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On λ-linear functionals arising from p-adic integrals on $\mathbb{Z}_p$

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Abstract

The aim of this paper is to determine the λ-linear functionals sending any given polynomial $p(x)$ with coefficients in $\mathbb{C}_p$ to the p-adic invariant integral of $P(x)$ on $\mathbb{Z}_p$ and also to that of $P(x_1 + \cdots + x_r)$ on $\mathbb{Z}_p^r$. We show that the former is given by the generating function of degenerate Bernoulli polynomials and the latter by that of degenerate Bernoulli polynomials of order r. For this purpose, we use the λ-umbral algebra which has been recently introduced by Kim and Kim (J. Math. Anal. Appl. 493(1):124521 2021).

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1 Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$.

The umbral calculus can be developed over any field of characteristic zero. Here the field is $\mathbb{C}_p$. Recently, λ-umbral calculus has been introduced to answer the question what if the usual exponential function in (17) is replaced with the degenerate exponential functions in (11). As we can see in Sect. 2, it corresponds to replacing linear functionals, differential operators, and Sheffer sequences respectively with λ-linear functionals, λ-differential operator, and λ-Sheffer sequences. The impetus for studying λ-umbral calculus was the recent regained interest in degenerate special numbers and polynomials.

This paper is concerned with linear functionals on the space of polynomials in one variable over the field $\mathbb{C}_p$ arising from p-adic invariant integrals on $\mathbb{Z}_p$ and on $\mathbb{Z}_p^r$. As such linear functionals are given by formal power series, we have to determine those series corresponding to such linear functionals arising from p-adic invariant integrals. It turns out that the one for $\mathbb{Z}_p$ is given by the generating function of the degenerate Bernoulli numbers (see Theorem 2) and the other for $\mathbb{Z}_p^r$ by that of the degenerate Bernoulli polynomials of order r (see Theorem 4). In addition, we show that λ-differentiations of any polynomial by such generating functions can be expressed by p-adic invariant integrals on $\mathbb{Z}_p$ or

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on $\mathbb{Z}_p^r$ (see Theorems 3, 5). Summarizing, this paper is the first one that treats λ-linear functionals arising from p-adic invariant integrals on $\mathbb{Z}_p$ or on a finite product of $\mathbb{Z}_p$.

The outline of this paper is as follows. In Sect. 1, we recall the necessary facts that are needed throughout this paper. In Sect. 2, we briefly go over the λ-umbral calculus. In addition, we recall the usual umbral calculus in order to state Theorem 1 which shows the differences between the λ-umbral calculus and the umbral calculus. In Sect. 3, we determine the λ-linear functional associated with degenerate Bernoulli numbers. In Sect. 4, we consider the λ-linear functional associated with higher-order degenerate Bernoulli numbers. In Sect. 5, we conclude the paper.

Let $UD(\mathbb{Z}_p)$ be the space of all $\mathbb{C}_p$-valued uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral of a function f on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) &= \lim_{n\to\infty}\sum_{k=0}^{p^n-1} f(k)\mu_0\bigl(k+p^n\mathbb{Z}_p\bigr) \\ &= \lim_{n\to\infty}\frac{1}{p^n}\sum_{k=0}^{p^n-1} f(k) \quad \text{(see [7, 15, 16])}.\end{aligned} \tag{1}$$

From (1), we note that

$$\int_{\mathbb{Z}_p} f(x+1)\,d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = f'(0) \quad \bigl(\text{see [7, 15, 16]}\bigr). \tag{2}$$

For any nonzero real number λ, the degenerate exponential functions are defined by

$$\begin{aligned} &e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \\ &e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}} \quad \bigl(\text{see [2, 5, 8, 9, 11, 12]}\bigr),\end{aligned} \tag{3}$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), (n \geq 1)$.

For $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, by (2), we get

$$\int_{\mathbb{Z}_p} e_\lambda^{x+y}(t)\,d\mu_0(y) = \frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}e_\lambda^x(t) \quad \bigl(\text{see [7, 10]}\bigr),$$

In [10], the degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ are defined by

$$\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}.$$

Note that the degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ here are different from those introduced by Carlitz [2] whose generating function is given by $\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)$. Thus we have

$$\int_{\mathbb{Z}_p} e_\lambda^{x+y}(t)\,d\mu_0(y) = \frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{4}$$

$$\int_{\mathbb{Z}_p}(x+y)_{n,\lambda}\,d\mu_0(y) = \beta_{n,\lambda}(x). \tag{5}$$

For $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. Then we have

$$\int_{\mathbb{Z}_p} e_\lambda^y(t)\,d\mu_0(y) = \frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} = \sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}, \tag{6}$$

$$\int_{\mathbb{Z}_p} (y)_{n,\lambda}\,d\mu_0(y) = \beta_{n,\lambda}. \tag{7}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by $\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$.

2 λ-umbral calculus

Let

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C}_p \right\}$$

be the algebra of all formal power series in t with coefficients in $\mathbb{C}_p$. Let $\mathbb{P} = \mathbb{C}_p[x]$ be the ring of all polynomials in x with coefficients in $\mathbb{C}_p$, and let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$ (see [6]). Let $\langle L|P(x)\rangle$ denote the action of the linear functional L on the polynomial $P(x)$.

The formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F} \tag{8}$$

defines the λ-linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|(x)_{n,\lambda}\rangle_\lambda = a_n \quad (n \geq 0), \bigl(\text{see [6, 11]}\bigr). \tag{9}$$

Thus, by (8) and (9), we get

$$\langle t^k|(x)_{n,\lambda}\rangle_\lambda = n!\delta_{n,k} \quad (n,k \geq 0), \bigl(\text{see [5, 6]}\bigr), \tag{10}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Here, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all λ-linear functionals on $\mathbb{P}$, so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a λ-linear functional. We shall call $\mathcal{F}$ the λ-umbral algebra. The λ-umbral calculus is the study of λ-umbral algebra. The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer k for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series; if $o(f(t)) = 1$, then $f(t)$ is said to be a delta series (see [1, 4–6, 13]).

For $f(t), g(t) \in \mathcal{F}$, with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $S_{n,\lambda}(x)$ ($\deg S_{n,\lambda}(x) = n$) such that $\langle g(t)(f(t))^k|S_{n,\lambda}(x)\rangle_\lambda = n!\delta_{n,k}$, $(n,k \geq 0)$. Such a sequence $S_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ (see [6]). In [6], we note that

$$S_{n,\lambda}(x) \sim \bigl(g(t), f(t)\bigr)_\lambda \quad\Longleftrightarrow\quad \frac{1}{g(\bar{f}(t))} e_\lambda^x\bigl(\bar{f}(t)\bigr) = \sum_{n=0}^{\infty} S_{n,\lambda}(x)\frac{t^n}{n!}, \tag{11}$$

where $\bar{f}(t)$ is the compositional inverse function of $f(t)$ with $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

By (8), (9), and (10), we easily get

$$f(t)=\sum_{k=0}^{\infty}\frac{\langle f(t)|(x)_{k,\lambda}\rangle_{\lambda}}{k!}t^{k},\qquad P(x)=\sum_{k=0}^{\infty}\frac{\langle t^{k}|P(x)\rangle_{\lambda}}{k!}(x)_{k,\lambda}\quad\bigl(\text{see [5, 6]}\bigr).\tag{12}$$

The formal power series $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$ defines the λ-differential operator $(f(t))_{\lambda}$ on $\mathbb{P}$, which is given by

$$\bigl(f(t)\bigr)_{\lambda}(x)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}a_k(x)_{n-k,\lambda}\quad(n\geq 0),\tag{13}$$

and by linear extension (see [5, 6, 11]).

For $k\geq 0$, by (13), we easily get

$$\bigl(t^{k}\bigr)_{\lambda}(x)_{n,\lambda}=\begin{cases}(n)_k(x)_{n-k,\lambda} & \text{if } k\leq n,\\ 0 & \text{if } k>n,\end{cases}\quad\bigl(\text{see [6]}\bigr).\tag{14}$$

Before proceeding further, we would like to say a little about the differences between the λ-umbral calculus and umbral calculus. Facts on umbral calculus are obtained from the corresponding ones on λ-umbral calculus by letting $\lambda\to 0$ and then suppressing 0s from everywhere. Here we mention a few of those.

The formal power series

$$f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$$

defines the linear functional on $\mathbb{P}$ by setting

$$\bigl\langle f(t)|x^{n}\bigr\rangle=a_n\quad(n\geq 0).\tag{15}$$

In particular, we get

$$\bigl\langle t^{k}|x^{n}\bigr\rangle=n!\delta_{n,k}\quad(n,k\geq 0),\bigl(\text{see [5, 6]}\bigr),\tag{16}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f(t),g(t)\in\mathcal{F}$ with $o(f(t))=1$ and $o(g(t))=0$, there exists a unique sequence $S_n(x)$ ($\deg S_n(x)=n$) such that $\langle g(t)(f(t))^{k}|S_n(x)\rangle=n!\delta_{n,k},(n,k\geq 0)$. Such a sequence $S_n(x)$ is called the Sheffer sequence for $(g(t),f(t))$, which is denoted by $S_n(x)\sim(g(t),f(t))$. Moreover, we have

$$S_n(x)\sim\bigl(g(t),f(t)\bigr)\quad\Longleftrightarrow\quad\frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)}=\sum_{n=0}^{\infty}S_n(x)\frac{t^n}{n!}.\tag{17}$$

By (15) and (16), we get

$$f(t)=\sum_{k=0}^{\infty}\frac{\langle f(t)|x^{k}\rangle}{k!}t^{k},\qquad P(x)=\sum_{k=0}^{\infty}\frac{\langle t^{k}|P(x)\rangle}{k!}x^{k}\quad\bigl(\text{see [5, 6]}\bigr).\tag{18}$$

The formal power series $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$ defines the differential operator on $\mathbb{P}$, which is given by

$$f(t)x^n=\sum_{k=0}^{n}\binom{n}{k}a_kx^{n-k}\quad(n\geq 0),\tag{19}$$

and by linear extension (see [5, 6, 11]).

In particular, for $k\geq 0$, by (19) we get

$$t^kx^n=\begin{cases}(n)_kx^{n-k} & \text{if } k\leq n,\\ 0 & \text{if } k>n,\end{cases}\quad\bigl(\text{see [6]}\bigr),\tag{20}$$

where $(x)_0=1,(x)_n=x(x-1)\cdots(x-n+1),(n\geq 1)$.

For further details on umbral calculus, we let the reader refer to [13, 14]. The next theorem is important for our discussion in the following and contains results not addressed in [6].

Theorem 1 *Let $\mathcal{E}_\lambda(t)=\frac{1}{\lambda}(e^{\lambda t}-1)$. Let $f(t),g(t)\in\mathcal{F}$ with $o(f(t))=1$ and $o(g(t))=0$. Then we have the following*:

(a) $S_{n,\lambda}(x)\sim(g(t),f(t))_\lambda\iff S_{n,\lambda}(x)\sim(g(\mathcal{E}_\lambda(t)),f(\mathcal{E}_\lambda(t)))$.

(b) *For any $P(x)\in\mathbb{P}=\mathbb{C}_p[x]$, we have*

$$\bigl(e_\lambda^y(t)\bigr)_\lambda P(x)=P(x+y),\qquad\bigl\langle e_\lambda^y(t)|P(x)\bigr\rangle_\lambda=P(y).$$

(c) *Let $S_{n,\lambda}(x)\sim(g(t),f(t))_\lambda$. Then we have*

$$\bigl(f(t)\bigr)_\lambda S_{n,\lambda}(x)=f\bigl(\mathcal{E}_\lambda(t)\bigr)S_{n,\lambda}(x)=nS_{n-1,\lambda}(x).$$

(d) *For any $f(t)\in\mathcal{F}$ and $P(x)\in\mathbb{P}=\mathbb{C}_p[x]$, we have*

$$\bigl(f(t)\bigr)_\lambda P(x)=f\bigl(\mathcal{E}_\lambda(t)\bigr)P(x).$$

(e) *Let $S_{n,\lambda}(x)\sim(g(t),f(t))_\lambda$, and let $P_{n,\lambda}(x)=(g(t))_\lambda S_{n,\lambda}(x)\sim(1,f(t))_\lambda$. Then we have*

$$\left(\frac{1}{g(t)}\right)_\lambda P_{n,\lambda}(x)=\left(\frac{1}{g(\mathcal{E}_\lambda(t))}\right)P_{n,\lambda}(x)=S_{n,\lambda}(x).$$

(f) *If $S_{n,\lambda}(x)\sim(g(t),t)_\lambda$ (that is, $S_{n,\lambda}(x)$ is a λ-Appell sequence for $g(t)$), then we have*

$$\left(\frac{1}{g(t)}\right)_\lambda(x)_{n,\lambda}=\left(\frac{1}{g(\mathcal{E}_\lambda(t))}\right)(x)_{n,\lambda}=S_{n,\lambda}(x).$$

Proof (a) Observe first that $\bar{\mathcal{E}}_\lambda(t)=\frac{1}{\lambda}\log(1+\lambda t)$. Then, from (17) and (11), we have

$$\begin{aligned}S_{n,\lambda}(x)\sim\bigl(g\bigl(\mathcal{E}_\lambda(t)\bigr),f\bigl(\mathcal{E}_\lambda(t)\bigr)\bigr)&\iff\sum_{n=0}^{\infty}S_{n,\lambda}(x)\frac{t^n}{n!}=\frac{1}{g(\bar{f}(t))}e^{x\bar{\mathcal{E}}_\lambda(\bar{f}(t))}=\frac{1}{g(\bar{f}(t))}e_\lambda^x\bigl(\bar{f}(t)\bigr)\\&\iff S_{n,\lambda}(x)\sim\bigl(g(t),f(t)\bigr)_\lambda.\end{aligned}$$

(b) It suffices to show these for $P(x) = (x)_{n,\lambda}$. The second one is immediate from definition (9). The first one follows from (13) as follows:

$$\begin{aligned}\big(e_\lambda^y(t)\big)_\lambda (x)_{n,\lambda} &= \left(\sum_{l=0}^{\infty}(y)_{l,\lambda}\frac{t^l}{l!}\right)_\lambda (x)_{n,\lambda}\\ &= \sum_{k=0}^{n}\binom{n}{k}(y)_{k,\lambda}(x)_{n-k,\lambda} = (x+y)_{n,\lambda}.\end{aligned}$$

(c) Here we show $(f(t))_\lambda S_{n,\lambda}(x) = nS_{n-1,\lambda}(x)$. Once we show (d), the middle equality will follow.

$$\begin{aligned}\big\langle g(t)f(t)^k\big|\big(f(t)\big)_\lambda S_{n,\lambda}(x)\big\rangle_\lambda &= \big\langle g(t)f(t)^{k+1}\big|S_{n,\lambda}(x)\big\rangle_\lambda = n!\delta_{n,k+1}\\ &= n(n-1)!\delta_{n-1,k} = \big\langle g(t)f(t)^k\big|nS_{n-1,\lambda}(x)\big\rangle_\lambda.\end{aligned}$$

Now, the result follows from the uniqueness of λ-Sheffer sequence. (d) By linear extension, it is enough to show that $(t^k)_\lambda(x)_{n,\lambda} = (\mathcal{E}_\lambda(t))^k(x)_{n,\lambda}$ for $0 \le k \le n$.

As $\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} = e_\lambda^x(t) = e^{x\bar{\mathcal{E}}_\lambda(t)}$, $(x)_{n,\lambda} \sim (1,\mathcal{E}_\lambda(t))$, and hence $\mathcal{E}_\lambda(t)(x)_{n,\lambda} = n(x)_{n-1,\lambda}$, by Theorem 2.3.7 in [13]. Now, from (14) we have

$$\big(t^k\big)_\lambda (x)_{n,\lambda} = (n)_k(x)_{n-k,\lambda} = \big(\mathcal{E}_\lambda(t)\big)^k(x)_{n,\lambda}.$$

(e) From (a), we note that $S_{n,\lambda}(x) \sim (g(\mathcal{E}_\lambda(t)), f(\mathcal{E}_\lambda(t)))$, and $P_{n,\lambda}(x) = g(\mathcal{E}_\lambda(t))S_{n,\lambda}(x) \sim (1, f(\mathcal{E}_\lambda(t))$. Then, from p.107 of [13], we have $S_{n,\lambda}(x) = \frac{1}{g(\mathcal{E}_\lambda(t))}P_{n,\lambda}(x) = (\frac{1}{g(t)})_\lambda P_{n,\lambda}(x)$.

(f) This follows from (e) by noting that $(x)_{n,\lambda} \sim (1,t)_\lambda$. □

3 The λ-linear functional associated with degenerate Bernoulli numbers

Let us determine the λ-linear functional $f(t)$ that satisfies

$$\big\langle f(t)\big|P(x)\big\rangle_\lambda = \int_{\mathbb{Z}_p} P(x)\,d\mu_0(x) \tag{21}$$

for all polynomials $P(x) \in \mathbb{P} = \mathbb{C}_p[x]$.

From (12), we note that

$$\begin{aligned}f(t) &= \sum_{k=0}^{\infty}\frac{\langle f(t)|(x)_{k,\lambda}\rangle_\lambda}{k!}t^k = \sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}(x)_{k,\lambda}\,d\mu_0(x)\frac{t^k}{k!}\\ &= \int_{\mathbb{Z}_p}\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}\,d\mu_0(x) = \int_{\mathbb{Z}_p} e_\lambda^x(t)\,d\mu_0(x)\\ &= \frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}.\end{aligned} \tag{22}$$

Therefore, by (22), we obtain the following theorem.

Theorem 2 *For $P(x) \in \mathbb{P}$, we have*

$$\left\langle \int_{\mathbb{Z}_p} e_\lambda^y(t)\,d\mu_0(y)\middle|P(x)\right\rangle_\lambda = \int_{\mathbb{Z}_p} P(x)\,d\mu_0(x).$$

That is,

$$\left\langle \frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \middle| P(x) \right\rangle_\lambda = \int_{\mathbb{Z}_p} P(x)\, d\mu_0(x).$$

In particular, for $P(x) = (x)_{n,\lambda}$*, we have*

$$\beta_{n,\lambda} = \left\langle \int_{\mathbb{Z}_p} e_\lambda^y(t)\, d\mu_0(y) \middle| (x)_{n,\lambda} \right\rangle_\lambda \quad (n \geq 0).$$

From (4), we note that

$$\beta_{n,\lambda}(x) \sim \left(g(t) = \frac{\lambda(e_\lambda(t)-1)}{\log(1+\lambda t)}, t \right)_\lambda. \tag{23}$$

From (23), Theorem 1(c), and the fact $e^{yt}P(x) = P(x+y)$ (see [13], p.14), we have

$$(t)_\lambda \beta_{n,\lambda}(x) = \mathcal{E}_\lambda(t)\beta_{n,\lambda}(x) = \frac{1}{\lambda}\big(\beta_{n,\lambda}(x+\lambda) - \beta_{n,\lambda}(x)\big) = n\beta_{n-1,\lambda}(x).$$

In addition, from Theorem 1(f), (5), and (6) and noting that $g(\mathcal{E}_\lambda(t)) = \frac{e^t-1}{t}$, we obtain

$$\begin{aligned} \beta_{n,\lambda}(x) &= \int_{\mathbb{Z}_p} (x+y)_{n,\lambda}\, d\mu_0(y) \\ &= \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)_\lambda (x)_{n,\lambda} \\ &= \left(\int_{\mathbb{Z}_p} e_\lambda^y(t)\, d\mu_0(y) \right)_\lambda (x)_{n,\lambda} \\ &= \left(\frac{t}{e^t-1} \right) (x)_{n,\lambda}. \end{aligned} \tag{24}$$

Therefore, by linear extension, from (24) we obtain the following theorem.

Theorem 3 *For any* $P(x) \in \mathbb{P}$*, we have*

$$\begin{aligned} \int_{\mathbb{Z}_p} P(x+y)\, d\mu_0(y) &= \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)_\lambda P(x) \\ &= \left(\int_{\mathbb{Z}_p} e_\lambda^y(t)\, d\mu_0(y) \right)_\lambda P(x) \\ &= \left(\frac{t}{e^t-1} \right) P(x). \end{aligned}$$

Examples (a) Choosing $P(x) = x^n$ in Theorem 3, we get

$$\int_{\mathbb{Z}_p} (x+y)^n\, d\mu_0(y) = \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)_\lambda x^n = \left(\frac{t}{e^t-1} \right) x^n = B_n(x).$$

(b) Let $P(x) = \frac{2}{n}\sum_{l=0}^{n-2} \frac{1}{n-l}\binom{n}{l} B_{n-l} x^l + \frac{2}{n} H_{n-1} x^n$ $(n \geq 2)$ in Theorem 3. Here $B_n = B_n(0)$ are Bernoulli numbers and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ are harmonic numbers. Then, by making use of (a), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} P(x+y)\,d\mu_0(y) &= \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}\right)_\lambda P(x)\\ &= \frac{2}{n}\sum_{l=0}^{n-2}\frac{1}{n-l}\binom{n}{l}B_{n-l}B_l(x)+\frac{2}{n}H_{n-1}B_n(x)\\ &= \sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)B_{n-k}(x),\end{aligned}$$

where the last equality is derived in [12] and can be modified so as to yield a variant of Miki's and Faber–Pandharipande–Zagier (FPZ) identities.

(c) Recall that the Euler polynomials are given by $\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$. By putting $P(x)=\frac{4E_{n+1}}{n^2(n+1)}-\frac{4}{n}\sum_{l=1}^{n}\frac{\binom{n}{l}(H_{n-1}-H_{n-l})}{n-l+1}E_{n-l+1}x^l$ in Theorem 3 and using (a), we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} P(x+y)\,d\mu_0(y) &= \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}\right)_\lambda P(x)\\ &= \frac{4E_{n+1}}{n^2(n+1)}-\frac{4}{n}\sum_{l=1}^{n}\frac{\binom{n}{l}(H_{n-1}-H_{n-l})}{n-l+1}E_{n-l+1}B_l(x)\\ &= \sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x),\end{aligned}$$

where the last equality is deduced in [12].

(d) Let $P(x)=-2\sum_{r=1}^{m}\binom{m}{r}E_r\frac{x^{m+n-r+1}}{m+n-r+1}-2\sum_{s=1}^{n}\binom{n}{s}E_s\frac{x^{m+n-s+1}}{m+n-s+1}+2(-1)^{n+1}\frac{m!n!}{(m+n+1)!}E_{m+n+1}$. Then, by Theorem 3 and using (a), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} P(x+y)\,d\mu_0(y) &= \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}\right)_\lambda P(x)\\ &= -2\sum_{r=1}^{m}\binom{m}{r}E_r\frac{B_{m+n-r+1}(x)}{m+n-r+1}-2\sum_{s=1}^{n}\binom{n}{s}E_s\frac{B_{m+n-s+1}(x)}{m+n-s+1}\\ &\quad+2(-1)^{n+1}\frac{m!n!}{(m+n+1)!}E_{m+n+1}\\ &= E_m(x)E_n(x),\end{aligned}$$

where the last equality is shown in [3].

4 The λ-linear functional associated with higher-order degenerate Bernoulli numbers

For $r\in\mathbb{N}$, we consider the degenerate Bernoulli polynomials of order r given by

$$\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{25}$$

From (2), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r+x}(t)\,d\mu_0(x_1)\cdots d\mu_0(x_r) = \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}\right)^r e_\lambda^x(t) \\ = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{26}$$

This shows that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{n,\lambda}\,d\mu_0(x_1)\cdots d\mu_0(x_r) = \beta_{n,\lambda}^{(r)}(x). \tag{27}$$

In the special case of $x=0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the degenerate Bernoulli numbers of order r.

From (27), we derive the following equation:

$$\begin{aligned}\beta_{n,\lambda}^{(r)} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_r)_{n,\lambda}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &= \sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\ldots,i_r}\int_{\mathbb{Z}_p}(x_1)_{i_1,\lambda}\,d\mu_0(x_1)\cdots\int_{\mathbb{Z}_p}(x_r)_{i_r,\lambda}\,d\mu_0(x_r)\\ &= \sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\ldots,i_r}\beta_{i_1,\lambda}\beta_{i_2,\lambda}\cdots\beta_{i_r,\lambda}.\end{aligned} \tag{28}$$

For $r\in\mathbb{N}$, let

$$g_r(t) = \left(\frac{\lambda(e_\lambda(t)-1)}{\log(1+\lambda t)}\right)^r. \tag{29}$$

Then, by (26) and (11), we get

$$\beta_{n,\lambda}^{(r)}(x) \sim \bigl(g_r(t),t\bigr)_\lambda. \tag{30}$$

Hence, from (30) and Theorem 1(c) we have

$$\frac{1}{g_r(t)}e_\lambda^x(t) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r+x}(t)\,d\mu_0(x_1)\cdots d\mu_0(x_r) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \tag{31}$$

and

$$(t)_\lambda\beta_{n,\lambda}^{(r)}(x) = \mathcal{E}_\lambda(t)\beta_{n,\lambda}^{(r)}(x) = \frac{1}{\lambda}\bigl(\beta_{n,\lambda}^{(r)}(x+\lambda)-\beta_{n,\lambda}^{(r)}(x)\bigr) = n\beta_{n-1,\lambda}^{(r)}(x).$$

Let us determine the λ-linear functional $f_r(t)$ that satisfies

$$\langle f_r(t)|P(x)\rangle_\lambda = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}P(x_1+\cdots+x_r)\,d\mu_0(x_1)\cdots d\mu_0(x_r) \tag{32}$$

for all polynomials $P(x)\in\mathbb{P}=\mathbb{C}_p[x]$.

From (12), we note that

$$\begin{aligned}
f_r(t) &= \sum_{k=0}^{\infty} \frac{\langle f_r(t) | (x)_{k,\lambda} \rangle_\lambda}{k!} t^k \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)_{k,\lambda} \, d\mu_0(x_1) \cdots d\mu_0(x_r) \frac{t^k}{k!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) \, d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r.
\end{aligned} \tag{33}$$

Therefore, by (33), we obtain the following theorem.

Theorem 4 *For any $P(x) \in \mathbb{P} = \mathbb{C}_p[x]$, we have*

$$\begin{aligned}
&\left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) \, d\mu_0(x_1) \cdots d\mu_0(x_r) \middle| P(x) \right\rangle_\lambda \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1 + \cdots + x_r) \, d\mu_0(x_1) \cdots d\mu_0(x_r).
\end{aligned}$$

That is,

$$\left\langle \left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \middle| P(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1 + \cdots + x_r) \, d\mu_0(x_1) \cdots d\mu_0(x_r).$$

In particular, for $P(x) = (x)_{n,\lambda}$, we get

$$\beta_{n,\lambda}^{(r)} = \left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) \, d\mu_0(x_1) \cdots d\mu_0(x_r) \middle| (x)_{n,\lambda} \right\rangle_\lambda.$$

From Theorem 1(f), (29), and (30) and noting that $g(\mathcal{E}_\lambda(t)) = \frac{e^t-1}{t}$, we obtain

$$\begin{aligned}
\beta_{n,\lambda}^{(r)}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_{n,\lambda} \, d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \left(\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \right)_\lambda (x)_{n,\lambda} \\
&= \left(\int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) \, d\mu_0(x_1) \cdots d\mu_0(x_r) \right)_\lambda (x)_{n,\lambda} \\
&= \left(\frac{t}{e^t-1} \right)^r (x)_{n,\lambda}.
\end{aligned} \tag{34}$$

Therefore, by linear extension, we obtain the following theorem.

Theorem 5 *For any $P(x) \in \mathbb{P}$, we have*

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1 + \cdots + x_r + x) \, d\mu_0(x_1) \cdots d\mu_0(x_r)$$

$$= \left(\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \right)_\lambda P(x)$$
$$= \left(\int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t)\, d\mu_0(x_1)\cdots d\mu_0(x_r) \right)_\lambda P(x)$$
$$= \left(\frac{t}{e^t-1} \right)^r P(x).$$

Examples (a) Choosing $P(x) = x^n$ in Theorem 5, we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n\, d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$= \left(\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \right)_\lambda x^n = \left(\frac{t}{e^t-1} \right)^r x^n = B_n^{(r)}(x),$$

where $B_n^{(r)}(x)$ are the Bernoulli polynomials of order r given by $(\frac{t}{e^t-1})^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}$.

(b) By putting $P(x) = (x)_n$ in Theorem 5, and from (34), we obtain

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_n\, d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$= \left(\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \right)_\lambda (x)_n$$
$$= \sum_{k=0}^{n} S_{1,\lambda}(n,k) \left(\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \right)_\lambda (x)_{k,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)\beta_{k,\lambda}^{(r)}(x),$$

where $S_{1,\lambda}(n,k)$, given by $(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}$, are the degenerate Stirling numbers of the first kind.

(c) Recall that the Euler polynomials of order r are given by $(\frac{2}{e^t+1})^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}$. By letting $P(x) = E_n^{(r)}(x)$ in Theorem 3, we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_n^{(r)}(x_1 + \cdots + x_r + x)\, d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$= \left(\left(\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)} \right)^r \right)_\lambda E_n^{(r)}(x)$$
$$= \left(\frac{t}{e^t-1} \right)^r E_n^{(r)}(x) = \left(\frac{t}{e^t-1} \right)^r \left(\frac{2}{e^t+1} \right)^r x^n = \left(\frac{2t}{e^{2t}-1} \right)^r x^n = 2^n B_n^{(r)}\left(\frac{x}{2} \right).$$

5 Conclusion

In this paper, we are concerned with linear functionals on $\mathbb{C}_p[x]$ arising from p-adic invariant integrals on $\mathbb{Z}_p$ and on $\mathbb{Z}_p^r$. Indeed, we determined the linear functional given by $P(x) \to \int_{\mathbb{Z}_p} P(x)\, d\mu_0(x)$ and that given by $P(x) \to \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1 + \cdots + x_r)\, d\mu_0(x_1)\cdots d\mu_0(x_r)$. This means that we have to determine those series corresponding to the two linear functionals arising from p-adic invariant integrals. It turned out that the one for $\mathbb{Z}_p$ is given by the generating function of the degenerate Bernoulli numbers (see Theorem 2) and the other for $\mathbb{Z}_p^r$ by that of the degenerate Bernoulli polynomials of order r (see Theorem 4). In addition, we showed that λ-differentiations of any polynomial by such generating functions can be expressed by p-adic invariant integrals on $\mathbb{Z}_p$ or on $\mathbb{Z}_p^r$ (see Theorems 3, 5). Further, we illustrated Theorems 3 and 5 with some examples. As tools of our research, we used both λ-umbral calculus and umbral calculus as well as p-adic invariant integrals. The differences of the two umbral calculi are stated in Theorem 1.

Our result here shows nice connections between λ-umbral calculus and p-adic invariant integrals. We would like to continue to study λ-umbral calculus and to find possible applications of λ-umbral calculus to other disciplines as well as to mathematics.

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The authors declare that they have no competing interests.

Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; TK and DSK wrote the paper; DSK and TK completed the revision of the article; SL, JK, and SP checked the errors of the article. All authors have read and agreed to the published version of the manuscript.

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