Khan et al. *Advances in Difference Equations* (2021) 2021:471
https://doi.org/10.1186/s13662-021-03629-w

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

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Some applications of q-difference operator involving a family of meromorphic harmonic functions

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Abstract

In this paper, we establish certain new subclasses of meromorphic harmonic functions using the principles of q-derivative operator. We obtain new criteria of sense preserving and univalency. We also address other important aspects, such as distortion limits, preservation of convolution, and convexity limitations. Additionally, with the help of sufficiency criteria, we estimate sharp bounds of the real parts of the ratios of meromorphic harmonic functions to their sequences of partial sums.

Keywords: Quantum derivative operator; Meromorphic harmonic starlike functions; Janowski functions

1 Introduction and definitions

Univalent harmonic functions are a new research area that was initially developed by Clunie and Sheil-Small [15]; see also [40]. The significance of such functions is attributed to their usage in the analysis of minimal surfaces and in problems relevant to applied mathematics. Hengartner and Schober [18] introduced and analyzed some specific types of harmonic functions in the region $\widetilde{\mathfrak{D}}=\{z \in \mathbb{C} : |z| > 1\}$. They proved that a harmonic complex-valued sense-preserving univalent mapping f defined in $\widetilde{\mathfrak{D}}$ and obeying $f(\infty) = \infty$ must satisfy the following representation:

$$f(z) = \mathfrak{G}_1(z) + \overline{\mathfrak{G}_2(z)} + A \log|z|, \tag{1.1}$$

where

$$\mathfrak{G}_1(z) = \mu_1 z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad \mathfrak{G}_2(z) = \mu_2 \bar{z} + \sum_{n=1}^{\infty} b_n \bar{z}^{-n}$$

with $0 \leq |\mu_2| < |\mu_1|$ and $A \in \mathbb{C}$. In 1999, Jahangiri and Silverman [26] gave adequate coefficient criteria for functions of type (1.1) to be univalent. They also provided necessary and sufficient coefficient criteria within certain constraints for functions to be harmonic and starlike. Using this idea, the authors of [24] contributed a certain family of harmonic close-to-convex functions involving the Alexander integral transform. In 2000, Jahangiri [22] and Murugusundaramoorthy [35, 36] analyzed the families of meromorphic harmonic function in $\widetilde{\mathfrak{D}}$. In [12, 14] the authors used the technique developed by Zou and his coauthors in [55] to examine the natures of meromorphic harmonic starlike functions with respect to symmetrical conjugate points in the punctured disc $\mathfrak{D}^*=\{z \in \mathbb{C} : 0 < |z| < 1\} = \mathfrak{D}\backslash\{0\}$. Particularly, in [14] a sharp approximation of the coefficients and a structural description of these functions are also determined. To understand the basics in a more clear way, we denote by $\mathcal{H}$ the family of harmonic functions f that can be represented in the series form

$$f(z) = \mathfrak{h}(z) + \overline{\mathfrak{g}(z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\bigl(a_n z^n + b_n \overline{z}^n\bigr) \quad \bigl(z \in \mathfrak{D}^*\bigr), \tag{1.2}$$

where $\mathfrak{h}$ and $\mathfrak{g}$ are holomorphic functions in $\mathfrak{D}^*$ and $\mathfrak{D}$ of the form

$$\mathfrak{h}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \bigl(z \in \mathfrak{D}^*\bigr) \quad \text{and} \quad \mathfrak{g}(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathfrak{D}) \tag{1.3}$$

and

$$|a_n| \geq 1, \qquad |b_n| \geq 1 \quad (n = 2, 3, \ldots).$$

Also, let us denote by $\mathcal{M}_{\mathcal{H}}$ the set of complex-valued functions $f \in \mathcal{H}$ that are sense preserving and univalent in $\mathfrak{D}^*$. Clearly, if $\mathfrak{g}(z) \equiv 0$ $(z \in \mathfrak{D})$, then $\mathcal{M}_{\mathcal{H}}$ matches with the collection $\mathcal{M}$ of holomorphic univalent normalized functions in $\mathfrak{D}$. The above foundational papers opened a new door for the researchers to add some input in this area of function theory. In this regard, we consider the collections of meromorphic harmonic starlike and meromorphic harmonic convex functions in $\mathfrak{D}^*$

$$\mathcal{MS}^*_{\mathcal{H}} = \left\{f \in \mathcal{M}_{\mathcal{H}} : -\frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+z}{1-z}\bigl(z \in \mathfrak{D}^*\bigr)\right\}$$

and

$$\mathcal{MS}^c_{\mathcal{H}} = \left\{f \in \mathcal{M}_{\mathcal{H}} : -\frac{\mathcal{D}_{\mathcal{H}}(\mathcal{D}_{\mathcal{H}} f(z))}{\mathcal{D}_{\mathcal{H}} f(z)} \prec \frac{1+z}{1-z}\bigl(z \in \mathfrak{D}^*\bigr)\right\},$$

where the notation $\prec$ shows the familiar subordination between the holomorphic functions, and

$$\mathcal{D}_{\mathcal{H}} f(z) = z\mathfrak{h}'(z) - \overline{z\mathfrak{g}'(z)}.$$

Furthermore, many subfamilies of meromorphic harmonic functions have also been established by some well-known researchers; for example, see Bostanci [11], Bostanci and Öztürk [13], Öztürk and Bostanci [38], Wang et al. [54], Al-dweby and Darus [3], Al-Shaqsi and Darus [4], Ponnusamy and Rajasekaran [39], Ahuja and Jahangiri [2], Al-Zkeri and Al-Oboudi [5], Stephen et al. [53], and Khan et al. [32].

The investigation of q-calculus (q stands for quantum) fascinated and inspired many scholars due its use in various areas of the quantitative sciences. Jackson [20, 21] was among the key contributors of all the scientists who introduced and developed the q-calculus theory. Just like q-calculus was used in other mathematical sciences, the formulations of this idea are commonly used to examine the existence of various structures of function theory. The first paper in which a link was established between certain geometric nature of the analytic functions and the q-derivative operator is due to the authors [19]. For the usage of q-calculus in function theory, a solid and comprehensive foundation is given by Srivastava [43]. After this development, many researchers introduced and studied some useful operators in q-analog with applications of convolution concepts. For example, Kanas and Răducanu [27] established the q-differential operator and then examined the behavior of this operator in function theory. For more applications of this operator, see [1, 7, 17]. This operator was generalized further for multivalent analytic functions by Arif et al. [8] and later studied in [30, 41, 51]. Analogous to q-differential operator Arif et al. [9] and Khan et al. [33] contributed the integral operators for analytic and multivalent functions, respectively. Similarly, in [6] the authors developed and analyzed operators in q-analog for meromorphic functions. Also, see the survey-type paper [44] on quantum calculus and its applications. In 2021, Srivastava, Arif, and Raza [46] introduced and studied a generalized convolution q-derivative operator for meromorphic harmonic functions. Using these operators, many researchers contributed some good papers in this direction in geometric function theory; see [16, 23, 25, 28, 29, 31, 37, 45, 50, 52].

Definition 1.1 Let $q \in]0,1[$. Then the q-analog derivative of f is

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} \quad (z \in \mathfrak{D}). \tag{1.4}$$

See also [10, 48, 49], and [47] for some recent applications of the q-difference operators in the theory of q-series and q-polynomials.

By means of (1.2) and (1.4) we obtain

$$\begin{aligned} D_q f(z) &= D_q \mathfrak{h}(z) + \overline{D_q \mathfrak{g}(z)} \\ &= -\frac{1}{qz^2} + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} + \sum_{n=1}^{\infty} [n]_q \overline{b_n z^{n-1}} \quad \text{for } n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{k=1}^{n-1} q^k, \quad \text{and} \quad [0]_q = 0.$$

To prevent repetition, we will assume, unless otherwise stated, that

$$-1 \le M < L \le 1 \quad \text{and} \quad q \in]0,1[.$$

Definition 1.2 By $\mathcal{MS}^*_{\mathcal{H}}(q, L, M)$ we denote the set of functions $f \in \mathcal{M}_{\mathcal{H}}$ such that

$$-\frac{q\mathcal{D}^q_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+Lz}{1+Mz} \quad \left(z \in \mathfrak{D}^*\right),$$

where

$$\mathcal{D}^q_{\mathcal{H}} f(z) = zD_q\mathfrak{h}(z) - \overline{zD_q\mathfrak{g}(z)}.$$

Similarly, we denote

$$\mathcal{MS}^c_{\mathcal{H}}(q,L,M) := \{f \in \mathcal{M}_{\mathcal{H}} : \mathcal{D}^q_{\mathcal{H}} f(z) \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M)(z \in \mathfrak{D}^*)\}.$$

In this paper, we learn some nice properties for the currently established families including distortion limits, univalency criteria, partial-sum problems, sufficiency criteria, convexity conditions, and preserving convolutions.

2 Necessary and sufficient conditions

Theorem 2.1 *If $f \in \mathcal{H}$ is described by the series of the form (1.2) and if*

$$\sum_{n=1}^{\infty}\bigl(\rho_n|a_n| + \sigma_n|b_n|\bigr) \le L - M, \tag{2.1}$$

then $f \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M)$ with

$$\rho_n = \bigl|\bigl(q[n]_q + 1\bigr)\bigr| + \bigl|\bigl(Mq[n]_q + L\bigr)\bigr|, \tag{2.2}$$

$$\sigma_n = \bigl|\bigl(q[n]_q - 1\bigr)\bigr| + \bigl|\bigl(Mq[n]_q - L\bigr)\bigr|. \tag{2.3}$$

Proof If $f(z) = \frac{1}{z}$, then we have $\mathfrak{h}(z) = \frac{1}{z}$ and $\mathfrak{g}(z) = 0$. This implies that

$$\bigl|\mathfrak{h}'(z)\bigr| - \bigl|\mathfrak{g}'(z)\bigr| > 0.$$

Hence by the result of Lewy [34] the function f in $\mathfrak{D}^*$ is locally univalent and orientation-preserving. Now we show that f is univalent in $\mathfrak{D}^*$. Let $z_1, z_2 \in \mathfrak{D}^*$ with $z_1 \neq z_2$. Then

$$\bigl|f(z_1) - f(z_2)\bigr| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right| = \frac{|z_2 - z_1|}{|z_1 z_2|} > 0.$$

To show that $f \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M)$, we have to establish that

$$\left|\frac{q\mathcal{D}^q_{\mathcal{H}} f(z) + f(z)}{Lf(z) + Mq\mathcal{D}^q_{\mathcal{H}} f(z)}\right| < 1.$$

It is easy to find that $q\mathcal{D}^q_{\mathcal{H}} f(z) = -\frac{1}{z}$ and $L - M > 0$. This indicates that

$$\left|\frac{q\mathcal{D}^q_{\mathcal{H}} f(z) + f(z)}{Lf(z) + Mq\mathcal{D}^q_{\mathcal{H}} f(z)}\right| = \left|\frac{-\frac{1}{z} + \frac{1}{z}}{L - M}\right| = 0 < 1.$$

Hence $f \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M)$. Now let $f \in \mathcal{H}$ have be of the form (1.2), and let us choose $n \ge 1$ such that $a_n \neq 0$ or $b_n \neq 0$. Also, by using

$$q[n]_q = q\left(1 + \sum_{k=1}^{n-1} q^k\right) > q \quad \text{for } 0 < q < 1$$

we have

$$\begin{aligned}\frac{\sigma_n}{L-M} &= \frac{|(q[n]_q-1)|+|(Mq[n]_q-L)|}{L-M}\\ &> \frac{|(q-n)|+|(Mq-Ln)|}{L-M} = \frac{(n-q)+(Ln-Mq)}{L-M}\\ &> \frac{(n-1)+(Ln-M)}{L-M} = \frac{(1+L)n-(1+M)}{L-M}\\ &> \frac{(1+L)n-(1+M)n}{L-M} = n \quad \text{for all } n \geq 1.\end{aligned}$$

Similarly, $\frac{\rho_n}{L-M} \geq n$ for $n \geq 1$. Thus using (2.1) together with the above evidence, we get

$$\sum_{n=1}^{\infty}\bigl(n|a_n|+n|b_n|\bigr) \leq 1, \tag{2.4}$$

and therefore

$$\begin{aligned}\bigl|\mathfrak{h}'(z)\bigr|-\bigl|\mathfrak{g}'(z)\bigr| &\geq \frac{1}{|z|^2}-\sum_{n=1}^{\infty}n|a_n||z|^{n-1}-\sum_{n=1}^{\infty}n|b_n||z|^{n-1}\\ &\geq \frac{1}{|z|^2}\left(1-|z|\sum_{n=1}^{\infty}\bigl(n|a_n|+n|b_n|\bigr)\right)\\ &\geq \frac{1}{|z|^2}\left(1-\frac{|z|}{L-M}\sum_{n=1}^{\infty}\bigl(\rho_n|a_n|+\sigma_n|b_n|\bigr)\right)\\ &\geq \frac{1}{|z|^2}\bigl(1-|z|\bigr) > 0 \quad \bigl(z \in \mathfrak{D}^*\bigr).\end{aligned}$$

Therefore by Lewy's result [34] the function f in $\mathfrak{D}^*$ is sense-preserving and locally univalent. Moreover, if $z_1, z_2 \in \mathfrak{D}^*$ with$z_1 \neq z_2$, then

$$\left|\frac{z_1^n-z_2^n}{z_1-z_2}\right| = \sum_{k=1}^{n}|z_1|^{k-1}|z_2|^{k-1} \leq n \quad \text{for } n \geq 2.$$

Hence by (2.4) we have

$$\begin{aligned}\bigl|f(z_1)-f(z_2)\bigr| &\geq \bigl|\mathfrak{h}(z_1)-\mathfrak{h}(z_2)\bigr|-\bigl|\mathfrak{g}(z_1)-\mathfrak{g}(z_2)\bigr|\\ &= \left|\frac{1}{z_1}-\frac{1}{z_2}-\sum_{n=1}^{\infty}a_n\bigl(z_1^n-z_2^n\bigr)\right|-\left|\sum_{n=1}^{\infty}b_n\bigl(\overline{z_1^n-z_2^n}\bigr)\right|\\ &\geq |z_1-z_2|\left(\frac{1}{|z_1z_2|}-\sum_{n=1}^{\infty}\bigl(n|a_n|+n|b_n|\bigr)\right)\\ &\geq |z_1-z_2|\left(\frac{1}{|z_1z_2|}-1\right) > 0.\end{aligned}$$

This shows that f is univalent in $\mathfrak{D}^*$, and thus $f \in \mathcal{M}_{\mathcal{H}}$. Therefore $f \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M)$ if and only if there exists a holomorphic function u with $u(0) = 0$ and $|u(z)| < 1$ such that

$$-\frac{q\mathcal{D}^q_{\mathcal{H}}f(z)}{f(z)} = \frac{1+Lu(z)}{1+Mu(z)}$$

or, alternatively,

$$\left|\frac{q\mathcal{D}^q_{\mathcal{H}}f(z)+f(z)}{Lf(z)+Mq\mathcal{D}^q_{\mathcal{H}}f(z)}\right| < 1. \tag{2.5}$$

To prove (2.5), it suffices to show that

$$\left|q\mathcal{D}^q_{\mathcal{H}}f(z)+f(z)\right| - \left|Lf(z)+Mq\mathcal{D}^q_{\mathcal{H}}f(z)\right| < 0$$

for $z \in \mathfrak{D}$. Putting $|z| = r$ $(0 < r < 1)$, we attain

$$\begin{aligned}
&\left|q\mathcal{D}^q_{\mathcal{H}}f(z)+f(z)\right| - \left|Lf(z)+Mq\mathcal{D}^q_{\mathcal{H}}f(z)\right| \\
&\quad\le \left|\sum_{n=1}^{\infty}\bigl(q[n]_q+1\bigr)a_n z^n - \sum_{n=1}^{\infty}\bigl(q[n]_q-1\bigr)\overline{b_n z^n}\right| \\
&\qquad - \left|\frac{(L-M)}{z} + \sum_{n=1}^{\infty}\bigl(L+Mq[n]_q\bigr)a_n z^n - \sum_{n=1}^{\infty}\bigl(Mq[n]_q-L\bigr)\overline{b_n z^n}\right| \\
&\quad\le \left\{\sum_{n=1}^{\infty}\bigl|\bigl(q[n]_q+1\bigr)\bigr||a_n| + \sum_{n=1}^{\infty}\bigl|\bigl(q[n]_q-1\bigr)\bigr||b_n|\right\} \\
&\qquad - \frac{1}{r}\left\{\bigl|(L-M)\bigr| - \sum_{n=1}^{\infty}\bigl|\bigl(Mq[n]_q+L\bigr)\bigr||a_n| - \sum_{n=1}^{\infty}\bigl|\bigl(Mq[n]_q-L\bigr)\bigr||b_n|\right\} \\
&\quad\le \frac{1}{r}\left\{-|L-M| + \sum_{n=1}^{\infty}\Bigl(\bigl|\bigl(q[n]_q+1\bigr)\bigr| + \bigl|\bigl(Mq[n]_q+L\bigr)\bigr|\Bigr)|a_n|\right. \\
&\qquad \left. + \sum_{n=1}^{\infty}\Bigl(\bigl|\bigl(q[n]_q-1\bigr)\bigr| + \bigl|\bigl(Mq[n]_q-L\bigr)\bigr|\Bigr)|b_n|\right\} \\
&\quad\le \frac{1}{r}\left\{-(L-M) + \sum_{n=1}^{\infty}\bigl(\rho_n|a_n| + \sigma_n|b_n|\bigr)\right\}
\end{aligned}$$

due inequality (2.1). Thus $f \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M)$. □

By substituting specific values of the parameters included in this result we obtain the following corollaries.

Corollary 2.2 *Let $f \in \mathcal{H}$ be of the form* (1.2). *If*

$$\sum_{n=1}^{\infty}\bigl(\rho_n|a_n| + \sigma_n|b_n|\bigr) \le (1+q)$$

with

$$\rho_n = \left|\left(q[n]_q + 1\right)\right| + \left|\left(q^2[n]_q - 1\right)\right|,$$
$$\sigma_n = \left|\left(q[n]_q - 1\right)\right| + \left|\left(q^2[n]_q + 1\right)\right|,$$

then $f \in \mathcal{MS}^*_{\mathcal{H}}(q,1,-q)$

Proof The result is obtained by setting $L = 1$ and $M = -q$ in the last theorem. □

Corollary 2.3 *Let* $f \in \mathcal{H}$ *be given in* (1.2). *If*

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) \leq 1,$$

then $f \in \mathcal{MS}^*_{\mathcal{H}}(1,1,-1)$.

Proof Taking the limit as $q \to 1-$ in the above corollary, we get the needed result. □

Influenced by Silverman's paper [42], the set ϑ^λ, $\lambda \in \{0,1\}$, of functions $f \in \mathcal{H}$ of type (1.2) is now described as

$$a_n = -|a_n|, \qquad b_n = (-1)^\lambda |b_n| \quad \left(\text{for } n \in \mathbb{N}\backslash\{1\}\right).$$

Hence (1.2) and (1.3) give $f(z) = \mathfrak{h}(z) + \overline{\mathfrak{g}(z)}$ with

$$\mathfrak{h}(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n, \qquad \overline{\mathfrak{g}(z)} = (-1)^\lambda \sum_{n=1}^{\infty} |b_n| \overline{z^n} \quad (z \in \mathfrak{D}). \tag{2.6}$$

Using the above facts, we now define the families

$$\mathcal{MS}^*_{\mathcal{H}_\vartheta}(q,L,M) = \vartheta^0 \cap \mathcal{MS}^*_{\mathcal{H}}(q,L,M),$$
$$\mathcal{MS}^c_{\mathcal{H}_\vartheta}(q,L,M) = \vartheta^1 \cap \mathcal{MS}^c_{\mathcal{H}}(q,L,M).$$

Let us now prove that condition (2.1) is also appropriate for $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}$.

Theorem 2.4 *Let* $f \in \vartheta^0$ *have expansion* (2.6). *Then* $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q,L,M)$ *if and only if* (2.1) *is true.*

Proof To achieve the result, it is sufficient to determine that $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q,L,M)$ validates relationship (2.1). Let $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q,L,M)$. Then inequality (2.5) holds, that is, for $z \in \mathfrak{D}^*$,

$$\left|\frac{\sum_{n=1}^{\infty}(q[n]_q + 1)a_n z^n + \sum_{n=1}^{\infty}(q[n]_q - 1)\overline{b_n z^n}}{\frac{L-M}{z} - \sum_{n=1}^{\infty}(qM[n]_q + L)a_n z^n + \sum_{n=1}^{\infty}(qM[n]_q - L)\overline{b_n z^n}}\right| < 1.$$

Setting $z = r$ ($r \in (0,1)$), we obtain

$$\frac{\sum_{n=1}^{\infty}(|(q[n]_q + 1)||a_n| + |(q[n]_q - 1)||b_n|)r^{n+1}}{(L-M) - \sum_{n=1}^{\infty}(|(qM[n]_q + L)||a_n| + |(qM[n]_q - L)||b_n|)r^{n+1}} < 1. \tag{2.7}$$

Obviously, in case of $r \in (0,1)$, the left-hand side denominator of (2.7) cannot be zero. In addition, this is positive when $r = 0$. Thus, using (2.7), we get

$$\sum_{n=1}^{\infty}\left(\rho_n|a_n| + \sigma_n|b_n|\right)r^{n+1} \le (L-M) \quad (0 \le r < 1). \tag{2.8}$$

It is straightforward that the partial-sum sequence $\{S_n\}$ attached with the series $\sum_{n=1}^{\infty}(\rho_n|a_n| + \sigma_n|b_n|)$ is nondecreasing sequence, and by (2.8) it is bounded by $(L-M)$. So $\{S_n\}$ is a convergent sequence, and

$$\sum_{n=1}^{\infty}\left(\rho_n|a_n| + \sigma_n|b_n|\right)r^{n+1} = \lim_{n\to\infty} S_n \le (L-M),$$

which gives assumption (2.1). □

Example 2.5 Let us choose the function

$$T(z) = \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{L-M}{\rho_n}\frac{1}{2^n}z^n + \frac{L-M}{\sigma_n}\frac{1}{2^n}\overline{z^n}\right) \quad \left(z \in \mathfrak{D}^*\right).$$

Then we easily get

$$\sum_{n=1}^{\infty}\left(\rho_n|a_n| + \sigma_n|b_n|\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}(L-M) = (L-M).$$

Thus $T \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q,L,M)$.

By using the above-mentioned theorem along with the notion of class $\mathcal{MS}^c_{\mathcal{H}}(q,L,M)$ we can easily derive the following results.

Corollary 2.6 *Let $f \in \mathcal{H}$ be written in the form of Taylor expansion* (1.2). *If*

$$\sum_{n=1}^{\infty}[n]_q\left(\rho_n|a_n| + \sigma_n|b_n|\right) \le (L-M), \tag{2.9}$$

then $f \in \mathcal{MS}^c_{\mathcal{H}}(q,L,M)$.

Proof From inequality (2.9), Theorem 2.1, and Alexander-type relation

$$f \in \mathcal{MS}^c_{\mathcal{H}}(q,L,M) \quad \Longleftrightarrow \quad \mathcal{D}^q_{\mathcal{H}}f \in \mathcal{MS}^*_{\mathcal{H}}(q,L,M) \tag{2.10}$$

we easily get the desired result. □

Corollary 2.7 *Let $f \in \vartheta^1$ be written in the series form* (2.6). *Then $f \in \mathcal{MS}^c_{\mathcal{H}_\vartheta}(q,L,M)$ if and only if inequality* (2.9) *is fulfilled.*

Proof Using relationship (2.10) and Theorem 2.4, we get the desired result. □

3 Investigation of partial-sum problems

In this section, we investigate problems of partial sums of certain meromorphic harmonic functions belonging to $\mathcal{MS}^*_{\mathcal{H}}(q,L,M)$. We produce some new findings that connect the meromorphic harmonic functions with their partial-sum sequences. Let $f=\mathfrak{h}+\overline{\mathfrak{g}}$ with $\mathfrak{h}$ and $\mathfrak{g}$ given in (1.3). Then the partial-sum sequences of f are specified by

$$\mathcal{M}_{\mathcal{S}_t}(f)=\frac{1}{z}+\sum_{n=1}^{t}a_nz^n+\sum_{n=1}^{\infty}b_n\overline{z}^n:=\mathcal{M}_{\mathcal{S}_t}(\mathfrak{h})+\overline{\mathfrak{g}},$$

$$\mathcal{M}_{\mathcal{S}_l}(f)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n+\sum_{n=1}^{l}b_n\overline{z}^n:=\mathcal{M}_{\mathcal{S}_l}(\overline{\mathfrak{g}})+\mathfrak{h},$$

$$\mathcal{M}_{\mathcal{S}_{t,l}}(f)=\frac{1}{z}+\sum_{n=1}^{t}a_nz^n+\sum_{n=1}^{l}b_n\overline{z}^n:=\mathcal{M}_{\mathcal{S}_t}(\mathfrak{h})+\mathcal{M}_{\mathcal{S}_l}(\overline{\mathfrak{g}}).$$

Now we find sharp lower bounds for

$$\operatorname{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_t}(f)}\right)\quad\text{and}\quad\operatorname{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_t}(f)}{f(z)}\right),\qquad\operatorname{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_l}(f)}\right)\quad\text{and}\quad\operatorname{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_l}(f)}{f(z)}\right),$$

and

$$\operatorname{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_{t,l}}(f)}\right)\quad\text{and}\quad\operatorname{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_{t,l}}(f)}{f(z)}\right).$$

Theorem 3.1 *Let f have the form* (1.2). *If f fulfills* (2.1), *then*

$$\operatorname{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_t}(f)}\right)\geq\frac{\mathcal{I}_{t+1}-L+M}{\mathcal{I}_{t+1}}\tag{3.1}$$

and

$$\operatorname{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_t}(f)}{f(z)}\right)\geq\frac{\mathcal{I}_{t+1}}{\mathcal{I}_{t+1}-L+M},\tag{3.2}$$

where

$$\mathcal{I}_n=\min(\rho_n,\sigma_n)\tag{3.3}$$

and

$$\mathcal{I}_n\geq\begin{cases}L-M & \text{for } n=1,2,\dots,t,\\ \mathcal{I}_{t+1} & \text{for } n=t+1,\dots.\end{cases}\tag{3.4}$$

The findings above are best suited for the function

$$f(z)=\frac{1}{z}+\frac{L-M}{\mathcal{I}_{t+1}}z^{t+1},\tag{3.5}$$

where $\mathcal{I}_{t+1}$ is given by (3.4).

Proof Let us represent

$$\begin{aligned}\Theta_1(z) &= \frac{\mathcal{I}_{t+1}}{L-M}\left\{\frac{f(z)}{\mathcal{M}_{\mathcal{S}_t}(f)} - \left(1 - \frac{L-M}{\mathcal{I}_{t+1}}\right)\right\} \\ &= 1 + \frac{\frac{\mathcal{I}_{t+1}}{L-M}\sum_{n=1}^{\infty} a_n z^n}{\frac{1}{z} + \sum_{n=1}^{t} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n}.\end{aligned}$$

Inequality (3.1) will be acquired if we are able to show that $\mathrm{Re}\{\Theta_1(z)\} > 0$, and for this, we required to conclude that

$$\left|\frac{\Theta_1(z) - 1}{\Theta_1(z) + 1}\right| \le 1.$$

Alternatively, we have the following inequalities:

$$\left|\frac{\Theta_1(z) - 1}{\Theta_1(z) + 1}\right| \le \frac{\frac{\mathcal{I}_{t+1}}{L-M}\sum_{n=t+1}^{\infty} |a_n|}{2 - 2(\sum_{n=1}^{t} |a_n| + \sum_{n=1}^{\infty} |b_n|) - \frac{\mathcal{I}_{t+1}}{L-M}\sum_{n=t+1}^{\infty} |a_n|} \le 1$$

if and only if

$$\sum_{n=1}^{t} |a_n| + \sum_{n=1}^{\infty} |b_n| + \frac{\mathcal{I}_{t+1}}{L-M}\sum_{n=t+1}^{\infty} |a_n| \le 1. \tag{3.6}$$

From (2.1) we have that it suffices to guarantee that the left-hand side of (3.6) is bounded above by

$$\sum_{n=1}^{\infty} \frac{\mathcal{I}_n}{L-M}|a_n| + \sum_{n=1}^{\infty} \frac{\mathcal{I}_n}{L-M}|b_n|,$$

which is exactly equivalent to

$$\sum_{n=1}^{t} \frac{\mathcal{I}_n - L + M}{L-M}|a_n| + \sum_{n=1}^{\infty} \frac{\mathcal{I}_n - L + M}{L-M}|b_n| + \sum_{n=t+1}^{\infty} \frac{\mathcal{I}_n - \mathcal{I}_{t+1}}{L-M}|a_n| \ge 0,$$

and this is true because of (3.4). We observe that the function

$$f(z) = \frac{1}{z} + \frac{L-M}{\mathcal{I}_{t+1}} z^{t+1}$$

offers the best possible outcome. We see for $z = re^{i\frac{\pi}{t}}$ that

$$\frac{f(z)}{\mathcal{M}_{\mathcal{S}_t}(f)} = 1 + \frac{L-M}{\mathcal{I}_{t+1}} z^{t+2} \to 1 - \frac{L-M}{\mathcal{I}_{t+1}} r^{t+2} = \frac{\mathcal{I}_{t+1} - L + M}{\mathcal{I}_{t+1}}.$$

To examine (3.2), let us write

$$\Theta_2(z) = \frac{\mathcal{I}_{t+1} + L - M}{L-M}\left\{\frac{\mathcal{M}_{\mathcal{S}_t}(f)}{f(z)} - \left(1 - \frac{L-M}{\mathcal{I}_{t+1} + L - M}\right)\right\}$$

$$= 1 - \frac{\frac{\mathcal{I}_{t+1}+L-M}{L-M}\sum_{n=t+1}^{\infty} a_n z^n}{\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n}.$$

Then

$$\left|\frac{\Theta_2(z)-1}{\Theta_2(z)+1}\right| \le \frac{\frac{\mathcal{I}_{t+1}+L-M}{L-M}\sum_{n=t+1}^{\infty}|a_n|}{2-2(\sum_{n=1}^{t}|a_n| + \sum_{n=1}^{\infty}|b_n|) - \frac{\mathcal{I}_{t+1}+L-M}{L-M}\sum_{n=t+1}^{\infty}|a_n|} \le 1$$

if and only if

$$\sum_{n=1}^{t}|a_n| + \sum_{n=1}^{\infty}|b_n| + \frac{\mathcal{I}_{t+1}}{L-M}\sum_{n=t+1}^{\infty}|a_n| \le 1. \tag{3.7}$$

Inequality (3.7) is valid if the left-hand side of this inequality is bounded above by

$$\sum_{n=1}^{\infty}\frac{\mathcal{I}_n}{L-M}|a_n| + \sum_{n=1}^{\infty}\frac{\mathcal{I}_n}{L-M}|b_n|,$$

and thus the proof is accomplished by using (2.1). □

Theorem 3.2 *Let $f = \mathfrak{h} + \overline{\mathfrak{g}}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are given by* (1.3). *If f fulfills* (2.1), *then*

$$\mathrm{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_l}(f)}\right) \ge \frac{\mathcal{I}_{l+1} - L + M}{\mathcal{I}_{l+1}} \tag{3.8}$$

and

$$\mathrm{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_l}(f)}{f(z)}\right) \ge \frac{\mathcal{I}_{l+1}}{\mathcal{I}_{l+1} - L + M}, \tag{3.9}$$

where $\mathcal{I}_n$ is given by (3.3), *and*

$$\mathcal{I}_n \ge \begin{cases} L-M & \text{for } n = 1,2,\ldots,l, \\ \mathcal{I}_{l+1} & \text{for } n = l+1,\ldots. \end{cases} \tag{3.10}$$

The equalities are achieved by considering the function

$$f(z) = \frac{1}{z} + \frac{L-M}{\mathcal{I}_{t+1}}\overline{z}^{l+1}. \tag{3.11}$$

Proof The proof for this specific outcome is similar to that of Theorem 3.1 and is thus omitted. □

Theorem 3.3 *Let $f = \mathfrak{h} + \overline{\mathfrak{g}}$ have the power series form* (1.3). *If f meets inequality* (2.1), *then*

$$\mathrm{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_{t,l}}(f)}\right) \ge \frac{\mathcal{I}_{t+1} - (L-M)}{\mathcal{I}_{t+1}} \tag{3.12}$$

and

$$\operatorname{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_{t,l}}(f)}{f(z)}\right) \geq \frac{\mathcal{I}_{t+1}}{\mathcal{I}_{t+1}+(L-M)}, \tag{3.13}$$

where $\mathcal{I}_n$ is given by (3.3)*. The equalities are easily achieved by using* (3.5)*.*

Proof To establish (3.12), let us consider

$$\begin{aligned}\Theta_3(z) &= \frac{\mathcal{I}_{t+1}}{L-M}\left\{\frac{f(z)}{\mathcal{S}_{t,l}(f)} - \left(1-\frac{L-M}{\mathcal{I}_{t+1}}\right)\right\} \\ &= 1 + \frac{\frac{\mathcal{I}_{t+1}}{L-M}(\sum_{n=t+1}^{\infty} a_n z^n + \sum_{n=l+1}^{\infty} b_n \overline{z}^n)}{\frac{1}{z} + \sum_{n=1}^{t} a_n z^n + \sum_{n=1}^{l} b_n \overline{z}^n}.\end{aligned}$$

Therefore, to show inequality (3.12), it is sufficient to prove the inequality

$$\left|\frac{\Theta_3(z)-1}{\Theta_3(z)+1}\right| \leq 1.$$

Now recalling the left-hand side of the above-mentioned inequality, by easy calculations we get

$$\left|\frac{\Theta_3(z)-1}{\Theta_3(z)+1}\right| \leq \frac{\frac{\mathcal{I}_{t+1}}{L-M}\sum_{n=t+1}^{\infty}|a_n|}{2-2(\sum_{n=1}^{t}|a_n| + \sum_{n=1}^{l}|b_n|) - \frac{\mathcal{I}_{t+1}}{L-M}(\sum_{n=t+1}^{\infty}|a_n| + \sum_{n=l+1}^{\infty}|b_n|)}.$$

Since we observe that from (2.1) that the denominator of the last inequality is positive. The right-hand side of the last inequality is also constrained by one if and only if the following inequality is maintained:

$$\sum_{n=1}^{t}|a_n| + \sum_{n=1}^{l}|b_n| + \frac{\mathcal{I}_{t+1}}{L-M}\left(\sum_{n=t+1}^{\infty}|a_n| + \sum_{n=t+1}^{\infty}|b_n|\right) \leq 1. \tag{3.14}$$

Eventually, to verify inequality (3.12), it suffices to show that the left-hand side of (3.14) is bounded above by

$$\sum_{n=1}^{\infty}\frac{\mathcal{I}_n}{L-M}|a_n| + \sum_{n=1}^{\infty}\frac{\mathcal{I}_n}{L-M}|b_n|,$$

which is further analogous to

$$\begin{aligned}&\sum_{n=1}^{t}\frac{\mathcal{I}_n-(L-M)}{L-M}|a_n| + \sum_{n=1}^{l}\frac{\mathcal{I}_n-(L-M)}{L-M}|b_n| + \sum_{n=l+1}^{\infty}\frac{\mathcal{I}_n-\mathcal{I}_{t+1}}{L-M}\left(\sum_{n=t+1}^{\infty}|a_n| + \sum_{n=t+1}^{\infty}|b_n|\right) \\ &\quad \geq 0,\end{aligned}$$

and this is true due to (3.10). Now let us choose

$$f(z) = \frac{1}{z} + \frac{L-M}{\mathcal{I}_{t+1}}z^{t+1},$$

which delivers a sharp result. We observe that for $z = re^{i\frac{\pi}{t}}$,

$$\frac{f(z)}{\mathcal{M}_{\mathcal{S}_t}(f)} = 1 + \frac{L-M}{\mathcal{I}_{t+1}} z^{t+2} \to 1 - \frac{L-M}{\mathcal{I}_{t+1}} r^{t+2} (r \to 1).$$

Similarly, we obtain (3.9). □

Theorem 3.4 *Let $f = \mathfrak{h} + \overline{\mathfrak{g}}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are expressed by (1.3). If f meets (2.1), then*

$$\operatorname{Re}\left(\frac{f(z)}{\mathcal{M}_{\mathcal{S}_{t,l}}(f)}\right) \geq \frac{\mathcal{I}_{l+1} - (L-M)}{\mathcal{I}_{l+1}}$$

and

$$\operatorname{Re}\left(\frac{\mathcal{M}_{\mathcal{S}_{t,l}}(f)}{f(z)}\right) \geq \frac{\mathcal{I}_{l+1}}{\mathcal{I}_{l+1} + (L-M)},$$

where $\mathcal{I}_n$ is given by (3.3). The equalities are obtained for the function stated in (3.11).

Proof The proof is very similar to that of Theorem 3.3 and is therefore omitted. □

4 Further properties of the class $\mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$

Theorem 4.1 *If $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$, then for $|z| = r$,*

$$\left|f(z)\right| \leq \frac{1}{r} + \frac{L-M}{\sigma_1} r \tag{4.1}$$

and

$$\left|f(z)\right| \geq \frac{1}{r} - \frac{L-M}{\sigma_1} r. \tag{4.2}$$

Proof Let $f = \mathfrak{h} + \overline{\mathfrak{g}} \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$ with $\mathfrak{h}$ and $\mathfrak{g}$ of the series form (1.3). Then by Theorem 2.4 we have

$$\begin{aligned}
\left|f(z)\right| &\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} \left(|a_n| + |b_n|\right) \left|z^n\right| \\
&\leq \frac{1}{|z|} + \frac{1}{\sigma_1} |z| \sum_{n=1}^{\infty} \left(\rho_n |a_n| + \sigma_n |b_n|\right) \\
&\leq \frac{1}{|z|} + \frac{L-M}{\sigma_1} |z|.
\end{aligned}$$

This completes the proof of (4.1). By similar arguments we easily obtain (4.2). □

Theorem 4.2 *A function $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} (\mathfrak{X}_n \mathfrak{h}_n + \mathfrak{Y}_n \mathfrak{g}_n), \tag{4.3}$$

where

$$\mathfrak{h}(z) = \frac{1}{z},$$

$$\mathfrak{h}_n(z) = \frac{1}{z} - \frac{L-M}{\rho_n} z^n \quad \textit{for } n \in \mathbb{N},$$

$$\mathfrak{g}_n(z) = \frac{1}{z} - \frac{L-M}{\sigma_n} \overline{z}^n \quad \textit{for } n \in \mathbb{N},$$

and $\mathfrak{X}_n, \mathfrak{Y}_n \geq 0$ *for* $n \in \mathbb{N}$ *are such that*

$$\sum_{n=1}^{\infty} (\mathfrak{X}_n + \mathfrak{Y}_n) = 1. \tag{4.4}$$

In particular, the points $\{\mathfrak{h}_n\}$, $\{\mathfrak{g}_n\}$ *are called the extreme points of the closed convex hull of the set* $\mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$ *denoted by* $clco\mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$.

Proof Let f be specified by (4.3). Then from (4.4) we get

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{L-M}{\rho_n} \mathfrak{X}_n z^n + \frac{L-M}{\sigma_n} \mathfrak{Y}_n \overline{z}^n \right),$$

which by Theorem 2.4 indicates that $f \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$, since

$$\sum_{n=1}^{\infty} \left(\frac{\rho_n}{L-M} \frac{L-M}{\rho_n} \mathfrak{X}_n + \frac{\sigma_n}{L-M} \frac{L-M}{\sigma_n} \mathfrak{Y}_n \right) = \sum_{n=1}^{\infty} (\mathfrak{X}_n + \mathfrak{Y}_n) = 1.$$

Thus $f \in clco\mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$. For the converse part, let $f = \mathfrak{h} + \overline{\mathfrak{g}} \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$. Put

$$\mathfrak{X}_n = \frac{\rho_n}{L-M} |a_n|, \qquad \mathfrak{Y}_n = \frac{\sigma_n}{L-M} |b_n|.$$

Then utilizing (4.4) together with the hypothesis, we have

$$\begin{aligned}
f(z) &= \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \\
&= \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n - \sum_{n=1}^{\infty} |b_n| \overline{z}^n \\
&= \frac{1}{z} - \sum_{n=1}^{\infty} \mathfrak{X}_n \frac{L-M}{\rho_n} z^n - \sum_{n=1}^{\infty} \mathfrak{Y}_n \frac{L-M}{\sigma_n} \overline{z}^n \\
&= \frac{1}{z} - \sum_{n=1}^{\infty} \mathfrak{X}_n \left\{ \frac{1}{z} - \mathfrak{h}_n \right\} - \sum_{n=1}^{\infty} \mathfrak{Y}_n \left\{ \frac{1}{z} - \mathfrak{g}_n \right\} \\
&= \left(1 - \sum_{n=1}^{\infty} (\mathfrak{X}_n + \mathfrak{Y}_n) \right) \frac{1}{z} + \sum_{n=1}^{\infty} \{ \mathfrak{X}_n \mathfrak{h}_n(z) + \mathfrak{Y}_n \mathfrak{g}_n(z) \} \\
&= \sum_{n=1}^{\infty} \{ \mathfrak{X}_n \mathfrak{h}_n(z) + \mathfrak{Y}_n \mathfrak{g}_n(z) \},
\end{aligned}$$

which is the needed form (4.3). Thus the proof of Theorem 4.2 is completed. □

Theorem 4.3 *Let $f_1, f_2 \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$. Then $f_1 * f_2 \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$.*

Proof Let

$$f_1(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n - |b_n| \overline{z}^n$$

and

$$f_2(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |A_n| z^n - |B_n| \overline{z}^n.$$

Then

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |A_n||a_n| z^n - |B_n||b_n| \overline{z}^n.$$

Now if $f_2 \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$, then by Theorem 2.4 we have $|A_n| \leq 1$ and $|B_n| \leq 1$. Thus

$$\frac{1}{L-M} \sum_{n=1}^{\infty} \bigl(\rho_n |A_n||a_n| + \sigma_n |B_n||b_n|\bigr) \leq \frac{1}{L-M} \sum_{n=1}^{\infty} \bigl(\rho_n |a_n| + \sigma_n |b_n|\bigr) \leq 1.$$

By Theorem 2.4 this gives that $f_1 * f_2 \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$. □

Theorem 4.4 *The family $\mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$ is closed by a convex combination.*

Proof For $k \in \mathbb{N}$, let $f_k \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$ be represented by

$$f_k(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_{n,k}| z^n - \sum_{n=1}^{\infty} |b_{n,k}| \overline{z}^n.$$

Then by (2.1) we have

$$\sum_{n=1}^{\infty} \left\{ \frac{\rho_n |a_{n,k}| + \sigma_n |b_{n,k}|}{L-M} \right\} \leq 1.$$

For $\sum_{k=1}^{\infty} \xi_k = 1$, $0 \leq \xi_k < 1$, the convex combination of f_k is

$$\sum_{k=1}^{\infty} \xi_k f_k(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \xi_k |a_{n,k}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \xi_k |b_{n,k}| \right) \overline{z}^n.$$

Then by Theorem 2.4 we can write

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \rho_n \xi_k |a_{n,k}| + \sum_{k=1}^{\infty} \sigma_n \xi_k |b_{n,k}| \right) &\leq \sum_{k=1}^{\infty} \xi_k \left\{ \sum_{n=1}^{\infty} \rho_n |a_{n,k}| + \sum_{n=1}^{\infty} \sigma_n |b_{n,k}| \right\} \\ &\leq (L-M) \sum_{k=1}^{\infty} \xi_k = L-M, \end{aligned}$$

and so $\sum_{k=1}^{\infty} \xi_k f_k(z) \in \mathcal{MS}^*_{\mathcal{H}_\vartheta}(q, L, M)$. □

5 Conclusion

Utilizing the principles of quantum calculus, we have added some new subfamilies of meromorphic harmonic mappings linked to a circular domain. We learned also certain important problems for the newly specified function families, namely necessary and sufficient conditions, problems for partial sums, distortion limits, convexity conditions, and convolution preserving. For these families, other problems, such as topological properties, fundamental mean inequality, and their implications are open problems for the scholars to investigate.

As pointed out in the survey-cum-expository review paper by Srivastava [44, p. 340], any attempt to produce the so-called (p,q)-variation of the q-results, which we have presented in this paper, will be trivial and inconsequential because the additional parameter p is obviously redundant or superfluous.

Acknowledgements
Not applicable.

Funding
Not Applicable.

Availability of data and materials
Not applicable.

Declarations

Conflicts of interest
The authors declare that they have no conflicts of interest.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors equally contributed to this manuscript and approved the final version.

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Publisher's Note

Received: 15 July 2021 Accepted: 3 October 2021 Published online: 24 October 2021

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