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A note on the approximate controllability of second-order integro-differential evolution control systems via resolvent operators

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Abstract
The approximate controllability of second-order integro-differential evolution control systems using resolvent operators is the focus of this work. We analyze approximate controllability outcomes by referring to fractional theories, resolvent operators, semigroup theory, Gronwall's inequality, and Lipschitz condition. The article avoids the use of well-known fixed point theorem approaches. We have also included one example of theoretical consequences that has been validated.

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1 Introduction

The memory effect of the system must be accounted for in numerous disciplines, such as nuclear reactor dynamics and thermoelasticity. The impact of history is overlooked when differential equations, which involve functions at any specific time and space, are used to model such systems. As a result, an integro-differential system is created by adding an integration term to the differential system to include the memory possessions in these frameworks. Integro-differential systems have been widely employed in viscoelastic mechanics, fluid dynamics, thermoelastic contact, control theory, heat conduction, industrial mathematics, financial mathematics, biological models, and other domains, one can refer to [1–7].

Grimmer started and showed the existence of integro-differential systems using resolvent operators in [1, 8, 9]. For solving integro-differential equations, the resolvent operator via fixed-point technique is very easy and most suitable one [10–37]. We recommend readers to [1, 2, 8, 9, 38–42] and the sources referenced therein for more information on resolvent operators and integro-differential systems. Very recently in [4], the author presented the controllability of integro-differential inclusions via resolvent operators by employing the facts connected with resolvent operators and Bohnenblust–Karlin's fixed point approach. Also very recently in [2], the authors proved the existence and controllability results for the integro-differential frameworks by applying resolvent operator theories and various fixed point theorems.

Because it is linked to pole assignment, quadratic optimal control, observer design, and other ideas, controllability is significant in mathematical control theories and technical sectors. In infinite-dimensional systems, the two fundamental principles of controllability that can be distinguished are exact and approximation controllability. This is because there are non-closed linear subspaces in infinite-dimensional spaces. Exact controllability enables the system to be directed to any ultimate state, whereas approximate controllability enables it to be derived to any smaller neighborhood of the ultimate state. Many researchers have published about the controllability debate for fractional and integer order frameworks, see [5, 6, 43–60].

Let us consider the following nonlinear differential evolution equations with control:

$$
\begin{aligned}
z''(t) = A(t)z(t) + \int_0^t \mathscr{B}(t,s)z(s)\,ds + Bv(t)& \\
+ F\bigl(t,z(t)\bigr), \quad t \in J = [0,c],& \qquad (1.1)\\
z(0) = z_0, \quad z'(0) = z_1,& \qquad (1.2)
\end{aligned}
$$

where $A(t) : D(A(t)) \subseteq X \to X$ and $\mathscr{B}(t,s) : D(\mathscr{B}) \subseteq X \to X$ are closed linear operators in the Hilbert space X. $v(\cdot) \in L^2(J,U)$ is a Hilbert space of admissible control functions corresponding to Hilbert space U. Additionally, the linear operator $B : U \to X$ is bounded and $F : J \times X \to X$. Let us consider that $D(\mathscr{B})$ is independent of (t,s).

The linear type system for the above nonlinear differential evolution equations with control (1.1)–(1.2) is presented by

$$
\begin{aligned}
z''(t) = A(t)z(t) + \int_0^t \mathscr{B}(t,s)z(s)\,ds + Bv(t), \quad t \in J = [0,c],& \qquad (1.3)\\
z(0) = z_0, \quad z'(0) = z_1.& \qquad (1.4)
\end{aligned}
$$

Motivations and contributions:

- We study the necessary conditions for the approximate controllability of the proposed system (1.1)–(1.2) by using two different conditions.
- In the first condition, we use B in system (1.1)–(1.2) as an I (identity operator), the sufficient conditions for controllability of integro-differential system are discussed.
- In the second condition, we use B in system (1.1)–(1.2) is any bounded operator, the sufficient conditions for controllability of integro-differential system are discussed.
- Results are obtained with the help of Gronwalls' inequality and Lipschitz condition on nonlinearity.
- It is assumed that the resolvent operator is compact, and consequently the associated linear control system is not exactly controllable but only approximately controllable.
- We show that our result has no analog for the concept of complete controllability. To the best of our knowledge, an approximate controllability result has not been studied in this connection.

- The research focused on the approximate controllability of the proposed system (1.1)–(1.2) under consideration that has not been addressed in the literature to our knowledge, and it supports the current findings.
- Finally, we give an example of the system which is not completely controllable, but approximately controllable.

The structure of the article will now be presented as follows:

1. Sect. 2 discusses some fundamental theories on resolvent operators as well as control theory results.
2. We demonstrate the approximate controllability of integro-differential systems using $B = I$ in Sect. 3.
3. We demonstrate the approximate controllability of integro-differential systems using $B \neq I$ in Sect. 4.
4. Sect. 5 gives an example of how the acquired hypotheses can be validated.

2 Preliminaries

In this section, we mention a few results, notations, and lemmas needed to establish our main results. We introduce certain notations which will be used throughout the article without any further mention. The remainder of this content is structured as follows: $(X, \|\cdot\|)$ is a Banach space and $A(t)$, $\mathscr{B}(t,s)$ for $0 \le s \le t$ are closed linear operators determined on $D(A)$ and $D(\mathscr{B})$, respectively. We assume that $D(A)$ is dense in X.

The space $D(A)$ provided with the graph norm induced by $A(t)$ is a Banach space. We will assume that all of these norms are equivalent. A simple condition for obtaining this property is that there exists $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, so that $(\lambda I - A(t))^{-1} : X \to D(A)$ is a bounded linear operator.

Nowadays, there has been an expanding enthusiasm for examining the second order initial value problem,

$$z''(t) = A(t)z(t) + f(t), \quad 0 \le s, t \le c, \tag{2.1}$$

$$z(s) = z^0, \quad z'(s) = z^1, \tag{2.2}$$

where $A(t) : D(A) \subseteq X \to X$, $t \in [0, c]$, is a densely defined closed linear operator. Additionally, $f : [0, c] \to X$ is an appropriate function. For discussion of this kind, we refer the readers to [3, 61–68]. In many of the articles, the authors discussed the existence of (2.1)–(2.2) connected with $S(t,s)$ has the form

$$z''(t) = A(t)z(t), \quad 0 \le t \le c.$$

Let us assume that, for every $z \in D(A)$, $t \mapsto A(t)z$ is continuous. Now, we assume $A(\cdot)$ generates $(S(t,s))_{0\le s\le t\le c}$, which is discussed by Kozak [64], Definition 2.1 (refer also to Henriquez [69], Definition 1.1).

We refer to these works for a careful study of this issue. We only regard here that $S(\cdot)z$ is continuously differentiable for all $z \in X$ with derivative uniformly bounded on bounded intervals, which in particular implies that there exists $M_1 > 0$ such that

$$\|S(t+h,s) - S(t,s)\| \le M_1|h|$$

for all $s, t, t+h \in [0,c]$. We now determine the operator $C(t,s) = -\frac{\partial S(t,s)}{\partial s}$. Consider $f : [0,c] \to X$ is an integrable function. We now define the mild solution $z : [0,c] \to X$ of system (2.1)–(2.2) is as follows:

$$z(t) = C(t,s)z^0 + S(t,s)z^1 + \int_0^t S(t,\xi)f(\xi)\,d\xi.$$

Next we consider the second-order integro-differential system

$$z''(t) = A(t)z(t) + \int_s^t \mathscr{B}(t,\xi)z(\xi)\,d\xi, \quad s \le t \le c, \tag{2.3}$$

$$z(s) = 0, \quad z'(s) = x \in X, \tag{2.4}$$

for $0 \le s \le c$. This problem was discussed in [3]. We denote $\Delta = \{(t,s) : 0 \le s \le t \le c\}$.

We now introduce some conditions that the operator $B(\cdot)$ presented in [3] fulfills.

(B1) For each $0 \le s \le t \le c$, $\mathscr{B}(t,s) : [D(A)] \to X$ is a bounded linear operator, additionally for every $z \in D(A)$, $\mathscr{B}(\cdot,\cdot)z$ is continuous and

$$\|\mathscr{B}(t,s)z\| \le b\|z\|_{[D(A)]}$$

for $b > 0$ which is independent of $s, t \in \Delta$.

(B2) There exists $L_{\mathscr{B}} > 0$ such that

$$\|\mathscr{B}(t_2,s)z - \mathscr{B}(t_1,s)z\| \le L_{\mathscr{B}}|t_2 - t_1|\|z\|_{[D(A)]}$$

for all $z \in D(A)$, $0 \le s \le t_1 \le t_2 \le c$.

(B3) There exists $b_1 > 0$ such that

$$\left\| \int_\zeta^t S(t,s)\mathscr{B}(s,\zeta)z\,ds \right\| \le b_1\|z\|$$

for all $z \in D(A)$ and $0 \le \zeta \le t \le c$.

Under these conditions, it has been established that there exists $(\mathcal{R}(t,s))_{t\ge s}$ associated with problem (2.3)–(2.4). From now, we are going to consider that such a resolvent operator exists, and we adopt its properties as a definition.

Definition 2.1 ([3]) A family of bounded linear operators $(\mathcal{R}(t,s))_{t\ge s}$ on X is said to be a resolvent operator for system (2.3)–(2.4) if it satisfies:

(a) The map $\mathcal{R} : \Delta \to \mathcal{L}(X)$ is strongly continuous, $\mathcal{R}(t,\cdot)z$ is continuously differentiable for all $z \in X$, $\mathcal{R}(s,s) = 0$, $\frac{\partial}{\partial t}\mathcal{R}(t,s)|_{t=s} = I$, and $\frac{\partial}{\partial s}\mathcal{R}(t,s)|_{s=t} = -I$.

(b) Assume $x \in D$. The function $\mathcal{R}(\cdot,s)x$ is a solution of system (2.3)–(2.4). This means that

$$\frac{\partial^2}{\partial t^2}\mathcal{R}(t,s)x = A(t)\mathcal{R}(t,s)x + \int_s^t \mathscr{B}(t,\xi)\mathcal{R}(\xi,s)x\,d\xi$$

for all $0 \le s \le t \le c$.

It follows from condition (a) that there are constants $M > 0$ and $\widetilde{M} > 0$ such that

$$\|\mathcal{R}(t,s)\| \le M, \qquad \left\|\frac{\partial}{\partial s}\mathcal{R}(t,s)\right\| \le \widetilde{M}, \quad (t,s) \in \Delta.$$

Moreover, the linear operator

$$G(t,\zeta)x = \int_{\zeta}^{t} \mathscr{B}(t,s)\mathcal{R}(s,\zeta)x\,ds, \quad x \in D(A), 0 \le \zeta \le t \le c,$$

can be extended to X. Portraying this expansion by the similar notation $G(t,\zeta)$, $G : \Delta \to \mathcal{L}(X)$ is strongly continuous, and it is verified that

$$\mathcal{R}(t,\zeta)x = S(t,\zeta) + \int_{\zeta}^{t} S(t,s)G(s,\zeta)x\,ds \quad \text{for all } x \in X.$$

The resulting property is that $\mathcal{R}(\cdot)$ is uniformly Lipschitz continuous, that is, there exists $L_{\mathcal{R}} > 0$ such that

$$\|\mathcal{R}(t+h,\zeta) - \mathcal{R}(t,\zeta)\| \le L_{\mathcal{R}}|h| \quad \text{for all } t, t+h, \zeta \in [0,c].$$

Let $g : J \to X$ be an integrable function. The nonhomogeneous problem

$$z''(t) = A(t)z(t) + \int_{0}^{t} \mathscr{B}(t,s)z(s)\,ds + g(t), \quad t \in J = [0,c], \tag{2.5}$$

$$z(0) = x^0, \qquad z'(0) = x^1, \tag{2.6}$$

was discussed in [3]. We now introduce the mild solution for system (2.5)–(2.6).

Definition 2.2 ([3]) Assume $x^0, x^1 \in X$. The function $z : [0,c] \to X$ given by

$$z(t) = -\frac{\partial \mathcal{R}(t,s)x^0}{\partial s}\bigg|_{s=0} + \mathcal{R}(t,0)x^1 + \int_{0}^{t} \mathcal{R}(t,s)g(s)\,ds$$

is said to be the mild solution for system (2.5)–(2.6).

It is clear that $z(\cdot)$ in Definition 2.2 is a continuous function.

Definition 2.3 A continuous function $z : [0,c] \to X$ is said to be a mild solution for system (1.1)–(1.2) if $z(0) = z_0$, $z'(0) = z_1$, and

$$\begin{aligned} z(t) = {} & -\frac{\partial \mathcal{R}(t,s)x^0}{\partial s}\bigg|_{s=0} + \mathcal{R}(t,0)z_1 + \int_{0}^{t} \mathcal{R}(t,s)Bv(s)\,ds \\ & + \int_{0}^{t} \mathcal{R}(t,s)F\bigl(s,z(s)\bigr)\,ds, \quad t \in J, \end{aligned}$$

is fulfilled.

Definition 2.4 The reachable set of system (1.1)–(1.2) given by

$$K_c(F) = \{z(c) \in X : z(t) \text{ represents a mild solution of system (1.1)–(1.2)}\}.$$

In case $F \equiv 0$, system (1.1)–(1.2) reduces to the corresponding linear system. The reachable set in this case is denoted by $K_c(0)$.

Definition 2.5 If $\overline{K_c(F)} = X$, then the semilinear control system is approximately controllable on $[0, c]$. Here $\overline{K_c(F)}$ represents the closure of $K_c(F)$. Clearly, if $\overline{K_c(0)} = X$, then the linear system is approximately controllable.

Assume that $\Psi = L^2(J, X)$. Define the operator $\aleph : \Psi \to \Psi$ as follows:

$$[\aleph z](t) = F\bigl(t, z(t)\bigr); \quad 0 < t \leq c.$$

3 Controllability results when $B = I$

For this discussion, it is shown that the approximate controllability of the linear system reaches from the semilinear system under specific requirements on the nonlinear term. Clearly, $X = U$.

Assume the following linear system

$$w''(t) = A(t)w(t) + \int_0^t \mathscr{B}(t,s)w(s)\,ds + u(t), \quad t \in J = [0, c], \tag{3.1}$$

$$z(0) = z_0, \qquad z'(0) = z_1, \tag{3.2}$$

and the semilinear system

$$z''(t) = A(t)z(t) + \int_0^t \mathscr{B}(t,s)z(s)\,ds + v(t) + F\bigl(t, z(t)\bigr), \quad t \in J, \tag{3.3}$$

$$z(0) = z_0, \qquad z'(0) = z_1. \tag{3.4}$$

For proving the primary task of this section, that is, the approximate controllability of system (3.3)–(3.4), we need to introduce the following hypotheses:

$(\mathbf{H_1})$ Linear system (3.1)–(3.2) is approximately controllable.

$(\mathbf{H_2})$ $F(t, z(t))$ is a nonlinear function which fulfills the Lipschitz condition in z, that is,

$$\bigl\|F(t, z_1) - F(t, z_2)\bigr\|_X \leq l\bigl(\|z_1 - z_2\|_X\bigr), \quad l > 0, \text{ for all } z_1, z_2 \in X, t \in J.$$

Theorem 3.1 *If hypotheses* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *are satisfied, then* (3.3)–(3.4) *is approximately controllable.*

Proof Assume that $w(t)$ is the mild solution of system (3.1)–(3.2), along with the control u. Assume the following semilinear system:

$$\begin{aligned} z''(t) = {} & A(t)z(t) + \int_0^t \mathscr{B}(t,s)z(s)\,ds + F\bigl(t, z(t)\bigr) \\ & + u(t) - F\bigl(t, w(t)\bigr), \quad t \in J, \end{aligned} \tag{3.5}$$

$$z(0) = z_0, \qquad z'(0) = z_1. \tag{3.6}$$

On comparing system (3.3)–(3.4) and system (3.5)–(3.6), we can see the control function $v(t)$ is assumed such that

$$v(t) = u(t) - F\bigl(t, w(t)\bigr).$$

The mild solution of system (3.1)–(3.2) is given by

$$w(t) = -\frac{\partial \mathcal{R}(t,s)x^0}{\partial s}\bigg|_{s=0} + \mathcal{R}(t,0)z_1 + \int_0^t \mathcal{R}(t,s)u(s)\,ds, \quad t \in J, \tag{3.7}$$

and the mild solution of system (3.5)–(3.6) is given by

$$\begin{aligned} z(t) = {}& -\frac{\partial \mathcal{R}(t,s)x^0}{\partial s}\bigg|_{s=0} + \mathcal{R}(t,0)z_1 + \int_0^t \mathcal{R}(t,s)\bigl[F\bigl(s,z(s)\bigr) + u(s) \\ & - F\bigl(s,w(s)\bigr)\bigr]\,ds, \quad t \in J. \end{aligned} \tag{3.8}$$

From equation (3.7) and equation (3.8), we get

$$w(t) - z(t) = \int_0^t \mathcal{R}(t,s)\bigl\{F\bigl(s,w(s)\bigr) - F\bigl(s,z(s)\bigr)\bigr\}\,ds. \tag{3.9}$$

Applying norm on both sides, we have

$$\begin{aligned} \bigl\|w(t) - z(t)\bigr\|_X &\le \int_0^t \bigl\|\mathcal{R}(t,s)\bigr\| \bigl\|F\bigl(s,w(s)\bigr) - F\bigl(s,z(s)\bigr)\bigr\|\,ds \\ &\le M \int_0^t \bigl\|F\bigl(s,w(s)\bigr) - F\bigl(s,z(s)\bigr)\bigr\|\,ds. \end{aligned}$$

Using hypothesis (H_2), we get

$$\bigl\|w(s) - z(s)\bigr\|_X \le Ml \int_0^t \bigl\|w(s) - z(s)\bigr\|\,ds.$$

By referring to Gronwall's inequality, $w(t) = z(t)$ for all $t \in [0,c]$. Thus, the solution w of the linear system along the control u is a solution of the semilinear system z along the control v, i.e., $K_c(F) \supset K_c(0)$. Because $K_c(0)$ is dense in X (by employing hypothesis (H_1)), $K_c(F)$ is dense in X too, which concludes the approximate controllability of system (3.3)–(3.4), and this concludes the proof. □

4 Controllability results of semilinear system when $B \neq I$

Now, the approximate controllability when $B \neq I$ is verified under certain conditions on A, B, and F.

Assume the following linear system

$$w''(t) = A(t)w(t) + \int_0^t \mathscr{B}(t,s)w(s)\,ds + Bu(t), \quad t \in J, \tag{4.1}$$

$$z(0) = z_0, \qquad z'(0) = z_1, \tag{4.2}$$

and the semilinear system

$$z''(t) = A(t)z(t) + \int_0^t \mathscr{B}(t,s)z(s)\,ds + Bv(t) + F\bigl(t,z(t)\bigr), \quad t \in J, \tag{4.3}$$

$$z(0) = z_0, \qquad z'(0) = z_1. \tag{4.4}$$

For proving the primary task of this section, that is, the approximate controllability of system (4.3)–(4.4), we have to introduce the following hypotheses:

($\mathbf{H_3}$) Linear system (4.1)–(4.2) is approximately controllable.

($\mathbf{H_4}$) Range of the operator $\aleph$ is a subset of the closure of range of B, i.e.,

$$\mathrm{Range}(\aleph) \subseteq \overline{\mathrm{Range}(B)}.$$

Theorem 4.1 *If hypotheses* ($\mathbf{H_2}$)–($\mathbf{H_4}$) *are fulfilled, then system* (1.1)–(1.2) *is approximately controllable.*

Proof The mild solution of system (4.1)–(4.2) corresponding to the control u is given by

$$w(t) = -\frac{\partial \mathcal{R}(t,s)x^0}{\partial s}\bigg|_{s=0} + \mathcal{R}(t,0)z_1 + \int_0^t \mathcal{R}(t,s)Bu(s)\,ds, \quad t \in J. \tag{4.5}$$

Assume the following semilinear system:

$$\begin{aligned} z''(t) = {}& A(t)z(t) + \int_0^t \mathscr{B}(t,s)z(s)\,ds + F\bigl(t,z(t)\bigr) + Bu(t) \\ & - F\bigl(t,w(t)\bigr), \quad t \in J, \\ z(0) = {}& z_0, \qquad z'(0) = z_1. \end{aligned}$$

Since $\aleph z \in \overline{\mathrm{Range}(B)}$, for given $\epsilon > 0$, there exists a control function $v \in L^2(J,U)$ such that

$$\|\aleph z - Bv\|_X \le \epsilon. \tag{4.6}$$

Now, assume that $z(t)$ is the mild solution of system (1.1)–(1.2) corresponding to $(u - v)$ given by

$$z(t) = -\frac{\partial \mathcal{R}(t,s)x^0}{\partial s}\bigg|_{s=0} + \mathcal{R}(t,0)z_1 + \int_0^t \mathcal{R}(t,s)\bigl\{B(u-v) + [\aleph z]\bigr\}(s)\,ds, \quad t \in J. \tag{4.7}$$

From equation (4.5) and equation (4.7), we have

$$\begin{aligned} w(t) - z(t) &= \int_0^t \mathcal{R}(t,s)[Bv - z](s)\,ds \\ &= \int_0^t \mathcal{R}(t,s)[Bv - \aleph w](s)\,ds + \int_0^t \mathcal{R}(t,s)[\aleph w - \aleph z](s)\,ds. \end{aligned}$$

Taking norm on both sides and using (4.6), we get

$$\begin{aligned}\|w(t)-z(t)\|_X &= \int_0^t \|\mathcal{R}(t,s)\| \|Bv(s)-\aleph w(s)\| \, ds \\ &\quad + \int_0^t \|\mathcal{R}(t,s)\| \|\aleph w(s)-\aleph z(s)\|_X \, ds \\ &\le M\left(\int_0^t ds\right)^{1/2}\left(\int_0^t \|Bv(s)-\aleph w(s)\|^2 \, ds\right)^{1/2} \\ &\quad + M\int_0^t \|\aleph w(s)-\aleph z(s)\|_X \\ &\le M\sqrt{t}\big(\|\aleph w - Bv\|_{L^2([0,c];X)}\big) + M\int_0^t \|[\aleph w](s)-[\aleph z](s)\|_X \, ds \\ &\le M\sqrt{c}\epsilon + Ml\int_0^t \|w(s)-z(s)\|_X \, ds.\end{aligned}$$

By referring to Gronwall's inequality, we have

$$\|w(t)-z(t)\|_X \le M\sqrt{c}\epsilon \exp(Mlc).$$

Since the right-hand side of the above inequality depends on $\epsilon > 0$ and ϵ is arbitrary, it is clear that $\|w(t)-z(t)\|_X$ becomes arbitrarily small by selecting a suitable control function ν. Clearly, the reachable set of system (1.1)–(1.2) is dense in the reachable set of system (4.1)–(4.2), which is dense in X due to hypothesis (H_3). Hence, the approximate controllability of system (4.1)–(4.2) implies that of the semilinear control system (1.1)–(1.2). □

5 Example

Assume that the integro-differential system with control has the form

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\chi(t,y) &= \frac{\partial^2}{\partial x^2}\chi(t,y) + a(t)\chi(t,y) + \int_0^t b(t-s)\frac{\partial^2\chi(s,y)}{\partial x^2}\, ds \\ &\quad + \mu(t,y) + \frac{\chi^2(t,y)}{(1+t)(1+t^2)}, \quad 0 \le t \le c, 0 \le y \le \pi, \qquad (5.1)\end{aligned}$$

$$\chi(t,0) = \chi(t,\pi) = 0, \qquad (5.2)$$

$$\chi(0,y) = \chi_0(y), \qquad \frac{\partial}{\partial t}\chi(0,y) = \chi_1(y) \quad 0 \le y \le \pi, \qquad (5.3)$$

where $a,b:[0,c]\to\mathbb{R}$, $c>0$ are continuous functions, $\chi_0(y), \chi_1(y) \in X = L^2([0,\pi])$ and the function $\mu : I \times [0,\pi] \to [0,\pi]$ is continuous.

We denote by A_0 the operator given by $A_0 z(\xi) = z''(x)$ with domain

$$D(A) = \{z \in H^2([0,\pi]) : z(0) = z(\pi) = 0\}.$$

Then A_0 is the infinitesimal generator of a cosine function of operators $(C_0(t))_{t\in\mathbb{R}}$ on H associated with sine function $(S_0(t))_{t\in\mathbb{R}}$. Additionally, A_0 has discrete spectrum which consists of eigenvalues $-n^2$ for $n \in \mathbb{N}$, with corresponding eigenvectors

$$w_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}, \quad n \in \mathbb{N}.$$

The set $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of H. Applying this idea, we can write

$$A_0 z = \sum_{n=1}^{\infty} -n^2 \langle z, w_n \rangle w_n$$

for $z \in D(A_0)$, $(C_0(t))_{t\in\mathbb{R}}$ is given by

$$C_0(t)z = \sum_{n=1}^{\infty} \cos(nt)\langle z, w_n \rangle w_n, \quad t \in \mathbb{R},$$

and the sine function is given by

$$S_0(t)z = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle z, w_n \rangle w_n, \quad t \in \mathbb{R}.$$

It is immediate from these representations that $\|C_0(t)\| \leq 1$ and that $S_0(t)$ is compact for all $t \in \mathbb{R}$.

We define $A(t)z = A_0 z + a(t)z$ on $D(A)$. Clearly, $A(t)$ is a closed linear operator. Therefore, $A(t)$ generates $(S(t,s))_{0\leq s\leq t\leq c}$ such that $S(t,s)$ is compact for all $0 \leq s \leq t \leq c$ ([3]).

We complete the terminology by defining $B(t,s) = b(t-s)A_0$ for $0 \leq s \leq t \leq c$ on $D(A)$.

We now assume the function $F : J \times X$ by

$$F(t,x) = \frac{\chi^2(t,x)}{(1+t)(1+t^2)},$$
$$Bv(t,\xi) = \mu(t,\xi).$$

Let us consider that the above functions meet the hypotheses condition with $B = I$ as shown above. Since all the requirements are fulfilled, then system (5.1)–(5.3) is approximately controllable.

6 Conclusion

We primarily focused on the approximate controllability of nonlinear resolvent integro-differential evolution control systems. We analyzed approximate controllability outcomes for the considered systems by referring to resolvent operators, semigroup theory, Gronwall's inequality, and Lipschitz condition. The article avoids the use of well-known fixed point theorem approaches. The discussion on the approximate controllability of nonlinear resolvent stochastic integro-differential evolution equations with impulses will be our future work.

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Declarations

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Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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