omega_{m,n}(a,x) = \sum_{i=0}^{-n} (-1)^i \binom{-n}{i} \frac{(\frac{1}{2}-m)_i(3a)_n}{(3a-m+\frac{1}{2})_{n+i}} \Omega_{m-i,0}(a,x).$$

2.3 $n > 0$

The next linear relation comes from a limiting case of a known one. Dividing by A^m equation (3.1) in [17, Lemma 3.1] and then letting $A \to \infty$, we get the following linearization lemma.

Lemma 5 (Linear representation) *For a natural number n and a variable y, the following linear relation holds*:

$$1 = \sum_{i=0}^{n} \langle B + y\rangle_{n-i} (3C + 3y)_i \mathrm{X}_n^i, \tag{3}$$

where the coefficients X_n^i *are independent of the variable y and given explicitly by the two expressions*

$$\begin{aligned}\mathrm{X}_n^i &= \sum_{j=0}^{i} \frac{(-1)^{n-i+j}}{i!} \binom{i}{j} \frac{3C - 3B + 3n - 2i}{3(C - B + \frac{j}{3})_{n-i+1}} \\ &= \sum_{j=0}^{n-i} \frac{(-1)^{n-i+j}}{(n-i)!} \binom{n-i}{j} \frac{3C - 3B + 3n - 2i}{(3C - 3B + 3j)_{i+1}}.\end{aligned}$$

Specifying in Lemma 5 the parameters

$$y = k, \qquad B = 3a + n - 1, \qquad C = a - \frac{1}{3},$$

the equality corresponding to (3) becomes

$$1 = \sum_{i=0}^{n} \langle 3a + n + k - 1\rangle_{n-i} (3a - 1 + 3k)_i \mathcal{X}_n^i \tag{4}$$

with the coefficients $\mathcal{X}_n^i$ being determined by

$$\begin{aligned}\mathcal{X}_n^i &= \sum_{j=0}^{i} \frac{(-1)^{n-i+j}}{i!} \binom{i}{j} \frac{2 - 6a - 2i}{3(\frac{2}{3} - 2a - n + \frac{j}{3})_{n-i+1}} \\ &= \sum_{j=0}^{n-i} \frac{(-1)^{n-i+j}}{(n-i)!} \binom{n-i}{j} \frac{2 - 6a - 2i}{(2 - 6a - 3n + 3j)_{i+1}}.\end{aligned} \tag{5}$$

By inserting this relation (5) in the $\Omega_{m,n}$-series, we get the double sum

$$\begin{aligned}\Omega_{m,n}(a,x) &= \sum_{k=0}^{\infty} \frac{(a)_k (a - \frac{1}{3})_k (a + \frac{1}{3})_k}{k! (\frac{1}{2} + m)_k (3a + n)_k} x^{2k} \\ &\quad \times \sum_{i=0}^{n} \langle 3a + n + k - 1\rangle_{n-i} (3a - 1 + 3k)_i \mathcal{X}_n^i \\ &= \sum_{i=0}^{n} (3a - 1)_i (3a + i)_{n-i} \mathcal{X}_n^i\end{aligned}$$

$$\times\sum_{k=0}^{\infty}\begin{bmatrix} a+\frac{i}{3}-\frac{1}{3}, & a+\frac{i}{3}, & a+\frac{i}{3}+\frac{1}{3} \\ 1, & \frac{1}{2}+m, & 3a+i \end{bmatrix}_k x^{2k}.$$

Expressing the last sum by $\Omega_{m,0}(a+\frac{i}{3},x)$, we have the third reduction formula.

Proposition 6 *Let $n \in \mathbb{N}$ and the connection coefficients $\{\mathcal{X}_n^i\}$ be given by* (5). *Then the following formula holds*:

$$\Omega_{m,n}(a,x)=\sum_{i=0}^{n}\mathcal{X}_n^i(3a-1)_i(3a+i)_{n-i}\Omega_{m,0}\left(a+\frac{i}{3},x\right).$$

3 Conclusive theorem and examples

For a given integer pair $\{m,n\}$, we can express the $\Omega_{m,n}$-series, by making use of Propositions 2, 4, and 6, in terms of $\Omega_{m',0}$-series with $m' \leq 0$. Therefore it remains to evaluate this last series. This will be done by utilizing differential operations. Suppose that $f(x)$ is a differentiable function. Define the operator δ by

$$\delta f(x)=\frac{d}{dx}\left\{\frac{f(x)}{x}\right\}.$$

Then it is not hard to check that

$$\delta\Omega_{0,0}(a,x)=\sum_{k=0}^{\infty}(2k-1)\begin{bmatrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ 1, & \frac{1}{2}, & 3a \end{bmatrix}_k x^{2k-2}=\frac{-1}{x^2}\Omega_{-1,0}(a,x),$$

$$\delta^2\Omega_{0,0}(a,x)=\sum_{k=0}^{\infty}(3-2k)\begin{bmatrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ 1, & -\frac{1}{2}, & 3a \end{bmatrix}_k x^{2k-4}=\frac{3}{x^4}\Omega_{-2,0}(a,x).$$

Proceeding by induction, we can show that

$$\begin{aligned}\delta^n\Omega_{0,0}(a,x)&=(-1)^{n-1}(2n-3)!!\sum_{k=0}^{\infty}\begin{bmatrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ 1, & \frac{3}{2}-n, & 3a \end{bmatrix}_k (2k-2n+1)x^{2k-2n}\\ &=\frac{(-1)^n(2n-1)!!}{x^{2n}}\Omega_{-n,0}(a,x).\end{aligned}$$

Recalling that

$$\Omega_{0,0}(a,x)=\frac{1}{2}\left\{(1+y_+)^{3a-1}+(1+y_-)^{3a-1}\right\}$$

and then relabeling n by $-m$, we get the following expression.

Proposition 7 *For $m<0$ and the three variables $\{x,y_\pm\}$ related by* (2), *the following formula holds*:

$$\Omega_{m,0}(a,x)=\frac{(-2/x^2)^m}{2(\frac{1}{2})_{-m}}\delta^{-m}\left\{(1+y_+)^{3a-1}+(1+y_-)^{3a-1}\right\}.$$

As an anonymous referee pointed out, instead of Proposition 4 the case $n < 0$ can be alternatively treated by repeatedly applying the operator δ to the initial function $x^{6a-1}\Omega_{0,0}(a,x)$.

Summing up, for any given pair of integers m and n, the series $\Omega_{m,n}(a,x)$ can be evaluated by carrying out the following procedure:

- *Step-A*: If $m > 0$, write $\Omega_{m,n}(a,x)$, by means of Proposition 2, in terms of $\Omega_{0,n'}(a-m,x)$; then go to *Step-B*.
- *Step-B*: For $m \le 0$ and $n \neq 0$, apply Propositions 4 and 6 to express $\Omega_{m,n}(a,x)$ as $\Omega_{m',0}(a',x)$ with $m' \le m$; then go to *Step-C*.
- *Step-C*: Finally, for $m \le 0$ and $n = 0$, evaluate $\Omega_{m,0}(a,x)$, according to Proposition 7, by differentiating $\Omega_{0,0}(a,x)$.

Therefore, we have shown the following general conclusion.

Theorem 8 *For all the $m,n \in \mathbb{Z}$, the nonterminating $\Omega_{m,n}$-series are always evaluable explicitly in a finite number of terms of algebraic functions in $y_\pm$.*

Based on Propositions 2, 4, 6, and 7, we have devised appropriately *Mathematica* commands that are employed to evaluate $\Omega_{m,n}$ in closed forms for any specific integer pair "m,n". Apart from the four formulae anticipated in the Introduction, we highlight further 26 elegant formulae as exemplification.

Example 1 ($m = 0$ and $n = 1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a+1 \end{matrix}\,\middle|\, x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{2(6a+1)}\{1+6a+y-3ay\}. $$

Example 2 ($m = 0$ and $n = 2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a+2 \end{matrix}\,\middle|\, x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{4(2a+1)(3a+2)(6a+1)}\left\{\begin{matrix} 4+96a^2+72a^3+4y+10ay-42a^2y \\ +38a-72a^3y-ay^2-3a^2y^2+18a^3y^2 \end{matrix}\right\}. $$

Example 3 ($m = 0$ and $n = -2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a-2 \end{matrix}\,\middle|\, x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}(8-12a-7y+6ay)}{(3a-2)(y-2)^3}. $$

Example 4 ($m = 0$ and $n = -3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a-3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{2(1+y)^{3a-1}}{(a-1)(3a-2)(2-y)^5}\left\{\begin{matrix} 16-40a+24a^2-29y+58ay \\ -24a^2y+15y^2-19ay^2+6a^2y^2 \end{matrix}\right\}. $$

Example 5 ($m = 1$ and $n = 0$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{3}{2}, & 3a \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}(4-12a-5y+6ay)}{6(1-2a)(6a-5)y}. $$

Example 6 ($m = 1$ and $n = 1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{3}{2}, & 3a+1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{-3(1+y)^{3a-1}}{32y(3a-\frac{5}{2})_4}\left\{\begin{matrix} 8a+24a^2-144a^3+5y+4ay-132a^2y \\ +144a^3y+5y^2-31ay^2+60a^2y^2-36a^3y^2 \end{matrix}\right\}. $$

Example 7 ($m = 1$ and $n = -3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{3}{2}, & 3a-3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{2(1+y)^{3a-1}(7-6a-5y+3ay)}{3y(a-1)(3a-2)(y-2)^3}. $$

Example 8 ($m = -1$ and $n = 0$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{2(2-y)}\{2+y-6ay\}. $$

Example 9 ($m = -1$ and $n = 1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a+1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{1}{2}(1+y)^{3a-1}\{1+y-3ay\}. $$

Example 10 ($m = -1$ and $n = 2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a+2 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{4(3a+2)}\left\{4+6a+4y-6ay-18a^2y-3ay^2+9a^2y^2\right\}. $$

Example 11 ($m = -1$ and $n = 3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a+3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{4(a+1)(6a+7)}\left\{\begin{matrix} 14-16ay-66a^2y-36a^3y-14ay^2+30a^2y^2 \\ +26a+12a^2+14y+36a^3y^2+ay^3-9a^3y^3 \end{matrix}\right\}. $$

Example 12 ($m = -1$ and $n = -1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a-1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{(2-y)^3}\left\{4-2y-12ay-3y^2+6ay^2\right\}. $$

Example 13 ($m = -1$ and $n = -2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a-2 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{(3a-2)(y-2)^5}\left\{\begin{matrix} 32-24ay+144a^2y+168ay^2-144a^2y^2 \\ -48a-48y+35y^3-72ay^3+36a^2y^3 \end{matrix}\right\}. $$

Example 14 ($m = 2$ and $n = -1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{5}{2}, & 3a-1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{3(1+y)^{3a-1}}{16y^3(3a-\frac{11}{2})_4}\left\{\begin{matrix} 20-192ay+72a^2y+120ay^2 \\ -24a+110y-99y^2-36a^2y^2 \end{matrix}\right\}. $$

Example 15 ($m = 2$ and $n = -2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{5}{2}, & 3a-2 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{3(1+y)^{3a-1}(2+11y-6ay)}{4y^3(2-3a)(3a-\frac{11}{2})_2}. $$

Example 16 ($m = 2$ and $n = -3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{5}{2}, & 3a-3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{6(1+y)^{3a-1}(1+5y-3ay)}{(3a-3)_2(6a-11)(y-2)y^3}. $$

Example 17 ($m = -2$ and $n = 0$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{3}{2}, & 3a \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{6(2-y)^3}\left\{\begin{matrix} 24-12y-72ay-14y^2+y^3 \\ +72ay^2+72a^2y^2-36a^2y^3 \end{matrix}\right\}. $$

Example 18 ($m = -2$ and $n = 1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{3}{2}, & 3a+1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{2(2-y)}\left\{2+y-6ay-y^2+ay^2+6a^2y^2\right\}. $$

Example 19 ($m = -2$ and $n = -1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{3}{2}, & 3a-1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{(2-y)^5} \left\{ \begin{matrix} 16 - 24y - 48ay + 88ay^2 + 48a^2y^2 + 20y^3 \\ -16ay^3 - 48a^2y^3 + y^4 - 8ay^4 + 12a^2y^4 \end{matrix} \right\}. $$

Example 20 ($m = -2$ and $n = 2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{3}{2}, & 3a+2 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{4(3a+2)} \left\{ \begin{matrix} 4 + 6a + 4y - 6ay - 18a^2y \\ -5ay^2 + 9a^2y^2 + 18a^3y^2 \end{matrix} \right\}. $$

Example 21 ($m = -2$ and $n = 3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{3}{2}, & 3a+3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{12(a+1)} \left\{ \begin{matrix} 6 + 6a + 6y - 12ay - 18a^2y - 8ay^2 \\ +18a^2y^2 + 18a^3y^2 + ay^3 - 9a^3y^3 \end{matrix} \right\}. $$

Example 22 ($m = -3$ and $n = 1$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{5}{2}, & 3a+1 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{10(2-y)^3} \left\{ \begin{matrix} 40 - 120ay - 30y^2 + 132ay^2 + 144a^2y^2 + 25y^3 - 22ay^3 \\ -20y - 180a^2y^3 - 72a^3y^3 - 5y^4 - ay^4 + 36a^2y^4 + 36a^3y^4 \end{matrix} \right\}. $$

Example 23 ($m = -3$ and $n = 2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{5}{2}, & 3a+2 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{20(3a+2)(2-y)} \left\{ \begin{matrix} 40 - 90ay - 180a^2y - 20y^2 - 24ay^2 + 180a^2y^2 + 35ay^3 \\ +60a + 20y + 216a^3y^2 - 33a^2y^3 - 180a^3y^3 - 108a^4y^3 \end{matrix} \right\}. $$

Example 24 ($m = -3$ and $n = 3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{5}{2}, & 3a+3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{(1+y)^{3a-1}}{20(a+1)} \begin{Bmatrix} 10 - 20ay - 30a^2y - 14ay^2 + 30a^2y^2 + 36a^3y^2 \\ +10a + 10y + 3ay^3 + 2a^2y^3 - 27a^3y^3 - 18a^4y^3 \end{Bmatrix}. $$

Example 25 ($m = 3$ and $n = -3$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{7}{2}, & 3a-3 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{-45(1+y)^{3a-1}}{8y^5(3a-3)_2(3a-\frac{17}{2})_4} \begin{Bmatrix} 22 + 187y - 168ay + 36a^2y + 425y^2 \\ -12a - 575ay^2 + 252a^2y^2 - 36a^3y^2 \end{Bmatrix}. $$

Example 26 ($m = 3$ and $n = -2$)

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{7}{2}, & 3a-2 \end{matrix}\middle| x^2\right] = w(y_+) + w(y_-), $$

where

$$ w(y) = \frac{45(1+y)^{3a-1}}{32y^5(2-3a)(3a-\frac{17}{2})_5} \times \begin{Bmatrix} 72 - 48a - 624ay + 144a^2y + 1190y^2 - 1916ay^2 + 936a^2y^2 \\ +612y - 144a^3y^2 - 1105y^3 + 1342ay^3 - 540a^2y^3 + 72a^3y^3 \end{Bmatrix}. $$

These identities are valid for all the x and y tied by (2) under the conditions $|x| < 1$ and $-1/4 < y < 2$. When x is assigned to particular values, they may produce strange evaluation formulae. We limit ourselves to recording three groups of such formulae.

- Series with $\{x, y_+, y_-\} = \{\sqrt{\frac{3^5}{7^3}}, \frac{3}{4}, -\frac{2}{9}\}$.

$$ {}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{1}{2}, & 3a \end{matrix}\middle| \frac{3^5}{7^3}\right] = \frac{1}{2}\left\{\left(\frac{7}{4}\right)^{3a-1} + \left(\frac{7}{9}\right)^{3a-1}\right\}, \qquad [\Omega_{0,0}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{3}{2}, & 3a-1 \end{matrix}\middle| \frac{3^5}{7^3}\right] = \frac{7^{3a-1}}{6(6a-5)}\left\{2^{5-6a} - 3^{5-6a}\right\}, \qquad [\Omega_{1,-1}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a+1 \end{matrix}\middle| \frac{3^5}{7^3}\right] = \frac{7^{3a-1}}{2(6a+1)}\left\{\frac{15a+7}{2^{6a}} + \frac{60a+7}{3^{6a}}\right\}, \qquad [\Omega_{0,1}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a+1 \end{matrix}\middle| \frac{3^5}{7^3}\right] = \frac{7^{3a-1}}{2}\left\{\frac{6a+7}{3^{6a}} - \frac{9a-7}{2^{6a}}\right\}. \qquad [\Omega_{-1,1}] $$

• Series with $\{x, y_+, y_-\} = \{2\sqrt{\frac{3^5}{13^3}}, \frac{4}{9}, -\frac{3}{16}\}$.

$$ {}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{1}{2}, & 3a-1 \end{matrix}\middle| \frac{4\cdot 3^5}{13^3}\right] = \frac{9}{14}\left(\frac{13}{9}\right)^{3a-1} + \frac{16}{35}\left(\frac{13}{16}\right)^{3a-1}, \qquad [\Omega_{0,-1}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a-\frac{1}{3}, & a+\frac{1}{3} \\ & \frac{3}{2}, & 3a-2 \end{matrix}\middle| \frac{4\cdot 3^5}{13^3}\right] = \frac{13^{3a-1}}{3a-2}\left\{\frac{3^{6-6a}}{56} - \frac{4^{6-6a}}{105}\right\}, \qquad [\Omega_{1,-2}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{3}{2}, & 3a \end{matrix}\middle| \frac{4\cdot 3^5}{13^3}\right] = \frac{13^{3a-1}}{24(3a-\frac{5}{2})_2}\left\{\frac{63a-12}{9^{3a-1}} - \frac{210a-79}{16^{3a-1}}\right\}, \qquad [\Omega_{1,0}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{5}{2}, & 3a-2 \end{matrix}\middle| \frac{4\cdot 3^5}{13^3}\right] = \frac{13^{3a-1}/(3a-2)}{2304(3a-\frac{11}{2})_2}\left\{\frac{24a-62}{27^{2a-3}} + \frac{18a-1}{64^{2a-3}}\right\}. \qquad [\Omega_{2,-2}] $$

• Series with $\{x, y_+, y_-\} = \{5\sqrt{\frac{3^5}{19^3}}, \frac{10}{9}, -\frac{6}{25}\}$.

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a-2 \end{matrix}\middle| \frac{25\cdot 3^5}{19^3}\right] = \frac{19^{3a-1}}{3a-2}\left\{\frac{24a-1}{256\cdot 3^{6a-6}} + \frac{168a-121}{87{,}808\cdot 5^{6a-6}}\right\}, \qquad [\Omega_{0,-2}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{3}{2}, & 3a-3 \end{matrix}\middle| \frac{25\cdot 3^5}{19^3}\right] $$
$$ = \frac{19^{3a-1}}{(3a-2)(a-1)}\left\{\frac{24a-13}{2560\cdot 3^{6a-7}} + \frac{205-168a}{1{,}580{,}544\cdot 5^{6a-8}}\right\}, \qquad [\Omega_{1,-3}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & -\frac{1}{2}, & 3a \end{matrix}\middle| \frac{25\cdot 3^5}{19^3}\right] = 19^{3a-1}\left\{\frac{7-15a}{4\cdot 9^{3a-1}} + \frac{9a+11}{28\cdot 25^{3a-1}}\right\}, \qquad [\Omega_{-1,0}] $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{5}{2}, & 3a-3 \end{matrix}\middle| \frac{25\cdot 3^5}{19^3}\right] $$
$$ = \frac{19^{3a-1}}{(3a-3)_2(6a-11)}\left\{\frac{30a-59}{4000\cdot 3^{6a-9}} + \frac{18a-5}{2016\cdot 5^{6a-8}}\right\}. \qquad [\Omega_{2,-3}] $$

To our knowledge, the formulae presented in this paper for $\Omega_{m,n}(a,x)$ (when x is a free variable) have not appeared previously. Exceptions are about $\Omega_{0,0}$, $\Omega_{1,-1}$, and $\Omega_{0,1}$. Their particular cases with $\{x, y_+, y_-\} = \{1, 2, -1/4\}$ have been recorded by Milgram in his compendium [21, Equations 25, 30, 31]:

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a \end{matrix}\middle| 1\right] = \frac{3^{3a-1}(1+4^{1-3a})}{2}, $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{3}{2}, & 3a-1 \end{matrix}\middle| 1\right] = \frac{3^{3a-1}}{(6a-5)}\left\{\frac{1}{2} - 4^{2-3a}\right\}, $$

$$ {}_3F_2\left[\begin{matrix} a, & a+\frac{1}{3}, & a-\frac{1}{3} \\ & \frac{1}{2}, & 3a+1 \end{matrix}\middle| 1\right] = \frac{27^a}{6a+1}\left\{\frac{1}{2} + \frac{9a+1}{2^{6a+1}}\right\}. $$

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