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An analysis on the controllability and stability to some fractional delay dynamical systems on time scales with impulsive effects

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Abstract

In this article, we establish a new class of mixed integral fractional delay dynamic systems with impulsive effects on time scales. We investigate the qualitative properties of the considered systems. In fact, the article contains three segments, and the first segment is devoted to investigating the existence and uniqueness results. In the second segment, we study the stability analysis, while the third segment is devoted to investigating the controllability criterion. We use the Leray–Schauder and Banach fixed point theorems to prove our results. Moreover, the obtained results are examined with the help of an example.

MSC: 26A33; 34A08; 34B27

Keywords: Time scales; Dynamic system; Hyers–Ulam stability; Controllability; Impulsive effects

1 Introduction

The notion of fractional differential equations (FDEs) has been a field of intense research for the last few decades. In 1695, the notion of FDEs was initiated with a coincidence between Leibniz and L'Hospital. Nowadays, FDEs play an important role in establishing mathematical modeling of many problems occurring in control theory, bioengineering, mathematical networks, aerodynamics, blood flows, engineering, physics, signal processing, etc. [1, 2].

We can analyze from different experiments that FDEs have innumerable prominent status than integer-order derivatives. Consequently, fractional calculus got incredible interest and received more attention from many specialists and researchers. It also set up a better sketch over hereditary properties of processes and various materials, consequently many monographs and research papers have been reported in this field [3–22].

Recently, the theory of stability analysis, like Lyapunov, exponential, Mittag-Leffler function, and finite time stability for various kinds of functional equations, has been investigated. Ulam and Hyers introduced most important and interesting kind of stability called Hyers–Ulam stability [23] in 1940. Ulam during his talk at Wisconsin University asked a question about the stability of homomorphisms between groups. In 1941, Hyers [24] replied to Ulam's problem positively under the hypothesis that groups are considered as Banach spaces ($\mathbb{BS}$), and such a stability was called Ulam–Hyers stability. For more information, see [25–30]. Impulsive differential equations are best tools to model a physical situation that contains abrupt changes at certain instants. These equations describe medicine, biotechnology process, population dynamics, biological systems, chemical energy, mathematical economy, pharmacokinetics, etc. [31–43].

In the past few decades, because of these applications in various fields of interest, impulsive differential equations got considerable attention. In order to unify the difference and differential calculus, Hilger [44] provided the idea of time scales at the end of the twentieth century, which is now a well-known subject. For more details, see [45–51]. Lupulescu and Zada [49] provided the basics and fundamental notions of linear impulsive systems on time scales in 2010.

In 1960, Kalman presented the notion of controllability, which is the principal notion in mathematical control theory. In general, controllability provides steering the state of a control dynamical equation to the desired terminal state from an arbitrary initial state by utilizing a suitable control function. Numerous researchers examined the controllability results of dynamical systems [52, 53]. Moreover, controllability results on time scales is a new area, and few results have been achieved [54, 55]. Especially, there are a few articles that examined the existence, controllability, and Ulam type stability regarding a mixed structure of the impulsive fractional dynamical system on time scales.

Inspired by the research conducted in [56], we study the following mixed integral fractional dynamic systems on the time scale $\mathbb{T}$:

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \qquad + G(\varsigma,\omega(\varsigma),\int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_1(\varsigma,s,\omega(s))\Delta s,\int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_2(\varsigma,s,\omega(s))\Delta s), \\ \qquad \varsigma\in\mathbb{T}'=\mathbb{T}\backslash\{\varsigma_1,\varsigma_2,\dots,\varsigma_m\}, \sigma=(0,1), \\ \omega(\varsigma_k^+)-\omega(\varsigma_k^-)=\Xi_k(\omega(\varsigma_k^-))+\Phi_k(\varsigma_k^-,\omega(\varsigma_k^-)), \quad k=1,\dots,m, \\ \omega(\varsigma_0)=\omega_0, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \qquad + \mathcal{G}(\varsigma,\omega(\varsigma),\int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_1(\varsigma,s,\omega(s))\Delta s,\int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_2(\varsigma,s,\omega(s))\Delta s), \\ \qquad \varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T}, i=1,\dots,m, \sigma=(0,1), \\ \omega(\varsigma)=\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s, \quad \varsigma\in(\varsigma_i,s_i]\cap\mathbb{T}, i=1,\dots,m, \\ \omega(\varsigma_0)=\omega_0. \end{cases} \tag{2}$$

Also, we discuss the controllability of the following systems:

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \qquad + G(\varsigma,\omega(\varsigma),\int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_1(\varsigma,s,\omega(s))\Delta s,\int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_2(\varsigma,s,\omega(s))\Delta s)+\mathcal{H}\zeta(\varsigma), \\ \qquad \varsigma\in\mathbb{T}'=\mathbb{T}\backslash\{\varsigma_1,\varsigma_2,\dots,\varsigma_m\}, \sigma=(0,1), \\ \omega(\varsigma_k^+)-\omega(\varsigma_k^-)=\Xi_k(\omega(\varsigma_k^-))+\Phi_k(\varsigma_k^-,\omega(\varsigma_k^-)), \quad k=1,\dots,m, \\ \omega(\varsigma_0)=\omega_0, \end{cases} \tag{3}$$

and

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \qquad + \mathcal{G}(\varsigma,\omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_1(\varsigma,s,\omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_2(\varsigma,s,\omega(s))\Delta s) + \mathcal{H}\zeta(\varsigma), \\ \qquad \varsigma \in (s_i,\varsigma_{i+1}] \cap \mathbb{T}, i = 1,\dots,m, \sigma = (0,1), \\ \omega(\varsigma) = \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s, \quad \varsigma \in (\varsigma_i,s_i] \cap \mathbb{T}, i = 1,\dots,m, \\ \omega(\varsigma_0) = \omega_0, \end{cases} \tag{4}$$

where ${}^{c,\mathbb{T}}\mathrm{D}^{\sigma}$ represents the classical Caputo derivative [1] of fractional order σ on time scales $\mathbb{T}$. The regressive square matrix $A(\varsigma)$ is piecewise continuous, and $\mathcal{H} : \mathbb{T} \to \mathbb{T}$ is a bounded linear operator. By assuming $\mathbb{R}$ as the real number, $\zeta \in \mathcal{L}^2(I,\mathbb{R})$ is a control map, $\mathbb{T}^0 := [\varsigma_0,\varsigma_f]_{\mathbb{T}}$, the pre-fixed numbers are $\varsigma_0 = s_0 < \varsigma_1 < s_1 < \varsigma_2 < \cdots < \varsigma_m < s_m < \varsigma_{m+1} = \varsigma_f$, and $\mathcal{F} : \mathbb{T}^0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{F}_1,\mathcal{F}_2 : \mathbb{T}^0 \times \mathbb{T}^0 \times \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{T}^0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\hbar_i : (\varsigma_i,s_i] \cap \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1,\dots,m$, $\mathbb{F} : (s_i,\varsigma_{i+1}] \cap \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 0,\dots,m$, $\mathcal{G} : (s_i,\varsigma_{i+1}] \cap \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1,\dots,m$, $\Xi_k : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_k : \mathbb{T}^0 \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings. In addition, we define the right limit and the left limit of $\omega(\varsigma)$ at ς_k as $\omega(\varsigma_k^+) = \lim_{\tau\to 0^+}\omega(\varsigma_k+\tau)$ and $\omega(\varsigma_k^-) = \lim_{\tau\to 0^-}\omega(\varsigma_k-\tau)$, respectively.

2 Auxiliary definitions and lemmas

Here, we provide definitions, basic notions, and preliminaries for this manuscript.

We define $\mathrm{C}(\mathrm{I},\mathbb{R})$ as a $\mathbb{BS}$ of all continuous mappings endowed with the norm $\|\omega\|_{\mathrm{C}} = \sup_{\varsigma\in\mathbb{T}}\|\omega(\varsigma)\|$. $\mathrm{PS} = \mathrm{C}(\mathrm{I},\mathbb{R}) \times \mathrm{C}(\mathrm{I},\mathbb{R}) \times \mathrm{C}(\mathrm{I},\mathbb{R})$ is the product space which is a $\mathbb{BS}$ furnished with the norm $\|(\omega,\zeta,\xi)\|_{\mathrm{C}} = \|\omega\|_{\mathcal{C}} + \|\zeta\|_{\mathcal{C}} + \|\xi\|_{\mathcal{C}}$. Also, we define a $\mathbb{BS}$ $\mathrm{C}^1(\mathrm{I},\mathbb{R}) = \{\omega \in \mathrm{C}(\mathrm{I},\mathbb{R}) : \omega^{\Delta} \in \mathrm{C}^1(\mathrm{I},\mathbb{R})\}$ with the norm $\|\omega\|_{\mathrm{C}^1} = \max\{\|\omega\|_{\mathcal{C}}, \|\omega^{\Delta}\|_{\mathcal{C}^1}\}$. Moreover, $\mathrm{PS}^1 = \mathrm{C}^1(\mathrm{I},\mathbb{R}) \times \mathrm{C}^1(\mathrm{I},\mathbb{R}) \times \mathrm{C}^1(\mathrm{I},\mathbb{R})$ is the product space via $\|(\omega,\zeta,\xi)\|_{\mathrm{C}^1} = \|\omega\|_{\mathcal{C}^1} + \|\zeta\|_{\mathcal{C}^1} + \|\xi\|_{\mathcal{C}^1}$.

A nonempty closed subset of $\mathbb{R}$ is known as a time scale ($\mathbb{T}$). We define a time scale interval as $[c,d]_{\mathbb{T}} = \{\varsigma \in \mathbb{T} : c \le \varsigma \le d\}$. Similarly, we can define $(c,d)_{\mathbb{T}}$, $[c,d)_{\mathbb{T}}$.

The forward and backward jump operators $\sigma : \mathbb{T} \to \mathbb{T}$, $\rho : \mathbb{T} \to \mathbb{T}$ are introduced as

$$\sigma(\varsigma) = \inf\{s \in \mathbb{T} : s > \varsigma\} \quad \text{and} \quad \rho(\varsigma) = \sup\{s \in \mathbb{T} : s < \varsigma\},$$

respectively. The operator $\eta : \mathbb{T} \to [0,\infty)$ formulated by $\eta(\varsigma) = \sigma(\varsigma) - \varsigma$ is applied to obtain the existing distance between two consecutive points. Along this, the derived version $\mathbb{T}^k$ of $\mathbb{T}$ is

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}),\sup\mathbb{T}], & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

The regressive (respectively positively regressive) function $\mathfrak{H} : \mathbb{T} \to \mathbb{R}$ is defined as $1 + \eta(\varsigma)\mathfrak{H}(\varsigma) \neq 0$ (respectively $1 + \eta(\varsigma)\mathfrak{H}(\varsigma) > 0$) for all $\varsigma \in \mathbb{T}^k$.

Definition 2.1 ([57]) At a point $\varsigma \in \mathbb{T}^k$, the delta derivative $g^{\Delta}(\varsigma)$ of a mapping $g : \mathbb{T} \to \mathbb{R}$ is a number (provided it exists) if, for $\epsilon > 0$, a neighborhood U of ς exists provided that

$$\bigl|\bigl[g(\sigma(\varsigma)) - g(\tau)\bigr] - g^{\Delta}(\varsigma)\bigl[\sigma(\varsigma) - \tau\bigr]\bigr| \le \epsilon\bigl|\sigma(\varsigma) - \tau\bigr|, \quad \text{for all } \tau \in \mathrm{U}.$$

Theorem 2.2 ([57]) *Let $c,d \in \mathbb{T}$ and $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, then*

1. $\mathbb{T} = \mathbb{R}$ *implies*

$$\int_c^d f(\varsigma)\Delta\varsigma = \int_c^d f(\varsigma)\,d\varsigma.$$

2. *If $[c,d)$ consists of only isolated points, then*

$$\int_c^d f(\varsigma)\Delta\varsigma = \begin{cases} \sum_{\varsigma\in[c,d)} \mu(\varsigma)f(\varsigma), & \text{if } c < d, \\ 0, & \text{if } c = d, \\ -\sum_{\varsigma\in[c,d)} \mu(\varsigma)f(\varsigma), & \text{if } c > d. \end{cases}$$

3. $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, $h > 0$, *implies*

$$\int_c^d f(\varsigma)\Delta\varsigma = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh), & \text{if } c < d, \\ 0, & \text{if } c = d, \\ -\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh), & \text{if } c > d. \end{cases}$$

Theorem 2.3 ([58]) *If $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing continuous and $c,d \in \mathbb{T}$, then*

$$\int_c^d f(\varsigma)\Delta\varsigma \le \int_c^d f(\varsigma)\,d\varsigma. \tag{5}$$

Definition 2.4 ([58]) Let $\phi : [c,d]_{\mathbb{T}} \to \mathbb{R}$ be an integrable mapping, then delta fractional integral is

$${}^{\Delta}\mathrm{I}_{a^+}^{\sigma}\phi(\varsigma) = \int_a^{\varsigma} \frac{(\varsigma - s)^{\sigma-1}}{\Gamma(\sigma)}\phi(s)\Delta s. \tag{6}$$

Definition 2.5 ([58]) The fractional Caputo derivative of a mapping $f : \mathbb{T} \to \mathbb{R}$ on the time scale is

$${}^{c,\mathbb{T}}\mathrm{D}_{a^+}^{\sigma}f(\varsigma) = \int_a^{\varsigma} \frac{(\varsigma - s)^{n-\sigma-1}}{\Gamma(n-\sigma)} f^{\Delta^n}(\varsigma)\Delta\varsigma, \tag{7}$$

where $n = [\sigma] + 1$ and the delta nth derivative of f is denoted by f^{Δ^n}.

- When $\mathbb{T} = \bigcup_{i=0}^{\infty}[2i, 2i+1]$. Then we get

$$\begin{aligned} {}^{c,\mathbb{T}}\mathrm{D}_{a^+}^{\sigma}f(\varsigma) &= \int_a^{\varsigma} \frac{(\varsigma - s)^{n-\sigma-1}}{\Gamma(n-\sigma)} f^{\Delta^n}(\varsigma)\Delta\varsigma \\ &= \frac{1}{\Gamma(n-\sigma)}\left[\sum_{k=0}^{i-1}\int_{2k}^{2k+1} (\varsigma - s)^{n-\sigma-1} f^{\Delta^n}\Delta s + \int_{2i}^{2i+1} (\varsigma - s)^{n-\sigma-1} f^{\Delta^n}\Delta s\right] \end{aligned}$$

  for $\varsigma \in [2i, 2i+1]$, $i = 0, 1, \ldots$.

- When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have

$$
\begin{aligned}
{}^{c,\mathbb{T}}\mathrm{D}^{\sigma}_{a^+} f(\varsigma) &= \int_a^{\varsigma} \frac{(\varsigma - s)^{n-\sigma-1}}{\Gamma(n-\sigma)} f^{\Delta^n}(\varsigma)\Delta\varsigma \\
&= \frac{1}{\Gamma(n-\sigma)} \sum_{k=0}^{\frac{\varsigma}{h}-1} h(\varsigma - ih)^{n-\sigma-1} f^{\Delta^n}(h), \quad \varsigma \in \mathbb{T}.
\end{aligned}
$$

- When $\mathbb{T} = \{p^n : p > 1, n \in \mathbb{Z}\} \cup \mathbb{Z}$, then

$$
\begin{aligned}
{}^{c,\mathbb{T}}\mathrm{D}^{\sigma}_{a^+} f(\varsigma) &= \int_a^{\varsigma} \frac{(\varsigma - s)^{n-\sigma-1}}{\Gamma(n-\sigma)} f^{\Delta^n}(\varsigma)\Delta\varsigma \\
&= \frac{1}{\Gamma(n-\sigma)} \sum_{\varsigma \in \mathbb{T}} \mu(s)(\varsigma - s)^{n-\sigma-1} f^{\Delta^n}(s).
\end{aligned}
$$

Regard the Mittag-Leffler function as

$$
E_{\sigma,\beta}(\varsigma) = \sum_{k=0}^{\infty} \frac{\varsigma^k}{\Gamma(k\sigma + \beta)} \quad \text{for } \sigma, \beta > 0.
$$

For $\beta = 1$,

$$
E_{\sigma,1}\left(\lambda\varsigma^{\sigma}\right) = E_{\sigma}\left(\lambda\varsigma^{\sigma}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k \varsigma^{k\sigma}}{\Gamma(\sigma k + 1)}, \quad \lambda, \varsigma \in \mathbb{C}
$$

has the interesting property ${}^{c}\mathrm{D}^{\sigma}_{0+} E_{\sigma}(\lambda\varsigma^{\sigma}) = \lambda E_{\sigma}(\lambda\varsigma^{\sigma})$.

Remark 2.1 ([59]) The solution of system (1) is of the form

$$
\omega(\varsigma) = \begin{cases}
E_{\sigma}(A\varsigma^{\sigma})\omega_0 + \int_{\varsigma_0}^{\varsigma_f} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \mathcal{F}(\varsigma, \omega(\varsigma))\Delta s \\
\quad + \int_{\varsigma_0}^{\varsigma_f} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \\
\quad \times G(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1(s, u, \omega(u))\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2(s, u, \omega(u))\Delta u)\Delta s, \\
\quad \varsigma \in (\varsigma_0, \varsigma_1], \\
E_{\sigma}(A\varsigma^{\sigma})\omega_0 + \int_{\varsigma_0}^{\varsigma_f} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \mathcal{F}(\varsigma, \omega(\varsigma))\Delta s \\
\quad + \int_{\varsigma_0}^{\varsigma_f} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \\
\quad \times G(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1(s, u, \omega(u))\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2(s, u, \omega(u))\Delta u)\Delta s \\
\quad + \sum_{j=1}^{i} (\Xi_j(\omega(\varsigma_j^-)) + \Phi_j(\varsigma_j^-, \omega(\varsigma_j^-))), \quad \varsigma \in (\varsigma_i, \varsigma_{i+1}], i = 1, \dots, m,
\end{cases}
$$

where $E_{\sigma}(A\varsigma^{\sigma})$ is the matrix representation of the aforesaid Mittag-Leffler function given by

$$
E_{\sigma}\left(A\varsigma^{\sigma}\right) = \sum_{j=1}^{i} \frac{A^{\sigma}\varsigma^{\kappa\sigma}}{\Gamma(1 + \kappa\sigma)}.
$$

To achieve our results, we consider the following:

(**A**): The mappings $G, \mathcal{G} : \mathbb{T}^0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and there exist $L_{\hbar_i}$, $\mathcal{L}_{\hbar_i}$, $i = 1, 2, 3$, as the positive constants such that ($i = 1, 2, 3$)

$$\big|G(\varsigma, q_1, q_2, q_3) - G(\varsigma, p_1, p_2, p_3)\big| \le \sum_{i=1}^{3} L_{\hbar_i}|q_i - p_i| \quad \text{for all } \varsigma \in \mathrm{I}, q_i, p_i \in \mathbb{R}^n,$$

$$\big|\mathcal{G}(\varsigma, q_1, q_2, q_3) - \mathcal{G}(\varsigma, p_1, p_2, p_3)\big| \le \sum_{i=1}^{3} \mathcal{L}_{\mathcal{G}_i}|q_i - p_i| \quad \text{for all } \varsigma \in \mathrm{I}, q_i, p_i \in \mathbb{R}^n.$$

(**B**): The mappings $G, \mathcal{G} : \mathbb{T}^0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and there exist l_i, m_i, $i = 1, 2, 3$, positive constants such that

$$\big|G(\varsigma, u, v, w)\big| \le l_0 + l_1|u| + l_2|v| + l_3|w| \quad \text{for all } \varsigma \in \mathrm{I}, u, v, w \in \mathbb{R}^n,$$

$$\big|\mathcal{G}(\varsigma, u, v, w)\big| \le m_0 + m_1|u| + m_2|v| + n_3|w| \quad \text{for all } \varsigma \in \mathrm{I}, u, v, w \in \mathbb{R}^n.$$

(**W**): The linear operator $({}^{\sigma}\mathcal{W}_{\varsigma_0}^{T}) : L^2(I, \mathbb{R}) \to \mathbb{R}$, defined by

$${}^{\sigma}\mathcal{W}_{\varsigma_0}^{T}\zeta = \int_{\varsigma_0}^{\varsigma_f} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s, \tag{8}$$

possesses a bounded invertible operator $({}^{\sigma}\mathcal{W}_{\varsigma_0}^{T})^{-1}$, and these operators admit values in $L^2(I, \mathbb{R})\setminus \ker({}^{\sigma}\mathcal{W}_{\varsigma_0}^{T})$. Also, there exists a positive constant provided that $\|({}^{\sigma}\mathcal{W}_{\varsigma_0}^{T})^{-1}\| \le M_{\mathcal{W}}^{\sigma}$. Also, $\mathcal{H} : \mathbb{T} \to \mathbb{T}$ is a continuous operator, and there exists a positive constant $M_{\mathcal{H}}$ provided that $\|\mathcal{H}\| \le M_{\mathcal{H}}$.

Using Theorem 2.2, equation (8) can be calculated for different $\mathbb{T}$.

- When $\mathbb{T} = h\mathbb{Z}$, $h > 0$:

$${}^{\sigma}\mathcal{W}_{\varsigma_0}^{T}\zeta = \int_{\varsigma_0}^{\varsigma_f} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s = \sum_{k=0}^{\frac{\varsigma}{h}-1} h(T - sh)^{\sigma-1}\mathcal{H}\zeta(sh).$$

- When $\mathbb{T} = \bigcup_{i=0}^{\infty}[2i, 2i + 1]$. Let $\varsigma \in [4, 5]$, then we have

$$\begin{aligned}
{}^{\sigma}\mathcal{W}_{\varsigma_0}^{T}\zeta &= \int_{\varsigma_0}^{\varsigma_f} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s \\
&= \int_{0}^{1} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s \\
&\quad + \int_{2}^{3} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s \\
&\quad + \int_{4}^{T} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s.
\end{aligned}$$

- When $\mathbb{T} = \{q^m : q > 1, m \in \mathbb{Z}\} \cup \mathbb{Z}$, then

$${}^{\sigma}\mathcal{W}_{\varsigma_0}^{T}\zeta = \int_{\varsigma_0}^{\varsigma_f} (T - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\mathcal{H}\zeta(s)\Delta s = \sum_{\varsigma\in[0,T]} \mu(T - \varsigma)^{\sigma-1}\mathcal{H}\zeta(\varsigma).$$

Throughout the manuscript, we set

$$
\begin{aligned}
&Q_1 := a_3\big(a_2L_{\mathcal{F}} + a_2L_{G_1} + a_2L_{G_2}L_{\mathcal{F}_1}(s_f - s_0) + a_2L_{G_3}L_{\mathcal{F}_2}(s_f - s_0) + M_G\big)(\varsigma_f - \varsigma_0);\\
&Q_2 := a_3\big(a_2L_{\mathcal{F}} + a_2L_{G_1} + a_2L_{G_2}L_{\mathcal{F}_1}(s_f - s_0) + a_2L_{G_3}L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0);\\
&Q_1^* := a_3\big(a_2L_F + a_2L_{\mathcal{G}_1} + a_2L_{\mathcal{G}_2}L_{F_1}(s_f - s_0) + a_2L_{\mathcal{G}_3}L_{F_2}(s_f - s_0) + \widehat{M}_{\mathcal{G}}\big)(\varsigma_f - \varsigma_0);\\
&Q_2^* := a_3\big(a_2L_F + a_2L_{G_1} + a_2L_{\mathcal{G}_2}L_{F_1}(s_f - s_0) + a_2L_{\mathcal{G}_3}L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0);\\
&Q_3 := N_2 + Q_2; \qquad Q_3^* = N_4 + Q_2^*;\\
&N_1 := \sum_{j=1}^{i} L_\Xi\delta + \sum_{j=1}^{i} L_\Phi\delta + a_1; \qquad N_2 := \sum_{j=1}^{i} L_\Xi + \sum_{j=1}^{i} L_\Phi;\\
&N_3 := \frac{1}{\Gamma(\sigma)}a_3L_g(\varsigma_f - \varsigma_0) + a_1; \qquad N_4 := \frac{1}{\Gamma(\sigma)}a_3L_g(\varsigma_f - \varsigma_0);\\
&a_1 := \sup_{\varsigma\in\mathbb{T}}\big\|E_\sigma\left(A\varsigma^\sigma\right)\omega_0\big\|; \qquad a_2 := \sup_{\varsigma\in\mathbb{T}}\big\|E_{\sigma,\sigma}\left(A(\varsigma - s)^\sigma\right)\big\|; \qquad a_3 = \sup_{\varsigma\in\mathbb{T}}\big\|(\varsigma - s)^{\sigma-1}\big\|.
\end{aligned}
$$

3 Existence of solution

Existence criteria are investigated here.

Theorem 3.1 *The mixed impulsive system* (1) *admits a unique solution if assertion* (**A**) *holds and*

$$\max_{1\le i\le 3}\{Q_i\} < 1. \tag{9}$$

Proof Let $\Omega \subseteq \mathrm{PS}$ and $\Omega = \{(\mathbb{X},\mathbb{Y},\mathbb{Z}) \in \mathrm{PS} : \|(\mathbb{X},\mathbb{Y},\mathbb{Z})\|_C \le \delta_2\}$ also $\delta_2 = \max\{\delta,\delta_1\}$ and $\delta,\delta_1 \in (0,1)$ provided that

$$\delta > \max\{N_1,N_2,N_3\},$$

and the remaining constants are introduced in the sequel. Now, we define $\Lambda_\sigma : \Omega \to \Omega$ as

$$
\Lambda_\sigma\big(\omega(\varsigma)\big) = \begin{cases}
E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s\\
\quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\\
\quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s,\\
\quad \varsigma \in (\varsigma_0,\varsigma_1],\\
E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s\\
\quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\\
\quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s\\
\quad + \sum_{j=1}^{i}(\Xi_j(\omega(\varsigma_j^-)) + \Phi_j(\varsigma_j^-,\omega(\varsigma_j^-))), \quad \varsigma \in (\varsigma_i,\varsigma_{i+1}], i = 1,\dots,m.
\end{cases} \tag{10}
$$

Assume that

$$\big\|\mathcal{F}(s,\omega)\big\| \le \big\|\mathcal{F}(s,\omega) - \mathcal{F}(s,0)\big\| + \big\|\mathcal{F}(s,0)\big\| \le L_{\mathcal{F}}\|\omega\| + M_{\mathcal{F}}$$

and

$$\begin{aligned}\big\|G(\varsigma,\mathbb{X},\mathbb{Y},\mathbb{Z})\big\| &\le \big\|G(\varsigma,\mathbb{X},\mathbb{Y},\mathbb{Z})-G(\varsigma,0,0,0)\big\| + \big\|G(\varsigma,0,0,0)\big\| \\ &\le L_{\hbar_1}\|\mathbb{X}\| + L_{\hbar_2}\|\mathbb{Y}\| + L_{\hbar_3}\|\mathbb{Z}\| + M_G,\end{aligned}$$

where $M_{\mathcal{F}} = \sup_{\varsigma\in\mathbb{T}}\|\mathcal{F}(s,0)\|$, $M_G = \sup_{\varsigma\in\mathbb{T}}\|G(\varsigma,0,0,0)\|$, and $\widehat{M}_G = M_{\mathcal{F}} + M_G$. In addition, $\mathbb{X} = \omega(s)$, $\mathbb{Y} = \int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u$, and $\mathbb{Z} = \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u$.

Now, we prove that $\Lambda_\sigma : \Omega \to \Omega$ is a self-mapping.

For $\varsigma \in (\varsigma_i,\varsigma_{i+1}]$, $i = 1,\dots,m$, one has

$$\begin{aligned}\big\|\Lambda_\sigma\big(\omega(\varsigma)\big)\big\| &\le \sum_{j=1}^{i}\big\|\Xi_j\big(\omega(\varsigma_j^-)\big)\big\| + \sum_{j=1}^{i}\big\|\Phi_j\big(\varsigma_j^-,\omega(\varsigma_j^-)\big)\big\| + \big\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\big\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\bigg\|\bigg(\mathcal{F}\big(s,\omega(s)\big) \\ &\quad + G\bigg(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\big(s,u,\omega(u)\big)\Delta u\bigg)\bigg)\bigg\|\Delta s \\ &\le \sum_{j=1}^{i}L_\Xi\big\|\omega(\varsigma_j^-)\big\| + \sum_{j=1}^{i}L_\Phi\big\|\omega(\varsigma_j^-)\big\| + \big\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\big\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\bigg(\big\|\mathcal{F}\big(s,\omega(s)\big)\big\| \\ &\quad + \bigg\|G\bigg(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\big(s,u,\omega(u)\big)\Delta u\bigg)\bigg\| + \widehat{M}_G\bigg)\Delta s \\ &\le \sum_{j=1}^{i}L_\Xi\big\|\omega(\varsigma_j^-)\big\| + \sum_{j=1}^{i}L_\Phi\big\|\omega(\varsigma_j^-)\big\| + \big\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\big\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\Big(L_{\mathcal{F}}\big\|\omega(s)\big\| + L_{\hbar_1}\big\|\omega(s)\big\| \\ &\quad + L_{\hbar_2}\int_{s_0}^{s_f}\big\|\mathcal{F}_1\big(s,u,\omega(u)\big)\big\|\Delta u + L_{\hbar_3}\int_{s_0}^{s_f}\big\|\mathcal{F}_2\big(s,u,\omega(u)\big)\big\|\Delta u + \widehat{M}_G\Big)\Delta s \\ &\le \sum_{j=1}^{i}L_\Xi\big\|\omega(\varsigma_j^-)\big\| + \sum_{j=1}^{i}L_\Phi\big\|\omega(\varsigma_j^-)\big\| + \big\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\big\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\Big(L_{\mathcal{F}}\big\|\omega(s)\big\| + L_{G_1}\big\|\omega(s)\big\| \\ &\quad + L_{G_2}L_{\mathcal{F}_1}\int_{s_0}^{s_f}\big\|\omega(u)\big\|\Delta u + L_{G_3}L_{\mathcal{F}_2}\int_{s_0}^{s_f}\big\|\omega(u)\big\|\Delta u + \widehat{M}_G\Big)\Delta s \\ &= \sum_{j=1}^{i}L_\Xi\big\|\omega(\varsigma_j^-)\big\| + \sum_{j=1}^{i}L_\Phi\big\|\omega(\varsigma_j^-)\big\| + \big\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\big\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\big(L_{\mathcal{F}}\big\|\omega(s)\big\| + L_{G_1}\big\|\omega(s)\big\| \\ &\quad + L_{G_2}L_{\mathcal{F}_1}\big\|\omega(u)\big\|(s_f-s_0) + L_{G_3}L_{\mathcal{F}_2}\big\|\omega(u)\big\|(s_f-s_0) + \widehat{M}_G\big)\Delta s\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{i} L_{\Xi} \sup_{\varsigma\in\mathbb{T}} \left\| \omega\left(\varsigma_j^-\right) \right\| + \sum_{j=1}^{i} L_{\Phi} \sup_{\varsigma\in\mathbb{T}} \left\| \omega\left(\varsigma_j^-\right) \right\| + \sup_{\varsigma\in\mathbb{T}} \left\| E_{\sigma}\left(A\varsigma^{\sigma}\right)\omega_0 \right\| \\
&\quad + \int_{\varsigma_0}^{\varsigma_f} \sup_{\varsigma\in\mathbb{T}} \left\| (\varsigma - s)^{\sigma-1} \right\| \sup_{\varsigma\in\mathbb{T}} \left\| E_{\sigma,\sigma}\left(A(\varsigma - s)^{\sigma}\right) \right\| \Big(L_{\mathcal{F}} \sup_{\varsigma\in\mathbb{T}} \left\| \omega(s) \right\| \\
&\quad + L_{G_1} \sup_{\varsigma\in\mathbb{T}} \left\| \omega(s) \right\| + L_{G_2} L_{\mathcal{F}_1} \sup_{\varsigma\in\mathbb{T}} \left\| \omega(u) \right\| (s_f - s_0) \\
&\quad + L_{G_3} L_{\mathcal{F}_2} \sup_{\varsigma\in\mathbb{T}} \left\| \omega(u) \right\| (s_f - s_0) + \widehat{M}_G \Big) \Delta s \\
&\leq \sum_{j=1}^{i} L_{\Xi} \|\omega\|_{\infty} + \sum_{j=1}^{i} L_{\Phi} \|\omega\|_{\infty} + a_1 + \int_{\varsigma_0}^{\varsigma_f} a_3 a_2 \big(L_{\mathcal{F}} \|\omega\|_{\infty} + L_{G_1} \|\omega\|_{\infty} \\
&\quad + L_{G_2} L_{\mathcal{F}_1} \|\omega\|_{\infty} (s_f - s_0) + L_{G_3} L_{\mathcal{F}_2} \|\omega\|_{\infty} (s_f - s_0) + \widehat{M}_G \big) \Delta s \\
&\leq \sum_{j=1}^{i} L_{\Xi} \delta + \sum_{j=1}^{i} L_{\Phi} \delta + a_1 + a_3 \big(\delta a_2 L_{\mathcal{F}} + \delta a_2 L_G + \delta a_2 L_G L_{\mathcal{F}_1} (s_f - s_0) \\
&\quad + \delta a_2 L_G L_{\mathcal{F}_2} (s_f - s_0) + \widehat{M}_G \big) \times \int_{\varsigma_0}^{\varsigma_f} \Delta s \\
&\leq \sum_{j=1}^{i} L_{\Xi} \delta + \sum_{j=1}^{i} L_{\Phi} \delta + a_1 + \delta a_3 \big(a_2 L_{\mathcal{F}} + a_2 L_{G_1} + a_2 L_{G_2} L_{\mathcal{F}_1} (s_f - s_0) \\
&\quad + a_2 L_{G_3} L_{\mathcal{F}_2} (s_f - s_0) + \widehat{M}_G \big) (\varsigma - \varsigma_f).
\end{aligned}$$

So

$$\left\| \Lambda_{\sigma}\left(\omega(\varsigma)\right) \right\| \leq N_1 + \delta Q_1 \leq \delta + \delta Q_1 = \delta_1,$$

where $\delta_1 = \delta + \delta Q_1$. Hence

$$\left\| \Lambda_{\sigma}\left(\omega(\varsigma)\right) \right\| \leq \delta_2. \tag{11}$$

Therefore, from (11), $\Lambda(\Omega) \subseteq \Omega$. Also, for $\varsigma \in (\varsigma_i, \varsigma_{i+1}]$, $i = 1, \dots, m$, with $\omega_0 = \widehat{\omega}_0$, one has

$$\begin{aligned}
&\left\| \Lambda_{\sigma}\left(\omega(\varsigma)\right) - \Lambda_{\sigma}\left(\widehat{\omega}(\varsigma)\right) \right\| \\
&\quad \leq \sum_{j=1}^{i} \left\| \Xi_j\left(\omega\left(\varsigma_j^-\right)\right) - \Xi_j\left(\widehat{\omega}\left(\varsigma_j^-\right)\right) \right\| \\
&\qquad + \sum_{j=1}^{i} \left\| \Phi_j\left(\varsigma_j^-, \omega\left(\varsigma_j^-\right)\right) - \Phi_j\left(\varsigma_j^-, \widehat{\omega}\left(\varsigma_j^-\right)\right) \right\| \\
&\qquad + \int_{\varsigma_0}^{\varsigma_f} \left\| (\varsigma - s)^{\sigma-1} \right\| \left\| E_{\sigma,\sigma}\left(A(\varsigma - s)^{\sigma}\right) \right\| \bigg\| \Big(\mathcal{F}\big(s, \omega(s)\big) \\
&\qquad + G\bigg(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s, u, \omega(u)\big) \Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s, u, \omega(u)\big) \Delta u \bigg) \bigg) \\
&\qquad - \Big(\mathcal{F}\big(s, \widehat{\omega}(s)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ G\Big(s, \widehat{\omega}(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s,u,\widehat{\omega}(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s,u,\widehat{\omega}(u)\big)\Delta u\Big)\Big)\Big\|\Delta s \\
\leq{}& \sum_{j=1}^{i} L_{\Xi}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| + \sum_{j=1}^{i} L_{\Phi}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| \\
&+ \int_{\varsigma_0}^{\varsigma_f} \big\|(\varsigma - s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\big\|\Big(\big\|\mathcal{F}\big(s,\omega(s)\big) - \mathcal{F}\big(s,\widehat{\omega}(s)\big)\big\| \\
&+ \Big\|G\Big(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s,u,\omega(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s,u,\omega(u)\big)\Delta u\Big) \\
&- G\Big(s, \widehat{\omega}(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s,u,\widehat{\omega}(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s,u,\widehat{\omega}(u)\big)\Delta u\Big)\Big\|\Big)\Delta s \\
\leq{}& \sum_{j=1}^{i} L_{\Xi}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| + \sum_{j=1}^{i} L_{\Phi}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| \\
&+ \int_{\varsigma_0}^{\varsigma_f} \big\|(\varsigma - s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\big\|\Big(L_F\big\|\omega(s) - \widehat{\omega}(s)\big\| \\
&+ L_{\mathcal{G}_1}\big\|\omega(s) - \widehat{\omega}(s)\big\| + L_{\mathcal{G}_2}\int_{s_0}^{s_f}\big\|\mathcal{F}_1\big(s,u,\omega(u)\big) - \mathcal{F}_1\big(s,u,\widehat{\omega}(u)\big)\big\|\Delta u \\
&+ L_{G_3}\int_{s_0}^{s_f}\big\|\mathcal{F}_2\big(s,u,\omega(u)\big) - \mathcal{F}_2\big(s,u,\widehat{\omega}(u)\big)\big\|\Delta u\Big)\Delta s \\
\leq{}& \sum_{j=1}^{i} L_{\Xi}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| + \sum_{j=1}^{i} L_{\Phi}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| \\
&+ \int_{\varsigma_0}^{\varsigma_f} \big\|(\varsigma - s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\big\|\Big(L_{\mathcal{F}}\big\|\omega(s) - \widehat{\omega}(s)\big\| \\
&+ L_{G_1}\big\|\omega(s) - \widehat{\omega}(s)\big\| + L_{G_2}L_{\mathcal{F}_1}\int_{s_0}^{s_f}\big\|\omega(u) - \widehat{\omega}(u)\big\|\Delta u \\
&+ L_{G_3}L_{\mathcal{F}_2}\int_{s_0}^{s_f}\big\|\omega(u) - \widehat{\omega}(u)\big\|\Delta u\Big)\Delta s \\
\leq{}& \sum_{j=1}^{i} L_{\Xi}\sup_{\varsigma\in\mathbb{T}}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| + \sum_{j=1}^{i} L_{\Phi}\sup_{\varsigma\in\mathbb{T}}\big\|\omega\big(\varsigma_j^-\big) - \widehat{\omega}\big(\varsigma_j^-\big)\big\| \\
&+ \int_{\varsigma_0}^{\varsigma_f} \sup_{\varsigma\in\mathbb{T}}\big\|(\varsigma - s)^{\sigma-1}\big\|\sup_{\varsigma\in\mathbb{T}}\big\|E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\big\|\Big(L_{\mathcal{F}}\sup_{\varsigma\in\mathbb{T}}\big\|\omega(s) - \widehat{\omega}(s)\big\| \\
&+ L_{G_1}\sup_{\varsigma\in\mathbb{T}}\big\|\omega(s) - \widehat{\omega}(s)\big\| + L_{G_2}L_{\mathcal{F}_1}\sup_{\varsigma\in\mathbb{T}}\big\|\omega(u) - \widehat{\omega}(u)\big\|(s_f - s_0) \\
&+ L_{G_3}L_{\mathcal{F}_2}\sup_{\varsigma\in\mathbb{T}}\big\|\omega(u) - \widehat{\omega}(u)\big\|(s_f - s_0)\Big)\Delta s \\
={}& \sum_{j=1}^{i} L_{\Xi}\|\omega - \widehat{\omega}\|_{\infty} + \sum_{j=1}^{i} L_{\Phi}\|\omega - \widehat{\omega}\|_{\infty} \\
&+ a_3a_2\big(L_{\mathcal{F}}\|\omega - \widehat{\omega}\|_{\infty} + L_{G_1}\|\omega - \widehat{\omega}\|_{\infty} + L_{G_2}L_{\mathcal{F}_1}\|\omega - \widehat{\omega}\|_{\infty}(s_f - s_0) \\
&+ L_{G_3}L_{\mathcal{F}_2}\|\omega - \widehat{\omega}\|_{\infty}(s_f - s_0)\big)\int_{\varsigma_0}^{\varsigma_f}\Delta s
\end{aligned}
$$

$$\begin{aligned}
&\leq \Bigg[\sum_{j=1}^{i} L_{\Xi} + \sum_{j=1}^{i} L_{\Phi} \\
&\quad + a_3\big(a_2 L_{\mathcal{F}} + a_2 L_{G_1} + a_2 L_{G_2} L_{\mathcal{F}_1}(s_f - s_0) + a_2 L_{G_3} L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma - \varsigma_f)\Bigg] \\
&\quad \times \|\omega - \widehat{\omega}\|_\infty.
\end{aligned}$$

It implies

$$\big\|\Lambda_\sigma\big(\omega(\varsigma)\big) - \Lambda_\sigma\big(\widehat{\omega}(\varsigma)\big)\big\| \leq (N_2 + Q_2)\|\omega - \widehat{\omega}\|_\infty = Q_3\|\omega - \widehat{\omega}\|_\infty.$$

Hence

$$\big\|\Lambda_\sigma\big(\omega(\varsigma)\big) - \Lambda_\sigma\big(\widehat{\omega}(\varsigma)\big)\big\| \leq Q_3\|\omega - \widehat{\omega}\|_\infty \quad (Q_3 < 1). \tag{12}$$

Therefore, from inequality (12) and (9), the operator Λ_σ is strictly contractive. Consequently, the mixed impulsive system (1) admits a unique solution via the Banach principle. □

Now, regarding the mixed impulsive system (2), we have the following result.

Theorem 3.2 *The mixed impulsive system* (2) *involves a unique solution if assertion* (**A**) *holds and*

$$\max_{1\leq i\leq 3}\{Q_i^*\} < 1. \tag{13}$$

Proof Let $\Omega \subseteq \mathrm{PS}$ and $\Omega = \{(\mathbb{X},\mathbb{Y},\mathbb{Z}) \in \mathrm{PS} : \|(\mathbb{X},\mathbb{Y},\mathbb{Z})\|_C \leq \delta_2\}$, also $\delta_2 = \max\{\delta,\delta_1\}$ and $\delta,\delta_1 \in (0,1)$ provided that

$$\delta > \max\{N_1, N_2, N_3\}.$$

Now, we define $\Psi_\sigma : \Omega \to \Omega$ as

$$\Psi_\sigma\big(\omega(\varsigma)\big) = \begin{cases}
E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)F(\varsigma,\omega(\varsigma))\Delta s \\
\quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\
\quad \times \mathcal{G}(s,\omega(s),\int_{s_0}^{s_f} F_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f} F_2(s,u,\omega(u))\Delta u)\Delta s, \\
\quad \varsigma \in (\varsigma_i, s_i] \cap \mathbb{T}, i = 1,\dots,m, \\
E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)F(\varsigma,\omega(\varsigma))\Delta s \\
\quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\
\quad \times \mathcal{G}(s,\omega(s),\int_{s_0}^{s_f} F_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f} F_2(s,u,\omega(u))\Delta u)\Delta s \\
\quad + \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}(\varsigma - s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s, \quad \varsigma \in (s_i, \varsigma_{i+1}] \cap \mathbb{T}, i = 1,\dots,m.
\end{cases} \tag{14}$$

Also, note that

$$\big\|F(s,\omega)\big\| \leq \big\|F(s,\omega) - F(s,0)\big\| + \big\|F(s,0)\big\| \leq L_F\|\omega\| + M_F$$

and

$$\begin{aligned}\left\|\mathcal{G}(\varsigma,\mathbb{X},\mathbb{Y},\mathbb{Z})\right\| &\le \left\|\mathcal{G}(\varsigma,\mathbb{X},\mathbb{Y},\mathbb{Z})-\mathcal{G}(\varsigma,0,0,0)\right\| + \left\|\mathcal{G}(\varsigma,0,0,0)\right\| \\ &\le L_{\mathcal{G}_1}\|\mathbb{X}\| + L_{\mathcal{G}_2}\|\mathbb{Y}\| + L_{\mathcal{G}_3}\|\mathbb{Z}\| + M_{\mathcal{G}},\end{aligned}$$

where $M_F = \sup_{\varsigma\in\mathbb{T}}\|F(s,0)\|$, $M_{\mathcal{G}} = \sup_{\varsigma\in\mathbb{T}}\|\mathcal{G}(\varsigma,0,0,0)\|$, and $\widehat{M}_{\mathcal{G}} = M_F + M_{\mathcal{G}}$. In addition, $\mathbb{X} = \omega(s)$, $\mathbb{Y} = \int_{s_0}^{s_f} F_1(s,u,\omega(u))\Delta u$, and $\mathbb{Z} = \int_{s_0}^{s_f} F_2(s,u,\omega(u))\Delta u$.

Now, we prove that $\Psi_\sigma : \Omega \to \Omega$ is a self-mapping.

For $\varsigma \in (s_i, \varsigma_{i+1}] \cap \mathbb{T}$, $i = 1,\dots,m$, one has

$$\begin{aligned}\left\|\Psi_\sigma\big(\omega(\varsigma)\big)\right\| &\le \left\|\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}(\varsigma-s)^{\sigma-1}\hbar_i\big(s,\omega(s)\big)\Delta s\right\| + \left\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\right\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\left\|(\varsigma-s)^{\sigma-1}\right\|\left\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\right\|\bigg\|\bigg(F\big(s,\omega(s)\big) \\ &\quad + \mathcal{G}\bigg(s,\omega(s),\int_{s_0}^{s_f}F_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}F_2\big(s,u,\omega(u)\big)\Delta u\bigg)\bigg)\bigg\|\Delta s \\ &\le \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\left\|(\varsigma-s)^{\sigma-1}\right\|\left\|\hbar_i\big(s,\omega(s)\big)\right\|\Delta s + \left\|E_\sigma\big(A\varsigma^\sigma\big)\right\|\|\omega_0\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\left\|(\varsigma-s)^{\sigma-1}\right\|\left\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\right\|\bigg(\left\|F\big(s,\omega(s)\big)\right\| \\ &\quad + \bigg\|\mathcal{G}\bigg(s,\omega(s),\int_{s_0}^{s_f}F_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}F_2\big(s,u,\omega(u)\big)\Delta u\bigg)\bigg\| + \widehat{M}_{\mathcal{G}}\bigg)\Delta s \\ &\le \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\left\|(\varsigma-s)^{\sigma-1}\right\|L_g\left\|\big(\omega(s)\big)\right\|\Delta s + \left\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\right\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\left\|(\varsigma-s)^{\sigma-1}\right\|\left\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\right\|\bigg(L_F\left\|\omega(s)\right\| + L_{\mathcal{G}_1}\left\|\omega(s)\right\| \\ &\quad + L_{\mathcal{G}_2}\int_{s_0}^{s_f}\left\|F_1\big(s,u,\omega(u)\big)\right\|\Delta u + L_{\mathcal{G}_3}\int_{s_0}^{s_f}\left\|F_2\big(s,u,\omega(u)\big)\right\|\Delta u + \widehat{M}_{\mathcal{G}}\bigg)\Delta s \\ &\le \frac{1}{\Gamma(\sigma)}\left\|(\varsigma-s)^{\sigma-1}\right\|L_g\left\|\big(\omega(s)\big)\right\|(s_i-\varsigma_i) + \left\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\right\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\left\|(\varsigma-s)^{\sigma-1}\right\|\left\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\right\|\bigg(L_F\left\|\omega(s)\right\| + L_{\mathcal{G}_1}\left\|\omega(s)\right\| \\ &\quad + L_{\mathcal{G}_2}L_{F_1}\int_{s_0}^{s_f}\left\|\omega(u)\right\|\Delta u + L_{\mathcal{G}_3}L_{F_2}\int_{s_0}^{s_f}\left\|\omega(u)\right\|\Delta u + \widehat{M}_{\mathcal{G}}\bigg)\Delta s \\ &= \frac{1}{\Gamma(\sigma)}\left\|(\varsigma-s)^{\sigma-1}\right\|L_g\left\|\big(\omega(s)\big)\right\|(s_i-\varsigma_i) + \left\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\right\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\left\|(\varsigma-s)^{\sigma-1}\right\|\left\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\right\|\big(L_F\left\|\omega(s)\right\| + L_{\mathcal{G}_1}\left\|\omega(s)\right\| \\ &\quad + L_{\mathcal{G}_2}L_{F_1}\left\|\omega(u)\right\|(s_f-s_0) + L_{\mathcal{G}_3}L_{F_2}\left\|\omega(u)\right\|(s_f-s_0) + \widehat{M}_{\mathcal{G}}\big)\Delta s \\ &\le \frac{1}{\Gamma(\sigma)}\sup_{\varsigma\in\mathbb{T}}\left\|(\varsigma-s)^{\sigma-1}\right\|L_g\sup_{\varsigma\in\mathbb{T}}\left\|\big(\omega(s)\big)\right\|(s_i-\varsigma_i) + \sup_{\varsigma\in\mathbb{T}}\left\|E_\sigma\big(A\varsigma^\sigma\big)\omega_0\right\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\sup_{\varsigma\in\mathbb{T}}\left\|(\varsigma-s)^{\sigma-1}\right\|\sup_{\varsigma\in\mathbb{T}}\left\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\right\|\Big(L_F\sup_{\varsigma\in\mathbb{T}}\left\|\omega(s)\right\|\end{aligned}$$

$$
\begin{aligned}
&+ L_{\mathcal{G}_1} \sup_{\varsigma\in\mathbb{T}} \|\omega(s)\| + L_{\mathcal{G}_2} L_{F_1} \sup_{\varsigma\in\mathbb{T}} \|\omega(u)\| (s_f - s_0) \\
&+ L_{\mathcal{G}_3} L_{F_2} \sup_{\varsigma\in\mathbb{T}} \|\omega(u)\| (s_f - s_0) + \widehat{M}_{\mathcal{G}} \Big) \Delta s \\
\le{}& \frac{1}{\Gamma(\sigma)} a_3 L_g \|\omega\|_\infty (s_i - \varsigma_i) + a_1 + \int_{\varsigma_0}^{\varsigma_f} a_3 a_2 \big(L_F \|\omega\|_\infty + L_{\mathcal{G}_1} \|\omega\|_\infty \\
&+ L_{\mathcal{G}_2} L_{F_1} \|\omega\|_\infty (s_f - s_0) + L_{\mathcal{G}_3} L_{F_2} \|\omega\|_\infty (s_f - s_0) + \widehat{M}_{\mathcal{G}} \big) \Delta s \\
\le{}& \frac{1}{\Gamma(\sigma)} a_3 L_g \delta (s_i - \varsigma_i) + a_1 + a_3 \big(\delta a_2 L_F + \delta a_2 L_{\mathcal{G}} + \delta a_2 L_{\mathcal{G}} L_{F_1} (s_f - s_0) \\
&+ \delta a_2 L_{\mathcal{G}} L_{F_2} (s_f - s_0) + \widehat{M}_{\mathcal{G}} \big) \times \int_{\varsigma_0}^{\varsigma_f} \Delta s \\
\le{}& \frac{1}{\Gamma(\sigma)} a_3 L_g \delta (s_i - \varsigma_i) + a_1 + \delta a_3 \big(a_2 L_F + a_2 L_{\mathcal{G}_1} + a_2 L_{\mathcal{G}_2} L_{F_1} (s_f - s_0) \\
&+ a_2 L_{\mathcal{G}_3} L_{F_2} (s_f - s_0) + \widehat{M}_{\mathcal{G}} \big) (\varsigma - \varsigma_f).
\end{aligned}
$$

Thus

$$
\big\| \Lambda_\sigma \big(\omega(\varsigma)\big) \big\| \le N_3 + \delta Q_1^* \le \delta + \delta Q_1^* = \delta_1,
$$

where $\delta_1 = \delta + \delta Q_1^*$. Hence

$$
\big\| \Lambda_\sigma \big(\omega(\varsigma)\big) \big\| \le \delta_2. \tag{15}
$$

Therefore, from (15), $\Psi_\sigma(\Omega) \subseteq \Omega$. Also, for $\varsigma \in (s_i, \varsigma_{i+1}] \cap \mathbb{T}$, $i = 1, \dots, m$, with $\omega_0 = \widehat{\omega}_0$, we have

$$
\begin{aligned}
&\big\| \Psi_\sigma \big(\omega(\varsigma)\big) - \Psi_\sigma \big(\widehat{\omega}(\varsigma)\big) \big\| \\
&\quad\le \left\| \frac{1}{\Gamma(\sigma)} \int_{\varsigma_i}^{s_i} (\varsigma - s)^{\sigma-1} \big(\hbar_i\big(s, \omega(s)\big) - \hbar_i\big(s, \widehat{\omega}(s)\big) \big) \Delta s \right\| \\
&\quad\quad+ \int_{\varsigma_0}^{\varsigma_f} \big\| (\varsigma - s)^{\sigma-1} \big\| \big\| E_{\sigma,\sigma} \big(A(\varsigma - s)^\sigma \big) \big\| \bigg\| \bigg(F\big(s, \omega(s)\big) \\
&\quad\quad+ \mathcal{G}\bigg(s, \omega(s), \int_{s_0}^{s_f} F_1\big(s, u, \omega(u)\big) \Delta u, \int_{s_0}^{s_f} F_2\big(s, u, \omega(u)\big) \Delta u \bigg) \bigg) \\
&\quad\quad- \bigg(F\big(s, \widehat{\omega}(s)\big) \\
&\quad\quad+ \mathcal{G}\bigg(s, \widehat{\omega}(s), \int_{s_0}^{s_f} F_1\big(s, u, \widehat{\omega}(u)\big) \Delta u, \int_{s_0}^{s_f} F_2\big(s, u, \widehat{\omega}(u)\big) \Delta u \bigg) \bigg) \bigg\| \Delta s \\
&\quad\le \frac{1}{\Gamma(\sigma)} \int_{\varsigma_i}^{s_i} \big\| (\varsigma - s)^{\sigma-1} \big\| \big\| \hbar_i\big(s, \omega(s)\big) - \hbar_i\big(s, \widehat{\omega}(s)\big) \big\| \Delta s \\
&\quad\quad+ \int_{\varsigma_0}^{\varsigma_f} \big\| (\varsigma - s)^{\sigma-1} \big\| \big\| E_{\sigma,\sigma} \big(A(\varsigma - s)^\sigma \big) \big\| \Big(\big\| F\big(s, \omega(s)\big) - F\big(s, \widehat{\omega}(s)\big) \big\| \\
&\quad\quad+ \bigg\| \mathcal{G}\bigg(s, \omega(s), \int_{s_0}^{s_f} F_1\big(s, u, \omega(u)\big) \Delta u, \int_{s_0}^{s_f} F_2\big(s, u, \omega(u)\big) \Delta u \bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \mathcal{G}\biggl(s,\widehat{\omega}(s),\int_{s_0}^{s_f} F_1\bigl(s,u,\widehat{\omega}(u)\bigr)\Delta u,\int_{s_0}^{s_f} F_2\bigl(s,u,\widehat{\omega}(u)\bigr)\Delta u\biggr)\biggr\|\biggr)\Delta s \\
&\leq \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|L_g\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\|\Delta s \\
&\quad + \int_{\varsigma_0}^{\varsigma_f}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma-s)^{\sigma}\bigr)\bigr\|\biggl(L_F\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\| \\
&\quad + L_{\mathcal{G}_1}\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\| + L_{\mathcal{G}_2}\int_{s_0}^{s_f}\bigl\|F_1\bigl(s,u,\omega(u)\bigr)-F_1\bigl(s,u,\widehat{\omega}(u)\bigr)\bigr\|\Delta u \\
&\quad + L_{\mathcal{G}_3}\int_{s_0}^{s_f}\bigl\|F_2\bigl(s,u,\omega(u)\bigr)-F_2\bigl(s,u,\widehat{\omega}(u)\bigr)\bigr\|\Delta u\biggr)\Delta s \\
&\leq \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|L_g\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\|\Delta s \\
&\quad + \int_{\varsigma_0}^{\varsigma_f}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma-s)^{\sigma}\bigr)\bigr\|\biggl(L_F\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\| \\
&\quad + L_{\mathcal{G}_1}\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\| + L_{\mathcal{G}_2}L_{F_1}\int_{s_0}^{s_f}\bigl\|\omega(u)-\widehat{\omega}(u)\bigr\|\Delta u \\
&\quad + L_{\mathcal{G}_3}L_{F_2}\int_{s_0}^{s_f}\bigl\|\omega(u)-\widehat{\omega}(u)\bigr\|\Delta u\biggr)\Delta s \\
&\leq \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\sup_{\varsigma\in\mathbb{T}}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|L_g\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\|\Delta s \\
&\quad + \int_{\varsigma_0}^{\varsigma_f}\sup_{\varsigma\in\mathbb{T}}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\sup_{\varsigma\in\mathbb{T}}\bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma-s)^{\sigma}\bigr)\bigr\|\Bigl(L_F\sup_{\varsigma\in\mathbb{T}}\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\| \\
&\quad + L_{\mathcal{G}_1}\sup_{\varsigma\in\mathbb{T}}\bigl\|\omega(s)-\widehat{\omega}(s)\bigr\| + L_{\mathcal{G}_2}L_{F_1}\sup_{\varsigma\in\mathbb{T}}\bigl\|\omega(u)-\widehat{\omega}(u)\bigr\|(s_f-s_0) \\
&\quad + L_{\mathcal{G}_3}L_{F_2}\sup_{\varsigma\in\mathbb{T}}\bigl\|\omega(u)-\widehat{\omega}(u)\bigr\|(s_f-s_0)\Bigr)\Delta s \\
&= \frac{1}{\Gamma(\sigma)}a_3L_g\|\omega-\widehat{\omega}\|_\infty\int_{\varsigma_0}^{\varsigma_f}\Delta s \\
&\quad + a_3a_2\bigl(L_F\|\omega-\widehat{\omega}\|_\infty + L_{\mathcal{G}_1}\|\omega-\widehat{\omega}\|_\infty + L_{\mathcal{G}_2}L_{F_1}\|\omega-\widehat{\omega}\|_\infty(s_f-s_0) \\
&\quad + L_{\mathcal{G}_3}L_{F_2}\|\omega-\widehat{\omega}\|_\infty(s_f-s_0)\bigr)\int_{\varsigma_0}^{\varsigma_f}\Delta s \\
&\leq \biggl[\frac{1}{\Gamma(\sigma)}a_3L_g(\varsigma_f-\varsigma_0) + a_3\bigl(a_2L_F + a_2L_{G_1} + a_2L_{\mathcal{G}_2}L_{F_1}(s_f-s_0) \\
&\quad + a_2L_{\mathcal{G}_3}L_{\mathcal{F}_2}(s_f-s_0)\bigr)(\varsigma-\varsigma_f)\biggr]\|\omega-\widehat{\omega}\|_\infty.
\end{aligned}
$$

It implies

$$
\bigl\|\Psi_\sigma\bigl(\omega(\varsigma)\bigr)-\Psi_\sigma\bigl(\widehat{\omega}(\varsigma)\bigr)\bigr\| \leq (N_4+Q_2)\|\omega-\widehat{\omega}\|_\infty \leq Q_3^*\|\omega-\widehat{\omega}\|_\infty.
$$

Hence

$$
\bigl\|\Psi_\sigma\bigl(\omega(\varsigma)\bigr)-\Psi_\sigma\bigl(\widehat{\omega}(\varsigma)\bigr)\bigr\| \leq Q_3^*\|\omega-\widehat{\omega}\|_\infty. \tag{16}
$$

Therefore, from (16) and (13), the operator Ψ_σ is strictly contractive. Consequently, the second impulsive system (2) admits a unique solution via the Banach principle. □

Next, for both mixed impulsive systems (1) and (2), we investigate the existence of at least one solution via the weaker condition (**B**) and the Leray–Schauder alternative fixed point method.

Theorem 3.3 *The mixed impulsive system* (1) *has at least one solution provided assumption* (**B**) *holds and* $\mathcal{K} > 0$ *exists so that*

$$a_1 + Q_3\mathcal{K} < \mathcal{K}. \tag{17}$$

Proof Firstly, we prove that Λ_σ defined by (10) is a completely continuous operator. We see that the continuity of the mappings Ξ, Φ, $\mathcal{F}$, and G provides that Λ_σ is a continuous operator. Also, assume that $\Omega_1 \subseteq PS$ along with the fact that the operators Ξ, Φ, $\mathcal{F}$, and G are bounded. Then there exist L_1, L_2, M_1, and M_2 (positive constants) such that $\sum_{j=1}^{i} \Xi_j(\omega) \leq L_1$, $\sum_{j=1}^{i} \Phi_j(\omega) \leq L_2$, $\mathcal{F}(\varsigma, \omega(\varsigma)) \leq M_1$, and $G(\varsigma, \mathbb{X}, \mathbb{Y}, \mathbb{Z}) \leq M_2$, where $p = \omega(s)$,

$$q = \int_{s_0}^{s_f} \mathcal{F}_1\bigl(s, u, \omega(u)\bigr)\Delta u,$$

and

$$r = \int_{s_0}^{s_f} \mathcal{F}_2\bigl(s, u, \omega(u)\bigr)\Delta u.$$

Note that we take $\mathcal{L} = L_4 + L_5 + a_1$, $\mathcal{M} = M_1 + M_2$, $\|(\varsigma - s)^{\sigma-1}\| \leq \mathcal{L}_1$, and $\mathcal{L} + \mathcal{L}_1 a_2 \mathcal{M}(\varsigma_f - \varsigma_0) = \mathfrak{G}$.

Then, for any $\omega \in \Omega_1$ and $\varsigma \in (\varsigma_i, \varsigma_{i+1}]$, $i = 1, \ldots, m$, we have

$$\begin{aligned}
\bigl\|\Lambda_\sigma\bigl(\omega(\varsigma)\bigr)\bigr\| &\leq \sum_{j=1}^{i} \bigl\|\Xi_j\bigl(\omega\bigl(\varsigma_j^-\bigr)\bigr)\bigr\| + \sum_{j=1}^{i} \bigl\|\Phi_j\bigl(\varsigma_j^-, \omega\bigl(\varsigma_j^-\bigr)\bigr)\bigr\| + \bigl\|E_\sigma\bigl(A\varsigma^\sigma\bigr)\omega_0\bigr\| \\
&\quad + \int_{\varsigma_0}^{\varsigma_f} \bigl\|(\varsigma - s)^{\sigma-1}\bigr\| \bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma - s)^\sigma\bigr)\bigr\| \biggl\| \biggl(\mathcal{F}\bigl(s, \omega(s)\bigr) \\
&\quad + G\biggl(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1\bigl(s, u, \omega(u)\bigr)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\bigl(s, u, \omega(u)\bigr)\Delta u\biggr)\biggr)\biggr\| \Delta s \\
&\leq L_4 + L_5 + a_1 + \mathcal{L}_1 a_2 (M_1 + M_2) \int_{\varsigma_0}^{\varsigma_f} \Delta s \\
&= \mathcal{L} + \mathcal{L}_1 a_2 \mathcal{M}(\varsigma_f - \varsigma_0).
\end{aligned}$$

It implies

$$\bigl\|\Lambda_\sigma\bigl(\omega(\varsigma)\bigr)\bigr\| \leq \mathfrak{G}. \tag{18}$$

Thus, from (18), we conclude that Λ is uniformly bounded.

Now, we prove that Λ_σ is completely continuous. For this, we discuss the following possibilities.

Case 1: Assume that all points on $\mathbb{T}$ are isolated, i.e., time scales consist of discrete points. Using Theorem 2.2, Λ_σ becomes

$$\Lambda_\sigma\big(\omega(\varsigma)\big) = \begin{cases} E_\sigma(A\varsigma^\sigma)\omega_0 + \sum_{\varsigma\in\mathbb{T}}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \sum_{\varsigma\in\mathbb{T}}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u), \\ \varsigma\in(\varsigma_0,\varsigma_1], \\ E_\sigma(A\varsigma^\sigma)\omega_0 + \sum_{\varsigma\in\mathbb{T}}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \sum_{\varsigma\in\mathbb{T}}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \sum_{j=1}^{i}(\Xi_j(\omega(\varsigma_j^-)) + \Phi_j(\varsigma_j^-,\omega(\varsigma_j^-))), \quad \varsigma\in(\varsigma_i,\varsigma_{i+1}], i=1,\dots,m. \end{cases} \tag{19}$$

Clearly, on a discrete finite set, (19) is a collection of summation operators. Further, the continuity of Ξ_j, Φ_j, $\mathcal{F}$, and G implies that Λ_σ is completely continuous.

Case 2: Assume that all the points of $\mathbb{T}$ are dense, i.e., $\mathbb{T}$ is continuous. Now, let $\varsigma_{f_1},\varsigma_{f_2}\in(\varsigma_i,\varsigma_{i+1}]$, $i=1,\dots,m$, such that $\varsigma_{f_1}<\varsigma_{f_2}$, then

$$\begin{aligned} &\big\|\Lambda_\sigma\big(\omega(\varsigma_{f_2})\big)-\Lambda_\sigma\big(\omega(\varsigma_{f_1})\big)\big\| \\ &\quad\le \Bigg\|\sum_{j=1}^{i}\big[\Xi_j\big(\omega(\varsigma_{f_{2_j}^-})\big)-\Xi_j\big(\omega(\varsigma_{f_{1_j}^-})\big)\big] \\ &\qquad + \sum_{j=1}^{i}\big[\Phi_j\big(\varsigma_{f_{2_j}^-},\omega(\varsigma_{f_{2_j}^-})\big)-\Phi_j\big(\varsigma_{f_{1_j}^-},\omega(\varsigma_{f_{1_j}^-})\big)\big] \\ &\qquad + \Bigg[\Bigg(\int_{\varsigma_0}^{\varsigma_{f_2}}(\varsigma_{f_2}-s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma_{f_2}-s)^\sigma\big)\Bigg(\mathcal{F}\big(s,\omega(s)\big) \\ &\qquad + G\Bigg(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\big(s,u,\omega(u)\big)\Delta u\Bigg)\Bigg)\Delta s\Bigg) \\ &\qquad - \Bigg(\int_{\varsigma_0}^{\varsigma_{f_1}}(\varsigma_{f_1}-s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma_{f_1}-s)^\sigma\big)\Bigg(\mathcal{F}\big(s,\omega(s)\big) \\ &\qquad + G\Bigg(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\big(s,u,\omega(u)\big)\Delta u\Bigg)\Bigg)\Bigg)\Bigg]\Delta s\Bigg\| \\ &\quad\le \sum_{j=1}^{i}\big\|\Xi_j\big(\omega(\varsigma_{f_{2_j}^-})\big)-\Xi_j\big(\omega(\varsigma_{f_{1_j}^-})\big)\big\| \\ &\qquad + \sum_{j=1}^{i}\big\|\Phi_j\big(\varsigma_{f_{2_j}^-},\omega(\varsigma_{f_{2_j}^-})\big)-\Phi_j\big(\varsigma_{f_{1_j}^-},\omega(\varsigma_{f_{1_j}^-})\big)\big\| \\ &\qquad + \Bigg\|\int_{\varsigma_0}^{\varsigma_{f_2}}\big[(\varsigma_{f_1}-s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma_{f_2}-s)^\sigma\big)-(\varsigma_{f_1}-s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma_{f_1}-s)^\sigma\big)\big] \\ &\qquad \times\mathcal{F}\big(s,\omega(s)\big)\Delta s\Bigg\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_{\varsigma_0}^{\varsigma_{f_2}}\left[(\varsigma_{f_1}-s)^{\sigma-1}E_{\sigma,\sigma}\left(A(\varsigma_{f_2}-s)^{\sigma}\right)-(\varsigma_{f_1}-s)^{\sigma-1}E_{\sigma,\sigma}\left(A(\varsigma_{f_1}-s)^{\sigma}\right)\right]\right.\\
&\left.\times G\left(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\left(s,u,\omega(u)\right)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\left(s,u,\omega(u)\right)\Delta u\right)\Delta s\right\|\\
&+\left\|\int_{\varsigma_{f_1}}^{\varsigma_{f_2}}\left[(\varsigma_{f_2}-s)^{\sigma-1}E_{\sigma,\sigma}\left(A(\varsigma_{f_2}-s)^{\sigma}\right)\right]\mathcal{F}\left(s,\omega(s)\right)\Delta s\right\|\\
&+\left\|\int_{\varsigma_{f_1}}^{\varsigma_{f_2}}\left[(\varsigma_{f_2}-s)^{\sigma-1}E_{\sigma,\sigma}\left(A(\varsigma_{f_2}-s)^{\sigma}\right)\right]\right.\\
&\left.\times G\left(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\left(s,u,\omega(u)\right)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\left(s,u,\omega(u)\right)\Delta u\right)\Delta s\right\|.
\end{aligned}
$$

Clearly, we observe from the above that it approaches 0 as $\varsigma_{f_2}\to\varsigma_{f_1}$. Hence, the operator Λ_σ is equicontinuous. Finally, using the Arzela–Ascoli theorem, we conclude that Λ_σ is completely continuous.

Case 3: Assume that $\mathbb{T}$ involves isolated points along with dense ones, i.e., continuous and discrete. Now, utilizing Theorem 2.2 for the isolated points, we can write Λ_σ as the summation operator which is completely continuous (discussed in case 1). For the dense points, one can prove that Λ_σ is a completely continuous operator (discussed in case 2). Consequently, Λ_σ can be written as a sum of two operators for isolated and dense points. As a result, we know that the sum of two operators which are completely continuous is also completely continuous. Thus, the operator Λ_σ is a completely continuous operator. Hence, by summarizing the above three possibilities, we arrive at the conclusion that Λ_σ is a completely continuous operator.

Finally, let $\beta\in[0,1]$, and there exists ω provided that $\omega(\varsigma)=\beta(\Lambda_\sigma(\omega)(\varsigma))$. Then, for $\varsigma\in(\varsigma_i,\varsigma_{i+1}]$, $i=1,\dots,m$, one obtains

$$
\begin{aligned}
\left\|\omega(\varsigma)\right\| &= \left\|\beta\left(\Lambda_\sigma\left(\omega(\varsigma_f)\right)\right)\right\|\\
&\le \left\|\beta\left[\sum_{j=1}^{i}\Xi_j\left(\omega(\varsigma_{f_j^-})\right)+\sum_{j=1}^{i}\Phi_j\left(\varsigma_{f_{2_j}^-},\omega(\varsigma_{f_{2_j}^-})\right)\right.\right.\\
&\quad+\int_{\varsigma_0}^{\varsigma_{f_2}}(\varsigma_{f_2}-s)^{\sigma-1}E_{\sigma,\sigma}\left(A(\varsigma_{f_2}-s)^{\sigma}\right)\left(\mathcal{F}\left(s,\omega(s)\right)\right.\\
&\quad\left.\left.\left.+G\left(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\left(s,u,\omega(u)\right)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\left(s,u,\omega(u)\right)\Delta u\right)\right)\Delta s\right]\right\|\\
&\le \sum_{j=1}^{i}L_\Xi\|\omega\|_\infty+\sum_{j=1}^{i}L_\Phi\|\omega\|_\infty+a_1+\left\|(\varsigma-s)^{\sigma-1}\right\|a_2\\
&\quad\times\left(L_{\mathcal{F}}\|\omega\|_\infty+L_{G_1}\|\omega\|_\infty+L_{G_2}L_{\mathcal{F}_1}\|\omega\|_\infty(s_f-s_0)\right.\\
&\quad\left.+L_{G_3}L_{\mathcal{F}_2}\|\omega\|_\infty(s_f-s_0)+\widehat{M}_G\right)(\varsigma_f-\varsigma_0)\\
&\le a_1+\left[\sum_{j=1}^{i}L_\Xi+\sum_{j=1}^{i}L_\Phi+\left\|(\varsigma-s)^{\sigma-1}\right\|a_2\right.\\
&\quad\left.\times\left(L_{\mathcal{F}}+L_{G_1}+L_{G_2}L_{\mathcal{F}_1}(s_f-s_0)+L_{G_3}L_{\mathcal{F}_2}(s_f-s_0)\right)(\varsigma_f-\varsigma_0)\right]\|\omega\|_\infty
\end{aligned}
$$

$$\leq a_1 + [N_2 + Q_2]\|\omega\|_\infty$$
$$\leq a_1 + Q_3\|\omega\|_\infty.$$

Hence

$$\frac{\|\omega\|_\infty}{a_1 + Q_3\|\omega\|_\infty} \leq 1.$$

Now, from (17), we get $\mathcal{K} > 0$ such that $\|\omega\|_\infty \neq \mathcal{K}$. Let us assume that

$$\Re = \{\omega \in \mathbb{T}, \|\omega\|_\infty < \mathcal{K}\}.$$

Then the operator $\Lambda_\sigma : \Re \to \mathbb{T}$ is continuous as well as completely continuous. Thus, from the choice of $\Re$, there is no $\omega \in \chi(\Re)$ provided that $\omega = \beta(\Lambda_\sigma((\omega)(\varsigma)))$, $\beta \in [0,1]$.

Therefore in the light of fixed point criterion due to nonlinear alternative of Leray–Schauder, Λ_σ admits a fixed point which is the solution of the mixed impulsive system (1). □

We have a similar conclusion for the mixed impulsive system (2).

Theorem 3.4 *The mixed impulsive system* (2) *admits at least one solution if assumption* (**B**) *is satisfied and* $\mathcal{K}^* > 0$ *exists such that*

$$a_1 + Q_3^*\mathcal{K}^* < \mathcal{K}^*. \tag{20}$$

Proof It is similar to the previous argument for Ψ_σ in Theorem 3.3. □

4 Stability analysis

Now, to start this section, we first consider the following inequalities:

$$\begin{cases} \|{}^{c,\mathbb{T}}\mathrm{D}^\sigma\omega(\varsigma) - A(\varsigma)\omega(\varsigma) - \mathcal{F}(\varsigma,\omega(\varsigma)) \\ \quad - G(\varsigma,\omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_1(\varsigma,s,\omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f}\mathcal{F}_2(\varsigma,s,\omega(s))\Delta s)\| \leq \epsilon; \quad \varsigma \in \mathbb{T}', \\ \|\omega(\varsigma_k^+) - \omega(\varsigma_k^-) - \Xi_k(\omega(\varsigma_k^-)) - \Phi_k(\varsigma_k^-,\omega(\varsigma_k^-))\| \leq \epsilon, \quad k = 1,\dots,m, \end{cases} \tag{21}$$

and

$$\begin{cases} \|{}^{c,\mathbb{T}}\mathrm{D}^\sigma\omega(\varsigma) - A(\varsigma)\omega(\varsigma) - F(\varsigma,\omega(\varsigma)) \\ \quad - \mathcal{G}(\varsigma,\omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f}F_1(\varsigma,s,\omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f}F_2(\varsigma,s,\omega(s))\Delta s)\| \leq \epsilon, \\ \quad \varsigma \in (s_i,\varsigma_{i+1}] \cap \mathbb{T}, i = 1,\dots,m, \\ \|\omega(\varsigma) - \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma - s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s\| \leq \epsilon, \quad \varsigma \in (\varsigma_i,s_i] \cap \mathbb{T}, i = 1,\dots,m, \end{cases} \tag{22}$$

for each $\epsilon > 0$.

Definition 4.1 The mixed impulsive system (1) is said to be UH stable on $\mathbb{T}$ if, for any $\omega \in PC^1(\mathbb{T},\mathbb{R}^n)$ fulfilling (21), there exists $\widehat{\omega} \in PC^1(\mathbb{T},\mathbb{R}^n)$ as a solution of (1) such that $\|\widehat{\omega}(s) - \omega(s)\| \leq C\epsilon$ for $C > 0$, $s \in \mathbb{T}$.

Definition 4.2 The mixed impulsive system (2) is termed as UH stable if, for any $\epsilon > 0$ and $\omega \in PC^1(D, \mathbb{R}^n)$ that fulfills (22), there exists, $\widehat{\omega} \in PC^1(D, \mathbb{R}^n)$ as a solution of (2) provided $\|\widehat{\omega}(\varsigma) - \omega(\varsigma)\| \le \varrho\epsilon$ for all $\varsigma \in D$. Here, $\varrho > 0$, and its value depends upon ϵ.

Remark 4.1 The solution $\omega \in PC^1(\mathbb{T}, \mathbb{R}^n)$ satisfies (21) iff $\exists f \in PC(\mathbb{T}, \mathbb{R}^n)$ together with the sequence f_k provided $\|f_k\| \le \epsilon$ so that

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^\sigma \omega(\varsigma) = A(\varsigma)\omega(\varsigma) + F(\varsigma, \omega(\varsigma)) \\ \qquad\qquad + G(\varsigma, \omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f} F_1(\varsigma, s, \omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f} F_2(\varsigma, s, \omega(s))\Delta s) + f(\varsigma), \\ \omega(\varsigma_0) = \omega_0, \quad \varsigma \in \mathbb{T}', \\ \omega(\varsigma_k^+) - \omega(\varsigma_k^-) = \Xi_k(\omega(\varsigma_k^-)) + \Phi_k(\varsigma_k^-, \omega(\varsigma_k^-)) + f_k. \end{cases}$$

Lemma 4.3 *Each function $\omega \in PC^1(\mathbb{T}, \mathbb{R}^n)$ that fulfills (21) also satisfies the following inequality*:

$$\begin{cases} \|\omega(\varsigma) - E_\sigma(A\varsigma^\sigma)\omega_0 - \sum_{k=1}^m (\Xi_k(\omega(\varsigma_k^-)) + \Phi_k(\varsigma_k^-, \omega(\varsigma_k^-))) \\ \quad - \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) F(s, \omega(s))\Delta s \\ \quad - \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\ \quad \times G(s, \omega(s), \int_{s_0}^{s_f} F_1(s, u, \omega(u))\Delta u, \int_{s_0}^{s_f} F_2(s, u, \omega(u))\Delta u)\Delta s\| \le \delta\epsilon \end{cases}$$

for $\varsigma \in (\varsigma_k, \varsigma_{k+1}] \subset \mathbb{T}$, where $\|E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\| \le a_2$ and $\delta = (m + a_3 a_2(\varsigma_f - \varsigma_0))$.

Proof If $\omega \in PC^1(\mathbb{T}, \mathbb{R}^n)$ satisfies (21), then via Remark 4.1

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^\sigma \omega(\varsigma) = A(\varsigma)\omega(\varsigma) + F(\varsigma, \omega(\varsigma)) \\ \qquad\qquad + G(\varsigma, \omega(\varsigma), \int_{\varsigma_0}^{\varsigma_f} F_1(\varsigma, s, \omega(s))\Delta s, \int_{\varsigma_0}^{\varsigma_f} F_2(\varsigma, s, \omega(s))\Delta s) + f(\varsigma), \\ \omega(\varsigma_0) = \omega_0, \quad \varsigma \in \mathbb{T}', \\ \omega(\varsigma_k^+) - \omega(\varsigma_k^-) = \Xi_k(\omega(\varsigma_k^-)) + \Phi_k(\varsigma_k^-, \omega(\varsigma_k^-)) + f_k, \quad k = 1, \ldots, m, \end{cases}$$

implies

$$\begin{aligned} \omega(\varsigma) = {} & E_\sigma\big(A\varsigma^\sigma\big)\omega_0 + \sum_{j=1}^m \big(\Xi\big(\omega\big(\varsigma_j^-\big)\big) + \Phi\big(\varsigma_j^-, \omega\big(\varsigma_j^-\big)\big)\big) + \sum_{i=1}^m f_i \\ & + \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^\sigma\big) F\big(s, \omega(s)\big)\Delta s \\ & + \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^\sigma\big) \\ & \times G\bigg(s, \omega(s), \int_{s_0}^{s_f} F_1\big(s, u, \omega(u)\big)\Delta u, \int_{s_0}^{s_f} F_2\big(s, u, \omega(u)\big)\Delta u\bigg)\Delta s \\ & + \int_{\varsigma_0}^{\varsigma} (\varsigma - s)^{\sigma-1} E_{\sigma,\sigma}\big(A(\varsigma - s)^\sigma\big) f(s)\Delta s. \end{aligned}$$

So,

$$\begin{aligned}
&\Bigg\|\omega(\varsigma)-E_\sigma\big(A\varsigma^\sigma\big)\omega_0-\sum_{j=1}^{m}\big(\Xi\big(\omega\big(\varsigma_j^-\big)\big)+\Phi\big(\varsigma_j^-,\omega\big(\varsigma_j^-\big)\big)\big)\\
&\quad-\int_{\varsigma_0}^{\varsigma}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)F\big(s,\omega(s)\big)\Delta s\\
&\quad-\int_{\varsigma_0}^{\varsigma}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\\
&\quad\times G\bigg(s,\omega(s),\int_{s_0}^{s_f}F_1\big(s,u,\omega(u)\big)\Delta u,\int_{\varsigma_0}^{\varsigma_f}F_2\big(s,u,\omega(u)\big)\Delta u\bigg)\Delta\omega\Bigg\|\\
&\leq\int_{\varsigma_0}^{\varsigma}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\big\|f(s)\big\|\Delta s+\sum_{i=1}^{m}\|f_i\|\\
&\leq\delta\epsilon,
\end{aligned}$$

and the argument is finished. □

Remark 4.2 The map $\omega\in PC^1(D,\mathbb{R}^n)$ fulfills inequality (22) iff there are $f\in PC^1(D,\mathbb{R}^n)$ as a map and bounded sequences $\{f_i:i=1,\dots,m\}\subset\mathbb{R}^n$ (depending upon ω) provided that $\|f(\varsigma)\|\leq\epsilon$ for each $\varsigma\in D$ and $\|f_i\|\leq\epsilon$ $\forall i=1,\dots,m$ such that

$$\begin{cases}
{}^{c,\mathbb{T}}\mathrm{D}^\sigma\omega(\varsigma)=A(\varsigma)\omega(\varsigma)+F(\varsigma,\omega(\varsigma))\\
\qquad\qquad+\mathcal{G}(\varsigma,\omega(\varsigma),\int_{\varsigma_0}^{\varsigma_f}F_1(\varsigma,s,\omega(s))\Delta s,\int_{\varsigma_0}^{\varsigma_f}F_2(\varsigma,s,\omega(s))\Delta s)+f(\varsigma),\\
\omega(\varsigma_0)=\omega_0,\quad \varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T},i=1,\dots,m,\\
\omega(\varsigma)=\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s+f_i,\quad i=1,\dots,m.
\end{cases}$$

Lemma 4.4 *Each map $\omega\in PC^1(D,\mathbb{R}^n)$ that fulfills (22) also satisfies the inequalities given below*:

$$\begin{cases}
\|\omega(\varsigma)-E_\sigma(A\varsigma^\sigma)\omega_0-\int_{s_i}^{\varsigma}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)F(s,\omega(s))\Delta s\\
\quad-\int_{s_i}^{\varsigma}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}F_2(s,u,\omega(u))\Delta u)\Delta s\\
\quad-\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s\|\leq(a_2\varsigma_f-a_2s_i+m)\epsilon,\\
\quad\varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T},i=1,\dots,m,
\end{cases}$$

and

$$\begin{cases}
\|\omega(\varsigma)-\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s\|\leq m\epsilon\quad(\textit{respectively } m\kappa),\\
\quad\varsigma\in(\varsigma_i,s_i]\cap\mathbb{T},i=1,\dots,m,
\end{cases}$$

in which $\|E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\|\leq a_2$.

Proof If $\omega \in PC^1(D,\mathbb{R}^n)$ satisfies (22), in this case, by virtue of Remark 4.2,

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\omega(\varsigma) = A(\varsigma)\omega(\varsigma) + F(\varsigma,\omega(\varsigma)) \\ \qquad + \mathcal{G}(\varsigma,\omega(\varsigma),\int_{\varsigma_0}^{\varsigma_f} F_1(\varsigma,s,\omega(s))\Delta s,\int_{\varsigma_0}^{\varsigma_f} F_2(\varsigma,s,\omega(s))\Delta s) \\ \qquad + f(\varsigma),\omega(\varsigma_0) = \omega_0, \quad \varsigma \in (s_i,\varsigma_{i+1}] \cap \mathbb{T}, i = 1,\ldots,m, \\ \omega(\varsigma) = \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma - s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s + f_i, \quad s,\varsigma \in (\varsigma_i,s_i] \cap \mathbb{T}, i = 1,\ldots,m. \end{cases} \tag{23}$$

Clearly, equation (23) implies that

$$\omega(\varsigma) = \begin{cases} E_{\sigma}(A\varsigma^{\sigma})\omega_0 + \int_{\varsigma_i}^{\varsigma} E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma})F(s,\omega(s))\Delta s \\ \quad + \int_{s_i}^{\varsigma} E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \\ \quad \times \mathcal{G}(s,\omega(s),\int_{s_0}^{s_f} F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f} F_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma - s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s, \quad \varsigma \in (s_i,\varsigma_{i+1}] \cap \mathbb{T}, i = 1,\ldots,m, \\ \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma - s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s + f_i, \quad \varsigma \in (\varsigma_i,s_i] \cap \mathbb{T}, i = 1,\ldots,m. \end{cases}$$

For $\varsigma \in (s_i,\varsigma_{i+1}] \cap \mathbb{T}$, $i = 1,\ldots,m$, one has

$$\begin{aligned} &\bigg\| \omega(\varsigma) - E_{\sigma}\big(A\varsigma^{\sigma}\big)\omega_0 - \int_{s_i}^{\varsigma}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)F\big(s,\omega(s)\big) \\ &\quad - \int_{s_i}^{\varsigma}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big) \\ &\quad \times \mathcal{G}\bigg(s,\omega(s),\int_{s_0}^{s_f} F_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f} F_2(s,u,\omega)\Delta u\bigg)\Delta s \\ &\quad - \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma - s)^{\sigma-1}\hbar_i\big(s,\omega(s)\big)\Delta s \bigg\| \\ &\le \int_{s_i}^{\varsigma}\big\|(\varsigma - s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma - s)^{\sigma}\big)\big\|\big\|f(s)\big\|\Delta s + \sum_{i=1}^{m}\|f_i\| \\ &\le \big(a_3a_2(\varsigma - s_i) + m\big)\epsilon. \end{aligned}$$

Using a similar method, we get

$$\bigg\| \omega(\varsigma) - \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{\varsigma}(\varsigma - s)^{\sigma-1}\hbar_i\big(s,\omega(s)\big)\Delta s \bigg\| \le m\epsilon, \quad \varsigma \in (\varsigma_i,s_i] \cap \mathbb{T}, i = 1,\ldots,m,$$

and this ends the argument. □

Now, we provide a sufficient condition for the UH stability of mixed impulsive systems (1) and (2).

Theorem 4.5 *The mixed impulsive system* (1) *is UH stable provided assumption* (**A**) *and inequality* (9) *are satisfied.*

Proof Let ω be the solution of the mixed impulsive system (1) and $\widetilde{\omega}$ be the solution of inequality (21). Therefore, from Theorem 3.1, we have

$$\omega(\varsigma)=\begin{cases} E_\sigma(A\varsigma^\sigma)\omega_0+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad +\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s, \\ \quad \varsigma\in(\varsigma_0,\varsigma_1], \\ E_\sigma(A\varsigma^\sigma)\omega_0+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad +\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad +\sum_{j=1}^{i}(\Xi_j(\omega(\varsigma_j^-))+\Phi_j(\varsigma_j^-,\omega(\varsigma_j^-))), \quad \varsigma\in(\varsigma_i,\varsigma_{i+1}], i=1,\dots,m, \end{cases}$$

where $E_\sigma(A\varsigma^\sigma)$ stands for the matrix representation of the Mittag-Leffler function. Using a similar approach as that in Theorem 3.1, we get

$$\begin{aligned} \|\widetilde{\omega}(\varsigma)-\omega(\varsigma)\| &\le \sum_{j=1}^{i}\big\|\Xi_j\big(\widetilde{\omega}\big(\varsigma_j^-\big)\big)-\Xi_j\big(\omega\big(\varsigma_j^-\big)\big)\big\| \\ &\quad +\sum_{j=1}^{i}\big\|\Phi_j\big(\varsigma_j^-,\widetilde{\omega}\big(\varsigma_j^-\big)\big)-\Phi_j\big(\varsigma_j^-,\omega\big(\varsigma_j^-\big)\big)\big\| \\ &\quad +\int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\big\|\bigg\|\bigg(\mathcal{F}\big(s,\widetilde{\omega}(s)\big) \\ &\quad +G\bigg(s,\widetilde{\omega}(s),\int_{s_0}^{s_f}\mathcal{F}_1\big(s,u,\widetilde{\omega}(u)\big)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\big(s,u,\widetilde{\omega}(u)\big)\Delta u\bigg)\bigg) \\ &\quad -\bigg(\mathcal{F}\big(s,\omega(s)\big) \\ &\quad +G\bigg(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\big(s,u,\omega(u)\big)\Delta u\bigg)\bigg)\bigg\|\Delta s \\ &\le a_3a_2(\varsigma_f-\varsigma_i)\epsilon+Q_3\|\widetilde{\omega}-\omega\|_\infty, \end{aligned}$$

which implies that

$$\|\widetilde{\omega}-\omega\|_\infty\le a_3a_2(\varsigma_f-\varsigma_i)\epsilon+Q_3\|\widetilde{\omega}-\omega\|_\infty.$$

Hence

$$\begin{aligned} \|\widetilde{\omega}-\omega\|_\infty &\le \big(a_3a_2(\varsigma_f-\varsigma_i)\big)\frac{\varepsilon}{1-Q_3} \\ &\le \mathcal{H}_{(a_3,a_2,L_{\mathcal{F}},L_{G_1},L_{G_2}L_{\mathcal{F}_1},L_{G_3},L_{\mathcal{F}_2},L_\Xi,L_\Phi)}\epsilon, \end{aligned}$$

where $\mathcal{H}_{(a_3,a_2,L_{\mathcal{F}},L_{G_1},L_{G_2}L_{\mathcal{F}_1},L_{G_3},L_{\mathcal{F}_2},L_{\Xi},L_{\Phi})} = \frac{a_3a_2(\varsigma_f-\varsigma_i)}{1-Q_3}$. Hence, the mixed impulsive system (1) is UH stable. Furthermore, if

$$\overline{\mathcal{H}}_{(a_3,a_2,L_{\mathcal{F}},L_{G_1},L_{G_2}L_{\mathcal{F}_1},L_{G_3},L_{\mathcal{F}_2},L_{\Xi},L_{\Phi})}(\epsilon) = \overline{\mathcal{H}}_{(a_3,a_2,L_{\mathcal{F}},L_{G_1},L_{G_2}L_{\mathcal{F}_1},L_{G_3},L_{\mathcal{F}_2},L_{\Xi},L_{\Phi})}(0) = 0,$$

then our impulsive system (1) becomes generalized UH stable. □

Theorem 4.6 *The mixed impulsive system* (2) *is UH stable provided that assumption* (**A**) *and inequality* (13) *are satisfied.*

Proof Let ω be the solution of the mixed impulsive system (2) and $\widetilde{\omega}$ be the solution of inequality (22). Therefore, from Theorem 3.2, we have

$$\omega(\varsigma) = \begin{cases} E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)F(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times \mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}F_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad \varsigma\in(\varsigma_i,s_i]\cap\mathbb{T}, i=1,\dots,m, \\ E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)F(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times \mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}F_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s, \quad \varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T}, i=1,\dots,m, \end{cases}$$

where $E_\sigma(A\varsigma^\sigma)$ stands for the matrix representation of the Mittag-Leffler function. Using the similar approach as in Theorem 3.2, we get

$$\begin{aligned} \|\widetilde{\omega}(\varsigma)-\omega(\varsigma)\| \le{}& \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\|(\varsigma-s)^{\sigma-1}\|\|\big(\hbar_i\big(s,\widetilde{\omega}(s)\big)-\hbar_i\big(s,\widetilde{\omega}(s)\big)\big)\|\Delta s \\ &+ \int_{\varsigma_0}^{\varsigma_f}\|(\varsigma-s)^{\sigma-1}\|\|E_{\sigma,\sigma}\big(A(\varsigma-s)^\sigma\big)\|\bigg\|\Big(F\big(s,\widetilde{\omega}(s)\big) \\ &+ \mathcal{G}\Big(s,\widetilde{\omega}(s),\int_{s_0}^{s_f}F_1\big(s,u,\widetilde{\omega}(u)\big)\Delta u,\int_{s_0}^{s_f}F_2\big(s,u,\widetilde{\omega}(u)\big)\Delta u\Big)\Big) \\ &- \Big(F\big(s,\omega(s)\big) \\ &+ \mathcal{G}\Big(s,\omega(s),\int_{s_0}^{s_f}F_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}F_2\big(s,u,\omega(u)\big)\Delta u\Big)\Big)\bigg\|\Delta s \\ \le{}& a_3a_2(\varsigma_f-\varsigma_i)\epsilon + Q_3\|\widetilde{\omega}-\omega\|_\infty, \end{aligned}$$

which implies

$$\|\widetilde{\omega}-\omega\|_\infty \le a_3a_2(\varsigma_f-\varsigma_i)\epsilon + Q_3^*\|\widetilde{\omega}-\omega\|_\infty.$$

Hence

$$\begin{aligned}\|\widetilde{\omega}-\omega\|_\infty &\le \bigl(a_3a_2(\varsigma_f-\varsigma_i)\bigr)\frac{\varepsilon}{1-Q_3^*}\\ &\le \mathcal{H}_{(a_3,a_2,L_F,L_{\mathcal{G}_1},L_{\mathcal{G}_2}L_{F_1},L_{\mathcal{G}_3},L_{F_2},L_g)}\epsilon,\end{aligned}$$

where $\mathcal{H}_{(a_3,a_2,L_F,L_{\mathcal{G}_1},L_{\mathcal{G}_2}L_{F_1},L_{\mathcal{G}_3},L_{F_2},L_g)} = \frac{a_3a_2(\varsigma_f-\varsigma_i)}{1-Q_3^*}$. Hence, the mixed impulsive system (2) is UH stable. Furthermore, if we take

$$\overline{\mathcal{H}}_{(a_3,a_2,L_F,L_{\mathcal{G}_1},L_{\mathcal{G}_2}L_{F_1},L_{\mathcal{G}_3},L_{F_2},L_g)}(\epsilon) = \overline{\mathcal{H}}_{(a_3,a_2,L_F,L_{\mathcal{G}_1},L_{\mathcal{G}_2}L_{F_1},L_{\mathcal{G}_3},L_{F_2},L_g)}(0) = 0,$$

then the mentioned system (2) is generalized UH stable. □

5 Controllability analysis

In the sequel, controllability analysis of given impulsive systems is conducted. At first, we have some definitions in this direction.

Definition 5.1 The function $\omega \in \mathbb{T}$ is said to be the solution of (3) if ω satisfies $\omega(0) = \omega_0$ and ω is the solution of the following integral equations:

$$\omega(\varsigma) = \begin{cases} E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{H}\zeta(\varsigma)\Delta s, \quad \varsigma\in(\varsigma_0,\varsigma_1], \\ E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma) \\ \quad \times G(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{H}\zeta(\varsigma)\Delta s \\ \quad + \sum_{j=1}^{i}(\Xi_j(\omega(\varsigma_j^-)) + \Phi_j(\varsigma_j^-,\omega(\varsigma_j^-))), \quad \varsigma\in(\varsigma_i,\varsigma_{i+1}], i=1,\dots,m, \end{cases} \tag{24}$$

where $E_\sigma(A\varsigma^\sigma)$ stands for the matrix representation of the Mittag-Leffler function.

Definition 5.2 The mixed impulsive system (3) is controllable on $\mathbb{T}$ if, for every $\omega_0,\omega_T \in \mathbb{T}$ where $\varsigma_{i+1} = T$, there exists an rd-continuous function $\zeta \in \mathcal{L}^2(I,\mathbb{R})$ such that the corresponding solution of (3) satisfies $\omega(0) = \omega_0$ and $\zeta(T) = \zeta_T$.

We set the following for simplicity:

$$\begin{aligned} Q_5 &:= a_3\bigl(a_2L_{\mathcal{F}} + a_2L_{G_1} + a_2L_{G_2}L_{\mathcal{F}_1}(s_f-s_0) + a_2L_{G_3}L_{\mathcal{F}_2}(s_f-s_0) + \widehat{M}_G\bigr) \\ &\quad \times (\varsigma_f-\varsigma_0)\bigl(1 + M_{\mathcal{H}}M_{\mathcal{W}}^\sigma(\varsigma_f-\varsigma_0)\bigr); \\ Q_6 &:= a_3\bigl(a_2L_F + a_2L_{\mathcal{G}} + a_2L_{\mathcal{G}}L_{F_1}(s_f-s_0) + a_2L_{\mathcal{G}}L_{F_2}(s_f-s_0)\bigr) \times (\varsigma_f-\varsigma_0); \end{aligned}$$

$$Q_7 := \big(1 + M_{\mathcal{H}} M^{\sigma}_{\mathcal{W}}(\varsigma_f - \varsigma_0)\big)\Bigg[\sum_{j=1}^{i} L_{\Xi} + \sum_{j=1}^{i} L_{\Phi}$$

$$+ \big(a_2 L_{\mathcal{F}} + a_2 L_{G_1} + a_2 L_{G_2} L_{\mathcal{F}_1}(s_f - s_0) + a_2 L_{G_3} L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0)\Bigg];$$

$$Q_8 := \big(1 + M_{\mathcal{H}} M^{\sigma}_{\mathcal{W}}(\varsigma - \varsigma_f)\big)\bigg[\frac{1}{\Gamma(\sigma)} a_3 L_g (s_i - \varsigma_i)$$

$$+ a_3\big(a_2 L_F + a_2 L_{\mathcal{G}} + a_2 L_{\mathcal{G}} L_{F_1}(s_f - s_0) + a_2 L_{\mathcal{G}} L_{F_2}(s_f - s_0)\big) \times (\varsigma_f - \varsigma_0)\bigg];$$

$$N_5 := \big(1 + M_{\mathcal{H}} M^{\sigma}_{\mathcal{W}}\big)\left(\sum_{j=1}^{i} L_{\Xi}\delta + \sum_{j=1}^{i} L_{\Phi}\delta + a_1\right) + M_{\mathcal{H}} M^{\sigma}_{\mathcal{H}} \|\omega_T\|;$$

$$N_6 := \big(1 + M_{\mathcal{H}} M^{\sigma}_{\mathcal{W}}\big)\left(\frac{1}{\Gamma(\sigma)} a_3 L_g \delta''(s_i - \varsigma_i) + a_1 + \widehat{M}_G\right) + M_{\mathcal{H}} M^{\sigma}_{\mathcal{H}} \|\omega_T\|.$$

Lemma 5.3 *If assertions* (**A**) *and* (**W**) *hold and $\omega_T \in \mathbb{T}$, where $\varsigma_{i+1} = T$ is any arbitrary point, then ω is a solution of* (3) *on $\mathbb{T}$ defined by* (24) *along with the control function*

$$\zeta(\varsigma) = \begin{cases} ({}^{\sigma}\mathcal{W}^{T}_{\varsigma_0})^{-1}[\omega_T - E_{\sigma}(A\varsigma^{\sigma})\omega_0 - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma})\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \\ \quad \times G(s,\omega(s), \int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s] \\ \quad \varsigma \in (\varsigma_0,\varsigma_1], \\ ({}^{\sigma}\mathcal{W}^{T}_{\varsigma_0})^{-1}[\omega_T - E_{\sigma}(A\varsigma^{\sigma})\omega_0 \\ \quad - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma})\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^{\sigma}) \\ \quad \times G(s,\omega(s), \int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad - \sum_{j=1}^{i}(\Xi_j(\omega(\varsigma_j^-)) - \Phi_j(\varsigma_j^-,\omega(\varsigma_j^-)))], \quad \varsigma \in (\varsigma_i,\varsigma_{i+1}], i = 1,\dots,m, \end{cases} \tag{25}$$

where $E_{\sigma}(A\varsigma^{\sigma})$ stands for the matrix representation of the Mittag-Leffler function, and $\omega(T) = \omega_T$ holds in which $\varsigma_{i+1} = T$. Also, the control function $\zeta(\varsigma)$ has the estimate $\|\zeta(\varsigma)\| \leq \aleph_{\zeta}$, where for $\varsigma \in (\varsigma_i,\varsigma_{i+1}]$, $i = 1,\dots,m$, we define

$$\aleph_{\zeta} = \aleph^{\sigma}_{\mathcal{W}}\Bigg[\|\omega_T\| + \big\|E_{\sigma}\big(A\varsigma^{\sigma}\big)\omega_0\big\| + \sum_{j=1}^{i} L_{\Xi}\|\omega\|_{\infty} + \sum_{j=1}^{i} L_{\Phi}\|\omega\|_{\infty}$$

$$+ \big(a_3 a_2\big(L_{\mathcal{F}}\|\omega\|_{\infty} + L_{G_1}\|\omega\|_{\infty} + L_{G_2}L_{\mathcal{F}_1}\|\omega\|_{\infty}(s_f - s_0)$$

$$+ L_{G_3}L_{\mathcal{F}_2}\|\omega\|_{\infty}(s_f - s_0) + \widehat{M}_G\big)(\varsigma_f - \varsigma_0)\big)\Bigg].$$

Proof Let ω be the solution of (3) on $\varsigma \in (\varsigma_i,\varsigma_{i+1}]$, $i = 1,\dots,m$, defined by (24). Then, for $\varsigma = T$, we have

$$\omega(T) = E_{\sigma}\big(AT^{\sigma}\big)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(T - s)^{\sigma-1}E_{\sigma,\sigma}\big(A(T - s)^{\sigma}\big)\mathcal{F}\big(s,\omega(s)\big)\Delta s$$

$$+ \int_{\varsigma_0}^{\varsigma_f}(T - s)^{\sigma-1}E_{\sigma,\sigma}\big(A(T - s)^{\sigma}\big)$$

$$
\begin{aligned}
&\times G\biggl(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\bigl(s,u,\omega(u)\bigr)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\bigl(s,u,\omega(u)\bigr)\Delta u\biggr)\Delta s\\
&+\sum_{j=1}^{i}\bigl(\Xi_j\bigl(\omega\bigl(T_j^-\bigr)\bigr)+\Phi_j\bigl(T_j^-,\omega\bigl(T_j^-\bigr)\bigr)\bigr)\\
&+\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(\varsigma-s)^\sigma\bigr)\mathcal{H}\zeta(\varsigma)\Delta s\\
={}&E_\sigma\bigl(AT^\sigma\bigr)\omega_0+\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\mathcal{F}\bigl(s,\omega(s)\bigr)\Delta s\\
&+\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\\
&\times G\biggl(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\bigl(s,u,\omega(u)\bigr)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\bigl(s,u,\omega(u)\bigr)\Delta u\biggr)\Delta s\\
&+\sum_{j=1}^{i}\bigl(\Xi_j\bigl(\omega\bigl(T_j^-\bigr)\bigr)+\Phi_j\bigl(T_j^-,\omega\bigl(T_j^-\bigr)\bigr)\bigr)\\
&+\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(\varsigma-s)^\sigma\bigr)\mathcal{H}\\
&\times\bigl({}^\sigma\mathcal{W}_{\varsigma_0}^T\bigr)^{-1}\biggl[\omega_T-E_\sigma\bigl(AT^\sigma\bigr)\omega_0\\
&-\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\mathcal{F}\bigl(s,\omega(s)\bigr)\Delta s\\
&-\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\\
&\times G\biggl(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\bigl(s,u,\omega(u)\bigr)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\bigl(s,u,\omega(u)\bigr)\Delta u\biggr)\Delta s\\
&-\sum_{j=1}^{i}\bigl(\Xi_j\bigl(\omega\bigl(T_j^-\bigr)\bigr)-\Phi_j\bigl(T_j^-,\omega\bigl(T_j^-\bigr)\bigr)\bigr)\biggr]\Delta\tau\\
={}&E_\sigma\bigl(AT^\sigma\bigr)\omega_0+\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\mathcal{F}\bigl(s,\omega(s)\bigr)\Delta s\\
&+\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\\
&\times G\biggl(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\bigl(s,u,\omega(u)\bigr)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\bigl(s,u,\omega(u)\bigr)\Delta u\biggr)\Delta s\\
&+\sum_{j=1}^{i}\bigl(\Xi_j\bigl(\omega\bigl(T_j^-\bigr)\bigr)+\Phi_j\bigl(T_j^-,\omega\bigl(T_j^-\bigr)\bigr)\bigr)\\
&\times\bigl({}^\sigma\mathcal{W}_{\varsigma_0}^T\bigr)\bigl({}^\sigma\mathcal{W}_{\varsigma_0}^T\bigr)^{-1}\biggl[\omega_T-E_\sigma\bigl(AT^\sigma\bigr)\omega_0\\
&-\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)\mathcal{F}\bigl(s,\omega(s)\bigr)\Delta s\\
&-\int_{\varsigma_0}^{\varsigma_f}(T-s)^{\sigma-1}E_{\sigma,\sigma}\bigl(A(T-s)^\sigma\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&\times G\biggl(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\bigl(s,u,\omega(u)\bigr)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\bigl(s,u,\omega(u)\bigr)\Delta u\biggr)\Delta s\\
&-\sum_{j=1}^{i}\bigl(\Xi_j\bigl(\omega\bigl(T_j^-\bigr)\bigr)-\Phi_j\bigl(T_j^-,\omega\bigl(T_j^-\bigr)\bigr)\bigr)\biggr]\\
={}&\omega_T.
\end{aligned}
$$

Also, for $\varsigma\in(\varsigma_i,\varsigma_{i+1}]$, $i=1,\dots,m$, the estimation

$$
\begin{aligned}
\bigl\|\zeta(\varsigma)\bigr\| ={}& \aleph_{\mathcal{W}}^{\sigma}\biggl[\|\omega_T\|-\bigl\|E_\sigma\bigl(A\varsigma^\sigma\bigr)\omega_0\bigr\|\\
&-\int_{\varsigma_0}^{\varsigma_f}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma-s)^\sigma\bigr)\bigr\|\bigl\|\mathcal{F}\bigl(\varsigma,\omega(\varsigma)\bigr)\bigr\|\Delta s\\
&-\int_{\varsigma_0}^{\varsigma_f}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma-s)^\sigma\bigr)\bigr\|\\
&\times\biggl\|G\biggl(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1\bigl(s,u,\omega(u)\bigr)\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2\bigl(s,u,\omega(u)\bigr)\Delta u\biggr)\biggr\|\Delta s\\
&-\sum_{j=1}^{i}\bigl(\bigl\|\Xi_j\bigl(\omega\bigl(\varsigma_j^-\bigr)\bigr)\bigr\|+\bigl\|\Phi_j\bigl(\varsigma_j^-,\omega\bigl(\varsigma_j^-\bigr)\bigr)\bigr\|\bigr)\biggr]
\end{aligned}
$$

implies that

$$
\begin{aligned}
\bigl\|\zeta(\varsigma)\bigr\| ={}& \aleph_{\mathcal{W}}^{\sigma}\biggl[\|\omega_T\|+\bigl\|E_\sigma\bigl(A\varsigma^\sigma\bigr)\omega_0\bigr\|+\sum_{j=1}^{i}L_\Xi\|\omega\|_\infty+\sum_{j=1}^{i}L_\Phi\|\omega\|_\infty\\
&+\bigl(a_3a_2\bigl(L_{\mathcal{F}}\|\omega\|_\infty+L_{G_1}\|\omega\|_\infty+L_{G_2}L_{\mathcal{F}_1}\|\omega\|_\infty(s_f-s_0)\\
&+L_{G_3}L_{\mathcal{F}_2}\|\omega\|_\infty(s_f-s_0)+\widehat{M}_G\bigr)(\varsigma_f-\varsigma_0)\bigr)\biggr]\\
={}&\aleph_\zeta,
\end{aligned}
$$

and the argument is completed. □

Definition 5.4 The function $\omega\in\mathbb{T}$ is said to be the solution of the mixed impulsive system (4) if ω satisfies $\omega(0)=\omega_0$ and ω is the solution of the following integral equations:

$$
\omega(\varsigma)=\begin{cases}
E_\sigma(A\varsigma^\sigma)\omega_0+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)F(\varsigma,\omega(\varsigma))\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\\
\quad\times\mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}F_2(s,u,\omega(u))\Delta u)\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{H}\zeta(\varsigma)\Delta s,\\
\quad\varsigma\in(\varsigma_i,s_i]\cap\mathbb{T}, i=1,\dots,m,\\
E_\sigma(A\varsigma^\sigma)\omega_0+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\\
\quad\times\mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^\sigma)\mathcal{H}\zeta(\varsigma)\Delta s\\
\quad+\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s,\quad\varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T}, i=1,\dots,m,
\end{cases}
\tag{26}
$$

where $E_\sigma(A\varsigma^\sigma)$ stands for the matrix representation of the Mittag-Leffler function.

Definition 5.5 The mixed impulsive system (4) is named as a controllable system on $\mathbb{T}$ if, for every $\omega_0, \omega_T \in \mathbb{T}$, where $\varsigma_{i+1} = T$, there exists an rd-continuous function $\zeta \in \mathcal{L}^2(I, \mathbb{R})$ such that the corresponding solution of (4) satisfies $\omega(0) = \omega_0$ and $\omega(T) = \omega_T$.

Lemma 5.6 *If assertions* (**A**) *and* (**W**) *hold and* $\omega_T \in \mathbb{T}$, *where* $\varsigma_{i+1} = T$ *is any arbitrary point, then* ω *is the solution of* (4) *on* $\varsigma \in (s_i, \varsigma_{i+1}] \cap \mathbb{T}$, $i = 1, \dots, m$, *defined by* (26) *along with the control function*

$$\zeta(\varsigma) = \begin{cases} ({}^\sigma\mathcal{W}_{\varsigma_0}^T)^{-1}[\omega_T - E_\sigma(A\varsigma^\sigma)\omega_0 - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)F(\varsigma, \omega(\varsigma))\Delta s \\ \quad - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\ \quad \times \mathcal{G}(s, \omega(s), \int_{s_0}^{s_f} F_1(s, u, \omega(u))\Delta u, \int_{s_0}^{s_f} F_2(s, u, \omega(u))\Delta u)\Delta s] \\ \quad \varsigma \in (\varsigma_i, s_i] \cap \mathbb{T}, i = 1, \dots, m, \\ ({}^\sigma\mathcal{W}_{\varsigma_0}^T)^{-1}[\omega_T - E_\sigma(A\varsigma^\sigma)\omega_0 - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)F(\varsigma, \omega(\varsigma))\Delta s \\ \quad - \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\ \quad \times \mathcal{G}(s, \omega(s), \int_{s_0}^{s_f} F_1(s, u, \omega(u))\Delta u, \int_{s_0}^{s_f} F_2(s, u, \omega(u))\Delta u)\Delta s \\ \quad - \frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}(\varsigma - s)^{\sigma-1}\hbar_i(s, \omega(s))\Delta s, \quad \varsigma \in (s_i, \varsigma_{i+1}] \cap \mathbb{T}, i = 1, \dots, m, \end{cases} \tag{27}$$

where $E_\sigma(A\varsigma^\sigma)$ *stands for the matrix representation of the Mittag-Leffler function and* $\omega(T) = \omega_T$ *holds, where* $\varsigma_{i+1} = T$. *Also, the control function* $\zeta(\varsigma)$ *has the estimate* $\|\zeta(\varsigma)\| \leq \aleph_\zeta^*$, *where for* $\varsigma \in (s_i, \varsigma_{i+1}] \cap \mathbb{T}$, $i = 1, \dots, m$, *we define*

$$\begin{aligned} \aleph_\zeta^* = \aleph_{\mathcal{W}}^\sigma \Bigg[&\|\omega_T\| + \left\|E_\sigma\left(A\varsigma^\sigma\right)\omega_0\right\| + \frac{1}{\Gamma(\sigma)}(s_i - \varsigma_i)a_1 L_g\|\omega\|_\infty \\ &+ \left(a_3 a_2\left(L_F\|\omega\|_\infty + L_{\mathcal{G}_1}\|\omega\|_\infty + L_{\mathcal{G}_2}L_{F_1}\|\omega\|_\infty(s_f - s_0)\right.\right. \\ &\left.\left. + L_{\mathcal{G}_3}L_{F_2}\|\omega\|_\infty(s_f - s_0) + \widehat{M}_{\mathcal{G}}\right)(\varsigma_f - \varsigma_0)\right) \Bigg]. \end{aligned}$$

Proof The proof is similar to that of Lemma 5.3. □

Theorem 5.7 *The mixed impulsive system* (3) *is controllable on* $\mathbb{T}$ *such that hypotheses* (**A**) *and* (**W**) *are satisfied and the following inequality holds*:

$$\max_{5 \leq i \leq 7}\{Q_i\} < 1. \tag{28}$$

Proof Let $\Omega'' \subseteq \mathrm{PS}$, provided that $\Omega'' = \{(\mathbb{X},\mathbb{Y},\mathbb{Z}) \in \mathrm{PS} : \|(\mathbb{X},\mathbb{Y},\mathbb{Z})\|_C \leq \delta_2''\}$, where $\delta_2'' = \max\{\delta'', \delta_1''\}$ such that $\delta'', \delta_1'' \in (0,1)$, and also $\delta'' > N_5$. Now, we define $\Lambda_\sigma'' : \Omega'' \to \Omega''$ as

$$\Lambda_\sigma''(\omega(\varsigma)) = \begin{cases} E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\ \quad \times G(s,\omega(s), \int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\mathcal{H}\zeta(\varsigma)\Delta s, \quad \varsigma \in (\varsigma_0, \varsigma_1], \\ E_\sigma(A\varsigma^\sigma)\omega_0 + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\mathcal{F}(\varsigma,\omega(\varsigma))\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma) \\ \quad \times G(s,\omega(s), \int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u)\Delta s \\ \quad + \int_{\varsigma_0}^{\varsigma_f}(\varsigma - s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\mathcal{H}\zeta(s)\Delta s \\ \quad + \sum_{j=1}^{i}(\Xi_j(\omega(\varsigma_j^-)) + \Phi_j(\varsigma_j^-, \omega(\varsigma_j^-))), \quad \varsigma \in (\varsigma_i, \varsigma_{i+1}], i = 1,\ldots,m. \end{cases} \tag{29}$$

Now, we prove $\Lambda_\sigma'' : \Omega'' \to \Omega''$ is a self-mapping.

For $\varsigma \in (\varsigma_i, \varsigma_{i+1}]$, $i = 1,\ldots,m$, we get

$$\begin{aligned} \|\Lambda_\sigma''(\omega(\varsigma))\| &\leq \sum_{j=1}^{i}\|\Xi_j(\omega(\varsigma_j^-))\| + \sum_{j=1}^{i}\|\Phi_j(\varsigma_j^-, \omega(\varsigma_j^-))\| + \|E_\sigma(A\varsigma^\sigma)\omega_0\| \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\|(\varsigma - s)^{\sigma-1}\|\|E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\|\bigg\|\Big(\mathcal{F}(s,\omega(s)) \\ &\quad + G\Big(s,\omega(s), \int_{s_0}^{s_f}\mathcal{F}_1(s,u,\omega(u))\Delta u, \int_{s_0}^{s_f}\mathcal{F}_2(s,u,\omega(u))\Delta u\Big)\Big)\bigg\|\Delta s \\ &\quad + \int_{\varsigma_0}^{\varsigma_f}\|(\varsigma - s)^{\sigma-1}\|\|E_{\sigma,\sigma}(A(\varsigma - s)^\sigma)\|\|\mathcal{H}\|\|\zeta(s)\|\Delta s \\ &\leq \sum_{j=1}^{i}L_\Xi\delta'' + \sum_{j=1}^{i}L_\Phi\delta'' + a_1 + \delta''a_3\big(a_2L_\mathcal{F} + a_2L_{G_1} \\ &\quad + a_2L_{G_2}L_{\mathcal{F}_1}(s_f - s_0) + a_2L_{G_3}L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0) \\ &\quad + M_\mathcal{H}M_\mathcal{W}^\sigma\Bigg[\sum_{j=1}^{i}L_\Xi\delta'' + \sum_{j=1}^{i}L_\Phi\delta'' + \|\omega\|_T + a_1 \\ &\quad + \delta''a_3\big(a_2L_\mathcal{F} + a_2L_{G_1} + a_2L_{G_2}L_{\mathcal{F}_1}(s_f - s_0) \\ &\quad + a_2L_{G_3}L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0)\Bigg] \\ &\leq N_5 + \delta''Q_5 \\ &\leq \delta'' + \delta''Q_5 = \delta_1''. \end{aligned}$$

Hence,

$$\|\Lambda_\sigma''(\omega(\varsigma))\| \leq \delta_2''. \tag{30}$$

Therefore, from (30), $\Lambda''_\sigma(\Omega'') \subseteq \Omega''$, also when $\varsigma \in (\varsigma_i, \varsigma_{i+1}]$, $i = 1, \dots, m$, with $\omega_0 = \widehat{\omega}_0$, we have

$$
\begin{aligned}
&\big\| \Lambda''_\sigma\big(\omega(\varsigma)\big) - \Lambda''_\sigma\big(\widehat{\omega}(\varsigma)\big)\big\| \\
&\quad \leq \sum_{j=1}^{i} \big\| \Xi_j\big(\omega\big(\varsigma_j^-\big)\big) - \Xi_j\big(\widehat{\omega}\big(\varsigma_j^-\big)\big)\big\| \\
&\qquad + \sum_{j=1}^{i} \big\| \Phi_j\big(\varsigma_j^-, \omega\big(\varsigma_j^-\big)\big) - \Phi_j\big(\varsigma_j^-, \widehat{\omega}\big(\varsigma_j^-\big)\big)\big\| \\
&\qquad + \int_{\varsigma_0}^{\varsigma_f} \big\|(\varsigma - s)^{\sigma-1}\big\| \big\| E_{\sigma,\sigma}\big(A(\varsigma - s)^\sigma\big)\big\| \bigg\| \bigg(\mathcal{F}\big(s, \omega(s)\big) \\
&\qquad + G\bigg(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s, u, \omega(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s, u, \omega(u)\big)\Delta u \bigg)\bigg) \\
&\qquad - \bigg(\mathcal{F}\big(s, \widehat{\omega}(s)\big) \\
&\qquad + G\bigg(s, \widehat{\omega}(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s, u, \widehat{\omega}(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s, u, \widehat{\omega}(u)\big)\Delta u \bigg)\bigg) \bigg\| \Delta s \\
&\qquad + \int_{\varsigma_0}^{\varsigma_f} \big\|(\varsigma - s)^{\sigma-1}\big\| \big\| E_{\sigma,\sigma}\big(A(\varsigma - s)^\sigma\big)\big\| \|\mathcal{H}\| \big\| \big({}^\sigma\mathcal{W}_{\varsigma_0}^T\big)^{-1} \big\| (\varsigma - \varsigma_f) \\
&\qquad \times \Bigg[\sum_{j=1}^{i} \big\| \Xi_j\big(\omega\big(\varsigma_j^-\big)\big) - \Xi_j\big(\widehat{\omega}\big(\varsigma_j^-\big)\big)\big\| \\
&\qquad + \sum_{j=1}^{i} \big\| \Phi_j\big(\varsigma_j^-, \omega\big(\varsigma_j^-\big)\big) - \Phi_j\big(\varsigma_j^-, \widehat{\omega}\big(\varsigma_j^-\big)\big)\big\| \\
&\qquad + \int_{\varsigma_0}^{\varsigma_f} \big\|(\varsigma - s)^{\sigma-1}\big\| \big\| E_{\sigma,\sigma}\big(A(\varsigma - s)^\sigma\big)\big\| \bigg\| \bigg(\mathcal{F}\big(s, \omega(s)\big) \\
&\qquad + G\bigg(s, \omega(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s, u, \omega(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s, u, \omega(u)\big)\Delta u \bigg)\bigg) \\
&\qquad - \bigg(\mathcal{F}\big(s, \widehat{\omega}(s)\big) \\
&\qquad + G\bigg(s, \widehat{\omega}(s), \int_{s_0}^{s_f} \mathcal{F}_1\big(s, u, \widehat{\omega}(u)\big)\Delta u, \int_{s_0}^{s_f} \mathcal{F}_2\big(s, u, \widehat{\omega}(u)\big)\Delta u \bigg)\bigg) \bigg\| \Delta s \Bigg] \Delta\tau \\
&\quad \leq \Bigg[\sum_{j=1}^{i} L_\Xi + \sum_{j=1}^{i} L_\Phi + \big(a_2 L_{\mathcal{F}} + a_2 L_{G_1} + a_2 L_{G_2} L_{\mathcal{F}_1}(s_f - s_0) \\
&\qquad + a_2 L_{G_3} L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0) \Bigg] \times \|\widetilde{\omega} - \omega\|_\infty \\
&\qquad + M_{\mathcal{H}} M_{\mathcal{W}}^\sigma (\varsigma_f - \varsigma_0) \Bigg[\sum_{j=1}^{i} L_\Xi + \sum_{j=1}^{i} L_\Phi \\
&\qquad + \big(a_2 L_{\mathcal{F}} + a_2 L_{G_1} + a_2 L_{G_2} L_{\mathcal{F}_1}(s_f - s_0) \\
&\qquad + a_2 L_{G_3} L_{\mathcal{F}_2}(s_f - s_0)\big)(\varsigma_f - \varsigma_0) \Bigg] \times \|\widetilde{\omega} - \omega\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
&\leq \bigl(1+M_{\mathcal{H}}M_{\mathcal{W}}^{\sigma}(\varsigma_f-\varsigma_0)\bigr)\Biggl[\sum_{j=1}^{i}L_{\Xi}+\sum_{j=1}^{i}L_{\Phi} \\
&\quad+\bigl(a_2L_{\mathcal{F}}+a_2L_{G_1}+a_2L_{G_2}L_{\mathcal{F}_1}(s_f-s_0) \\
&\quad+a_2L_{G_3}L_{\mathcal{F}_2}(s_f-s_0)\bigr)(\varsigma_f-\varsigma_0)\Biggr]\times\|\widetilde{\omega}-\omega\|_{\infty}.
\end{aligned}
$$

Hence

$$
\bigl\|\Lambda_{\sigma}''\bigl(\omega(\varsigma)\bigr)-\Lambda_{\sigma}''\bigl(\widehat{\omega}(\varsigma)\bigr)\bigr\|\leq Q_7\|\widetilde{\omega}-\omega\|_{\infty}. \tag{31}
$$

Therefore, the operator Λ_{σ}'' is strictly contractive. Therefore, by using the Banach fixed point theorem method, Λ_{σ}'' has only one fixed point, i.e., the mixed impulsive system (3) has a unique solution. Also, using Lemma 5.3, we conclude that $\omega(\varsigma)$ fulfills $\omega(T)=\omega_T$. Consequently, we conclude that the mixed impulsive system (3) is controllable. □

One can indicate a similar theorem for the mixed impulsive system (4).

Theorem 5.8 *The mixed impulsive system* (4) *is controllable on* $\mathbb{T}$ *such that hypotheses* (**A**) *and* (**W**) *are satisfied and the following inequality holds*:

$$
\max\{Q_i\}<1 \quad \textit{where } i=6,8. \tag{32}
$$

Proof Let $\Omega''\subseteq \mathrm{PS}$, provided that $\Omega''=\{(\mathbb{X},\mathbb{Y},\mathbb{Z})\in\mathrm{PS}:\|(\mathbb{X},\mathbb{Y},\mathbb{Z})\|_C\leq\delta_2''\}$, where $\delta_2''=\max\{\delta'',\delta_1''\}$ such that $\delta'',\delta_1''\in(0,1)$, and also $\delta''>\{N_6\}$. Now, we define $\Lambda_{\sigma}^{**}:\Omega^{**}\to\Omega^{**}$ as

$$
\Lambda_{\sigma}^{**}\bigl(\omega(\varsigma)\bigr)=
\begin{cases}
E_{\sigma}(A\varsigma^{\sigma})\omega_0+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^{\sigma})F(\varsigma,\omega(\varsigma))\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^{\sigma})\\
\quad\times\mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}F_2(s,u,\omega(u))\Delta u)\Delta s,\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^{\sigma})\mathcal{H}\zeta(s)\Delta s\\
\quad\varsigma\in(\varsigma_i,s_i]\cap\mathbb{T}, i=1,\dots,m,\\
E_{\sigma}(A\varsigma^{\sigma})\omega_0+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^{\sigma})F(\varsigma,\omega(\varsigma))\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^{\sigma})\\
\quad\times\mathcal{G}(s,\omega(s),\int_{s_0}^{s_f}F_1(s,u,\omega(u))\Delta u,\int_{s_0}^{s_f}F_2(s,u,\omega(u))\Delta u)\Delta s\\
\quad+\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}(\varsigma-s)^{\sigma-1}\hbar_i(s,\omega(s))\Delta s\\
\quad+\int_{\varsigma_0}^{\varsigma_f}(\varsigma-s)^{\sigma-1}E_{\sigma,\sigma}(A(\varsigma-s)^{\sigma})\mathcal{H}\zeta(s)\Delta s,\\
\quad\varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T}, i=1,\dots,m,
\end{cases} \tag{33}
$$

where $E_{\sigma}(A\varsigma^{\sigma})$ stands for the matrix representation of the Mittag-Leffler function. Now, we prove $\Lambda_{\sigma}^{**}:\Omega^{**}\to\Omega^{**}$ is a self-mapping.

For $\varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T}$, $i=1,\dots,m$, we have

$$
\begin{aligned}
\bigl\|\Lambda_{\sigma}^{**}\bigl(\omega(\varsigma)\bigr)\bigr\|&\leq\frac{1}{\Gamma(\sigma)}\int_{\varsigma_i}^{s_i}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\bigl\|\hbar_i\bigl(s,\omega(s)\bigr)\bigr\|\Delta s+\bigl\|E_{\sigma}\bigl(A\varsigma^{\sigma}\bigr)\omega_0\bigr\| \\
&\quad+\int_{\varsigma_0}^{\varsigma_f}\bigl\|(\varsigma-s)^{\sigma-1}\bigr\|\bigl\|E_{\sigma,\sigma}\bigl(A(\varsigma-s)^{\sigma}\bigr)\bigr\|\biggl\|\biggl(F\bigl(s,\omega(s)\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{G}\Big(s,\omega(s),\int_{s_0}^{s_f}F_1\big(s,u,\omega(u)\big)\Delta u,\int_{s_0}^{s_f}F_2\big(s,u,\omega(u)\big)\Delta u\Big)\Big)\Big\|\Delta s\\
&+\int_{\varsigma_0}^{\varsigma_f}\big\|(\varsigma-s)^{\sigma-1}\big\|\big\|E_{\sigma,\sigma}\big(A(\varsigma-s)^{\sigma}\big)\big\|\|\mathcal{H}\|\big\|\zeta(s)\big\|\Delta s\\
\le{}&\frac{1}{\Gamma(\sigma)}a_3L_g\delta(s_i-\varsigma_i)+a_1\\
&+a_3\big(\delta''a_2L_F+\delta''a_2L_{\mathcal{G}}+\delta''a_2L_{\mathcal{G}}L_{F_1}(s_f-s_0)\\
&+\delta''a_2L_{\mathcal{G}}L_{F_2}(s_f-s_0)\big)\times(\varsigma_f-\varsigma_0)\\
&+M_{\mathcal{H}}M_{\mathcal{W}}^{\sigma}\bigg[\frac{1}{\Gamma(\sigma)}a_3L_g\delta''(s_i-\varsigma_i)+a_1+\|\omega\|_T\\
&+\delta''a_3\big(a_2L_F+a_2L_{\mathcal{G}}+a_2L_{\mathcal{G}}L_{F_1}(s_f-s_0)\\
&+a_2L_{\mathcal{G}}L_{F_2}(s_f-s_0)\big)\times(\varsigma_f-\varsigma_0)\bigg]\\
\le{}&N_6+\delta''Q_6\le\delta''+\delta''Q_6=\delta''_1.
\end{aligned}
$$

Hence,

$$
\big\|\Lambda_\sigma^{**}\big(\omega(\varsigma)\big)\big\|\le\delta''_2. \tag{34}
$$

Therefore, from (34), $\Lambda_\sigma^{**}(\Omega^{**})\subseteq\Omega^{**}$. Also, for $\varsigma\in(s_i,\varsigma_{i+1}]\cap\mathbb{T}$, $i=1,\dots,m$, with $\omega_0=\widehat{\omega}_0$, we have

$$
\begin{aligned}
&\big\|\Lambda_\sigma^{**}\big(\omega(\varsigma)\big)-\Lambda_\sigma^{**}\big(\widehat{\omega}(\varsigma)\big)\big\|\\
&\quad\le\bigg[\frac{1}{\Gamma(\sigma)}a_3L_g(s_i-\varsigma_i)+a_3\big(a_2L_F+a_2L_{\mathcal{G}}\\
&\qquad+a_2L_{\mathcal{G}}L_{F_1}(s_f-s_0)+a_2L_{\mathcal{G}}L_{F_2}(s_f-s_0)\big)(\varsigma_f-\varsigma_0)\bigg]\|\widetilde{\omega}-\omega\|_\infty\\
&\qquad+M_{\mathcal{H}}M_{\mathcal{W}}^{\sigma}(\varsigma-\varsigma_f)\bigg[\frac{1}{\Gamma(\sigma)}a_3L_g(s_i-\varsigma_i)+a_3\big(a_2L_F+a_2L_{\mathcal{G}}\\
&\qquad+a_2L_{\mathcal{G}}L_{F_1}(s_f-s_0)+a_2L_{\mathcal{G}}L_{F_2}(s_f-s_0)\big)(\varsigma_f-\varsigma_0)\bigg]\|\widetilde{\omega}-\omega\|_\infty\\
&\quad\le\big(1+M_{\mathcal{H}}M_{\mathcal{W}}^{\sigma}(\varsigma-\varsigma_f)\big)\bigg[\frac{1}{\Gamma(\sigma)}a_3L_g(s_i-\varsigma_i)+a_3\big(a_2L_F+a_2L_{\mathcal{G}}\\
&\qquad+a_2L_{\mathcal{G}}L_{F_1}(s_f-s_0)+a_2L_{\mathcal{G}}L_{F_2}(s_f-s_0)\big)\times(\varsigma_f-\varsigma_0)\bigg]\|\widetilde{\omega}-\omega\|_\infty.
\end{aligned}
$$

Hence

$$
\big\|\Lambda_\sigma^{**}\big(\omega(\varsigma)\big)-\Lambda_\sigma^{**}\big(\widehat{\omega}(\varsigma)\big)\big\|\le Q_8\|\widetilde{\omega}-\omega\|_\infty. \tag{35}
$$

Therefore, the operator Λ_σ^{**} is strictly contractive. Thus, using the Banach fixed point theorem method, Λ_σ^{**} has a unique fixed point, which is the unique solution of the mixed impulsive system (4). Also, using Lemma 5.6, we conclude that $\omega(\varsigma)$ fulfills $\omega(T)=\omega_T$. Consequently, the mixed impulsive system (4) is controllable. □

6 Illustrative example

The last part of the manuscript is devoted to examining our results established in the previous steps.

Example 6.1 Consider the following mixed impulsive system:

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\omega(\varsigma) = \frac{2}{\varsigma-1}\omega(\varsigma) + \frac{2}{\varsigma-1.2}\mathbb{U}(\varsigma) + e_p(\varsigma,\chi(\omega(\varsigma))) + \int_0^{\varsigma} E_{\sigma,\beta}(\omega)\Delta \mathrm{s} + \mathbb{U}(\varsigma), \\ \omega(0) = 1, \quad \varsigma \in [0,3]_{\mathbb{T}}\backslash\{1,1.2\}, \\ \omega(\varsigma_k) = \Xi(\omega(\varsigma_k^-)) + \Gamma_k(\varsigma_k^-,\omega(\varsigma_k^-),\mathbb{U}(\varsigma_k^-)), \quad k = 1,2, \end{cases} \tag{36}$$

and its relevant inequality

$$\begin{cases} |{}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\widehat{\omega}(\varsigma) - \frac{2}{\varsigma-1}\widetilde{\omega}(\varsigma) - \frac{2}{\varsigma-1.2}\mathbb{U}(\varsigma) \\ \quad - e_p(\varsigma,\chi(\widetilde{\omega}(\varsigma))) - \int_0^{\varsigma} E_{\sigma,\beta}(\widehat{\omega})\Delta \mathrm{s} - \mathbb{U}(\varsigma)| \leq 1, \\ \quad \varsigma \in [0,3]_{\mathbb{T}}\backslash\{1,1.2\}, \\ |\Delta\widetilde{\omega}(\varsigma_k) - \Xi(\widetilde{\omega}(\varsigma_k^-)) - \Gamma_k(\varsigma_k^-,\widetilde{\omega}(\varsigma_k^-))| \leq 1, \quad k = 1,2. \end{cases} \tag{37}$$

We set $\mathbb{T}^k = [0,3]_{\mathbb{T}}\backslash\{1,1.2\}$, $\varsigma_1 = 1$, $\varsigma_2 = 1.2$, $\mathbb{X}(\varsigma) = \frac{2}{\varsigma-1}$, and $\mathbb{Y}(\varsigma) = \frac{2}{\varsigma-1.2}$, $E_{\sigma,\beta}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(k\sigma+\beta)}for\sigma,\beta > 0$. In addition, we set

$$F\big(\varsigma,\omega(\varsigma),J_{\omega}(\varsigma),\mathbb{U}(\varsigma)\big) = e_p\big(\varsigma,\chi\big(\omega(\varsigma)\big)\big) + \int_0^t E_{\sigma,\beta}(\omega)\Delta \mathrm{s} + \mathbb{U}(\varsigma),$$

where $J_{\omega}(\varsigma) = \int_0^t E_{\sigma,\beta}(\omega)\Delta \mathrm{s}$ and $\mathbb{U}(\varsigma)$ is a control map for $\varsigma \in \mathbb{T}^k$ and substitute $\epsilon = 1$. Let $\widetilde{\omega} \in PC^1([0,2]_{\mathbb{T}},\mathbb{R})$ fulfill (37), then there exists $\hbar \in PC^1([0,2]_{\mathbb{T}^k},\mathbb{R})$ with $\hbar_0 \in \mathbb{R}$ such that $|\hbar(\varsigma)| \leq 1$ $\forall$ $\varsigma \in \mathbb{T}^k$ and $|\hbar_0| \leq 1$, and so (37) implies that

$$\begin{cases} {}^{c,\mathbb{T}}\mathrm{D}^{\sigma}\widetilde{\omega}(\varsigma) = \frac{2}{\varsigma-1}\widetilde{\omega}(\varsigma) + \frac{2}{\varsigma-1.5}\mathbb{U}(\varsigma) + e_p(\varsigma,\chi(\widetilde{\omega}(\varsigma))) \\ \qquad\qquad + \int_0^{\varsigma} E_{\sigma,\beta}(\omega)\Delta \mathrm{s} + \mathbb{U}(\varsigma) + \hbar(\varsigma), \quad \varsigma \in \mathbb{T}^k, \\ \widetilde{\omega}(\varsigma_k) = \Xi_k(\widetilde{\omega}(\varsigma_k^-)) - \Gamma_k(\varsigma_k^-,\widetilde{\omega}(\varsigma_k^-),\mathbb{U}(\varsigma_k^-)) + \hbar_0, \quad k = 1,2. \end{cases}$$

So the solution of (36) is

$$\begin{aligned} \omega(\varsigma) = {} & \Xi_1\big(\omega\big(\varsigma_1^-\big)\big) + \Xi_2\big(\omega\big(\varsigma_2^-\big)\big) + \Gamma_1\big(\varsigma_1^-,\omega\big(\varsigma_1^-\big),\mathbb{U}\big(\varsigma_1^-\big)\big) + \Gamma_2\big(\varsigma_2^-,\omega\big(\varsigma_2^-\big),\mathbb{U}\big(\varsigma_2^-\big)\big) \\ & + \int_0^t e_p\big(\mathrm{s},\chi(\mathrm{s})\big)\bigg(e_p\big(\mathrm{s},\chi\big(\omega(\mathrm{s})\big)\big) + \int_0^s E_{\sigma,\beta}(\omega)\Delta u + \mathbb{U}(\mathrm{s})\bigg)\Delta \mathrm{s}. \end{aligned}$$

By our obtained results, the mixed impulsive system (36) has only one solution in $PC^1([0,2]_{\mathbb{T}},\mathbb{R})$ and is UH stable on $\mathbb{T}^k$.

7 Conclusion

In this article, we conducted our research on some mixed integral dynamic systems with impulsive effects on times scales in the fractional settings. We studied the existence and uniqueness successfully using a fixed point method for the considered systems. We established our results by using the Leray–Schauder and Banach fixed point theorems in this regard. In the next step, Ulam–Hyers stability and a generalized version of it were proved for the mentioned mixed impulsive systems. After that, we investigated the controllability property for the aforesaid systems. Lastly, an illustrative example was proposed to examine the results established in the previous sections. For future projects, the main aim of the authors is that these qualitative specifications can be checked and established on some real-world impulsive systems arising in mathematical models of brain.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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