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Well-posed results for nonlocal biparabolic equation with linear and nonlinear source terms

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Abstract

In this paper, we consider the biparabolic problem under nonlocal conditions with both linear and nonlinear source terms. We derive the regularity property of the mild solution for the linear source term while we apply the Banach fixed-point theorem to study the existence and uniqueness of the mild solution for the nonlinear source term. In both cases, we show that the mild solution of our problem converges to the solution of an initial value problem as the parameter epsilon tends to zero. The novelty in our study can be considered as one of the first results on biparabolic equations with nonlocal conditions.

Keywords: Biparabolic equation; Source term; Nonlocal condition; Mild solution; Existence; Uniqueness; Convergence

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we consider the following biparabolic equation:

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = F(u(x,t)) & \text{in } \Omega \times (0,T], \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 & \text{in } \Omega, \\ u_t(x,0) = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

under temporal nonlocal condition

$$u(x,0) + \varepsilon u(x,T) = f(x), \quad x \in \partial\Omega. \tag{1.2}$$

Here $u(x,t)$ is a function of temperature or concentration, $F(u)$ is a source function, ε is a parameter, and $f \in L^2(\Omega) \cap \mathbb{H}^s(\Omega)$. When $\varepsilon = 0$, the problem becomes an initial conditional problem.

The main equation of problem (1.1) is equivalent to

$$P^2 u = P(Pu) = \frac{\partial^2}{\partial t^2} u + 2\frac{\partial}{\partial t}\mathcal{A}u + \mathcal{A}^2 u = G(x,t;u),$$

where P is the main operator for the classical parabolic equation,

$$Pu = \left(\frac{\partial}{\partial t} + \mathcal{A}\right)u = G(x,t;u).$$

As mentioned by Fushchich, Galitsyn, and Polubinskii [9], the new fourth-order partial differential equation (1.1) is invariant with respect to the Galilei group. From the results in [9] we realize that the classical heat equations

$$u_t - \Delta u = 0$$

do not completely describe heat and mass transfer processes. Therefore, in many situations of heat conduction, it tends to replace the classical thermal equation by one of the hyperbolic form, such as problem (1.1). Problem (1.1) is a form of quadratic PDEs equations, which have a wide range of applications in various scientific and engineering disciplines, such as conduction of heat [7, 9, 24, 33], dynamics of filtration consolidation [6, 8], strongly damped wave equations [14, 23, 34], ice formulation and accretion problems on structures, ships, and aircraft [19–21], the transport of liquids and insoluble surfactant through the lung airways [11, 12], brain imaging for the detection and mapping of subtle abnormalities of shape and volume in the brains of patients with metastatic tumors [18, 26, 27], and so on.

Whereas there were a number of studies focused on parabolic equations [1–4, 10, 13, 15, 22, 25, 28], the studies on biparabolic equations are still limited. Let us mention previous works related to biparabolic equation (1.1). Lakhdari and Boussetila [16] applied Kozlov–Maz'ya iteration method for approximating the final value problem for biparabolic equation. Bulavatsky [7] studied some boundary value problems for biparabolic equations with nonlocal boundary conditions. Besma et al. [5] considered the problem of approximating a solution of an ill-posed biparabolic problem in the abstract Hilbert space. They introduced a modified quasi-boundary value method to get stable solutions for regularizing the ill-posedness of a biparabolic equation. Tuan et al. [32] studied the problem of finding the initial distribution for a linear inhomogeneous or nonlinear biparabolic equation. Recently, Phuong et al. [25] studied an inverse source problem of the biparabolic equation. Very recently, Tuan et al. [31] investigated two terminal value problems for stochastic biparabolic equations perturbed by a standard Brownian motion or a fractional Brownian motion.

The nonlocal problem focused in this paper is considered as one of the most interesting areas for the readers in various applications, such as chaos, chemistry, biology, and physics; see [30]. In comparison with the initial or final conditions, the nonlocal conditions are more difficult to handle. The novelty of our problem is the presence of condition of nonlocal type (1.2). In many real-world applications, it is difficult to collect accurate data at the beginning or at the end of a process. In addition, many processes happen so fast and in a short period, in which we only can observe the data at the beginning and the end of a process, not the data at a specific time in the range of (0,T). Therefore studies on nonlocal conditional problems can help us to track down a process in more detail and in an effective way.

To the best of our knowledge, up to date, there is still no any study considering problem (1.1) under the nonlocal condition (1.2). This motivates us to focus on problems (1.1)–(1.2). The main contributions of the paper are as follows.

- For the linear source function, we give the well-posedness and investigate the convergence of the mild solution to problem (1.1)–(1.2) as ε approaches 0. In more detail, we prove that the solution of problem (1.1)–(1.2) converges to a mild solution with the initial value problem for (1.1).
- For nonlinear source functions, we prove the existence and uniqueness of mild solutions. In the main analysis, we apply the Banach fixed point theorem. Our next aim is to demonstrate the convergence of the mild solution as the parameter ε tends to 0.

The main techniques to handle the above problem are based on the ideas of some recent publications [17, 29, 30]. We overcome some difficulties by setting up complex evaluations on Hilbert scale spaces. Choosing the right spaces for the input f and for the solution is also not simple task.

This paper is organized as follows. In Sect. 2, we provide some useful notations and the definition of a solution in the mild sense. In Sect. 3, we focus on the well-posed results for the linear case and discuss on what happens as $\varepsilon \to 0$. The well-posed results for the nonlinear source term are introduced in Sect. 4. Eventually, the results are summarized in Sect. 5.

2 Preliminary results and mild solution

In this section, we introduce the notation and the functional setting used in our paper. Recall that the spectral problem

$$\begin{cases} \Delta\psi_n(x) = -\lambda_n\psi_n(x), & x \in \Omega, \\ \psi_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are $\psi_n \in H_0^1(\Omega)$.

Definition 2.1 (Hilbert scale space) We recall the Hilbert scale space given as follows:

$$\mathbb{H}^s(\Omega) = \left\{ f \in L^2(\Omega), \sum_{n=1}^{\infty} \lambda_n^{2s} \left(\int_\Omega f(x)\psi_n(x)\,dx \right)^2 < \infty \right\}$$

for $s \ge 0$. It is well known that $\mathbb{H}^s(\Omega)$ is the Hilbert space corresponding to the norm

$$\|f\|_{\mathbb{H}^s(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} \left(\int_\Omega f(x)\psi_n(x)\,dx \right)^2 \right)^{1/2}, \quad f \in \mathbb{H}^s(\Omega).$$

Let us give an explicit formula of the mild solution. First, taking the inner product of both sides of (1.1) with $\psi_n(x)$, we find that

$$\begin{aligned} &\frac{d^2}{dt^2}\left(\int_\Omega u(x,t)\psi_n(x)\,dx \right) + 2\lambda_n \left(\int_\Omega u(x,t)\psi_n(x)\,dx \right) \\ &\quad + \lambda_n^2 \left(\int_\Omega u(x,t)\psi_n(x)\,dx \right) = \int_\Omega F\bigl(u(x,t)\bigr)\psi_n(x)\,dx. \end{aligned} \tag{2.1}$$

It is easy to see that the latter problem has a solution given by

$$\begin{aligned}\int_\Omega u(x,t)\psi_n(x)\,dx &= e^{-t\lambda_n}(1+t\lambda_n)\int_\Omega u(x,0)\psi_n(x)\,dx \\ &\quad + \int_0^t (t-r)e^{-(t-r)\lambda_n}\left(\int_\Omega F\big(u(x,r)\big)\psi_n(x)\,dx\right)dr. \end{aligned} \tag{2.2}$$

The condition $u(x,0)+\varepsilon u(x,T)=f(x)$ implies that

$$\begin{aligned}&\int_\Omega u(x,0)\psi_n(x)\,dx + \varepsilon\int_\Omega u(x,T)\psi_n(x)\,dx \\ &\quad = \big(1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)\big)\int_\Omega u(x,0)\psi_n(x)\,dx \\ &\qquad + \varepsilon\int_0^T (T-r)e^{-(T-r)\lambda_n}\left(\int_\Omega F\big(u(x,r)\big)\psi_n(x)\,dx\right)dr \\ &\quad = \int_\Omega f(x)\psi_n(x)\,dx. \end{aligned} \tag{2.3}$$

We rewrite it as

$$\begin{aligned}&\int_\Omega u(x,0)\psi_n(x)\,dx \\ &\quad = \frac{\int_\Omega f(x)\psi_n(x)\,dx - \varepsilon\int_0^T (T-r)e^{-(T-r)\lambda_n}(\int_\Omega F(u(x,r))\psi_n(x)\,dx)\,dr}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)}. \end{aligned} \tag{2.4}$$

Combining (2.2) and (2.4), we find that

$$\begin{aligned}\int_\Omega u(x,t)\psi_n(x)\,dx &= \frac{e^{-t\lambda_n}(1+t\lambda_n)}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)}\int_\Omega f(x)\psi_n(x)\,dx \\ &\quad - \frac{\varepsilon e^{-t\lambda_n}(1+t\lambda_n)\int_0^T (T-r)e^{-(T-r)\lambda_n}(\int_\Omega F(u(x,r))\psi_n(x)\,dx)\,dr}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)} \\ &\quad + \int_0^t (t-r)e^{-(t-r)\lambda_n}\left(\int_\Omega F\big(u(x,r)\big)\psi_n(x)\,dx\right)dr. \end{aligned} \tag{2.5}$$

For any $f\in L^2(\Omega)$, we define

$$\mathbf{Q}_\varepsilon(t)f = \sum_{n=1}^\infty \frac{e^{-t\lambda_n}(1+t\lambda_n)}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)\psi_n(x) \tag{2.6}$$

and

$$\mathbf{S}(t)f = e^{-t\Delta}f = \sum_{n=1}^\infty e^{-t\lambda_n}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)\psi_n(x). \tag{2.7}$$

From (2.5) we give an explicit formula of the solution to problem (1.1)–(1.2) in the mild setting:

$$\begin{aligned} u_\varepsilon(t) = \mathbf{Q}_\varepsilon(t)f + \int_0^t (t-r)\mathbf{S}(t-r)F\big(u_\varepsilon(r)\big)\,dr \\ - \varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big)\,dr. \end{aligned} \tag{2.8}$$

3 Well-posed results for linear case

In this section, we focus on the case $F(t,u) = F(t)$. Under the linear case, we recall the mild solution u_ε to problem (1.1)–(1.2):

$$u_\varepsilon(t) = \mathbf{Q}_\varepsilon(t)f + \int_0^t (t-r)\mathbf{S}(t-r)F(r)\,dr - \varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F(r)\,dr. \tag{3.1}$$

Lemma 3.1 *Let $f \in \mathbb{H}^s(\Omega)$.*
a) If $s < m+1$, then

$$\big\|\mathbf{Q}_\varepsilon(t)f\big\|_{\mathbb{H}^m(\Omega)} \le C(s,m)t^{s-m-1}\|f\|_{\mathbb{H}^s(\Omega)}. \tag{3.2}$$

b) If $s < m$, then

$$\big\|\mathbf{S}(t)f\big\|_{\mathbb{H}^m(\Omega)} \le C(s,m)t^{s-m}\|f\|_{\mathbb{H}^s(\Omega)}. \tag{3.3}$$

Proof Using Parseval's equality, we find that

$$\begin{aligned} \big\|\mathbf{Q}(t)f\big\|^2_{\mathbb{H}^m(\Omega)} &= \sum_{n=1}^\infty \lambda_n^{2m}\left(\frac{e^{-t\lambda_n}(1+t\lambda_n)}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)}\right)^2\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2 \\ &\le 2\sum_{n=1}^\infty \lambda_n^{2m}e^{-2t\lambda_n}\big(1+t^2\lambda_n^2\big)\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2 \\ &\le 2C_T\sum_{n=1}^\infty \lambda_n^{2m+2}e^{-2t\lambda_n}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2. \end{aligned} \tag{3.4}$$

In view of the inequality $e^{-z} \le C_\nu z^{-\nu}$ for all $\nu > 0$, we know that

$$\lambda_n^{2m+2}e^{-2t\lambda_n} \le C_\nu\lambda_n^{2m+2}(t\lambda_n)^{-2\nu} = t^{-2\nu}\lambda_n^{2m+2-2\nu}.$$

It follows from (3.4) that

$$\big\|\mathbf{Q}_\varepsilon(t)f\big\|^2_{\mathbb{H}^m(\Omega)} \le t^{-2\nu}\sum_{n=1}^\infty \lambda_n^{2m+2-2\nu}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2, \tag{3.5}$$

which gives the estimate

$$\big\|\mathbf{Q}_\varepsilon(t)f\big\|_{\mathbb{H}^m(\Omega)} \le C_T t^{-\nu}\|f\|_{\mathbb{H}^{m+1-\nu}(\Omega)}. \tag{3.6}$$

Setting $\nu = m + 1 - s > 0$, we know that

$$\left\|\mathbf{Q}_\varepsilon(t)f\right\|_{\mathbb{H}^m(\Omega)} \le C(m,s)t^{s-m-1}\|f\|_{\mathbb{H}^s(\Omega)}. \tag{3.7}$$

Using again $e^{-z} \le C_\nu z^{-\nu}$ for all $\nu > 0$, we find that

$$\begin{aligned}
\left\|\mathbf{S}(t)f\right\|^2_{\mathbb{H}^m(\Omega)} = \left\|e^{-t\Delta}f\right\|^2_{\mathbb{H}^m(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^{2m}e^{-2t\lambda_n}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2 \\
&\le C_\nu t^{-2\nu}\lambda_n^{2m-2\nu}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2 \\
&= C_\nu t^{-2\nu}\|f\|^2_{\mathbb{H}^{m-\nu}(\Omega)}.
\end{aligned} \tag{3.8}$$

Setting $\nu = m - s$ for $s < m$, we get

$$\left\|\mathbf{S}(t)f\right\|_{\mathbb{H}^m(\Omega)} \le C(s,m)t^{s-m}\|f\|_{\mathbb{H}^s(\Omega)}. \tag{3.9}$$

□

Theorem 3.1 *Let $F \in L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))$ and $f \in \mathbb{H}^s(\Omega)$. Then*

$$\|u_\varepsilon\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} \le C(T,s,m,\mu)\big(\|f\|_{\mathbb{H}^s(\Omega)} + \|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}\big). \tag{3.10}$$

Proof Applying Lemma 3.1 and noting that $m < s < m + 1$, we find that

$$\begin{aligned}
\left\|u_\varepsilon(\cdot,t)\right\|_{\mathbb{H}^m(\Omega)} &\le \left\|\mathbf{Q}_\varepsilon(t)f\right\|_{\mathbb{H}^m(\Omega)} + \int_0^t (t-r)\left\|\mathbf{S}(t-r)F(r)\right\|_{\mathbb{H}^m(\Omega)}\,dr \\
&\quad + \varepsilon\left\|\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F(r)\,dr\right\|_{\mathbb{H}^m(\Omega)} \\
&\le C(m,s)t^{s-m-1}\|f\|_{\mathbb{H}^s(\Omega)} + \int_0^t (t-r)^{s-m}\left\|F(r)\right\|_{\mathbb{H}^{s-1}(\Omega)}\,dr \\
&\quad + \varepsilon t^{s-m-1}\int_0^T (T-r)\left\|\mathbf{S}(T-r)F(r)\right\|_{\mathbb{H}^s(\Omega)}\,dr \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{3.11}$$

Let μ be such that $1 < \mu < \frac{1}{m+1-s}$. The first term I_1 is bounded by

$$\begin{aligned}
\|I_1\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} &\le \left(\int_0^T \left|C(m,s)t^{s-m-1}\|f\|_{\mathbb{H}^s(\Omega)}\right|^\mu dt\right)^{1/\mu} \\
&= \left(\frac{T^{(s-m-1)\mu+1}}{(s-m-1)\mu+1}\right)^{1/\mu} C(m,s)\|f\|_{\mathbb{H}^s(\Omega)}.
\end{aligned} \tag{3.12}$$

For the second term I_2, we easily observe that

$$I_2(t) \le \left(\int_0^t (t-r)^{s-m}\,dr\right)\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))} = \frac{T^{s-m+1}}{s-m+1}\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}. \tag{3.13}$$

Then we get the bound

$$\|I_2\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} \le \left(\int_0^T \left|\frac{T^{s-m+1}}{s-m+1}\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}\right|^\mu dt\right)^{1/\mu} = \frac{T^{s-m+1}}{s-m+1+1\mu}\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}. \tag{3.14}$$

For the third term I_3, using Lemma (3.1), we have that

$$\begin{aligned} I_3 &\le \varepsilon t^{s-m-1}\int_0^T (T-r)\big\|\mathbf{S}(T-r)F(r)\big\|_{\mathbb{H}^s(\Omega)}\,dr \\ &\le \varepsilon C(m,s)t^{s-m-1}\int_0^T (T-r)(T-r)^{-1}\big\|F(r)\big\|_{\mathbb{H}^{s-1}(\Omega)}\,dr \\ &\le \varepsilon C(m,s)T\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}t^{s-m-1}. \end{aligned} \tag{3.15}$$

This immediately implies that

$$\begin{aligned} \|I_3\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} &\le \left(\int_0^T \left|\varepsilon C(m,s)T\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}t^{s-m-1}\right|^\mu dt\right)^{1/\mu} \\ &= \varepsilon C(m,s)T\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}\left(\frac{T^{(s-m-1)\mu+1}}{(s-m-1)\mu+1}\right)^{1/\mu}. \end{aligned} \tag{3.16}$$

Combining (3.11), (3.12), (3.14), and (3.16), we find that

$$\begin{aligned} \|u_\varepsilon\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} &\le \|I_1\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} + \|I_2\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} + \|I_3\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} \\ &\le C(T,s,m,\mu)\big(\|f\|_{\mathbb{H}^s(\Omega)} + \|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}\big). \end{aligned} \tag{3.17}$$

Let us recall the formula

$$u(t) = \mathbf{S}(t)f + \int_0^t (t-r)\mathbf{S}(t-r)F(r)\,dr. \tag{3.18}$$

Since (3.1), we get that

$$u_\varepsilon(t) - u(t) = \big(\mathbf{Q}_\varepsilon(t) - \mathbf{S}(t)\big)f - \varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F(r)\,dr. \tag{3.19}$$

From (3.16) we know that

$$\begin{aligned} &\left\|\varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F(r)\,dr\right\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} \\ &\quad\le \varepsilon C(m,s)T\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}\left(\frac{T^{(s-m-1)\mu+1}}{(s-m-1)\mu+1}\right)^{1/\mu}. \end{aligned} \tag{3.20}$$

Our next aim is estimating the term $(\mathbf{Q}_\varepsilon(t) - \mathbf{S}(t))f$. We clearly see that

$$\big(\mathbf{Q}_\varepsilon(t) - \mathbf{S}(t)\big)f = \sum_{n=1}^\infty \left[\frac{e^{-t\lambda_n}(1+t\lambda_n)}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)} - e^{-t\lambda_n}(1+t\lambda_n)\right]\left(\int_\Omega f(x)\psi_n(x)\,dx\right)\psi_n(x)$$

$$= \sum_{n=1}^{\infty}\left[\frac{\varepsilon e^{-T\lambda_n}(1+T\lambda_n)e^{-t\lambda_n}(1+t\lambda_n)}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)}\right]\left(\int_\Omega f(x)\psi_n(x)\,dx\right)\psi_n(x).$$

Parseval's equality implies that

$$\begin{aligned}
&\left\|\left(\mathbf{Q}_\varepsilon(t)-\mathbf{S}(t)\right)f\right\|^2_{\mathbb{H}^m(\Omega)} \\
&\quad= \sum_{n=1}^{\infty}\lambda_n^{2m}\left[\frac{\varepsilon e^{-T\lambda_n}(1+T\lambda_n)e^{-t\lambda_n}(1+t\lambda_n)}{1+\varepsilon e^{-T\lambda_n}(1+T\lambda_n)}\right]^2\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2 \\
&\quad\le \varepsilon^2\sum_{n=1}^{\infty}\lambda_n^{2m}e^{-2T\lambda_n}(1+T\lambda_n)^2e^{-2t\lambda_n}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2. \qquad (3.21)
\end{aligned}$$

Using the inequality $e^{-z}\le C_\nu z^{-\nu}$ for all $\nu>0$, we arrive at

$$e^{-2t\lambda_n}\le C(m,s)(t\lambda_n)^{-2(m+1-s)}. \qquad (3.22)$$

It is obvious that

$$e^{-2T\lambda_n}(1+T\lambda_n)^2\le C.$$

It follows from (3.21) that

$$\begin{aligned}
\left\|\left(\mathbf{Q}_\varepsilon(t)-\mathbf{S}(t)\right)f\right\|^2_{\mathbb{H}^m(\Omega)} &\le C\varepsilon^2t^{2m-2s}\sum_{n=1}^{\infty}\lambda_n^{2s-2}\left(\int_\Omega f(x)\psi_n(x)\,dx\right)^2 \\
&= C(m,s)\varepsilon^2t^{2s-2m-2}\|f\|^2_{\mathbb{H}^{s-1}(\Omega)}. \qquad (3.23)
\end{aligned}$$

This implies that

$$\begin{aligned}
\left\|(\mathbf{Q}_\varepsilon-\mathbf{S})f\right\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} &\le C(m,s)\varepsilon\|f\|_{\mathbb{H}^{s-1}(\Omega)}\left(\int_0^T t^{(s-m-1)\mu}\,dt\right)^{1/\mu} \\
&= C(m,s)\varepsilon\|f\|_{\mathbb{H}^{s-1}(\Omega)}\left(\frac{T^{(s-m-1)\mu+1}}{(s-m-1)\mu+1}\right)^{1/\mu}, \qquad (3.24)
\end{aligned}$$

where we recall that $1<\mu<\frac{1}{m+1-s}$. Combining (3.19), (3.20), and (3.24), we arrive at

$$\begin{aligned}
&\|u_\varepsilon-u\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} \\
&\quad\le \left\|(\mathbf{Q}_\varepsilon-\mathbf{S})f\right\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))}+\left\|\varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T(T-r)\mathbf{S}(T-r)F(r)\,dr\right\|_{L^\mu(0,T;\mathbb{H}^m(\Omega))} \\
&\quad\le C(m,s)\varepsilon\|f\|_{\mathbb{H}^{s-1}(\Omega)}\left(\frac{T^{(s-m-1)\mu+1}}{(s-m-1)\mu+1}\right)^{1/\mu} \\
&\qquad+\varepsilon C(m,s)T\|F\|_{L^\infty(0,T;\mathbb{H}^{s-1}(\Omega))}\left(\frac{T^{(s-m-1)\mu+1}}{(s-m-1)\mu+1}\right)^{1/\mu}. \qquad (3.25)
\end{aligned}$$

□

4 Well-posed results for nonlinear case

Theorem 4.1 *Let $f \in \mathbb{H}^s(\Omega)$ for $s \geq p$. Let F be such that*

$$\big\|F(\varphi) - F(\psi)\big\|_{\mathbb{H}^q(\Omega)} \leq K_f \|\varphi - \psi\|_{\mathbb{H}^p(\Omega)} \tag{4.1}$$

for all $\varphi, \psi \in \mathbb{H}^p(\Omega)$ and $p < q < p + 1$. Then for any $\varepsilon > 0$ and K_f small enough, problem (1.1)–(1.2) has a unique mild solution in $\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))$, which satisfies

$$\begin{aligned} u_\varepsilon(t) = \mathbf{Q}_\varepsilon(t)f &+ \int_0^t (t-r)\mathbf{S}(t-r)F\big(u_\varepsilon(r)\big)\,dr \\ &- \varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big)\,dr, \end{aligned} \tag{4.2}$$

where

$$\max(0, p+1-s) \leq a < 1. \tag{4.3}$$

In addition,

$$\|u_\varepsilon\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} \leq \frac{2C_T T^{\frac{1}{\mu}+s-p-1}}{(1-a\mu)^{1/\mu}}\|f\|_{\mathbb{H}^s(\Omega)} \tag{4.4}$$

for $1 < \mu < \frac{1}{a}$.

Proof We look for the solution in the space $\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))$. Let us define the function

$$\begin{aligned} B_\varepsilon(\psi)(t) = \mathbf{Q}_\varepsilon(t)f &+ \int_0^t (t-r)\mathbf{S}(t-r)F\big(\psi(r)\big)\,dr \\ &- \varepsilon\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F\big(\psi(r)\big)\,dr. \end{aligned} \tag{4.5}$$

If $\psi = 0$, then by the assumption $F(0) = 0$ we have that

$$t^a\big\|B_\varepsilon\psi(t)\big\|_{\mathbb{H}^p(\Omega)} = t^a\big\|\mathbf{Q}_\varepsilon(t)f\big\|_{\mathbb{H}^p(\Omega)} \leq C_T t^{a-\nu}\|f\|_{\mathbb{H}^{p+1-\nu}(\Omega)}. \tag{4.6}$$

Since $s < p + 1$, we set $\nu = p + 1 - s$. Then it follows from (4.6) that

$$t^a\big\|B_\varepsilon\psi(t)\big\|_{\mathbb{H}^p(\Omega)} \leq C_T t^{s+a-p-1}\|f\|_{\mathbb{H}^s(\Omega)}. \tag{4.7}$$

Under the assumption $p + 1 \leq s + a$, if $\psi = 0$, then we find that for any $0 \leq t \leq T$,

$$t^a\big\|B_\varepsilon\big(\psi(t) = 0\big)\big\|_{\mathbb{H}^p(\Omega)} \leq C_T T^{s+a-p-1}\|f\|_{\mathbb{H}^s(\Omega)}, \tag{4.8}$$

which allows us to derive that $B_\varepsilon\psi$ belongs to the space $\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))$ if $\psi = 0$.

Let $\varphi, \psi \in \mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))$. It is obvious that

$$B_\varepsilon(\psi)(t) - B_\varepsilon(\varphi)(t) = \int_0^t (t-r)\mathbf{S}(t-r)\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\,dr$$

$$\begin{aligned} & - \varepsilon \mathbf{Q}_\varepsilon(t) \int_0^T (T-r)\mathbf{S}(T-r)\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\,dr \\ &= J_1(t) + J_2(t). \end{aligned} \tag{4.9}$$

By the second part of Lemma 3.1 the term J_1 is bounded by

$$\begin{aligned} \big\|J_1(t)\big\|_{\mathbb{H}^p(\Omega)} &\le \int_0^t (t-r)\big\|\mathbf{S}(t-r)\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\big\|_{\mathbb{H}^p(\Omega)}\,dr \\ &\le \int_0^t (t-r)(t-r)^{q-p}\big\|F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big\|_{\mathbb{H}^q(\Omega)}\,dr, \end{aligned} \tag{4.10}$$

where we note that $p > q$. Since F is globally Lipschitz as in (4.1), we infer that

$$\begin{aligned} \big\|J_1(t)\big\|_{\mathbb{H}^p(\Omega)} &\le K_f \int_0^t (t-r)^{q-p+1}\big\|\psi(r) - \varphi(r)\big\|_{\mathbb{H}^p(\Omega)}\,dr \\ &\le K_f\bigg(\int_0^t (t-r)^{q-p+1} r^{-a}\,dr\bigg)\Big(\sup_{0\le t\le T} r^a\big\|\psi(r) - \varphi(r)\big\|_{\mathbb{H}^p(\Omega)}\Big) \\ &= K_f B(2+q-p, 1-a) t^{2-a+q-p}\|\psi - \varphi\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}, \end{aligned} \tag{4.11}$$

where we note that $q + 2 > p$ and $a < 1$. This implies that

$$\begin{aligned} t^a\big\|J_1(t)\big\|_{\mathbb{H}^p(\Omega)} &\le K_f B(2+q-p, 1-a) t^{2+q-p}\|\psi - \varphi\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \\ &\le K_f B(2+q-p, 1-a) T^{2+q-p}\|\psi - \varphi\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}. \end{aligned} \tag{4.12}$$

The right-hand side of this expression is independent of t, and we deduce that

$$\|J_1\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \le K_f B(2+q-p, 1-a) T^{2+q-p}\|\psi - \varphi\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}. \tag{4.13}$$

Since $q < p + 1$ and $a > 0$, we can choose a real number s' such that

$$\max(p + 1 - a, q) \le s' < p + 1.$$

Then we find that

$$\begin{aligned} & \bigg\|\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\,dr\bigg\|_{\mathbb{H}^p(\Omega)} \\ &\quad \le t^{s'-p-1}\bigg\|\int_0^T (T-r)\mathbf{S}(T-r)\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\,dr\bigg\|_{\mathbb{H}^s(\Omega)}. \end{aligned} \tag{4.14}$$

Since $s' > q$, we get that

$$\begin{aligned} & \bigg\|\int_0^T (T-r)\mathbf{S}(T-r)\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\,dr\bigg\|_{\mathbb{H}^s(\Omega)} \\ &\quad \le \int_0^T (T-r)^{q-s'+1}\big\|\big(F\big(\psi(r)\big) - F\big(\varphi(r)\big)\big)\big\|_{\mathbb{H}^q(\Omega)}\,dr \end{aligned}$$

$$
\begin{aligned}
&\le K_f \int_0^T (T-r)^{q-s'+1} \big\| \psi(r) - \varphi(r) \big\|_{\mathbb{H}^p(\Omega)} \, dr \\
&= K_f \int_0^T (T-r)^{q-s'+1} r^{-a} r^a \big\| \psi(r) - \varphi(r) \big\|_{\mathbb{H}^p(\Omega)} \, dr \\
&\le K_f B\big(2+q-s', 1-a\big) T^{2+q-s'-a} \| \psi - \varphi \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}. \qquad (4.15)
\end{aligned}
$$

Combining (4.14) and (4.15) and noting that $s' + a \ge p + 1$, we obtain that

$$
\begin{aligned}
t^a \big\| J_2(t) \big\|_{\mathbb{H}^p(\Omega)} &\le \varepsilon t^{a+s'-p-1} K_f B\big(2+q-s', 1-a\big) T^{2+q-s'-a} \| \psi - \varphi \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \\
&\le \varepsilon K_f B\big(2+q-s', 1-a\big) T^{q+1-p} \| \psi - \varphi \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}. \qquad (4.16)
\end{aligned}
$$

The condition $q + 1 > p$ ensures that the right-hand side is defined. Therefore we can deduce that

$$
\| J_2 \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \le \varepsilon K_f B\big(2+q-s', 1-a\big) T^{q+1-p} \| \psi - \varphi \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}. \qquad (4.17)
$$

Combining (4.9), (4.13), and (4.17), we arrive at

$$
\begin{aligned}
&\big\| B_\varepsilon(\psi) - B_\varepsilon(\varphi) \big\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \\
&\quad \le \| J_1 \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} + \| J_2 \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \\
&\quad \le K_f B(2+q-p, 1-a) T^{2+q-p} \| \psi - \varphi \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \\
&\qquad + \varepsilon K_f B\big(2+q-s', 1-a\big) T^{q+1-p} \| \psi - \varphi \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))}. \qquad (4.18)
\end{aligned}
$$

Let K_f be small enough such that

$$
M_T = K_f B(2+q-p, 1-a) T^{2+q-p} + \varepsilon K_f B\big(2+q-s', 1-a\big) T^{q+1-p} < 1/2.
$$

It follows from (4.7) that

$$
B_\varepsilon\big(\mathbf{X}^{a,\infty}\big((0,T];\mathbb{H}^p(\Omega)\big)\big) \subset \mathbf{X}^{a,\infty}\big((0,T];\mathbb{H}^p(\Omega)\big),
$$

and together with (4.18), we find that B_ε is a contraction mapping. By using the Banach fixed point theorem we deduce that roblem (1.1)–(1.2) has a unique solution $u_\varepsilon \in \mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))$.

It follows from (4.8) that

$$
\begin{aligned}
\| u_\varepsilon \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} &= \big\| B_\varepsilon(u_\varepsilon) \big\|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \\
&\le M_T \| u_\varepsilon \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} + C_T T^{s+a-p-1} \| f \|_{\mathbb{H}^s(\Omega)}. \qquad (4.19)
\end{aligned}
$$

Therefore we get that

$$
\| u_\varepsilon \|_{\mathbf{X}^{a,\infty}((0,T];\mathbb{H}^p(\Omega))} \le \frac{C_T T^{s+a-p-1} \| f \|_{\mathbb{H}^s(\Omega)}}{1 - M_T} \le 2 C_T T^{s+a-p-1} \| f \|_{\mathbb{H}^s(\Omega)}. \qquad (4.20)
$$

This estimate implies that

$$\left\| u_\varepsilon(t) \right\|_{\mathbb{H}^p(\Omega)} \le 2C_T T^{s+a-p-1} t^{-a} \|f\|_{\mathbb{H}^s(\Omega)}. \tag{4.21}$$

Since $a < 1$, we can find that $0 < \mu < \frac{1}{a}$. Thus we arrive at

$$\begin{aligned} \|u_\varepsilon\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} &= \left(\int_0^T \left\| u_\varepsilon(t) \right\|^\mu_{\mathbb{H}^p(\Omega)} \, dt \right)^{1/\mu} \\ &\le 2C_T T^{s+a-p-1} \|f\|_{\mathbb{H}^s(\Omega)} \left(\int_0^T t^{-a\mu} \, dt \right)^{1/\mu}, \end{aligned} \tag{4.22}$$

which allows us to get that

$$\|u_\varepsilon\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} \le \frac{2C_T T^{\frac{1}{\mu}+s-p-1}}{(1-a\mu)^{1/\mu}} \|f\|_{\mathbb{H}^s(\Omega)}. \tag{4.23}$$

The proof is completed. □

Theorem 4.2 *Let F be as in (4.1). Let $f \in \mathbb{H}^s(\Omega)$ for $p < s < p+1$. Let K_f be small enough such that $K_f T^{q-p+2} \le \frac{1}{2}$. Then*

$$\begin{aligned} \|u_\varepsilon - u\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} &\le 2C(p,s)\varepsilon \|f\|_{\mathbb{H}^{s-1}(\Omega)} \left(\frac{T^{(s-p-1)\mu+1}}{(s-p-1)\mu+1} \right)^{1/\mu} \\ &\quad + 4\varepsilon K_f T^{q-p} \|f\|_{\mathbb{H}^s(\Omega)} \left(\frac{T^{(s-p-1)\mu+1}}{(s-p-1)\mu+1} \right)^{1/\mu}, \end{aligned} \tag{4.24}$$

where $1 < \mu < \frac{1}{p+1-s}$.

Proof Let us recall that

$$u(t) = \overline{S}(t)f + \int_0^t (t-r)\mathbf{S}(t-r)F\big(u(r)\big) \, dr, \tag{4.25}$$

where we recall that

$$\overline{S}(t)f = \sum_{n=1}^\infty e^{-t\lambda_n}(1+t\lambda_n) \left(\int_\Omega f(x)\psi_n(x) \, dx \right) \psi_n(x).$$

By (4.2) we immediately have the result on the difference between $u_\varepsilon(t)$ and $u(t)$ which is split as the sum of three terms

$$\begin{aligned} u_\varepsilon(t) - u(t) &= \mathbf{Q}_\varepsilon(t)f - \overline{S}(t)f + \int_0^t (t-r)\mathbf{S}(t-r)\big(F\big(u_\varepsilon(r)\big) - F\big(u(r)\big)\big) \, dr \\ &\quad - \varepsilon \mathbf{Q}_\varepsilon(t) \int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big) \, dr \\ &= H_1(t) + H_2(t) + H_3(t). \end{aligned} \tag{4.26}$$

Let us first treat the first term $H_1(t)$. By applying (3.24) we find that

$$\begin{aligned}\big\|(\mathbf{Q}_\varepsilon - \mathbf{S})f\big\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} &\le C(p,s)\varepsilon\|f\|_{\mathbb{H}^{s-1}(\Omega)}\left(\int_0^T t^{(s-p-1)\mu}\,dt\right)^{1/\mu}\\ &= C(p,s)\varepsilon\|f\|_{\mathbb{H}^{s-1}(\Omega)}\left(\frac{T^{(s-p-1)\mu+1}}{(s-p-1)\mu+1}\right)^{1/\mu},\end{aligned}\tag{4.27}$$

where we recall that $p+1>s>p$ and $1<\mu<\frac{1}{p+1-s}$.

The second term $H_2(t)$ by the second part of Lemma 3.1 is bounded by

$$\begin{aligned}\big\|H_2(t)\big\|_{\mathbb{H}^p(\Omega)} &\le \int_0^t (t-r)\big\|\mathbf{S}(t-r)\big(F\big(u_\varepsilon(r)\big)-F\big(u(r)\big)\big)\big\|_{\mathbb{H}^p(\Omega)}\,dr\\ &\le \int_0^t (t-r)(t-r)^{q-p}\big\|F\big(u_\varepsilon(r)\big)-F\big(u(r)\big)\big\|_{\mathbb{H}^q(\Omega)}\,dr,\end{aligned}\tag{4.28}$$

where we note that $p>q$. Since F is globally Lipschitz as in (4.1), we infer that

$$\begin{aligned}&\int_0^t (t-r)(t-r)^{q-p}\big\|F\big(u_\varepsilon(r)\big)-F\big(u(r)\big)\big\|_{\mathbb{H}^q(\Omega)}\,dr\\ &\quad\le K_f\int_0^t (t-r)^{q-p+1}\big\|u_\varepsilon(r)-u(r)\big\|_{\mathbb{H}^p(\Omega)}\,dr.\end{aligned}$$

This implies that

$$\begin{aligned}\big\|H_2(t)\big\|_{\mathbb{H}^p(\Omega)} &\le K_f\int_0^t (t-r)^{q-p+1}\big\|u_\varepsilon(r)-u(r)\big\|_{\mathbb{H}^p(\Omega)}\,dr\\ &\le K_f T^{q-p+1}\int_0^t \big\|u_\varepsilon(r)-u(r)\big\|_{\mathbb{H}^p(\Omega)}\,dr\\ &\le K_f T^{q-p+2}\left(\int_0^t \big\|u_\varepsilon(r)-u(r)\big\|^\mu_{\mathbb{H}^p(\Omega)}\,dr\right)^{1/\mu}\\ &\le K_f T^{q-p+2}\|u_\varepsilon-u\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))}.\end{aligned}\tag{4.29}$$

Thus we obtain that

$$\begin{aligned}\|H_2\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} &\le \left(\int_0^T \big(K_f T^{q-p+2}\|u_\varepsilon-u\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))}\big)^\mu\right)^{1/\mu}\\ &= K_f T^{q-p+2+\frac{1}{\mu}}\|u_\varepsilon-u\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))}.\end{aligned}\tag{4.30}$$

For the third term $H_3(t)$, we apply Lemma 3.1 (noting that $s<p+1$) to get that

$$\begin{aligned}&\left\|\mathbf{Q}_\varepsilon(t)\int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big)\,dr\right\|_{\mathbb{H}^p(\Omega)}\\ &\quad\le t^{s-p-1}\left\|\int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big)\,dr\right\|_{\mathbb{H}^s(\Omega)}.\end{aligned}\tag{4.31}$$

Since $s > q$, it follows from this estimate that

$$\left\| \int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big)\,dr \right\|_{\mathbb{H}^s(\Omega)} \le \int_0^T (T-r)^{q-s+1} \big\| F\big(u_\varepsilon(r)\big) \big\|_{\mathbb{H}^q(\Omega)}\,dr \le K_f \int_0^T (T-r)^{q-s+1} \big\| u_\varepsilon(r) \big\|_{\mathbb{H}^p(\Omega)}\,dr, \tag{4.32}$$

where in the last line, we have used that F is globally Lipschitz. Recalling (4.21), we find that the right-hand side of (4.32) is bounded by

$$\begin{aligned} &K_f \int_0^T (T-r)^{q-s+1} \big\| u_\varepsilon(r) \big\|_{\mathbb{H}^p(\Omega)}\,dr \\ &\quad \le 2K_f C_T T^{s+a-p-1} \|f\|_{\mathbb{H}^s(\Omega)} \int_0^T (T-r)^{q-s+1} r^{-a}\,dr \\ &\quad = 2K_f C_T T^{s+a-p-1} \|f\|_{\mathbb{H}^s(\Omega)} T^{q-s+1-a} B(q-s+2, 1-a) = 2K_f T^{q-p} \|f\|_{\mathbb{H}^s(\Omega)}. \end{aligned} \tag{4.33}$$

Combining (4.31), (4.32), and (4.33), we arrive at

$$\begin{aligned} \big\| H_3(t) \big\|_{\mathbb{H}^p(\Omega)} &\le \varepsilon \left\| \int_0^T (T-r)\mathbf{S}(T-r)F\big(u_\varepsilon(r)\big)\,dr \right\|_{\mathbb{H}^s(\Omega)} \\ &\le 2\varepsilon K_f T^{q-p} t^{s-p-1} \|f\|_{\mathbb{H}^s(\Omega)}. \end{aligned} \tag{4.34}$$

This leads to

$$\begin{aligned} \|H_3\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} &\le 2\varepsilon K_f T^{q-p} \|f\|_{\mathbb{H}^s(\Omega)} \left(\int_0^T t^{(s-p-1)\mu}\,dt \right)^{1/\mu} \\ &= 2\varepsilon K_f T^{q-p} \|f\|_{\mathbb{H}^s(\Omega)} \left(\frac{T^{(s-p-1)\mu+1}}{(s-p-1)\mu+1} \right)^{1/\mu}, \end{aligned} \tag{4.35}$$

where we recall that $p+1 > s > p$ and $1 < \mu < \frac{1}{p+1-s}$. Combining (4.26), (4.27), (4.30), and (4.35), we deduce that

$$\begin{aligned} \|u_\varepsilon - u\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} &\le \sum_{j=1}^{3} \|H_j\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))} \\ &\le C(p,s)\varepsilon \|f\|_{\mathbb{H}^{s-1}(\Omega)} \left(\frac{T^{(s-p-1)\mu+1}}{(s-p-1)\mu+1} \right)^{1/\mu} \\ &\quad + 2\varepsilon K_f T^{q-p} \|f\|_{\mathbb{H}^s(\Omega)} \left(\frac{T^{(s-p-1)\mu+1}}{(s-p-1)\mu+1} \right)^{1/\mu} \\ &\quad + K_f T^{q-p+2} \|u_\varepsilon - u\|_{L^\mu(0,T;\mathbb{H}^p(\Omega))}. \end{aligned} \tag{4.36}$$

Let K_f be small enough such that $K_f T^{q-p+2} \le \frac{1}{2}$. Then from (4.36) the desired result follows. The proof is completed. □

5 Conclusion

In this paper, we considered a biparabolic equation under temporal nonlocal conditions with linear and nonlinear source terms. We derived the regularity of the mild solution for the linear source term and applied the Banach fixed point theorem to study the existence and uniqueness of a mild solution for the nonlinear source term. In both cases, we demonstrated that the mild solution of our problem converges to the solution of an initial value problem as the parameter $\varepsilon \to 0$. The most compelling findings of our study can be considered as one of the first results on biparabolic equations with nonlocal conditions.

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