Arora et al. *Advances in Difference Equations* (2021) 2021:455
https://doi.org/10.1186/s13662-021-03620-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Results on exact controllability of second-order semilinear control system in Hilbert spaces

Urvashi Arora[1], V. Vijayakumar[2], Anurag Shukla[3], Kottakkaran Sooppy Nisar[4*], Shahram Rezapour[5,6*] and Wasim Jamshed[7]

*Correspondence:
n.sooppy@psau.edu.sa;
sh.rezapour@azaruniv.ac.ir;
sh.rezapour@mail.cmuh.org.tw;
rezapourshahram@yahoo.ca
[4]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia
[5]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
Full list of author information is available at the end of the article

Abstract

In our manuscript, we extend the controllability outcomes given by Bashirov (Math. Methods Appl. Sci. 44(9):7455–7462, 2021) for a family of second-order semilinear control system by formulating a sequence of piecewise controls. This approach does not involve large estimations which are required to apply fixed point theorems. Therefore, we avoid the use of fixed point theory and the contraction mapping principle. We establish that a second-order semilinear system drives any starting position to the required final position from the domain of the system. To achieve the required results, we suppose that the linear system is exactly controllable at every non-initial time period, the norm of the inverse of the controllability Grammian operator increases as the time approaches zero with the slower rate in comparison to the reciprocal of the square function, and the nonlinear term is bounded. Finally, an example has been presented to validate the results.

MSC: Primary 34K30; secondary 34K35; 93C25

Keywords: Mild solution; Controllability; Second-order system; Cosine family

1 Introduction

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time is known or postulated. This is illustrated in classical mechanics where the motion of a body is described by its position and velocity as time varies. For the studies related to the existence of solution for integer- and fractional-order systems, one can refer to [2–30]. The concept of controllability is one of the underlying ideas in mathematical control theory. Controllability analysis is used in several real-life problems which include, but are not limited to, rocket launching problems for satellite and aircraft control, missiles and anti-missiles problems in defense, regulating inflation rate in the economy, controlling sugar level in the blood, etc. A systematic study of controllability was initiated by Kalman [31] in 1963 when the theory of controllability for time-invariant and time-varying control systems in state-space form was developed.

Several engineering and scientific problems can be expressed by infinite-dimensional differential equations. Therefore, it becomes necessary to discuss the controllability results for infinite-dimensional systems. The controllability problems for finite-dimensional nonlinear systems have been immensely analyzed by several authors. Many authors have advanced the idea of controllability from finite-dimensional systems to infinite-dimensional systems and determined appropriate requirements for the controllability of nonlinear systems. Different techniques have been practiced for the examination of controllability including fixed point theorems [32–34]. For more problems on controllability and recent progresses on fractional calculus and its applications, please refer to [35–47].

Second-order differential equations represent abstract mathematical interpretations of several partial differential equations which occur in many applications related to the oscillation of fastened bars, the transverse motion of an extensible beam, and several other real-world physical phenomena. Hence it becomes really important to determine the controllability outcomes for this kind of system. Controllability discussions for various second-order nonlinear systems have been widely studied by several authors [33, 48–58]. Fixed point theorems have been utilized greatly in determining the existence and controllability results for different first-order and second-order systems which involve large estimations on system constants see [33, 59, 60]. Recently Bashirov [1] obtained the exact controllability results for first-order semilinear systems using a new technique that is based on the piecewise formulation of driving controls and without using fixed point theory. Earlier the same approach was applied in terms of approximate controllability [61]. Motivated by Bashirov [1], we extend the controllability results for second-order semilinear systems excluding the use of fixed point theory. It is centered on a piecewise formulation of steering controls and does not involve large estimations which are required to apply fixed point theorems. By constructing a piecewise sequence of controls, we determine that a second-order semilinear system is exactly controllable to the domain of the system operator, that is, it drives the system from any starting position to the required final position from the domain of the system. To achieve the required results, we suppose the subsequent requirements:

(a) The corresponding linear system is exactly controllable at every non-initial time period.
(b) The norm of the inverse of the controllability Grammian operator increases as the time approaches zero with the slower rate in comparison to the reciprocal of the square function.
(c) The nonlinear term is bounded.

Let us consider $Z = L_2[0,b;\mathbb{X}]$ and $Y = L_2[0,b;\mathbb{U}]$ as the function spaces defined on $J = [0,b]$, $0 \le b < \infty$, where $\mathbb{X}$ and $\mathbb{U}$ are two Hilbert spaces. Consider the following second-order semilinear control system:

$$\begin{cases} p''(t) = Ap(t) + Bv(t) + r(t,p(t)), \quad \text{for } t \in (0,b], \\ p(0) = p_0, \\ p'(0) = q_0, \end{cases} \tag{1.1}$$

where

1 $p(t)$ represents the state having values in Hilbert space $\mathbb{X}$.
2 Control function v is defined from $[0,b] \to \mathbb{U}$.

3 B is a bounded and linear operator from $\mathbb{U}$ into $\mathbb{X}$.
4 The function $r : [0, b] \times \mathbb{X} \to \mathbb{X}$ is a purely nonlinear function which produces nonlinearity in the system.
5 $A : \mathrm{dom}(A) \subseteq \mathbb{X} \to \mathbb{X}$ is linear, closed where $\mathrm{dom}(A)$ is a dense subset of X.

The linear system corresponding to (1.1) with state vector $q(t)$ and control v is defined by

$$\begin{cases} q''(t) = \mathrm{Aq}(t) + Bv(t), \quad \text{for } t \in (0, b], \\ q(0) = p_0, \\ q'(0) = q_0. \end{cases} \tag{1.2}$$

The article is structured in the subsequent manner:
1 Sect. 2 presents a few basic results related to control theory and second-order systems.
2 Sect. 3 provides the assumptions which are required to obtain the controllability results.
3 Sect. 4 discusses the controllability results using the new technique.
4 Sect. 5 presents an example to verify the established outcomes.

2 Auxiliary results

Here, we will review fundamental theories and a few definitions which would be helpful for further discussions.

Definition 2.1 [62] A one-parameter family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators mapping the Hilbert space $\mathbb{X}$ into itself is called a strongly continuous cosine family if and only if
1 $C(0) = I$;
2 $C(s + t) + C(s - t) = 2C(s)C(t)$;
3 $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in \mathbb{X}$.

If $\{C(t), t \in \mathbb{R}\}$ is a strongly continuous cosine family in $\mathbb{X}$, then $\{S(t), t \in \mathbb{R}\}$ is a one-parameter family of operators in $\mathbb{X}$ defined by

$$S(t) = \int_0^t C(s)\,ds, \quad t \in \mathbb{R}.$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ is the operator $A : \mathbb{X} \to \mathbb{X}$ defined by

$$Ax = \frac{d^2}{dt^2} C(0)x.$$

The domain of operator A is defined as

$$\mathrm{dom}(A) = \{x \in \mathbb{X} : C(t)x \text{ is a twice continuously differentiable function of } t\}.$$

These cosine and sine families defined above and generator A fulfill the following properties.

Lemma 2.2 ([28]) *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) : t \in \mathbb{R}\}$. Then the following hold*:

(1) *There exist $M' \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq M'e^{\omega|t|}$, and hence $\|S(t)\| \leq M'e^{\omega|t|}$.*
(2) *$A\int_s^r S(u)x\,du = [C(r) - C(s)]x$ for all $0 \leq s \leq r < \infty$.*
(3) *There exists $N' \geq 1$ such that $\|S(s) - S(r)\| \leq N'|\int_s^r e^{\omega|s|}\,ds|$ for all $0 \leq s \leq r < \infty$.*

The uniform boundedness principle together with (1) implies that both $\{C(t) : t \in J\}$ and $\{S(t) : t \in J\}$ are uniformly bounded and $M = M'e^{\omega|b|}$.

Proposition 2.3 ([62]) *Let $\{C(t), t \in \mathbb{R}\}$ be a strongly continuous cosine family in $\mathbb{X}$ with infinitesimal generator A. The following are true*:

1 $S(0) = 0$.
2 *$C(t) = C(-t)$ and $S(t) = -S(-t)$ for all $t \in \mathbb{R}$.*
3 *If $x \in E$, then $S(t)x, C(t)x \in \operatorname{dom}(A)$ and $\frac{d}{dt}S(t)x = C(t)x$, and $\frac{d}{dt}C(t)x = AS(t)x$, where $E = \{x : C(t)x$ is once continuously differentiable function of $t\}$.*
4 *If $x \in \operatorname{dom}(A)$, then $S(t)x \in \operatorname{dom}(A)$ and $AS(t)x = S(t)Ax$.*
5 *If $x \in \operatorname{dom}(A)$, then $C(t)x \in \operatorname{dom}(A)$ and $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$.*
6 *If $x \in E$, then $\lim_{t\to 0} AS(t)x = 0$.*

Proposition 2.4 ([62]) *Let $\{C(t), t \in \mathbb{R}\}$ be a strongly continuous cosine family in $\mathbb{X}$. The operator $\hat{A} : \mathbb{X} \to \mathbb{X}$ defined by*

$$\hat{A}x = \lim_{t\to 0} \frac{(C(2t)x - x)}{2t^2},$$

with domain $x \in \mathbb{X}$ for which this limit exists, is the infinitesimal generator of the cosine family $\{C(t), t \in \mathbb{R}\}$.

Suppose $U_{ad} = L_2[0, b; \mathbb{U}]$, which is the set of admissible controls. We define the mild solution of the given semi linear system (1.1) and its corresponding linear system as follows.

Definition 2.5 The mild solution of system (1.1) is defined by a function $p(\cdot) \in \mathbb{X}$ which satisfies the following integral equation:

$$\begin{cases} p(t) = C(t)p_0 + S(t)q_0 + \int_0^t S(t-s)\{Bv(s) + r(s, p(s))\}\,ds, & t \in (0, b], \\ p(0) = p_0, \\ p'(0) = q_0, \end{cases} \tag{2.1}$$

and the mild solution of the corresponding linear system (1.2) is described by the following integral equation:

$$\begin{cases} q(t) = C(t)p_0 + S(t)q_0 + \int_0^t S(t-s)Bv(s)\,ds, & t \in (0, b], \\ q(0) = p_0, \\ q'(0) = q_0. \end{cases} \tag{2.2}$$

Definition 2.6 ([1]) System (1.1) is said to be approximately controllable in the time interval $[0, b]$ if, for the given starting position $(p_0, q_0) \in \mathbb{X}$ and the required final position

$(p_F, q_F) \in \mathbb{X}$ and $\epsilon > 0$, there exists a control function $\nu \in U_{ad}$ such that the solution of (1.1) satisfies

$$\|p(b) - p_F\| < \epsilon, \qquad \|p'(b) - q_F\| < \epsilon,$$

where $p(b)$ is the state value of system (1.1) at time $t = b$. If $p(b) = p_F$, then the system is said to be exactly controllable. The system is said to be exactly controllable to $\mathrm{dom}(A)$ on $[0, b]$ if, for the given starting position $(p_0, q_0) \in \mathbb{X}$ and the required final position $(p_F, q_F) \in \mathrm{dom}(A)$, there exists a control function $\nu \in U_{ad}$ such that the solution of the system satisfies $p(b) = p_F$, $p'(b) = q_F$.

Remark 2.7 Note that the exact controllability to $\mathrm{dom}(A)$ lies in between the exact and approximate controllability. Therefore, it is a weaker concept than the exact controllability. In real life applications, sometimes we are more concerned with attaining the points from $\mathrm{dom}(A)$. If it is possible to reach the points from $\mathbb{X} \setminus \mathrm{dom}(A)$ as well, then it can be considered as an additional capability of the system.

3 Assumptions

Let us introduce the controllability Grammian operator W associated with linear system (1.2) by

$$W(t) = \int_0^t S(s)BB^*S^*(s)\,ds, \quad 0 \le t \le b,$$

where $S^*(s)$ denotes the adjoint of $S(s)$.

Theorem 3.1 *The corresponding linear system* (1.2) *is exactly controllable on the interval* $[h, b]$ *iff* $W(b-h)$ *is coercive. The control which drives the system from the starting position* $(q(h), q'(h)) \in X$ *to the final position* $(p_F, q_F) \in X$ *is given by*

$$\nu(t) = B^*S^*(b-t)\bigl(W(b-h)\bigr)^{-1}\bigl(p_F - C(b-h)q(h) - S(b-h)q'(h)\bigr) \quad (h \le t \le b). \tag{3.1}$$

Proof The result can be seen in [59]. Moreover, it can be easily verified by substituting the above defined control in the mild solution of the corresponding linear system that it transfers $(q(h), q'(h))$ to (p_F, q_F) on the interval $[h, b]$. □

Remark 3.2 The coercivity of $W(t)$ indicates that $(W(t))^{-1}$ is a bounded linear operator. We say that $W(t)$ is coercive if there exists $\gamma > 0$ such that $\langle W(t)x, x\rangle \ge \gamma \|x\|^2$ for all $x \in X$. Here $W(0) = 0$ and, therefore, it fails to be coercive. But it may be coercive for $0 < t \le b$. Therefore, the above result holds on $[h, b]$.

To determine the main result, we make the following assumptions on the controllability Grammian operator W and the nonlinear function $r(t, p)$:

(I) $W(t)$ is coercive for all $0 < t \le b$.

(II) There exists some $N \ge 0$ such that

$$t^{1+\alpha}\bigl\|\bigl(W(t)\bigr)^{-1}\bigr\| \le N \quad \text{for all } 0 < t \le b, 0 \le \alpha < 1.$$

That is, $\|(W(t))^{-1}\| \to \infty$ as $t \to 0^+$ with the slower rate in comparison to the reciprocal of the square function as

$$\left\|\left(W(t)\right)^{-1}\right\| \leq \frac{N}{t^{1+\alpha}} < \frac{N}{t^2},$$

for small values of t.

(III) The nonlinear function r is Lebesgue measurable in t.

(IV) r is Lipschitz continuous in p.

(V) r is bounded in $[0,b] \times \mathbb{X}$, *i.e.*, there exists $K > 0$ such that

$$\left\|r(t,p)\right\| \leq K \quad \text{for all } (t,p) \in [0,b] \times \mathbb{X}.$$

4 Results on controllability

In this section, we primarily focus on the study of exact controllability of the assumed system.

Theorem 4.1 *System* (1.1) *is exactly controllable to* $\operatorname{dom}(A)$ *on the interval* $[0,b]$ *for every* $b > 0$ *provided assumptions* (I)–(V) *hold.*

Proof: We construct a piecewise sequence of driving controls to formulate the required control function v which drives the given system from the starting position $(p_0,q_0) \in \mathbb{X}$ to the final position $(p_F,q_F) \in \operatorname{dom}(A)$ in the following manner.

For this, consider the sequence $\{h_n\}$ which is defined by $h_n = \frac{b}{2^n}$ for $n = 1,2,\ldots.$

We have $\sum_{n=1}^{\infty} h_n = b$. For the sake of simplicity, let us take $h_0 = 0$ and

$$b_0 = h_0, \qquad b_1 = h_0 + h_1, \qquad \ldots, \qquad b_n = \sum_{k=0}^{n} h_k, \qquad \ldots$$

Then $\lim_{n\to\infty} b_n = \sum_{k=0}^{\infty} h_k = b$.

Using Theorem 3.1, the corresponding linear system (1.2) is exactly controllable on $[b_0,b_1]$ along with the control

$$v_1(\varrho) = B^*S^*(b_1-\varrho)\left(W(h_1)\right)^{-1}\left(C(h_1)(p_F-p_0) + S(h_1)(q_F-q_0)\right), \quad b_0 \leq \varrho \leq b_1,$$

which steers the initial state p_0 to $C(h_1)p_F + S(h_1)q_F$.

That is,

$$C(h_1)p_F + S(h_1)q_F = C(h_1)p_0 + S(h_1)q_0 + \int_{b_0}^{b_1} S(b_1-s)Bv_1(s)\,ds.$$

Define v on $[b_0,b_1]$ by letting $v(\varrho) = v_1(\varrho)$. Then, from (2.1), we obtain

$$p(b_1) = C(h_1)p_F + S(h_1)q_F + \int_{b_0}^{b_1} S(b_1-s)r\left(s,p(s)\right)ds.$$

For brevity, let $p(b_1) = p_1$. Next, consider (1.2) on $[b_1,b_2]$. By Theorem 3.1, the control

$$v_2(\varrho) = B^*S^*(b_2-\varrho)\left(W(h_2)\right)^{-1}\left(C(h_2)(p_F-p_1) + S(h_1)\left(q_F-p'(b_1)\right)\right), \quad b_1 \leq \varrho \leq b_2,$$

steers p_1 to $C(h_2)p_F + S(h_2)q_F$. Writing $p'(b_1) = q_1$, then the control $u_2(\varrho)$ can be written as

$$v_2(\varrho) = B^*S^*(b_2 - \varrho)\big(W(h_2)\big)^{-1}\big(C(h_2)(p_F - p_1) + S(h_2)(q_F - q_1)\big), \quad b_1 \le \varrho \le b_2.$$

That is,

$$C(h_2)p_F + S(h_2)q_F = C(h_2)p_1 + S(h_2)q_1 + \int_{b_1}^{b_2} S(b_2 - s)Bv_2(s)\,ds.$$

Define v on $(b_1, b_2]$ by letting $v(\varrho) = v_2(\varrho)$. Then, from (2.1), we obtain

$$p(b_2) = C(h_2)p_F + S(h_2)q_F + \int_{b_1}^{b_2} S(b_2 - s)r\big(s, p(s)\big)\,ds.$$

For the sake of convenience, let $p(b_2) = p_2$. Progressing in this fashion, we acquire a sequence of driving controls

$$\begin{aligned} &v_n(\varrho) = B^*S^*(b_n - \varrho)\big(W(h_n)\big)^{-1}\big(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\big), \\ &\quad b_{n-1} \le \varrho \le b_n, \end{aligned} \tag{4.1}$$

where $q_{n-1} = p'(b_{n-1})$.

After combining the above sequence of controls, we get the control function as follows:

$$v(\varrho) = \begin{cases} v_1(\varrho) & \text{if } b_0 \le \varrho \le b_1, \\ v_2(\varrho) & \text{if } b_1 < \varrho \le b_2, \\ \dots & \dots \\ v_n(\varrho) & \text{if } b_{n-1} < \varrho \le b_n, \\ \dots & \dots \end{cases}$$

and

$$p_n = p(b_n) = C(h_n)p_F + S(h_n)q_F + \int_{b_{n-1}}^{b_n} S(b_n - s)r\big(s, p(s)\big)\,ds. \tag{4.2}$$

Now by using the assumption (V), we get

$$\begin{aligned} \|p_n - p_F\| &\le \big\|C(h_n)p_F - p_F\big\| + \big\|S(h_n)q_F\big\| + \int_{b_{n-1}}^{b_n} \big\|S(b_n - s)\big\| \big\|r\big(s, p(s)\big)\big\|\,ds \\ &\le \big\|C(h_n)p_F - p_F\big\| + \big\|S(h_n)q_F\big\| + KMh_n, \quad n = 1, 2, \dots, \end{aligned} \tag{4.3}$$

where $M = \sup_{[0,b]} \|C(\varrho)\|$ and $K = \sup_{[0,b]\times X} \|r(\varrho, p)\|$.

Since $C(\varrho)$ is strongly continuous, $S(0) = 0$ and $\lim_{n\to\infty} h_n = 0$.

Therefore, $\lim_{n\to\infty} p_n = p_F$. Also,

$$q_n = p'(b_n) = AS(h_n)p_F + C(h_n)q_F + \int_{b_{n-1}}^{b_n} C(b_n - s)r\big(s, p(s)\big)\,ds.$$

Thus, we have

$$\begin{aligned} \|q_n - q_F\| &\le \big\|AS(h_n)p_F\big\| + \big\|C(h_n)q_F - q_F\big\| + \int_{b_{n-1}}^{b_n} \big\|C(b_n - s)\big\| \big\|r\big(s, p(s)\big)\big\| \, ds \\ &\le \big\|AS(h_n)p_F\big\| + \big\|C(h_n)q_F - q_F\big\| + KMh_n, \quad n = 1, 2, \ldots . \end{aligned} \tag{4.4}$$

Since $C(\varrho)$ is strongly continuous, $S(0) = 0$ and $\lim_{n\to\infty} h_n = 0$.

Therefore, $\lim_{n\to\infty} q_n = q_F$.

Next we prove that $v \in \mathcal{U}_{ad}$. Since every v_n of v is continuous on the interval $(b_{n-1}, b_n]$ for $n = 0, 1, 2, \ldots$, hence v is measurable. Also,

$$\begin{aligned} &\int_{b_{n-1}}^{b_n} \big\|v_n(\varrho)\big\|^2 \, d\varrho \\ &\quad = \int_{b_{n-1}}^{b_n} \big\|B^* S^*(b_n - \varrho)\big(W(h_n)\big)^{-1}\big(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\big)\big\|^2 \, d\varrho \\ &\quad = \int_{b_{n-1}}^{b_n} \big\langle S(b_n - \varrho)BB^* S^*(b_n - \varrho)\big(W(h_n)\big)^{-1}\big(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\big), \\ &\qquad \big(W(h_n)\big)^{-1}\big(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\big)\big\rangle \, d\varrho \\ &\quad = \big\langle \big(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\big), \\ &\qquad \big(W(h_n)\big)^{-1}\big(C(h_n)(p_F - p_{n-1}) + S(h_n)(q_F - q_{n-1})\big)\big\rangle \\ &\quad \le M^2 \big\|\big(W(h_n)\big)^{-1}\big\| \big(\|p_F - p_{n-1}\|^2 + \|q_F - q_{n-1}\|^2\big), \quad n = 1, 2, \ldots \end{aligned}$$

Therefore, by (4.3) and (4.4),

$$\begin{aligned} &\int_0^b \big\|v(\varrho)\big\|^2 \, d\varrho \\ &\quad = \sum_{n=0}^{\infty} \int_{b_n}^{b_{n+1}} \big\|v_{n+1}(\varrho)\big\|^2 \, d\varrho \\ &\quad \le M^2 \sum_{n=0}^{\infty} \big\|\big(W(h_{n+1})\big)^{-1}\big\| \big(\|p_F - p_n\|^2 + \|q_F - q_n\|^2\big) \\ &\quad \le M^2 \sum_{n=0}^{\infty} \big\|\big(W(h_{n+1})\big)^{-1}\big\| \big[\big(\big\|C(h_n)p_F - p_F\big\| + \big\|S(h_n)q_F\big\| + KMh_n\big)^2 \\ &\qquad + \big(\big\|AS(h_n)p_F\big\| + |\big\|C(h_n)q_F - q_F\big\| + KMh_n\big)^2\big]. \end{aligned}$$

Since $C(h_0)p_F = p_F$ and $KMh_0 = 0$, we obtain

$$\begin{aligned} &\int_0^b \big\|v(\varrho)\big\|^2 \, d\varrho \\ &\quad \le M^2 \sum_{n=1}^{\infty} \big\|\big(W(h_{n+1})\big)^{-1}\big\| . \big(\big\|C(h_n)p_F - p_F\big\| + \big\|S(h_n)q_F\big\| + KMh_n\big)^2 \end{aligned}$$

$$
\begin{aligned}
&+ M^2 \sum_{n=1}^{\infty} \bigl\| \bigl(W(h_{n+1})\bigr)^{-1} \bigr\| . \bigl(\| AS(h_n)p_F \| + \| C(h_n)q_F - q_F \| + KMh_n \bigr)^2 \\
&\le M^2 \sum_{n=1}^{\infty} (h_{n+1})^{(1+\alpha)} \bigl\| \bigl(W(h_{n+1})\bigr)^{-1} \bigr\| \left(\frac{h_n}{h_{n+1}} \right)^2 . \left(\frac{1}{(h_{n+1})^{\alpha-1}} \right) \left(\frac{\|C(h_n)p_F - p_F\|}{h_n} \right. \\
&\quad \left. + \frac{\|S(h_n)q_F\|}{h_n} + KM \right)^2 \\
&\quad + M^2 \sum_{n=1}^{\infty} (h_{n+1})^{(1+\alpha)} \bigl\| \bigl(W(h_{n+1})\bigr)^{-1} \bigr\| \left(\frac{h_n}{h_{n+1}} \right)^2 \left(\frac{1}{(h_{n+1})^{\alpha-1}} \right) . \left(\frac{\|AS(h_n)p_F\|}{h_n} \right. \\
&\quad \left. + \frac{\|C(h_n)q_F - q_F\|}{h_n} + KM \right)^2 \\
&= 4M^2 \sum_{n=1}^{\infty} (h_{n+1})^{1+\alpha} \bigl\| \bigl(W(h_{n+1})\bigr)^{-1} \bigr\| \left(\frac{1}{(h_{n+1})^{\alpha-1}} \right) \left(\frac{h_n}{2} . \frac{2\|C(h_n)p_F - p_F\|}{h_n^2} \right. \\
&\quad \left. + \frac{\|S(h_n)q_F\|}{h_n} + KM \right)^2 \\
&\quad + 4M^2 \sum_{n=1}^{\infty} (h_{n+1})^{1+\alpha} \bigl\| \bigl(W(h_{n+1})\bigr)^{-1} \bigr\| \left(\frac{1}{(h_{n+1})^{\alpha-1}} \right) \left(\frac{\|S(h_n)Ap_F\|}{h_n} \right. \\
&\quad \left. + \frac{h_n}{2} \frac{2\|C(h_n)q_F - q_F\|}{h_n^2} + KM \right)^2 .
\end{aligned}
$$

Since $\lim_{\varrho\to 0} \frac{C(2t)p_F - p_F}{2t^2} = \hat{A}p_F$, using proposition (2) results in

$$
\lim_{n\to\infty} \frac{2(C(h_n)p_F - p_F)}{h_n^2} = \hat{A}p_F \quad \text{and} \quad \lim_{n\to\infty} \frac{2(C(h_n)q_F - q_F)}{h_n^2} = \hat{A}q_F,
$$

which implies

$$
\frac{2\|C(h_n)p_F - p_F\|}{h_n^2} \le P \quad \text{and} \quad \frac{2\|C(h_n)q_F - q_F\|}{h_n^2} \le P'
$$

for some $P, P' > 0$.

Then, by using assumption (II),

$$
\begin{aligned}
\int_0^b \bigl\| \nu(\varrho) \bigr\|^2 \, d\varrho &\le 4M^2 N \sum_{n=1}^{\infty} \left(\frac{1}{(h_{n+1})^{\alpha-1}} \right) \left(P\frac{h_n}{2} + C + KM \right)^2 \\
&\quad + 4M^2 N \sum_{n=1}^{\infty} \left(\frac{1}{(h_{n+1})^{\alpha-1}} \right) \left(L + P'\frac{h_n}{2} + KM \right)^2 ,
\end{aligned}
$$

where $\|q_F\| \le C$ and $\|Ap_F\| \le L$ for some $C, L > 0$.

Since $0 < \alpha < 1$, therefore the series

$$
\sum_{n=1}^{\infty} \frac{1}{(h_{n+1})^{\alpha-1}},
$$

$\sum_{n=1}^{\infty} \frac{h_n}{(h_{n+1})^{\alpha-1}}$ and $\sum_{n=1}^{\infty} \frac{h_n^2}{(h_{n+1})^{\alpha-1}}$ are convergent.

This shows that $\nu \in U_{ad}$. Therefore, p is continuous and

$$\lim_{t\to b} p(\varrho) = \lim_{n\to\infty} p_n = p_F \quad \text{and} \quad \lim_{t\to b} p'(\varrho) = \lim_{n\to\infty} q_n = q_F.$$

Thus, $p(b) = p_F$ and $p'(b) = q_F$.

5 Example

Consider the partial differential equation:

$$\left.\begin{array}{l} \frac{\partial^2}{\partial t^2} u(\varrho,y) = u_{xx}(\varrho,y) + \gamma(\varrho,y) + \sigma(\varrho,u(\varrho,y)); \\ u(\varrho,0) = u(\varrho,\pi) = 0 \quad \text{for } t > 0. \end{array}\right\} \tag{5.1}$$

Let $\mathbb{X} = L_2[0,\pi]$ and $\gamma : [0,b] \times (0,\pi) \to \mathbb{R}$ be a continuous control function in ϱ. Define the operator $A : D(A) \to \mathbb{X}$ by

$$A\eta = \eta''; \quad \eta \in D(A)$$

with $D(A) = \{\eta \in \mathbb{X} : \eta, \eta' \text{ are absolutely continuous } \eta'' \in \mathbb{X}, \eta(0) = \eta(\pi) = 0\}$. A is an infinitesimal generator of a strongly continuous cosine family $C(\varrho)$ on X. Moreover, the spectrum of A consists of eigenvalues $-n^2$ for $n = 1, 2, 3, \ldots$, with the associated normalized eigenvectors $\eta_n(s) = (2/\pi)^{1/2} \sin(ns)$. In particular,

$$A\eta = \sum_{n=1}^{\infty} \bigl(-n^2\bigr)(\eta, \eta_n)\eta_n, \quad \eta \in D(A).$$

The cosine function $C(\varrho)$ and the sine function $S(\varrho)$ are defined in the following way:

$$C(\varrho)\eta = \sum_{n=1}^{+\infty} \cos n\varrho (\eta, \eta_n)\eta_n, \quad \eta \in \mathbb{X},$$

$$S(\varrho)\eta = \sum_{n=1}^{+\infty} \frac{1}{n} \sin n\varrho (\eta, \eta_n)\eta_n, \quad \eta \in \mathbb{X},$$

respectively. Define $r : J \times \mathbb{X} \to \mathbb{X}$ by

$$r(\varrho, u)(y) = \sigma\bigl(\varrho, u(y)\bigr); \quad u \in \mathbb{X}, y \in [0,\pi].$$

Let $\nu : [0,b] \to \mathbb{U}$ be defined by

$$\nu(\varrho)(y) = \gamma(\varrho, y); \quad y \in [0,\pi].$$

Define the controllability operator in the following way:

$$\begin{aligned} W(\varrho) &= \int_0^{\varrho} S(s)BB^*S^*(s)\,ds, \quad 0 \le \varrho \le b \\ &= \int_0^{\varrho} S(s)S^*(s)\,ds, \quad \text{as } B = I. \end{aligned}$$

Let us choose the system operator A in such a way that $W(\varrho)$ is coercive for all $0 < \varrho \leq b$ and there exists some $N \geq 0$ such that

$$\varrho^{1+\alpha} \left\| \left(W(\varrho)\right)^{-1} \right\| \leq N \quad \forall 0 < \varrho \leq b \text{ and } 0 \leq \alpha < 1.$$

Also, the nonlinear function σ can be considered satisfying conditions (III)–(V).

The considered PDE (5.1) can be converted to (1.1). Therefore, system (5.1) is exactly controllable to dom(A).

6 Conclusion

In the present manuscript, the exact controllability to dom(A) for a second-order semilinear system has been discussed using a new technique which avoids fixed point theorems and does not involve large estimations on the system constants. The control function has been formulated by the piecewise construction of steering controls. These results can be further extended for systems with delay or deviated arguments with impulses and fractional-order systems.

Acknowledgements
The fifth author was supported by Azarbaijan Shahid Madani University. The authors express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Bennett University, Greater Noida, India. [2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, 632 014 Tamil Nadu, India. [3]Department of Applied Sciences, Rajkiya Engineering College Kannauj, Aher, India. [4]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia. [5]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. [6]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan. [7]Department of Mathematics, Capital University of Science and Technology (CUST), Islamabad, 44000, Pakistan.

Received: 2 August 2021 Accepted: 4 October 2021 Published online: 16 October 2021

References
1. Bashirov, A.E.: On exact controllability of semilinear systems. Math. Methods Appl. Sci. **44**(9), 7455–7462 (2021). https://doi.org/10.1002/mma.6265

2. Afshari, H., Shojaat, H., Moradi, M.S.: Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions. Results Nonlinear Anal. **4**(3), 186–193 (2021). https://doi.org/10.53006/rna.938851
3. Afshari, H., Kalantari, S., Karapinar, E.: Solution of fractional differential equations via coupled fixed point. Electron. J. Differ. Equ. **15**, 286 (2015) http://ejde.math.txstate.edu
4. Afshari, H., Gholamyan, H., Zhai, C.B.: New applications of concave operators to existence and uniqueness of solutions for fractional differential equations. Math. Commun. **25**(1), 157–169 (2020)
5. Abdeljawad, T., Agarwal, R.P., Karapinar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Symmetry **11**(5), 686 (2019). https://doi.org/10.3390/sym11050686
6. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652
7. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. **20**(2), 313–333 (2021)
8. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **2021**, 155 (2021). https://doi.org/10.1007/s13398-021-01095-3
9. Sabetghadam, F., Masiha, H.P.: Fixed-point results for multi-valued operators in quasi-ordered metric spaces. Appl. Math. Lett. **25**(11), 1856–1861 (2012). https://doi.org/10.1016/j.aml.2012.02.046
10. Masiha, H.P., Sabetghadam, F., Shahzad, N.: Fixed point theorems in partial metric spaces with an application. Filomat **27**(4), 617–624 (2013)
11. Sabetghadam, F., Masiha, H.P., Altun, I.: Fixed-point theorems for integral-type contractions on partial metric spaces. Ukr. Math. J. **68**, 940–949 (2016). https://doi.org/10.1007/s11253-016-1267-5
12. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705
13. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of $CD4^{+}$ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. **2020**, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w
14. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. Chaos Solitons Fractals **144**, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668
15. Rezapour, S., Mohammadi, H., Jajarmi, A.: A new mathematical model for Zika virus transmission. Adv. Differ. Equ. **2020**, 589 (2020). https://doi.org/10.1186/s13662-020-03044-7
16. Rezapour, S., Henriquez, H.R., Vijayakumar, V., Nisar, K.S., Shukla, A.: A note on existence of mild solutions for second-order neutral integro-differential evolution equations with state-dependent delay. Fractal Fract. **5**(3), 126 (2021). https://doi.org/10.3390/fractalfract5030126
17. Rezapour, S., Imran, A., Hussain, A., Martinez, F., Etemad, S., Kaabar, M.K.A.: Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs. Symmetry **13**(3), 469 (2021). https://doi.org/10.3390/sym13030469
18. Rezapour, S., Azzaoui, B., Tellab, B., Etemad, S., Masiha, H.P.: An analysis on the positivesSolutions for a fractional configuration of the Caputo multiterm semilinear differential equation. J. Funct. Spaces **2021**, Article ID 6022941 (2021). https://doi.org/10.1155/2021/6022941
19. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. **59**(5), 3019–3027 (2020). https://doi.org/10.1016/j.aej.2020.04.053
20. Panda, S.K., Karapinar, E., Atangana, A.: A numerical schemes and comparisons for fixed point results with applications to solutions of Volterra integral equations in dislocated extended b-metric space. Alex. Eng. J. **59**(2), 815–827 (2020). https://doi.org/10.1016/j.aej.2020.02.007
21. Hazarika, B., Karapinar, E., Arab, R., Rabbani, M.: Metric-like spaces to prove existence of solution for nonlinear quadratic integral equation and numerical method to solve it. J. Comput. Appl. Math. **328**, 302–313 (2018). https://doi.org/10.1016/j.cam.2017.07.012
22. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo-Fabrizio derivative. Bound. Value Probl. **2018**, 90 (2018). https://doi.org/10.1186/s13661-018-1008-9
23. Baleanu, D., Mousalou, A., Rezapour, S.: On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations. Bound. Value Probl. **2017**, 145 (2017). https://doi.org/10.1186/s13661-017-0867-9
24. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation. Bound. Value Probl. **2019**, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0
25. Samei, M.E., Rezapour, S.: On a system of fractional q-differential inclusions via sum of two multi-term functions on a time scale. Bound. Value Probl. **2020**, 135 (2020). https://doi.org/10.1186/s13661-020-01433-1
26. Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular pointwise defined fractional q-integro-differential equation. Bound. Value Probl. **2020**, 38 (2020). https://doi.org/10.1186/s13661-020-01342-3
27. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0
28. Fattorini, H.O.: Second Order Linear Differential Equations in Banach Spaces. North-Holland, Amsterdam (1995)
29. Neshati, J., Masiha, H.P., Jafarian, M.: Electrochemical noise analysis for estimation of corrosion rate of carbon steel in crude oil. Anti-Corros. Methods Mater. **54**(1), 27–33 (2007). https://doi.org/10.1108/00035590710717366
30. Neshati, J., Masiha, H.P., Mahjani, M.G., Jafarian, M.: Electrochemical noise analysis for estimation of corrosion rate of carbon steel in crude oil. Corros. Eng. Sci. Technol. **42**(4), 371–376 (2007). https://doi.org/10.1179/174327807X214879
31. Kalman, R.: Lectures on Controllability and Observability. Springer, Berlin (2010). https://doi.org/10.1007/978-3-642-11063-4_1

32. Arthi, G., Balachandran, K.: Controllability of second-order impulsive evolution systems with infinite delay. Nonlinear Anal. Hybrid Syst. **11**, 139–153 (2014). https://doi.org/10.1016/j.nahs.2013.08.001
33. Sukavanam, S., Sukavanam, N.: Controllability of second-order systems with nonlocal conditions in Banach spaces. Numer. Funct. Anal. Optim. **35**(4), 423–431 (2014). https://doi.org/10.1080/01630563.2013.814067
34. Leiva, H.: Rothe's fixed point theorem and controllability of semilinear nonautonomous systems. Syst. Control Lett. **67**, 14–18 (2014). https://doi.org/10.1016/j.sysconle.2014.01.008
35. Bashirov, A.E., Mahmudov, N.I.: On concepts of controllability for deterministic and stochastic systems. SIAM J. Control Optim. **37**(6), 1808–1821 (1999). https://doi.org/10.1137/S036301299732184X
36. Chalishajar, D.N.: Controllability of mixed Volterra-Fredholm-type integro-differential systems in Banach space. J. Franklin Inst. **344**(1), 12–21 (2007). https://doi.org/10.1016/j.jfranklin.2006.04.002
37. Kavitha, K., Vijayakumar, V., Udhayakumar, R.: Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness. Chaos Solitons Fractals **139**, 110035 (2020). https://doi.org/10.1016/j.chaos.2020.110035
38. Klamka, J.: Controllability of dynamical systems, a survey. Bull. Pol. Acad. Sci., Tech. Sci. **61**(2), 335–342 (2013). https://doi.org/10.2478/bpasts-2013-0031
39. Klamka, J., Babiarz, A., Niezbitowski, M.: Banach fixed-point theorem in semilinear controllability problems, a survey. Bull. Pol. Acad. Sci., Tech. Sci. **64**(1), 21–35 (2016). https://doi.org/10.1515/bpasts-2016-0004
40. Klamka, J., Wyrwal, J., Zawiski, R.: On controllability of second order dynamical systems, a survey. Bull. Pol. Acad. Sci., Tech. Sci. **65**(3), 279–295 (2017). https://doi.org/10.1515/bpasts-2017-0032
41. Naito, K.: Controllability of semilinear control systems dominated by the linear part. SIAM J. Control Optim. **25**(3), 715–722 (1987). https://doi.org/10.1137/0325040
42. Raja, M.M., Vijayakumar, V., Udhayakumar, R., Zhou, Y.: A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces. Chaos Solitons Fractals **141**, 110310 (2020). https://doi.org/10.1016/j.chaos.2020.110310
43. Shukla, A., Patel, R.: Controllability results for fractional semilinear delay control systems. J. Appl. Math. Comput. **65**, 861–875 (2021). https://doi.org/10.1007/s12190-020-01418-4
44. Vijayakumar, V., Udhayakumar, R., Kavitha, K.: On the approximate controllability of neutral integro-differential inclusions of Sobolev-type with infinite delay. Evol. Equ. Control Theory **10**(2), 271–296 (2021). https://doi.org/10.3934/eect.2020066
45. Kumar, V., Malik, M.: Total controllability and observability for dynamic systems with non-instantaneous impulses on time scales. Asian J. Control **23**(2), 847–859 (2021). https://doi.org/10.1002/asjc.2268
46. Vijayakumar, V., Murugesu, R., Poongodi, R., Dhanalakshmi, S.: Controllability of second order impulsive nonlocal Cauchy problem via measure of noncompactness. Mediterr. J. Math. **14**(1), 1–23 (2017). https://doi.org/10.1007/s00009-016-0813-6
47. Vijayakumar, V., Murugesu, R., Selvan, M.T.: Controllability for a class of second order functional evolution differential equations without uniqueness. IMA J. Math. Control Inf. **36**(1), 225–246 (2019). https://doi.org/10.1093/imamci/dnx048
48. Balachandran, K., Anthoni, S.M.: Controllability of second-order semilinear neutral functional differential systems in Banach spaces. Comput. Math. Appl. **41**(10–11), 1223–1235 (2001). https://doi.org/10.1016/S0898-1221(01)00093-1
49. Haq, A., Sukavanam, N.: Controllability of second-order nonlocal retarded semilinear systems with delay in control. Appl. Anal. **99**(16), 2741–2754 (2020). https://doi.org/10.1080/00036811.2019.1582031
50. Henriquez, H.R., Hernandez, E.: Approximate controllability of second-order distributed implicit functional systems. Nonlinear Anal., Theory Methods Appl. **70**(2), 1023–1039 (2009). https://doi.org/10.1016/j.na.2008.01.029
51. Sukavanam, S., Tomar, N.K.: Mild solution and controllability of second-order non-local retarded semilinear systems. IMA J. Math. Control Inf. **37**(1), 39–49 (2020). https://doi.org/10.1093/imamci/dny037
52. Mahmudov, N.I., Udhayakumar, R., Vijayakumar, V.: On the approximate controllability of second-order evolution hemivariational inequalities. Results Math. **75**, 160 (2020). https://doi.org/10.1007/s00025-020-01293-2
53. Mahmudov, N.I., Vijayakumar, V., Murugesu, R.: Approximate controllability of second-order evolution differential inclusions in Hilbert spaces. Mediterr. J. Math. **13**(5), 3433–3454 (2016). https://doi.org/10.1007/s00009-016-0695-7
54. Sakthivel, R., Mahmudov, N.I., Kim, J.H.: On controllability of second order nonlinear impulsive differential systems. Nonlinear Anal., Theory Methods Appl. **71**(1–2), 44–52 (2009). https://doi.org/10.1016/j.na.2008.10.029
55. Shukla, A., Sukavanam, N., Pandey, D.N., Arora, U.: Approximate controllability of second-order semilinear control system. Circuits Syst. Signal Process. **35**, 3339–3354 (2016). https://doi.org/10.1007/s00034-015-0191-5
56. Shukla, A., Patel, R.: Existence and optimal control results for second-order semilinear system in Hilbert spaces. Circuits Syst. Signal Process. **40**, 4246–4258 (2021). https://doi.org/10.1007/s00034-021-01680-2
57. Vijayakumar, V., Murugesu, R.: Controllability for a class of second-order evolution differential inclusions without compactness. Appl. Anal. **98**(7), 1367–1385 (2019). https://doi.org/10.1080/00036811.2017.1422727
58. Vijayakumar, V., Udhayakumar, R., Dineshkumar, C.: Approximate controllability of second order nonlocal neutral differential evolution inclusions. IMA J. Math. Control Inf. **38**(1), 192–210 (2021). https://doi.org/10.1093/imamci/dnaa001
59. Curtain, R.F., Zwart, H.J.: An Introduction to Infinite Dimensional Linear Systems Theory. Springer, New York (1995)
60. Zhou, H.X.: Approximate controllability for a class of semilinear abstract equations. Asian J. Control **21**(4), 551–565 (1983). https://doi.org/10.1137/0321033
61. Bashirov, A.E., Ghahramanlou, N., Lam, J.: On partial approximate controllability of semilinear systems. Cogent Eng. **1**(1), 965947 (2014). https://doi.org/10.1080/23311916.2014.965947
62. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung. **32**, 75–96 (1978). https://doi.org/10.1007/BF01902205