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Fuzzy fixed point results of generalized almost $\mathbf{F}$-contractions in controlled metric spaces

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Abstract

In this paper, we derive some common α-fuzzy fixed point results for fuzzy mappings under generalized almost **F**-contractions in the context of a controlled metric space, which generalize many preexisting results in the literature. As an application, we establish some multivalued fixed point results. For justification of our results, we provide a nontrivial example.

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1 Introduction

The Banach fixed point theorem (BFPT) [1] is an important tool in fixed point theory. It guarantees the existence and uniqueness of a fixed point of certain self-mappings on metric spaces. It has various applications in several branches of mathematics. There are many extensions and generalizations of the BFPT in the literature; see [2–7]. Berinde [8, 9] studied various contractive-type mappings and introduced the concept of almost contractions.

Definition 1.1 ([8]) A mapping $T:\mathcal{W}\to\mathcal{W}$ on a metric space $(\mathcal{W},d)$ is called an almost contraction if there exist $0\le\lambda<1$ and $Ł\ge 0$ such that

$$d(T\omega_1,T\omega_2)\le\lambda d(\omega_1,\omega_2)+Ł d(\omega_2,T\omega_1) \tag{1}$$

for all $\omega_1,\omega_2\in\mathcal{W}$.

Further, Berinde [9] generalized Definition 1.1 in the following way.

Definition 1.2 ([9]) A mapping $T:\mathcal{W}\to\mathcal{W}$ on a metric space $(\mathcal{W},d)$ is called a generalized almost contraction if there exist $0\le\lambda<1$ and $Ł\ge 0$ such that

$$\begin{aligned} d(T\omega_1,T\omega_2)) &\le \lambda d(\omega_1,\omega_2) \\ &\quad + Ł\min\bigl\{d\bigl(\omega_1,T(\omega_1)\bigr),d\bigl(\omega_2,T(\omega_2)\bigr),d\bigl(\omega_1,T(\omega_2)\bigr),d\bigl(\omega_2,T(\omega_1)\bigr)\bigr\} \end{aligned} \tag{2}$$

for all $\omega_1,\omega_2\in\mathcal{W}$.

Wardowski [10] introduced a new type of contractions, called **F**-contractions, and established a related fixed point theorem in the context of complete metric spaces.

Definition 1.3 ([10]) A mapping $T:\mathcal{W}\to\mathcal{W}$ on a metric space $(\mathcal{W},d)$ is called an **F**-contraction if there exists $\Omega>0$ such that

$$d(T\omega_1,T\omega_2)>0 \quad\Longrightarrow\quad \Omega+\mathbf{F}\bigl(d(T\omega_1,T\omega_2)\bigr)\le\mathbf{F}\bigl(d(\omega_1,\omega_2)\bigr) \tag{3}$$

for all $\omega_1,\omega_2\in\mathcal{W}$, where $\mathbf{F}:(0,\infty)\to\mathbb{R}$ is a function satisfying the following axioms:

(C1) **F** is strictly nondecreasing;

(C2) for each sequence $\{a_n\}\subset(0,\infty)$ of positive real numbers, $\lim_{n\to\infty}a_n=0$ if and only if $\lim_{n\to\infty}\mathbf{F}(a_n)=-\infty$;

(C3) for each sequence $\{a_n\}\subset(0,\infty)$ such that $\lim_{n\to\infty}a_n=0$, there exists $l\in(0,1)$ such that $\lim_{n\to\infty}(a_n)^l\mathbf{F}(a_n)=0$.

The following works deal with F-contractions: [11–16]. Afterward, Altun et al. [17] modified Definition 1.3 by adding the following condition:

(C4) $\mathbf{F}(\inf\mathbf{A})=\inf\mathbf{F}(\mathbf{A})$ for all $\mathbf{A}\subset(0,\infty)$ with $\inf\mathbf{A}>0$.

We denote by $\mathcal{F}$ the family of all functions **F** satisfying (C1)–(C4).

Nadler [18] derived the multivalued version of Banach fixed point theorem by using the Hausdorff metric over the family of nonempty closed bounded subsets of a complete metric space. We denote by $CLB(\mathcal{W})$ the family of nonempty closed bounded subsets and by $CLD(\mathcal{W})$ the family of nonempty closed subsets of $\mathcal{W}$. Recently, Kamran et al. [19] introduced the concept of an extended b-metric space, which generalized the notion of a b-metric space [20, 21] by replacing the constant with a function depending on two variables.

Definition 1.4 ([19]) Let $\mathcal{W}$ be a nonempty set, and let $\sigma:\mathcal{W}\times\mathcal{W}\to[1,\infty)$. Then a function $d_\sigma:\mathcal{W}\times\mathcal{W}\to[0,\infty)$ is called an extended b-metric if for all $\omega_1,\omega_2,\omega_3\in\mathcal{W}$, it satisfies the following axioms:

(i) $d_\sigma(\omega_1,\omega_2)=0$ iff $\omega_1=\omega_2$,

(ii) $d_\sigma(\omega_1,\omega_2)=d_\sigma(\omega_2,\omega_1)$,

(iii) $d_\sigma(\omega_1,\omega_3)\le\sigma(\omega_1,\omega_3)[d_\sigma(\omega_1,\omega_2)+d_\sigma(\omega_2,\omega_3)]$.

The pair $(\mathcal{W},d_\sigma)$ is called an extended b-metric space.

Later on, several researchers worked on fixed point results in the context of extended b-metric spaces; see [22–25]. In the same direction, Mlaiki et al. [26] gave the idea of a controlled-type metric space (for further extensions, see [27]), which generalizes the notion of a b-metric space.

Definition 1.5 ([26]) Let $\mathcal{W}$ be a nonempty set, and let $\sigma : \mathcal{W} \times \mathcal{W} \to [1,\infty)$. Then a function $d_\sigma : \mathcal{W} \times \mathcal{W} \to [0,\infty)$ is called a controlled metric if for all $\omega_1, \omega_2, \omega_3 \in \mathcal{W}$, it satisfies the following axioms:

(i) $d_\sigma(\omega_1,\omega_2) = 0$ iff $\omega_1 = \omega_2$,

(ii) $d_\sigma(\omega_1,\omega_2) = d_\sigma(\omega_2,\omega_1)$,

(iii) $d_\sigma(\omega_1,\omega_3) \leq \sigma(\omega_1,\omega_2)d_\sigma(\omega_1,\omega_2) + \sigma(\omega_2,\omega_3)d_\sigma(\omega_2,\omega_3)$.

The pair $(\mathcal{W}, d_\sigma)$ is called a controlled metric space.

Remark 1.1 Every controlled metric space is a generalization of a b-metric space and is different from an extended b-metric space.

Example 1.1 Let $\mathcal{W} = [0,\infty)$. Define $d_\sigma : \mathcal{W} \times \mathcal{W} \to [0,\infty)$ as

$$d_\sigma(\omega_1,\omega_2) = \begin{cases} 0 & \text{if } \omega_1 = \omega_2, \\ \frac{1}{\omega_1} & \text{if } \omega_1 \geq 1 \text{ and } \omega_2 \in [0,1), \\ \frac{1}{\omega_2} & \text{if } \omega_2 \geq 1 \text{ and } \omega_1 \in [0,1), \\ 1 & \text{otherwise.} \end{cases}$$

Hence $(\mathcal{W}, d_\sigma)$ is a controlled metric space, where $\sigma : \mathcal{W} \times \mathcal{W} \to [1,\infty)$ is defined by

$$\sigma(\omega_1,\omega_2) = \begin{cases} 1 & \text{if } \omega_1, \omega_2 \in [0,1), \\ \max\{\omega_1,\omega_2\} & \text{otherwise.} \end{cases}$$

For other definitions and information on the topology induced by d_σ, see [26]. In [28], Alamgir et al. established a Pompieu–Hausdorff metric over the family of nonempty closed subsets of a controlled metric space W as follows.

Definition 1.6 ([28]) Let $\mathbf{A}$, $\mathbf{B}$ be nonempty closed subsets of a controlled metric space $(\mathcal{W}, d_\sigma)$. Define $H_\sigma : CLD(\mathcal{W}) \times CLD(\mathcal{W}) \to [0,\infty]$ by

$$H_\sigma(\mathbf{A},\mathbf{B}) = \begin{cases} \max\{\sup_{a\in\mathbf{A}} d_\sigma(a,\mathbf{B}), \sup_{b\in\mathbf{B}} d_\sigma(b,\mathbf{A})\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 1.1 ([28]) *Let $(\mathcal{W}, d_\sigma)$ be a controlled metric space. Then the mapping $H_\sigma : CLD(\mathcal{W}) \times CLD(\mathcal{W}) \to [0,\infty]$ is a Pompieu–Hausdorff controlled metric on $CLD(\mathcal{W})$.*

On the other hand, in 1981, Heilpern [29] used fuzzy sets [30] to introduce a class of fuzzy mappings, which is a generalization of multivalued mappings and proved a fixed point theorem for fuzzy contraction mappings in metric spaces. The result introduced by Heilpern is a fuzzy generalization of the Banach fixed point theorem. Consequently, several authors studied and generalized fuzzy fixed point theorems in many directions; see [31–38]. In this paper, we prove some common α-fuzzy fixed point results for fuzzy mappings under generalized almost $\mathbf{F}$-contractions in the context of controlled metric spaces, which generalize many preexisting results in the literature. At the end, we give an example for the justification of our main result.

2 Main results

In this section, we define fuzzy sets, fuzzy mappings, and α-fuzzy fixed points and prove some common α fuzzy fixed point results in the context of controlled metric spaces.

Definition 2.1 Let $(\mathcal{W}, d_\sigma)$ be a controlled metric space with $\sigma : \mathcal{W} \times \mathcal{W} \to [1, \infty)$. Then a fuzzy set $\mathbf{A}_\sigma$ in $\mathcal{W}$ is characterized by a membership function

$$\mathbb{F}_{\mathbf{A}_\sigma} : \mathcal{W} \to [0, 1],$$

which assigns to every member of $\mathcal{W}$ a membership grade in $\mathbf{A}_\sigma$.

We denote by $\mathbb{F}_\sigma(\mathcal{W})$ the collection of all fuzzy sets in $\mathcal{W}$. Let $\mathbf{A}_\sigma \in \mathbb{F}_\sigma(\mathcal{W})$ and $\alpha \in [0, 1]$. Then the α-level set of $\mathbf{A}_\sigma$ is denoted by $[\mathbf{A}_\sigma]_\alpha$ and is defined as

$$\begin{aligned} [\mathbf{A}_\sigma]_\alpha &= \bigl\{\mu \in \mathcal{W} : \mathbf{A}_\sigma(\mu) \geq \alpha\bigr\}, \quad \alpha \in (0, 1], \\ [\mathbf{A}_\sigma]_0 &= \overline{\bigl\{\mu \in \mathcal{W} : \mathbf{A}_\sigma(\mu) > 0\bigr\}}, \end{aligned}$$

where $\overline{\mathbf{B}}$ denotes the closure of $\mathbf{B}$. Clearly, $[\mathbf{A}_\sigma]_\alpha$ and $[\mathbf{A}_\sigma]_0$ are subsets of the controlled metric space $\mathcal{W}$. For $\mathbf{A}_\sigma, \mathbf{B}_\sigma \in \mathbf{F}_\sigma(\mathcal{W})$, a fuzzy set $\mathbf{A}_\sigma$ is said to be more accurate than a fuzzy set $\mathbf{B}_\sigma$, denoted by $\mathbf{A}_\sigma \subset \mathbf{B}_\sigma$, if $f_{\mathbf{A}_\sigma}(\mu) \leq f_{\mathbf{B}_\sigma}(\mu)$ for each $\mu \in \mathcal{W}$. Now, for $\mu \in \mathcal{W}$, $\mathbf{A}_\sigma, \mathbf{B}_\sigma \in \mathbb{F}_\sigma(\mathcal{W})$, $\alpha \in [0, 1]$, and $[\mathbf{A}_\sigma]_\alpha, [\mathbf{B}_\sigma]_\alpha \in CLB(\mathcal{W})$, define

$$\begin{aligned} &\rho_\alpha\bigl(\mu, [\mathbf{A}_\sigma]_\alpha\bigr) = \inf\bigl\{d(\mu, a) : a \in [\mathbf{A}_\sigma]_\alpha\bigr\}, \\ &\rho_\alpha\bigl([\mathbf{A}_\sigma]_\alpha, [\mathbf{B}_\sigma]_\alpha\bigr) = \inf\bigl\{d(a, b) : a \in [\mathbf{A}_\sigma]_\alpha, b \in [\mathbf{B}_\sigma]_\alpha\bigr\}, \\ &\rho\bigl([\mathbf{A}_\sigma]_\alpha, [\mathbf{B}_\sigma]_\alpha\bigr) = \sup_\alpha \rho_\alpha\bigl([\mathbf{A}_\sigma]_\alpha, [\mathbf{B}_\sigma]_\alpha\bigr). \end{aligned}$$

Remark 2.1 By Theorem 1.1 the function $H_\sigma : CLB(\mathcal{W}) \times CLB(\mathcal{W}) \to [0, \infty]$ defined by

$$\begin{aligned} &H_\sigma\bigl([\mathbf{A}_\sigma]_\alpha, [\mathbf{B}_\sigma]_\alpha\bigr) \\ &\quad = \begin{cases} \max\{\sup_{a \in [\mathbf{A}_\sigma]_\alpha} d(a, [\mathbf{B}_\sigma]_\alpha), \sup_{b \in [\mathbf{B}_\sigma]_\alpha} d(b, [\mathbf{A}_\sigma]_\alpha)\} & \text{if the maximum exists,} \\ \infty & \text{otherwise,} \end{cases} \end{aligned}$$

is a generalized Hausdorff controlled fuzzy metric on $CLB(\mathcal{W})$.

Definition 2.2 Let $\mathbf{S}$, $\mathbf{T}$ be fuzzy mappings from $\mathcal{W}$ into $\Gamma(\mathcal{W})$. Then

(i) An element $\mu \in \mathcal{W}$ is called an α-fuzzy fixed point of $\mathbf{T}$ if there exists $\alpha_{\mathbf{T}}(\mu) \in (0, 1]$ such that $\mu \in [\mathbf{T}\mu]_{\alpha_{\mathbf{T}}(\mu)}$.

(ii) An element $\mu \in \mathcal{W}$ is called a common α-fuzzy fixed point of $\mathbf{S}$ and $\mathbf{T}$ if there exist $\alpha_{\mathbf{S}}(\mu), \alpha_{\mathbf{T}}(\mu) \in (0, 1]$ such that $\mu \in [\mathbf{S}\mu]_{\alpha_{\mathbf{S}}(\mu)} \cap [\mathbf{T}\mu]_{\alpha_{\mathbf{T}}(\mu)}$.

(iii) For $\alpha = 1$, μ is called a common fixed point of fuzzy mappings.

Lemma 2.1 *Let $(\mathcal{W}, d_\sigma)$ be a controlled metric space, and let $\mathbf{A}, \mathbf{B} \in CLB(\mathcal{W})$. Then for each $a \in \mathbf{A}$,*

$$d_\sigma(a, \mathbf{B}) \leq H_\sigma(\mathbf{A}, \mathbf{B}).$$

Proof Let us suppose on the contrary that for each $a \in \mathbf{A}$,

$$d_\sigma(a, \mathbf{B}) > H_\sigma(\mathbf{A}, \mathbf{B}). \tag{4}$$

From Definition 1.6 we have that for each $a \in \mathbf{A}$,

$$d_\sigma(a, \mathbf{B}) \leq H_\sigma(\mathbf{A}, \mathbf{B}). \tag{5}$$

Hence from equations (4) and (5) we get

$$H_\sigma(\mathbf{A}, \mathbf{B}) < d_\sigma(a, \mathbf{B}) \leq H_\sigma(\mathbf{A}, \mathbf{B}),$$

a contradiction. □

Theorem 2.1 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{S}$, $\mathbf{T}$ be fuzzy mappings from $\mathcal{W}$ into $\Gamma(\mathcal{W})$. Suppose for each $\omega_1 \in \mathcal{W}$, there exist $\alpha_{\mathbf{S}}(\omega_1), \alpha_{\mathbf{T}}(\omega_2) \in (0, 1]$ such that $[\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}$, $[\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}$ are nonempty closed subsets of $\mathcal{W}$. Suppose that there exist some $\mathbf{F} \in \mathcal{F}$, $\Omega > 0$, and $Ł \geq 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma\big([\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) + Ł\big(M(\omega_1, \omega_2)\big) \tag{6}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma([\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}) > 0$, where

$$\begin{aligned} M(\omega_1, \omega_2) = \min\big\{&d_\sigma\big(\omega_1, [\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}\big), d_\sigma\big(\omega_2, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big), \\ &d_\sigma\big(\omega_1, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big), d_\sigma\big(\omega_2, [\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}\big)\big\}. \end{aligned}$$

Then there exists a common α-fuzzy fixed point of $\mathbf{S}$ and $\mathbf{T}$.

Proof Let us take an arbitrary $\omega_0 \in \mathcal{W}$. Then by the hypothesis there exists $\alpha_{\mathbf{S}}(\omega_0) \in (0, 1]$ such that $[\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}$ is a nonempty closed subset of $\mathcal{W}$. Let $\omega_1 \in [\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}$. For such ω_1, there exists $\alpha_{\mathbf{T}}(\omega_1) \in (0, 1]$ such that $[\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}$ is a nonempty closed subset of $\mathcal{W}$. From Lemma 2.1, condition $(C1)$ of Definition 1.3, and (6) we can write

$$\begin{aligned} \Omega + \mathbf{F}(d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big) &\leq \Omega + \mathbf{F}(H_\sigma\big([\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big) \\ &\leq \mathbf{F}\big(d_\sigma(\omega_0, \omega_1)\big) + Ł\big(M(\omega_0, \omega_1)\big), \end{aligned} \tag{7}$$

where

$$\begin{aligned} M(\omega_0, \omega_1) = \min\big\{&d_\sigma\big(\omega_0, [\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}\big), d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), \\ &d_\sigma\big(\omega_0, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), d_\sigma\big(\omega_1, [\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}\big)\big\}. \end{aligned}$$

From condition $(C4)$ we can write

$$\mathbf{F}(d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big) = \inf_{y \in [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}} \mathbf{F}\big(d_\sigma(\omega_1, y)\big).$$

Thus we have

$$\begin{aligned}
&\Omega + \inf_{y\in[\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}} \mathbf{F}(d_\sigma(\omega_1, y) \\
&\quad \le \mathbf{F}\big(d_\sigma(\omega_0,\omega_1)\big) + Ł \min\big\{d_\sigma\big(\omega_0, [\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}\big), d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), \\
&\qquad d_\sigma\big(\omega_0, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), d_\sigma\big(\omega_1, [\mathbf{S}\omega_0]_{\alpha_{\mathbf{S}}(\omega_0)}\big)\big\}.
\end{aligned}$$

Then there exists $\omega_2 \in [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}$ such that

$$\begin{aligned}
&\Omega + \mathbf{F}(d_\sigma(\omega_1, \omega_2) \\
&\quad \le \mathbf{F}\big(d_\sigma(\omega_0,\omega_1)\big) + Ł \min\big\{d_\sigma(\omega_0,\omega_1), d_\sigma(\omega_1,\omega_2), d_\sigma(\omega_0,\omega_2), d_\sigma(\omega_1,\omega_1)\big\} \\
&\quad = \mathbf{F}\big(d_\sigma(\omega_0,\omega_1)\big).
\end{aligned}$$

For this ω_2, there exists $\alpha_{\mathbf{S}}(\omega_2) \in (0,1]$ such that $[\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}$ is a nonempty closed subset of $\mathcal{W}$. From Lemma 2.1, condition $(C1)$ of Definition 1.3, and (6) we have

$$\begin{aligned}
\Omega + \mathbf{F}(d_\sigma\big(\omega_2, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}\big) &\le \Omega + \mathbf{F}(H_\sigma\big([\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}\big) \\
&\le \Omega + \mathbf{F}(H_\sigma\big([\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big) \\
&\le \mathbf{F}\big(d_\sigma(\omega_2,\omega_1)\big) + Ł\big(M(\omega_2,\omega_1)\big),
\end{aligned}$$

where

$$\begin{aligned}
M(\omega_2,\omega_1) = \min\big\{&d_\sigma\big(\omega_2, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}\big), d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), \\
&d_\sigma\big(\omega_2, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), d_\sigma\big(\omega_1, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}\big)\big\}.
\end{aligned}$$

From condition $(C4)$, we can write

$$\mathbf{F}\big(d_\sigma\big(\omega_2, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}}(\omega_2)\big)\big) = \inf_{y'\in[\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}} \mathbf{F}\big(d_\sigma\big(\omega_2, y'\big)\big).$$

Then we have

$$\begin{aligned}
&\Omega + \inf_{y'\in[\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}} \mathbf{F}\big(d_\sigma\big(\omega_2, y'\big)\big) \\
&\quad \le \mathbf{F}\big(d_\sigma(\omega_2,\omega_1)\big) + Ł \min\big\{d_\sigma\big(\omega_2, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}\big), d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), \\
&\qquad d_\sigma\big(\omega_2, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), d_\sigma\big(\omega_1, [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}\big)\big\}.
\end{aligned}$$

Thus there exists $\omega_3 \in [\mathbf{S}\omega_2]_{\alpha_{\mathbf{S}}(\omega_2)}$ such that

$$\begin{aligned}
&\Omega + \mathbf{F}\big(d_\sigma(\omega_2,\omega_3)\big) \\
&\quad \le \mathbf{F}\big(d_\sigma(\omega_1,\omega_2)\big) + Ł \min\big\{d_\sigma(\omega_2,\omega_3), d_\sigma(\omega_1,\omega_2), d_\sigma(\omega_2,\omega_2), d_\sigma(\omega_1,\omega_3)\big\}.
\end{aligned}$$

This implies that

$$\Omega + \mathbf{F}\big(d_\sigma(\omega_2,\omega_3)\big) \le \mathbf{F}\big(d_\sigma(\omega_1,\omega_2)\big).$$

By continuing the same procedure recursively we obtain a sequence $\{\omega_n\}_{n=0}^{\infty}$ in $\mathcal{W}$ such that $\omega_{2n+1} \in [\mathbf{S}\omega_{2n}]_{\alpha_{\mathbf{S}}(\omega_{2n})}$, $\omega_{2n+2} \in [\mathbf{T}\omega_{2n+1}]_{\alpha_{\mathbf{T}}(\omega_{2n+1})}$. Also,

$$\Omega + \mathbf{F}\big(d_\sigma(\omega_{2n+1}, \omega_{2n+2})\big) \leq \mathbf{F}\big(d_\sigma(\omega_{2n}, \omega_{2n+1})\big), \tag{8}$$

and

$$\Omega + \mathbf{F}\big(d_\sigma(\omega_{2n+2}, \omega_{2n+3})\big) \leq \mathbf{F}\big(d_\sigma(\omega_{2n+1}, \omega_{2n+2})\big) \tag{9}$$

for all $n \in \mathbb{N}$. From equations (8) and (9) we have

$$\Omega + \mathbf{F}\big(d_\sigma(\omega_n, \omega_{n+1})\big) \leq \mathbf{F}\big(d_\sigma(\omega_{n-1}, \omega_n)\big).$$

Therefore

$$\begin{aligned} \mathbf{F}\big(d_\sigma(\omega_n, \omega_{n+1})\big) &\leq \mathbf{F}\big(d_\sigma(\omega_{n-1}, \omega_n)\big) - \Omega \leq \mathbf{F}\big(d_\sigma(\omega_{n-2}, \omega_{n-1})\big) - 2\Omega \leq \cdots \\ &\leq \mathbf{F}\big(d_\sigma(\omega_0, \omega_1)\big) - n\Omega. \end{aligned} \tag{10}$$

By taking the limit as $n \to \infty$ in equation (10) we get $\lim_{n\to\infty} \mathbb{F}(d_\sigma(\omega_n, \omega_{n+1})) = -\infty$. Next, from condition $(C2)$ of Definition 1.3 we have

$$\lim_{n\to\infty} d_\sigma(\omega_n, \omega_{n+1}) = 0.$$

Also, by condition $(C3)$ of Definition 1.3 there exists $l \in (0,1)$ such that

$$\lim_{n\to\infty} \big(d_\sigma(\omega_n, \omega_{n+1})\big)^l \mathbf{F}\big(d_\sigma(\omega_n, \omega_{n+1})\big) = 0.$$

From equation (10) we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} &\big(d_\sigma(\omega_n, \omega_{n+1})\big)^l \mathbf{F}\big(d_\sigma(\omega_n, \omega_{n+1})\big) - \big(d_\sigma(\omega_n, \omega_{n+1})\big)^l \mathbf{F}\big(d_\sigma(\omega_0, \omega_1)\big) \\ &\quad \leq -\big(d_\sigma(\omega_n, \omega_{n+1})\big)^l n\Omega \leq 0. \end{aligned} \tag{11}$$

By letting $n \to \infty$ in (11) we obtain

$$\lim_{n\to\infty} n\big(d_\sigma(\omega_n, \omega_{n+1})\big)^l = 0. \tag{12}$$

By equation (12) there exists $n_1 \in \mathbb{N}$ such that $n(\mathbf{F}(d_\sigma(\omega_n, \omega_{n+1})))^l \leq 1$ for all $n \geq n_1$. Thus, for all $n \geq n_1$, we have

$$d_\sigma(\omega_n, \omega_{n+1}) \leq \frac{1}{n^{\frac{1}{l}}}. \tag{13}$$

From the triangle inequality and equation (13) for $m > n \geq n_1$, we have

$$\begin{aligned} d_\sigma(\omega_n, \omega_m) &\leq \sigma(\omega_n, \omega_{n+1}) d_\sigma(\omega_n, \omega_{n+1}) + \sigma(\omega_{n+1}, \omega_m) d_\sigma(\omega_{n+1}, \omega_m) \\ &\leq \sigma(\omega_n, \omega_{n+1}) d_\sigma(\omega_n, \omega_{n+1}) + \sigma(\omega_n, \omega_m)\sigma(\omega_{n+1}, \omega_{n+2}) d_\sigma(\omega_{n+1}, \omega_{n+2}) \end{aligned}$$

$$
\begin{aligned}
&+\sigma(\omega_n,\omega_m)\sigma(\omega_{n+2},\omega_m)d_\sigma(\omega_{n+2},\omega_m)\\
&\vdots\\
&\le \sigma(\omega_n,\omega_{n+1})d_\sigma(\omega_n,\omega_{n+1})+\sum_{i=1}^{m-2}\left(\prod_{j=1}^{i}\sigma(\omega_j,\omega_m)\right)\sigma(\omega_i,\omega_{i+1})d_\sigma(\omega_i,\omega_{i+1})\\
&\quad+\prod_{j=1}^{m-1}\sigma(\omega_j,\omega_m)\sigma(\omega_{m-1},\omega_m)d_\sigma(\omega_{m-1},\omega_m)\\
&\le \sigma(\omega_n,\omega_{n+1})d_\sigma(\omega_n,\omega_{n+1})+\sum_{i=1}^{m-1}\left(\prod_{j=1}^{i}\sigma(\omega_j,\omega_m)\right)\sigma(\omega_i,\omega_{i+1})d_\sigma(\omega_i,\omega_{i+1})\\
&\le \sigma(\omega_n,\omega_{n+1})\frac{1}{n^{\frac{1}{l}}}+\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\sigma(\omega_j,\omega_m)\right)\sigma(\omega_i,\omega_{i+1})\frac{1}{i^{\frac{1}{l}}}.
\end{aligned}
$$

Since $\lim_{n,m\to\infty}\sigma(\omega_{n+1},\omega_m)l<1$ for all $\omega_n,\omega_m\in\mathcal{W}$, the series $\sum_{i=1}^{\infty}(\prod_{j=1}^{i}\sigma(\omega_j,\omega_m))\sigma(\omega_i,\omega_{i+1})\frac{1}{i^{\frac{1}{l}}}$ converges by the ratio test for each $m\in\mathbb{N}$. Therefore, by taking the limit as $n\to\infty$ in the above inequality we get $d_\sigma(\omega_n,\omega_m)\to 0$. Since $\mathcal{W}$ is complete, there exists $\rho\in\mathcal{W}$ such that $\lim_{n\to\infty}\omega_n=\rho$. Next, we prove that ρ is a fixed point of $\mathbf{T}$. Suppose on the contrary that ρ is not a fixed point of $\mathbf{T}$. Then there exist $\mathbb{N}_0\in\mathbb{N}$ and a subsequence $\{\omega_{n_r}\}$ of $\{\omega_n\}$ such that $d_\sigma(\omega_{2n_r},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)})>0$ for all $n_r\ge\mathbb{N}_0$. As $d_\sigma(\omega_{2n_r},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)})>0$ for all $n_r\ge\mathbb{N}_0$, from Lemma 2.1, condition (1) of Definition 1.3, and (6) we have

$$
\begin{aligned}
&\Omega+\mathbf{F}\big(d_\sigma\big(\omega_{2n_r},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big)\big)\\
&\quad\le \Omega+\mathbf{F}\big(H_\sigma\big([\mathbf{S}\omega_{2n_r-1}]_{\alpha_{\mathbf{S}}(\omega_{2n_r-1})},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big)\big)\\
&\quad\le \mathbf{F}\big(d_\sigma(\omega_{2n_r-1},\rho)\big)+Ł\min\big\{d_\sigma\big(\omega_{2n_r-1},[\mathbf{S}\omega_{2n_r-1}]_{\alpha_{\mathbf{S}}(\omega_{2n_r-1})}\big),\\
&\qquad d_\sigma\big(\rho,[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),d_\sigma\big(\omega_{2n_r-1},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),d_\sigma\big(\rho,[\mathbf{S}\omega_{2n_r-1}]_{\alpha_{\mathbf{S}}(\omega_{2n_r-1})}\big)\big\}\\
&\quad\le \mathbf{F}(d_\sigma(\omega_{2n_r-1},\rho)+Ł\min\big\{d_\sigma(\omega_{2n_r-1},\omega_{2n_r}),d_\sigma\big(\rho,[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),\\
&\qquad d_\sigma\big(\omega_{2n_r-1},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),d_\sigma(\rho,\omega_{2n_r})\big\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\mathbf{F}\big(d_\sigma\big(\omega_{2n_r},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big)\big)&\le \mathbf{F}\big(d_\sigma(\omega_{2n_r-1},\rho)\big)+Ł\min\big\{d_\sigma(\omega_{2n_r-1},\omega_{2n_r}),d_\sigma\big(\rho,[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),\\
&\qquad d_\sigma\big(\omega_{2n_r-1},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),d_\sigma(\rho,\omega_{2n_r})\big\}-\Omega\\
&< \mathbf{F}(d_\sigma(\omega_{2n_r-1},\rho)+Ł\min\big\{d_\sigma(\omega_{2n_r-1},\omega_{2n_r}),d_\sigma\big(\rho,[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),\\
&\qquad d_\sigma\big(\omega_{2n_r-1},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),d_\sigma(\rho,\omega_{2n_r})\big\}.
\end{aligned}
$$

As $\mathbf{F}$ is strictly increasing, we have

$$
\begin{aligned}
d_\sigma\big(\omega_{2n_r},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big)&< d_\sigma(\omega_{2n_r-1},\rho)+Ł\min\big\{d_\sigma(\omega_{2n_r-1},\omega_{2n_r}),d_\sigma\big(\rho,[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),\\
&\qquad d_\sigma\big(\omega_{2n_r-1},[\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big),d_\sigma(\rho,\omega_{2n_r})\big\}.
\end{aligned}
$$

By taking the limit as $n \to \infty$ we get

$$d_\sigma\big(\rho, [\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}\big) \leq 0.$$

Thus $\rho \in [\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)}$. By a similar procedure we can prove that $\rho \in [\mathbf{S}\rho]_{\alpha_{\mathbf{S}}(\rho)}$. Hence $\rho \in [\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)} \cap [\mathbf{S}\rho]_{\alpha_{\mathbf{S}}(\rho)}$. □

Theorem 2.2 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{S}$, $\mathbf{T}$ be fuzzy mappings from $\mathcal{W}$ into $\Gamma(\mathcal{W})$. Suppose that for each $\omega_1 \in \mathcal{W}$, there exist $\alpha_{\mathbf{S}}(\omega_1), \alpha_{\mathbf{T}}(y) \in (0,1]$ such that $[\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}$, $[\mathbf{T}y]_{\alpha_{\mathbf{T}}(y)}$ are nonempty closed subsets of $\mathcal{W}$. If there exist $\mathbf{F} \in \mathcal{F}$ and $\Omega > 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma\big([\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) \tag{14}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma([\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}) > 0$, then there exists a common α-fuzzy fixed point of $\mathbf{S}$ and $\mathbf{T}$.

Proof By taking $Ł = 0$ in Theorem 2.1 we get the proof. □

Corollary 2.1 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{T}$ be a fuzzy mapping from $\mathcal{W}$ into $\Gamma(\mathcal{W})$. Suppose that for each $\omega_1 \in \mathcal{W}$, there exist $\alpha_{\mathbf{T}}(\omega_1), \alpha_{\mathbf{T}}(\omega_2) \in (0,1]$ such that $[\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}$, $[\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}$ are nonempty closed subsets of $\mathcal{W}$. If there exist $\mathbf{F} \in \mathcal{F}$, $\Omega > 0$, and $Ł \geq 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma\big([\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) + Ł\big(M(\omega_1, \omega_2)\big) \tag{15}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma([\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}) > 0$, where

$$\begin{aligned} M(\omega_1, \omega_2) = \min\big\{&d_\sigma\big(\omega_1, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big), d_\sigma\big(\omega_2, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big), \\ &d_\sigma\big(\omega_1, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big), d_\sigma\big(\omega_2, [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}\big)\big\}, \end{aligned}$$

then there exists an α-fuzzy fixed point of $\mathbf{T}$.

Proof By taking $\mathbf{S} = \mathbf{T}$ in Theorem 2.1 we get the proof. □

Corollary 2.2 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{T}$ be a fuzzy mapping from $\mathcal{W}$ into $\Gamma(\mathcal{W})$. Suppose that for each $\omega_1 \in \mathcal{W}$, there exist $\alpha_{\mathbf{T}}(\omega_1), \alpha_{\mathbf{T}}(\omega_2) \in (0,1]$ such that $[\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}$, $[\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}$ are nonempty closed subsets of $\mathcal{W}$. Assume there exist $\mathbf{F} \in \mathcal{F}$ and $\Omega > 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma\big([\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\big) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) \tag{16}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma([\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}) > 0$. Then there exists an α-fuzzy fixed point of $\mathbf{T}$.

Proof By taking $\mathbf{S} = \mathbf{T}$ and $Ł = 0$ in Theorem 2.1 we get the proof. □

Remark 2.2

(i) Theorem 2.1 generalizes Theorem 2.1 of [39].
(ii) Theorem 2.2 generalizes Theorem 6 of [40].
(iii) Corollary 2.1 (resp., Corollary 2.2) generalizes Corollary 2.3 (resp., Corollary 2.4) of [39].

Corollary 2.3 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{A}, \mathbf{B} : \mathcal{W} \to CLB(\mathcal{W})$ be multivalued mappings. Assume that there exist $\mathbf{F} \in \mathcal{F}$, $\Omega > 0$, and $Ł \geq 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma(\mathbf{A}\omega_1, \mathbf{B}\omega_2) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) + Ł\big(M(\omega_1, \omega_2)\big) \tag{17}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma(\mathbf{A}\omega_1, \mathbf{B}\omega_2) > 0$, where

$$M(\omega_1, \omega_2) = \min\big\{d_\sigma\big(\omega_1, \mathbf{A}(\omega_1)\big), d_\sigma\big(\omega_2, \mathbf{B}(\omega_2)\big), d_\sigma\big(\omega_1, \mathbf{B}(\omega_2)\big), d_\sigma\big(\omega_2, \mathbf{A}(\omega_1)\big)\big\}.$$

Then there is a common fixed point of $\mathbf{A}$ and $\mathbf{B}$.

Proof Let $\alpha : \mathcal{W} \to (0, 1]$ be an arbitrary mapping and define the mappings $\mathbf{S}, \mathbf{T} : \mathcal{W} \to \mathbf{F}(\mathcal{W})$ by

$$\mathbf{S}(\omega_1)(\mathbf{T}) = \begin{cases} \alpha & \text{if } \mathbf{T} \in \mathbf{A}\omega_1, \\ 0 & \text{if } \mathbf{T} \notin \mathbf{A}\omega_1, \end{cases}$$

and

$$\mathbf{T}(\omega_1)(\mathbf{T}) = \begin{cases} \alpha & \text{if } \mathbf{T} \in \mathbf{B}\omega_1, \\ 0 & \text{if } \mathbf{T} \notin \mathbf{B}\omega_1. \end{cases}$$

Then we obtain

$$[\mathbf{S}\omega_1]_{\alpha(\omega_1)} = \big\{\mathbf{T} : \mathbf{S}(\omega_1)(\mathbf{T}) \geq \alpha\big\} = \mathbf{A}\omega_1 \quad \text{and}$$
$$[\mathbf{T}\omega_1]_{\alpha(\omega_1)} = \big\{\mathbf{T} : \mathbf{T}(\omega_1)(\mathbf{T}) \geq \alpha\big\} = \mathbf{B}\omega_1.$$

Therefore we can apply Theorem 2.1 to get a fixed point $\rho \in \mathcal{W}$ such that

$$\rho \in [\mathbf{S}\rho]_{\alpha_{\mathbf{S}}(\rho)} \cap [\mathbf{T}\rho]_{\alpha_{\mathbf{T}}(\rho)} = \mathbf{A}\rho \cap \mathbf{B}\rho. \qquad \square$$

Corollary 2.4 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{A}, \mathbf{B} : \mathcal{W} \to CLB(\mathcal{W})$ be multivalued mappings. Assume there exist $\mathbf{F} \in \mathcal{F}$ and $\Omega > 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma(\mathbf{A}\omega_1, \mathbf{B}\omega_2) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) \tag{18}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma(\mathbf{A}\omega_1, \mathbf{B}\omega_2) > 0$. Then there exists a common fixed point of $\mathbf{A}$ and $\mathbf{B}$.

Proof It suffices to take $Ł = 0$ in Corollary 2.3. $\square$

Corollary 2.5 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{A} : \mathcal{W} \to CLB(\mathcal{W})$ be a multivalued mapping. Assume there exist $\mathbf{F} \in \mathcal{F}$, $\Omega > 0$, and $Ł \geq 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma(\mathbf{A}\omega_1, \mathbf{A}\omega_2) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) + Ł\big(M(\omega_1, \omega_2)\big) \tag{19}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma(\mathbf{A}\omega_1, \mathbf{A}\omega_2) > 0$, where

$$M(\omega_1, \omega_2) = \min\big\{d_\sigma\big(\omega_1, \mathbf{A}(\omega_1)\big), d_\sigma\big(\omega_2, \mathbf{A}(\omega_2)\big), d_\sigma\big(\omega_1, \mathbf{A}(\omega_2)\big), d_\sigma\big(\omega_2, \mathbf{A}(\omega_1)\big)\big\}.$$

Then there exists a fixed point of $\mathbf{A}$.

Proof Take $\mathbf{A} = \mathbf{B}$ in Corollary 2.3. □

Corollary 2.6 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\mathbf{A} : \mathcal{W} \to CLB(\mathcal{W})$ be a multivalued mapping. Assume there exist $\mathbf{F} \in \mathcal{F}$ and $\Omega > 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma(\mathbf{A}\omega_1, \mathbf{A}\omega_2) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big) \tag{20}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma(\mathbf{A}\omega_1, \mathbf{A}\omega_2) > 0$. Then there exists a fixed point of $\mathbf{A}$.

Proof Take $\mathbf{A} = \mathbf{B}$ and $Ł = 0$ in Corollary 2.3. □

Remark 2.3

(i) Corollary 2.3 generalizes Corollary 2.5 of [39].
(ii) Corollary 2.4 generalizes Corollary 2.6.
(iii) Corollary 2.5 (resp., Corollary 2.6) generalizes Corollary 2.7 (resp., Corollary 2.8) of [39].

We further suppose that $\hat{\mathbf{T}}$ is a multivalued mapping induced by the fuzzy mapping $\mathbf{T} : \mathcal{W} \to \Gamma(\mathcal{W})$, that is,

$$\hat{\mathbf{T}}(\omega_1)(\mathbf{T}) = \Big\{\mu \in \mathcal{W} : \mathbf{T}(\omega_1)(\mu) = \max_{t \in \mathcal{W}} \mathbf{T}(\omega_1)(t)\Big\}.$$

Lemma 2.2 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, $\mu \in \mathcal{W}$, and let $\mathbf{T}$ be a fuzzy mapping from $\mathcal{W}$ into $\Gamma(\mathcal{W})$ such that $\hat{\mathbf{T}}(\omega_1)$ is a nonempty compact set for all $\omega_1 \in \mathcal{W}$. Then $\mu \in \hat{\mathbf{T}}(\mu)$ if and only if*

$$\mathbf{T}(\mu)(\mu) \geq \mathbf{T}(\mu)(\omega_1)$$

for all $\omega_1 \in \mathcal{W}$.

Proof Suppose that $\mu \in \hat{\mathbf{T}}(\mu)$. Then

$$\hat{\mathbf{T}}(\mu)(\mu) = \max_{\omega_1 \in \mathcal{W}} \mathbf{T}(\mu)(\omega_1).$$

This implies that

$$\hat{\mathbf{T}}(\mu)(\mu) \geq \mathbf{T}(\mu)(\omega_1) \quad \text{for all } \omega_1 \in \mathcal{W}.$$

Conversely, suppose that

$$\hat{\mathbf{T}}(\mu)(\mu) \geq \mathbf{T}(\mu)(\omega_1) \quad \text{for all } \omega_1 \in \mathcal{W}.$$

Then by the same steps we can show that $\mu \in \hat{\mathbf{T}}(\mu)$. □

Corollary 2.7 *Let $(\mathcal{W}, d_\sigma)$ be a complete controlled metric space, and let $\hat{\mathbf{S}}, \hat{\mathbf{T}} : \mathcal{W} \to \Gamma(\mathcal{W})$ be fuzzy mappings such that for each $\omega_1 \in \mathcal{W}$, $\hat{\mathbf{S}}(\omega_1)$ and $\hat{\mathbf{T}}(\omega_1)$ are nonempty closed subsets of $\mathcal{W}$. Assume there exist $\mathbf{F} \in \mathcal{F}$, $\Omega > 0$, and $Ł \geq 0$ such that*

$$\Omega + \mathbf{F}(H_\sigma(\hat{\mathbf{S}}(\omega_1), \hat{\mathbf{T}}(\omega_2))) \leq \mathbf{F}(d_\sigma(\omega_1, \omega_2)) + Ł(M(\omega_1, \omega_2)) \tag{21}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma(\hat{\mathbf{S}}(\omega_1), \hat{\mathbf{T}}(\omega_2)) > 0$, where

$$M(\omega_1, \omega_2) = \min\{d_\sigma(\omega_1, \hat{\mathbf{S}}(\omega_1)), d_\sigma(\omega_2, \hat{\mathbf{T}}(\omega_2)), d_\sigma(\omega_1, \hat{\mathbf{T}}(\omega_2)), d_\sigma(\omega_2, \hat{\mathbf{S}}(\omega_1))\}.$$

Then there exists $\mu \in \mathcal{W}$ such that $\mathbf{S}(\mu)(\mu) \geq \mathbf{S}(\mu)(\omega_1)$ and $\mathbf{T}(\mu)(\mu) \geq \mathbf{T}(\mu)(\omega_1)$ for all $\omega_1 \in \mathcal{W}$.

Proof By Corollary 2.3 there exists $\mu \in \mathcal{W}$ such that $\mu \in \hat{\mathbf{S}}(\mu) \cap \hat{\mathbf{T}}(\mu)$. Then from Lemma 2.2 we get

$$\mathbf{S}(\mu)(\mu) \geq \mathbf{S}(\mu)(\omega_1) \quad \text{and} \quad \mathbf{T}(\mu)(\mu) \geq \mathbf{T}(\mu)(\omega_1)$$

for all $\omega_1 \in \mathcal{W}$. □

Example 2.1 Let $\mathcal{W} = [0, 1]$. Define $d_\sigma : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ by

$$d_\sigma(\omega_1, \omega_2) = |\omega_1 - \omega_2|.$$

Then $(\mathcal{W}, d_\sigma)$ is a complete controlled metric space, where $\sigma : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ is defined by

$$\sigma(\omega_1, \omega_2) = \begin{cases} 1 & \text{if } \omega_1, \omega_2 \in [0, 0.5), \\ \omega_1 + \omega_2 + 2 & \text{otherwise.} \end{cases}$$

For $\alpha \in [0, 1)$ and $\omega_1 \in \mathcal{W}$, define the mappings $\mathbf{S}, \mathbf{T} : \mathcal{W} \to \Gamma(\mathcal{W})$ by

$$\mathbf{S}(\omega_1)(\mathbf{T}) = \begin{cases} \alpha & \text{if } 0 \leq \mathbf{T} \leq \frac{\omega_1}{50}, \\ \frac{\alpha}{2} & \text{if } \frac{\omega_1}{50} < \mathbf{T} \leq \frac{\omega_1}{40}, \\ \frac{\alpha}{3} & \text{if } \frac{\omega_1}{40} < \mathbf{T} \leq \frac{\omega_1}{30}, \\ \frac{\alpha}{4} & \text{if } \frac{\omega_1}{30} < \mathbf{T} \leq 1, \end{cases}$$

and

$$\mathbf{T}(\omega_1)(\mathbf{T}) = \begin{cases} \alpha & \text{if } 0 \leq \mathbf{T} \leq \frac{\omega_1}{20}, \\ \frac{\alpha}{4} & \text{if } \frac{\omega_1}{20} < \mathbf{T} \leq \frac{\omega_1}{10}, \\ \frac{\alpha}{5} & \text{if } \frac{\omega_1}{10} < \mathbf{T} \leq \frac{\omega_1}{5}, \\ \frac{\alpha}{7} & \text{if } \frac{\omega_1}{5} < \mathbf{T} \leq 1, \end{cases}$$

so that

$$[\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)} = \left[0, \frac{\omega_1}{50}\right] \quad \text{and} \quad [\mathbf{T}\omega_1]_{\alpha_{\mathbf{T}}(\omega_1)} = \left[0, \frac{\omega_1}{20}\right].$$

Let $\mathbf{F}(\mathbf{T}) = \ln(\mathbf{T})$. Then there exists $\Omega \in (0, \ln \frac{|\omega_2 - \omega_1|}{|\omega_2 - \frac{\omega_1}{2}| \frac{1}{50}})$ such that

$$\Omega + \mathbf{F}(H_\sigma\left([\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}\right) \leq \mathbf{F}\big(d_\sigma(\omega_1, \omega_2)\big)$$

for all $\omega_1, \omega_2 \in \mathcal{W}$ with $H_\sigma([\mathbf{S}\omega_1]_{\alpha_{\mathbf{S}}(\omega_1)}, [\mathbf{T}\omega_2]_{\alpha_{\mathbf{T}}(\omega_2)}) > 0$. Hence all the axioms of Theorem 2.1 are satisfied, and therefore $0 \in [\mathbf{S}0]_\alpha \cap [\mathbf{T}0]_\alpha$.

3 Conclusion

In this work, we introduced the concept of fuzzy mappings in a more general space, called a controlled metric space. Further, we derived the existence of common α-fuzzy fixed points for two fuzzy mappings under generalized almost $\mathbf{F}$-contractions in the setting of controlled metric spaces. Our results generalize many well-known results in the literature. For justification of the obtained results, we gave an illustrative example.

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