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Existence and nonexistence of entire k-convex radial solutions to Hessian type system

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Abstract

In this paper, a Hessian type system is studied. After converting the existence of an entire solution to the existence of a fixed point of a continuous mapping, the existence of entire k-convex radial solutions is established by the monotone iterative method. Moreover, a nonexistence result is also obtained.

Keywords: k-convex radial solution; Existence; Nonexistence; Hessian type system

1 Introduction

In this paper, we study the existence of entire k-convex radial solutions to the following problem of Hessian type system:

$$\begin{cases} \sigma_k(\lambda(D^2u+\mu|\nabla u|I)) = p(|x|)f_1(u)f_2(v), & x\in B_1(0),\\ \sigma_l(\lambda(D^2v+\nu|\nabla v|I)) = q(|x|)g_1(u)g_2(v), & x\in B_1(0),\\ u=v=0, & x\in \partial B_1(0), \end{cases} \tag{1.1}$$

where $k,l=1,2,\dots,N$, $\mu,\nu\geq 0$ are constants, $B_1(0)$ is the unit ball in $\mathbb{R}^N$, for any $N\times N$ real symmetric matrix A, $\lambda(A)$ denotes the eigenvalues of A, $D^2u(x)=(\frac{\partial^2 u(x)}{\partial x_i\partial x_j})$ denotes the Hessian matrix of the function $u\in C^2(\overline{B_1(0)})$, ∇u denotes the gradient of u, and $\sigma_k(\lambda)=\sum_{1\leq i_1<\cdots<i_k\leq N}\lambda_{i_1}\cdots\lambda_{i_k}$ denotes the kth elementary symmetric function of $\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb{R}^N$.

For p,q,f_1,f_2,g_1,g_2, we introduce the following conditions:

(H1) $p,q\in C([0,1],(0,+\infty))$. $f_1,f_2,g_1,g_2\in C((-\infty,0],[0,+\infty))$ are decreasing.

(H2) For any $a>0$, the integral $\int_{-\infty}^{-a}\frac{d\tau}{(f_1(\tau)f_2(\tau))^{\frac{1}{k}}+(g_1(\tau)g_2(\tau))^{\frac{1}{l}}}$ is divergent.

(H3) For any $a>0$, the integral $\int_{-a}^{0}\frac{d\tau}{(f_1(\tau)f_2(\tau))^{\frac{1}{k}}+(g_1(\tau)g_2(\tau))^{\frac{1}{l}}}$ is divergent.

Denote

$$\Gamma_k:=\bigl\{\lambda\in\mathbb{R}^N:\sigma_j(\lambda)>0, 1\leq j\leq k\bigr\}.$$

We say that a function $u\in C^2(\overline{B_1(0)})$ is k-convex in $B_1(0)$ if $\lambda(D^2u(x))\in\Gamma_k$ for all $x\in B_1(0)$.

In (1.1), if $\mu = 0$ and $f_2(v) \equiv 1$, the first equation in the system becomes the following k-Hessian type equation:

$$\sigma_k\bigl(\lambda\bigl(D^2 u\bigr)\bigr) = p\bigl(|x|\bigr)f_1(u); \tag{1.2}$$

if $\mu = \nu = 0$ and $f_1(u) = g_2(v) \equiv 1$, the system becomes the following coupling k-Hessian system:

$$\begin{cases} \sigma_k(\lambda(D^2 u)) = p(|x|)f_2(v), \\ \sigma_l(\lambda(D^2 v)) = q(|x|)g_1(u). \end{cases} \tag{1.3}$$

Related to k-Hessian equations, if $k = 1$ the k-Hessian equations become the well-known Laplacian equations, and if $k = N$ the k-Hessian equations become the Monge–Ampère equations. Concerning Laplacian equations and Monge–Ampère equations, there are a great number of research papers, see for examples [1, 6, 7, 22] and the references therein. Here we specially mention Keller [15], Osserman [21], and Lair and Wood [17] for Laplacian equations and Cheng and Yau [2] and Laser and McKenna [19] for Monge–Ampère equations. Similar situations occur for coupling k-Hessian system (1.3), although in this case there are not so many research papers. Here we only mention Lair and Wood [18] and Cîrstea and Rădulescu [3] for coupling Laplacian systems and Wang and An [24] and Zhang and Qi [26] for coupling Monge–Ampère systems.

For general k-Hessian equation (1.2), when $p \equiv 1$ and $f(u) = u^{\gamma k}$, $\gamma > 1$, Jin, Li, and Xu [13] showed the nonexistence of entire k-convex positive solutions. When $p \equiv 1$, Ji and Bao [11] gave necessary and sufficient conditions on the existence of entire positive k-convex radial solutions. If we generalize $p(|x|)f(u)$ to $f(x,u)$, de Oliveira, do Ó, and Ubilla obtained the existence of k-convex radial solutions in the case of supercritical nonlinearity by means of variational techniques (see [5] and the references therein for research in this direction). For general k-Hessian equation (1.2) and coupling k-Hessian system (1.3), Zhang and Zhou [27] obtained several results on the existence of entire positive k-convex radial solutions. We refer to the papers of Feng and Zhang [8] and Gao, He, and Ran [9] and the references therein for research on coupling k-Hessian system (1.3).

It is obvious that the k-Hessian type equation

$$\sigma_k\bigl(\lambda\bigl(D^2 u + \mu|\nabla u|I\bigr)\bigr) = p\bigl(|x|\bigr)f(u)$$

is a generalization of k-Hessian equation (1.2), but it is a special case of the following fully nonlinear Hessian equation:

$$F\bigl(\lambda\bigl(D^2 u + A(x,u,\nabla u)\bigr)\bigr) = f(x,u,\nabla u). \tag{1.4}$$

See Guan and Jiao [10] and Jiang and Trudinger [12] and the references therein for research on fully nonlinear Hessian equation (1.4). Here we also want to mention the work of Dai [4] for similar study.

Inspired by the works above, and as we know that now there are no papers on the problem of k-Hessian type system (1.1), we obtain the following results in this paper.

Theorem 1.1 *Under conditions* $(H1)$ *and* $(H2)$, *if* $f_1(0)g_2(0) \neq 0$ *and* $f_2(0) + g_1(0) \neq 0$, *then problem* (1.1) *admits an entire k-convex radial solution* $(u, v) \in C^2(\overline{B_1(0)}) \times C^2(\overline{B_1(0)})$.

Remark 1.1 In the case of $f_1(0)g_2(0) = 0$, if $f_1(0) = g_2(0) = 0$, then there is a trivial solution $(u, v) = (0, 0)$ to problem (1.1); if $f_1(0) = 0$ or $g_2(0) = 0$, then there is a semi-trivial solution $(u, v) = (0, v)$ or $(u, v) = (u, 0)$ to problem (1.1); moreover, the semi-trivial solution may become trivial if $f_1(0) = 0$ with $g_1(0) = 0$ or $g_2(0) = 0$ with $f_2(0) = 0$.

In the case of $f_2(0) + g_1(0) = 0$, there is a trivial solution $(u, v) = (0, 0)$ to problem (1.1).

Theorem 1.2 *Under conditions* $(H1)$ *and* $(H3)$, *problem* (1.1) *admits no entire k-convex radial solution* $(u, v) \in C^2(\overline{B_1(0)}) \times C^2(\overline{B_1(0)})$.

Remark 1.2 In this case, $f_1(0)f_2(0) = g_1(0)g_2(0) = 0$, and there is a trivial solution $(u, v) = (0, 0)$ to problem (1.1).

2 Preliminaries

In this section, we give some preliminary results which will be used to prove the main results in the next section.

Lemma 2.1 *Assume* $\varphi(r) \in C^2[0, 1]$ *with* $\varphi'(0) = 0$. *Then, for* $u(x) = \varphi(r)$, *there holds that* $u \in C^2(\overline{B_1(0)})$ *and*

$$\lambda\bigl(D^2u + \eta|\nabla u|I\bigr) = \begin{cases} (\varphi''(r) + \eta\varphi'(r), (\frac{1}{r} + \eta)\varphi'(r), \ldots, (\frac{1}{r} + \eta)\varphi'(r)), & r \in (0, 1], \\ (\varphi''(0), \varphi''(0), \ldots, \varphi''(0)), & r = 0, \end{cases}$$

and further

$$\begin{aligned} &\sigma_k\bigl(\lambda\bigl(D^2u + \eta|\nabla u|I\bigr)\bigr) \\ &\quad = \begin{cases} C_{N-1}^{k-1}(\varphi''(r) + \eta\varphi'(r))((\frac{1}{r} + \eta)\varphi'(r))^{k-1} + C_{N-1}^{k}((\frac{1}{r} + \eta)\varphi'(r))^k, & r \in (0, 1], \\ C_N^k(\varphi''(0))^k, & r = 0, \end{cases} \end{aligned}$$

where $C_N^k = \frac{N!}{k!(N-k)!}$.

Proof It is immediate that, for $x \neq 0$, $1 \le i, j \le N$,

$$\frac{\partial u(x)}{\partial x_i} = \left(\frac{\varphi'(r)}{r}\right)x_i$$

and

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \left(\frac{\varphi''(r)}{r^2}\right)x_i x_j - \left(\frac{\varphi'(r)}{r^3}\right)x_i x_j + \left(\frac{\varphi'(r)}{r}\right)\delta_{ij}.$$

Further if define

$$\frac{\partial u(0)}{\partial x_i} = 0, \qquad \frac{\partial^2 u(0)}{\partial x_i \partial x_j} = \varphi''(0)\delta_{ij},$$

then $u \in C^2(\overline{B_1(0)})$.

Now it is easy to show the two equalities for $\lambda(D^2u+\eta|\nabla u|I)$ and $\sigma_k(\lambda(D^2u+\eta|\nabla u|I))$. □

Lemma 2.2 *Let $f \in C(-\infty,0]$ be decreasing. Assume that $\varphi \in C^0[0,1]\cap C^1(0,1]$ is a solution of the Cauchy problem*

$$\begin{cases} \varphi'(r) = (\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\eta}(r)}\int_0^r \mathrm{e}^{\psi_{k,\eta}(s)}\frac{s^{k-1}p(s)}{(1+\eta s)^{k-1}}f(\varphi(s))\,ds)^{\frac{1}{k}}, \quad 0<r<1, \\ \varphi(1)=0, \end{cases}$$

where

$$\psi_{k,\eta}(r) = \frac{k}{C_{N-1}^{k-1}}\bigl(C_N^k\eta r + C_{N-1}^k \ln r\bigr).$$

Then $\varphi \in C^2[0,1]$, and it satisfies the problem

$$\begin{cases} C_{N-1}^{k-1}\varphi''(r)(\varphi'(r))^{k-1}r + (C_N^k\eta r + C_{N-1}^k)(\varphi'(r))^k = \frac{r^kp(r)}{(1+\eta r)^{k-1}}f(\varphi(r)), \quad 0<r<1, \\ \varphi'(0)=0. \end{cases}$$

Furthermore, if φ is nontrivial, i.e., $\varphi(r)<0$ for $0\le r<1$, then

$$\lambda_r := \left(\varphi''(r)+\eta\varphi'(r), \left(\frac{1}{r}+\eta\right)\varphi'(r), \dots, \left(\frac{1}{r}+\eta\right)\varphi'(r)\right) \in \Gamma_k$$

for $0\le r<1$.

Proof It is easy to see that $\varphi(r)\in C^2[0,1]$.

From

$$\varphi'(r) = \left(\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\eta}(r)}\int_0^r \mathrm{e}^{\psi_{k,\eta}(s)}\frac{s^{k-1}p(s)}{(1+\eta s)^{k-1}}f\bigl(\varphi(s)\bigr)\,ds\right)^{\frac{1}{k}}$$

we have

$$\bigl(\varphi'(r)\bigr)^k = \frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\eta}(r)}\int_0^r \mathrm{e}^{\psi_{k,\eta}(s)}\frac{s^{k-1}p(s)}{(1+\eta s)^{k-1}}f\bigl(\varphi(s)\bigr)\,ds,$$

and further differentiating with respect to r we have

$$C_{N-1}^{k-1}\varphi''(r)\bigl(\varphi'(r)\bigr)^{k-1}r + \bigl(C_N^k\eta r + C_{N-1}^k\bigr)\bigl(\varphi'(r)\bigr)^k = \frac{r^kp(r)}{(1+\eta r)^{k-1}}f\bigl(\varphi(r)\bigr).$$

If φ is nontrivial, it is easy to see that φ is increasing, so for $0\le r<1$ we conclude $\varphi(r)<\varphi(1)=0$, $f(\varphi(r))>f(\varphi(1))\ge 0$ and further

$$\sigma_k(\lambda_r) = f\bigl(\varphi(r)\bigr) > 0 \quad \text{for } 0\le r<1.$$

By the properties of kth elementary symmetric functions (see for example [20]), we know $\sigma_j(\lambda_r)>0$ for $1\le j<k$ and $0\le r<1$. Therefore we conclude the lemma. □

3 Proofs of the main results

In this section, we prove the main results in this paper, i.e., the existence and nonexistence of entire k-convex radial solutions for problem (1.1).

Proof of Theorem 1.1 From the system

$$\begin{cases} C_{N-1}^{k-1}u''(r)(u'(r))^{k-1}r + (C_N^k\mu r + C_{N-1}^k)(u'(r))^k = \frac{r^k p(r)}{(1+\mu r)^{k-1}}f_1(u(r))f_2(v(r)), \\ C_{N-1}^{l-1}v''(r)(v'(r))^{l-1}r + (C_N^l\nu r + C_{N-1}^l)(v'(r))^l = \frac{r^l q(r)}{(1+\nu r)^{l-1}}g_1(u(r))g_2(v(r)), \end{cases}$$

we get

$$\begin{cases} u'(r) = (\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\mu}(r)}\int_0^r \mathrm{e}^{\psi_{k,\mu}(s)}\frac{s^{k-1}p(s)}{(1+\mu s)^{k-1}}f_1(u(s))f_2(v(s))\,ds)^{\frac{1}{k}}, \\ v'(r) = (\frac{l}{C_{N-1}^{l-1}}\mathrm{e}^{-\psi_{l,\nu}(r)}\int_0^r \mathrm{e}^{\psi_{l,\nu}(s)}\frac{s^{l-1}q(s)}{(1+\nu s)^{l-1}}g_1(u(s))g_2(v(s))\,ds)^{\frac{1}{k}}, \end{cases}$$

furthermore we have

$$\begin{cases} u(r) = \int_1^r(\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\mu}(t)}\int_0^t \mathrm{e}^{\psi_{k,\mu}(s)}\frac{s^{k-1}p(s)}{(1+\mu s)^{k-1}}f_1(u(s))f_2(v(s))\,ds)^{\frac{1}{k}}\,dt, \\ v(r) = \int_1^r(\frac{l}{C_{N-1}^{l-1}}\mathrm{e}^{-\psi_{l,\nu}(t)}\int_0^t \mathrm{e}^{\psi_{l,\nu}(s)}\frac{s^{l-1}q(s)}{(1+\nu s)^{l-1}}g_1(u(s))g_2(v(s))\,ds)^{\frac{1}{k}}\,dt. \end{cases}$$

Define

$$\mathcal{L}(u,v)(r) = \begin{pmatrix} \int_1^r(\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\mu}(t)}\int_0^t \mathrm{e}^{\psi_{k,\mu}(s)}\frac{s^{k-1}p(s)}{(1+\mu s)^{k-1}}f_1(u(s))f_2(v(s))\,ds)^{\frac{1}{k}}\,dt \\ \int_1^r(\frac{l}{C_{N-1}^{l-1}}\mathrm{e}^{-\psi_{l,\nu}(t)}\int_0^t \mathrm{e}^{\psi_{l,\nu}(s)}\frac{s^{l-1}q(s)}{(1+\nu s)^{l-1}}g_1(u(s))g_2(v(s))\,ds)^{\frac{1}{k}}\,dt \end{pmatrix}^T,$$

then we need only to find a fixed point of $\mathcal{L}$. Here we use the monotone iterative method to find such a fixed point.

It is easy to show that $\mathcal{L}$ is a mapping from $C^2[0,1]\times C^2[0,1]$ to $C^2[0,1]\times C^2[0,1]$, and it is continuous on $C[0,1]\times C[0,1]$.

Let $\{u_n\}$ and $\{v_n\}$ be the sequence of continuous functions defined by

$$\begin{cases} u_0(r) = 0, \\ v_0(r) = 0, \\ u_n(r) = \int_1^r(\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\mu}(t)}\int_0^t \mathrm{e}^{\psi_{k,\mu}(s)}\frac{s^{k-1}p(s)}{(1+\mu s)^{k-1}}f_1(u_{n-1}(s))f_2(v_{n-1}(s))\,ds)^{\frac{1}{k}}\,dt, \\ v_n(r) = \int_1^r(\frac{l}{C_{N-1}^{l-1}}\mathrm{e}^{-\psi_{l,\nu}(t)}\int_0^t \mathrm{e}^{\psi_{l,\nu}(s)}\frac{s^{l-1}q(s)}{(1+\nu s)^{l-1}}g_1(u_{n-1}(s))g_2(v_{n-1}(s))\,ds)^{\frac{1}{k}}\,dt. \end{cases}$$

It is easy to see that u_n and v_n are decreasing on $[0,1]$ for $n>1$ and by induction $\{u_n\}$ and $\{v_n\}$ are decreasing as well, i.e., $u_{n+1}(r) < u_n(r)$ and $v_{n+1}(r) < v_n(r)$ for $0\le r<1$ and $n\ge 1$.

By condition (H1), for each $0<r<1$ and $n>1$,

$$\begin{aligned} 0 &< u_n'(r) \\ &= \left(\frac{k}{C_{N-1}^{k-1}}\mathrm{e}^{-\psi_{k,\mu}(r)}\int_0^r \mathrm{e}^{\psi_{k,\mu}(s)}\frac{s^{k-1}p(s)}{(1+\mu s)^{k-1}}f_1\bigl(u_{n-1}(s)\bigr)f_2\bigl(v_{n-1}(s)\bigr)\,ds\right)^{\frac{1}{k}} \end{aligned}$$

$$\begin{aligned}
&\le C(N,k,p)\bigl(f_1\bigl(u_n(r)\bigr)f_2\bigl(v_n(r)\bigr)\bigr)^{\frac{1}{k}} \\
&\le C(N,k,p)\bigl(f_1\bigl(u_n(r)+v_n(r)\bigr)f_2\bigl(u_n(r)+v_n(r)\bigr)\,ds\bigr)^{\frac{1}{k}},
\end{aligned}$$

where $C(N,k,p)$ is a constant dependent on N, k, and p.

Similarly,

$$0 < v_n'(r) \le C(N,l,q)\bigl(g_1\bigl(u_n(r)+v_n(r)\bigr)g_2\bigl(u_n(r)+v_n(r)\bigr)\bigr)^{\frac{1}{l}}$$

and further

$$\begin{aligned}
0 &< \bigl(u_n(r)+v_n(r)\bigr)' \\
&\le C(N,k,l,p,q)\bigl(\bigl(f_1\bigl(u_n(r)+v_n(r)\bigr)f_2\bigl(u_n(r)+v_n(r)\bigr)\bigr)^{\frac{1}{k}} \\
&\quad + \bigl(g_1\bigl(u_n(r)+v_n(r)\bigr)g_2\bigl(u_n(r)+v_n(r)\bigr)\bigr)^{\frac{1}{l}}\bigr),
\end{aligned} \tag{3.1}$$

i.e.,

$$\begin{aligned}
0 &< \frac{(u_n(r)+v_n(r))'}{(f_1(u_n(r)+v_n(r))f_2(u_n(r)+v_n(r)))^{\frac{1}{k}} + (g_1(u_n(r)+v_n(r))g_2(u_n(r)+v_n(r)))^{\frac{1}{l}}} \\
&\le C(N,k,l,p,q),
\end{aligned}$$

where $C(N,l,q)$ and $C(N,k,l,p,q)$ are constants dependent on N, l, q and N, k, l, p, q, respectively.

Integrating from 1 to r, we have

$$\int_0^{u_n(r)+v_n(r)} \frac{d\tau}{(f_1(\tau)f_2(\tau))^{\frac{1}{k}} + (g_1(\tau)g_2(\tau))^{\frac{1}{l}}} \ge -C(N,k,l,p,q). \tag{3.2}$$

By condition (H2), denote

$$F(w) = \int_0^{w} \frac{d\tau}{(f_1(\tau)f_2(\tau))^{\frac{1}{k}} + (g_1(\tau)g_2(\tau))^{\frac{1}{l}}},$$

then F is continuous and increasing on $(-\infty,0]$, and it has an inverse function F^{-1}. From (3.2), we have

$$F^{-1}\bigl(-C(N,k,l,p,q)\bigr) \le u_n(r) + v_(r) \le 0$$

for $0 \le r \le 1$ and $n \ge 1$.

By condition (H1) and (3.1), we have for $n \ge 1$

$$\begin{aligned}
0 &< \bigl(u_n(r)+v_n(r)\bigr)' \\
&\le C(N,k,l,p,q)\Bigl(\max_{F^{-1}(-C(N,k,l,p,q))\le w\le 0}\bigl(\bigl(f_1(w)f_2(w)\bigr)^{\frac{1}{k}} + \bigl(g_1(w)g_2(w)\bigr)^{\frac{1}{l}}\bigr)\Bigr) \\
&= C(N,k,l,p,q,f_1,f_2,g_1,g_2),
\end{aligned}$$

where $C(N,k,l,p,q,f_1,f_2,g_1,g_2)$ is a constant dependent on N, k, l, p, q, f_1, f_2, g_1, and g_2. So $\{u_n\}$ and $\{v_n\}$ are bounded in $C^1[0,1]$ and by Arzela–Ascoli theorem $\{u_n\}$ and $\{v_n\}$ have convergent subsequences (still denoted by $\{u_n\}$ and $\{v_n\}$) in $C[0,1]$. Denote

$$u(r) = \lim_{n\to+\infty} u_n(r),$$
$$v(r) = \lim_{n\to+\infty} v_n(r).$$

By the continuity of $\mathcal{L}$ on $C[0,1]\times C[0,1]$, from

$$(u_n, v_n) = \mathcal{L}(u_{n-1}, v_{n-1}),$$

we conclude that (u,v) is a fixed point of $\mathcal{L}$ after letting $n\to+\infty$. □

Proof of Theorem 1.2 We prove by contradiction. Suppose that (u,v) is a k-convex radial solution to problem (1.1). Then u and v are decreasing on $[0,1]$. For $0<r<1$, by Lemma 2.2 we can get

$$0 < u'(r) \le C(N,k,p)\bigl(f_1\bigl(u(r)+v(r)\bigr)g_2\bigl(u(r)+v(r)\bigr)\bigr)^{\frac{1}{k}},$$
$$0 < v'(r) \le C(N,l,q)\bigl(g_1\bigl(u(r)+v(r)\bigr)g_2\bigl(u(r)+v(r)\bigr)\bigr)^{\frac{1}{l}}.$$

So

$$\begin{aligned} 0 &< \frac{(u(r)+v(r))'}{(f_1(u(r)+v(r))f_2(u(r)+v(r)))^{\frac{1}{k}} + (g_1(u(r)+v(r))g_2(u(r)+v(r)))^{\frac{1}{l}}} \\ &\le C(N,k,l,p,q). \end{aligned}$$

Integrating for 0 to 1, we have

$$0 < \int_{u(0)+v(0)}^{0} \frac{d\tau}{(f_1(\tau)f_2(\tau))^{\frac{1}{k}} + (g_1(\tau)g_2(\tau))^{\frac{1}{l}}} \le C(N,k,l,p,q),$$

which contradicts condition (H3). Now we finish the proof. □

At the end of this section, we give some examples for the sake of clearly understanding the results in this paper.

Assume that α, β, α_1, β_1, α_2, and β_2 are positive.

Example 3.1 If $\alpha_1+\beta_1\le k$ and $\alpha_2+\beta_2\le l$, then the following problem admits an entire k-convex radial solution $(u,v)\in C^2(\overline{B_1(0)})\times C^2(\overline{B_1(0)})$:

$$\begin{cases} \sigma_k(\lambda(D^2u+\mu|\nabla u|I)) = (1+|x|)^{\alpha}(1+|u|)^{\alpha_1}|v|^{\beta_1}, & x\in B_1(0), \\ \sigma_l(\lambda(D^2v+\nu|\nabla v|I)) = (1+|x|)^{\beta}(1+|u|)^{\alpha_2}(1+|v|)^{\beta_2}, & x\in B_1(0), \\ u=v=0, & x\in\partial B_1(0). \end{cases}$$

Example 3.2 If $\alpha_1 + \beta_1 \geq k$ and $\alpha_2 + \beta_2 \geq l$, then the following problem admits no entire k-convex radial solution $(u, v) \in C^2(\overline{B_1(0)}) \times C^2(\overline{B_1(0)})$:

$$\begin{cases} \sigma_k(\lambda(D^2u + \mu|\nabla u|I)) = (1 + |x|)^{\alpha}|u|^{\alpha_1}|v|^{\beta_1}, & x \in B_1(0), \\ \sigma_l(\lambda(D^2v + \nu|\nabla v|I)) = (1 + |x|)^{\beta}|u|^{\alpha_2}|v|^{\beta_2}, & x \in B_1(0), \\ u = v = 0, & x \in \partial B_1(0). \end{cases}$$

4 Conclusion

In this paper, by converting the existence of an entire solution to the existence of a fixed point of a continuous mapping, we establish the existence of entire k-convex radial solutions for a Hessian type system. Moreover the nonexistence of entire k-convex radial solutions is also obtained. In the process of obtaining the existence of entire k-convex radial solutions, we utilize the monotone iterative method. By different fixed point theorems (such as the ones in [14] and [25]) or different methods (such as degree theory in [16] and the regularization method in [23]), we may get different results on Hessian type systems. In our opinion, it is interesting to fulfil this kind of works in the future.

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Authors' contributions
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References

1. Cavalheiro, A.C.: Existence results for Navier problems with degenerated (p,q)-Laplacian and (p,q)-biharmonic operators. Results Nonlinear Anal. **1**(2), 74–87 (2018)
2. Cheng, S.Y., Yau, S.T.: On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. Commun. Pure Appl. Math. **33**, 507–544 (1980)
3. Cîrstea, F.C., Rădulescu, V.: Entire solutions blowing up at infinity for semilinear elliptic systems. J. Math. Pures Appl. **81**, 827–846 (2002)
4. Dai, L.M.: Existence and nonexistence of subsolutions for augmented Hessian equations. Discrete Contin. Dyn. Syst. **40**(1), 579–596 (2020)
5. de Oliveira, J.F., do Ó, J.M., Ubilla, P.: Existence for a k-Hessian equation involving supercritical growth. J. Differ. Equ. **267**, 1001–1024 (2019)
6. Enache, C., Porru, G.: A note on Monge–Ampere equation in $\mathbb{R}^2$. Results Math. **76**(1), Article ID 29 (2021)
7. Feng, M.Q.: Convex solutions of Monge–Ampère equations and systems: existence, uniqueness and asymptotic behavior. Adv. Nonlinear Anal. **10**(1), 371–399 (2021)
8. Feng, M.Q., Zhang, X.M.: A coupled system of k-Hessian equations. Math. Methods Appl. Sci. **44**(9), 7377–7394 (2021)
9. Gao, C.H., He, X.Y., Ran, M.J.: On a power-type coupled system of k-Hessian equations. Quaest. Math. https://doi.org/10.2989/16073606.2020.1816586
10. Guan, B., Jiao, H.: Second order estimates for Hessian type fully nonlinear elliptic equations on Riemannian manifolds. Calc. Var. Partial Differ. Equ. **54**, 2693–2712 (2015)
11. Ji, X., Bao, J.: Necessary and sufficient conditions on solvability for Hessian inequalities. Proc. Am. Math. Soc. **138**, 175–188 (2010)
12. Jiang, F., Trudinger, N.S.: On the Dirichlet problem for general augmented Hessian equations. J. Differ. Equ. **269**, 5204–5227 (2020)

13. Jin, Q., Li, Y., Xu, H.: Nonexistence of positive solutions for some fully nonlinear elliptic equations. Methods Appl. Anal. **12**, 441–450 (2005)
14. Karapınar, E.: A fixed point theorem without a Picard operator. Results Nonlinear Anal. **4**(3), 127–129 (2021)
15. Keller, J.B.: On solutions of $\Delta u = f(u)$. Commun. Pure Appl. Math. **10**, 503–510 (1957)
16. Kim, I.S.: Semilinear problems involving nonlinear operators of monotone type. Results Nonlinear Anal. **2**(1), 25–35 (2019)
17. Lair, A.V., Wood, A.W.: Large solutions of semilinear elliptic problems. Nonlinear Anal. **37**, 805–812 (1999)
18. Lair, A.V., Wood, A.W.: Existence of entire large positive solutions of semilinear elliptic systems. J. Differ. Equ. **164**, 380–394 (2000)
19. Lazer, A.C., McKenna, P.J.: On singular boundary value problems for the Monge–Ampère operator. J. Math. Anal. Appl. **197**, 341–362 (1996)
20. Lieberman, G.: Second Order Parabolic Differential Equations. World Scientific, New Jersey (1996)
21. Osserman, R.: On the inequality $\Delta u \geq f(u)$. Pac. J. Math. **7**, 1641–1647 (1957)
22. Ourraou, A.: Existence and uniqueness of solutions for Steklov problem with variable exponent. Adv. Theory Nonlinear Anal. Appl. **5**(1), 158–166 (2021)
23. Phuong, N.D., Luc, N.H., Long, L.D.: Modified quasi boundary value method for inverse source problem of the bi-parabolic equation. Adv. Theory Nonlinear Anal. Appl. **4**(3), 132–142 (2020)
24. Wang, F., An, Y.: Triple nontrivial radial convex solutions of systems of Monge–Ampère equations. Appl. Math. Lett. **25**, 88–92 (2012)
25. Zhang, C., Chen, J.: Convergence analysis of variational inequality and fixed point problems for pseudo-contractive mapping with Lipschitz assumption. Results Nonlinear Anal. **2**(3), 102–112 (2019)
26. Zhang, Z., Qi, Z.: On a power-type coupled system of Monge–Ampère equations. Topol. Methods Nonlinear Anal. **46**, 717–729 (2015)
27. Zhang, Z., Zhou, S.: Existence of entire positive k-convex radial solutions to Hessian equations and systems with weights. Appl. Math. Lett. **50**, 48–55 (2015)