Rao and Ahmadini *Advances in Difference Equations* (2021) 2021:436
https://doi.org/10.1186/s13662-021-03591-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Multiple positive solutions for a system of (p_1, p_2, p_3)-Laplacian Hadamard fractional order BVP with parameters

Sabbavarapu Nageswara Rao[1*] and Abdullah Ali H. Ahmadini[1]

*Correspondence:
snrao@jazanu.edu.sa
[1]Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia

Abstract

In this article, we are pleased to investigate multiple positive solutions for a system of Hadamard fractional differential equations with (p_1, p_2, p_3)-Laplacian operator. The main results rely on the standard tools of different fixed point theorems. Finally, we demonstrate the application of the obtained results with the aid of examples.

MSC: 34B15; 34B18; 34B27; 34A08

Keywords: Hadamard fractional derivative; Parameters; Triple system; p-Laplacian; Fixed point theorems

1 Introduction

The majority of the aforesaid analysis on the topic is based upon fractional differential equations and Hadamard fractional derivatives involving many numerous applications in a variety of fields such as control theory, electrical circuits, biology, physics, and finance [1–10]. For example, Arafa et al. [8] proposed a fractional order into a model of HIV-1 infection of $CD4^+$ T-cells dynamics model:

$$\begin{cases} D^{\sigma_1}(T) = s - KVT - dT + bI, \\ D^{\sigma_2}(I) = KVT - (b+\delta)I, \\ D^{\sigma_3}(V) = N\delta I - cV, \end{cases}$$

where D^{σ_i} $(i = 1, 2, 3)$ are fractional order derivatives. Jesus et al. [10] studied the fractional electrical impedance of vegetables and fruits by using Bode and polar diagrams. In the modern decades, the results of multiplicity of positive solutions for a system of fractional differential equations which are subject to various levels of boundary conditions have been analyzed extensively by numerous researchers using a variety of methods and techniques [11–18]. Further analysis of positive solutions with p-Laplacian made an extensive contribution to amalgamate the study [19–31]. In the recent past, Hadamard fractional order problems under contrasting different boundary conditions were briefly discussed in the literature [32–36]. Contrarily, many researchers studied the theory of Hadamard fractional order along with p-Laplacian operator [37–40]. Our results were combined to generalize the study from the papers [30, 31]. In [31], Han et al. studied the boundary value problem with fractional differential equation involving the p-Laplacian operator:

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}x(t))) + a(t)f(x) = 0, \quad 0 < t < 1, \\ x(0) = \gamma x(\xi) + \lambda, \phi_p(D_{0^+}^{\alpha}x(0)) = \phi_p(D_{0^+}^{\alpha}x(0))' = \phi_p(D_{0^+}^{\alpha}x(0))'' = 0, \end{cases}$$

where $0 < \alpha \leq 1$, $2 < \beta \leq 3$ are real numbers, $0 \leq \gamma < 1$, $0 \leq \xi \leq 1$, $\lambda > 0$ is a parameter, and $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Caputo fractional derivatives. Under some assumptions, several new existence and nonexistence results for positive solutions in terms of different values of the parameter λ are obtained.

Inspired by the aforementioned works, here we have amalgamated the system for nonlinear Hadamard fractional differential equations for the existence of multiple positive solutions along with (p_1, p_2, p_3)-Laplacian operators:

$$\begin{aligned} &-{}^H\mathcal{D}_{1^+}^{\rho_1}\big(\phi_{p_1}\big({}^H\mathcal{D}_{1^+}^{\sigma_1}ß(t)\big)\big) = f_1\big(t, ß(t), \varpi(t), \omega(t)\big), \quad 1 < t < e, \\ &-{}^H\mathcal{D}_{1^+}^{\rho_2}\big(\phi_{p_2}\big({}^H\mathcal{D}_{1^+}^{\sigma_2}\varpi(t)\big)\big) = f_2\big(t, ß(t), \varpi(t), \omega(t)\big), \quad 1 < t < e, \\ &-{}^H\mathcal{D}_{1^+}^{\rho_3}\big(\phi_{p_3}\big({}^H\mathcal{D}_{1^+}^{\sigma_3}\omega(t)\big)\big) = f_3\big(t, ß(t), \varpi(t), \omega(t)\big), \quad 1 < t < e, \end{aligned} \tag{1}$$

subject to the two-point boundary conditions

$$\begin{aligned} &ß(1) = ß'(1) = 0, \qquad {}^H\mathcal{D}_{1^+}^{\sigma_1}ß(1) = 0, \qquad \lambda_1 ß(e) + \mu_1 {}^H\mathcal{D}_{1^+}^{\delta_1}ß(e) = \psi_1, \\ &\varpi(1) = \varpi'(1) = 0, \qquad {}^H\mathcal{D}_{1^+}^{\sigma_2}\varpi(1) = 0, \qquad \lambda_2\varpi(e) + \mu_2 {}^H\mathcal{D}_{1^+}^{\delta_2}\varpi(e) = \psi_2, \\ &\omega(1) = \omega'(1) = 0, \qquad {}^H\mathcal{D}_{1^+}^{\sigma_3}\omega(1) = 0, \qquad \lambda_3\omega(e) + \mu_3 {}^H\mathcal{D}_{1^+}^{\delta_3}\omega(e) = \psi_3, \end{aligned} \tag{2}$$

where $\sigma_i, \rho_i, \delta_i \in \mathbb{R}$, $\sigma_i \in (2,3]$, $\rho_i \in (0,1]$, $\delta_i \in (1,2]$, $i = 1,2,3$, λ_i, μ_i, $i = 1,2,3$, are real positive constants, $\psi_i > 0$ is a parameter for $i = 1,2,3$, ${}^H\mathcal{D}_{1^+}^{\dagger}$ denotes the Hadamard fractional derivative of order $\dagger$ for ($\dagger = \sigma_i, \rho_i, \delta_i$, $i = 1,2,3$), $p_1, p_2, p_3 > 1$, $\phi_{p_i}(s) = |s|^{p_i-2}s$, $\phi_{p_i}^{-1} = \phi_{q_i}$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 1,2,3$, and $f_i \in C([1,e] \times \mathbb{R}_+^3 \to \mathbb{R}_+)$, $i = 1,2,3$. Sufficient conditions for the existence of single and multiple positive solutions are derived by imposing growth conditions f_1, f_2 and on f_3 by applying various fixed point theorems in a cone. By a positive solution of problem (1)–(2), we mean a triplet of functions $(ß(t), \varpi(t), \omega(t)) \in (C([1,e], \mathbb{R}_+))^3$, $(\mathbb{R}_+ = [0,\infty))$ gratifying (1)–(2) with $ß(t), \varpi(t), \omega(t) \geq 0$ for all $t \in [1,e]$ and $(ß(t), \varpi(t), \omega(t)) \neq (0,0,0)$.

We assume the following hypotheses:

(H1) The function $f_i : [1,e] \times \mathbb{R}_+^3 \to \mathbb{R}_+$ is continuous.

(H2) $\mu_i, \lambda_i > 0$, $\sigma_i, \rho_i, \delta_i \in \mathbb{R}$, $2 < \sigma_i \leq 3$, $0 < \rho_i \leq 1$, $1 < \delta_i \leq 2$, $\mu_i(\delta_i - 1) > \frac{\lambda_i\Gamma(\sigma_i-\delta_i)}{\Gamma(\sigma_i)}$, and $\Lambda_i = \lambda_i\Gamma(\sigma_i - \delta_i) + \mu_i\Gamma(\sigma_i) > 0$, $\forall i = 1,2,3$.

(H3) $\mho_1$, $\mho_2$, $\mho_3$, $\aleph_1$, $\aleph_2$, $\aleph_3$ are positive constants such that $\frac{1}{\mho_1} + \frac{1}{\mho_2} + \frac{1}{\mho_3} + \frac{1}{\aleph_1} + \frac{1}{\aleph_2} + \frac{1}{\aleph_3} \leq 1$.

The rest of this paper is organized as follows. In Sect. 2, we provide some preliminaries and theorems to prove our main results. In Sect. 3, we construct the Green function and also give some properties of the Green function which are needed later. Section 4 is devoted to establishing the existence results of at least one or three positive solutions for system (1)–(2). In Sect. 5, as an application, two examples are presented to illustrate our main results.

2 Preliminaries

First, we provide the definitions of Hadamard fractional derivative and Hadamard fractional integral on a finite interval, the details of which can be found in the materials [41–43].

Definition 2.1 The Hadamard derivative of fractional order σ for a function $u : [1, +\infty) \to \mathbb{R}$ is defined as

$$^{H}\mathcal{D}^{\sigma} u(t) = \frac{1}{\Gamma(n-\sigma)} \left(t\frac{d}{dt}\right)^{n} \int_{1}^{t} \left(\log \frac{t}{s}\right)^{n-\sigma-1} u(s)\frac{ds}{s},$$

where $\sigma > 0$, $n = [\sigma] + 1$, and $[\sigma]$ denotes the largest integer which is less than or equal to σ and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 The Hadamard fractional integral of order σ for a function $u : [1, +\infty) \to \mathbb{R}$ is defined by

$$\mathcal{I}^{\sigma} u(t) = \frac{1}{\Gamma(\sigma)} \int_{1}^{t} \left(\log \frac{t}{s}\right)^{\sigma-1} u(s)\frac{ds}{s}, \quad \sigma > 0,$$

provided the integral exists.

Definition 2.3 Let $\mathcal{R}$ be a real Banach space. A nonempty closed convex set $\mathcal{M} \subset \mathcal{R}$ is called a cone if it satisfies the following conditions:

(i) $\tau \in \mathcal{M}$, $\ell \geq 0$ implies $\ell\tau \in \mathcal{M}$;

(ii) $\tau \in \mathcal{M}$, $-\tau \in \mathcal{M}$ implies $\tau = 0$.

Every cone $\mathcal{M} \subset \mathcal{R}$ induces an ordering in $\mathcal{R}$ given by $\tau \leq \zeta$ if and only if $\zeta - \tau \in \mathcal{M}$.

Definition 2.4 Let $\mathcal{M}$ be a cone in the real Banach space $\mathcal{R}$. A map $\chi : \mathcal{M} \to [0, \infty)$ is said to be a nonnegative continuous concave functional on a cone $\mathcal{M}$ if χ is continuous and

$$\chi\left(\ell\tau + (1-\ell)\zeta\right) \geq \ell\chi(\tau) + (1-\ell)\chi(\zeta), \quad \tau, \zeta \in \mathcal{M}, 0 \leq \ell \leq 1.$$

Definition 2.5 Let $\mathcal{M}$ be a cone in the real Banach space $\mathcal{R}$. A map $\pi : \mathcal{M} \to [0, \infty)$ is said to be a nonnegative continuous convex functional on a cone $\mathcal{M}$ if π is continuous and

$$\pi\left(\ell\tau + (1-\ell)\zeta\right) \leq \ell\pi(\tau) + (1-\ell)\pi(\zeta), \quad \tau, \zeta \in \mathcal{M}, 0 \leq \ell \leq 1.$$

Property 2.1 ([44]) Let $\mathcal{M}$ be a cone in a real Banach space $\mathcal{R}$, and let Θ be a bounded open subset of $\mathcal{R}$ with $0 \in \Theta$. Then continuous functional $\eta : \mathcal{M} \to [0, \infty)$ is said to satisfy Property K_1 if one of the following conditions holds:

(i) η is convex, $\eta(0) = 0$, $\eta(\flat) \neq 0$ if $\flat \neq 0$ and $\inf_{\flat \in \mathcal{M} \cap \partial\Theta} \eta(\flat) > 0$,

(ii) η is sublinear, $\eta(0) = 0$, $\eta(\flat) \neq 0$ if $\flat \neq 0$ and $\inf_{\flat \in \mathcal{M} \cap \partial\Theta} \eta(\flat) > 0$,
(iii) η is concave and unbounded.

Property 2.2 ([44]) Let $\mathcal{M}$ be a cone in a real Banach space $\mathcal{R}$, and let Θ be a bounded open subset of $\mathcal{R}$ with $0 \in \Theta$. Then the continuous functional $\xi : \mathcal{M} \to [0,\infty)$ is said to satisfy Property K_2 if one of the following conditions holds:
(i) ξ is convex, $\xi(0) = 0$, $\xi(\flat) \neq 0$ if $\flat \neq 0$,
(ii) ξ is sublinear, $\xi(0) = 0$, $\xi(\flat) \neq 0$ if $\flat \neq 0$,
(iii) $\xi(\flat + \tau) \geq \xi(\flat) + \xi(\tau)$ for all $\flat, \tau \in \mathcal{M}$, $\xi(0) = 0$, $\xi(\flat) \neq 0$ if $\flat \neq 0$.

In the proof of our existence results, we shall use the following fixed point theorems of the cone expansion and compression of functional type due to Avery et al. [44] and five functionals fixed point theorem [45].

Theorem 2.1 ([44]) *Let Θ_1 and Θ_2 be two bounded open sets in a Banach space $\mathcal{R}$ such that $0 \in \Theta_1$ and $\overline{\Theta_1} \subset \Theta_2$ in $\mathcal{R}$. Suppose that $\mathcal{L} : \mathcal{M} \cap (\overline{\Theta_2} \backslash \Theta_1) \to \mathcal{M}$ is a completely continuous operator, η and ξ are nonnegative continuous functionals on $\mathcal{M}$ and one of the two conditions holds*:
(i) *η satisfies Property 2.1 with $\eta(\mathcal{L}\flat) \geq \eta(\flat)$ for all $\flat \in \mathcal{M} \cap \partial\Theta_1$, and ξ satisfies Property 2.2 with $\xi(\mathcal{L}\flat) \leq \xi(\flat)$ for all $\flat \in \mathcal{M} \cap \partial\Theta_2$;*
(ii) *ξ satisfies Property 2.2 with $\xi(\mathcal{L}\flat) \leq \xi(\flat)$ for all $\flat \in \mathcal{M} \cap \partial\Theta_1$ and η satisfies Property 2.1 with $\eta(\mathcal{L}\flat) \geq \eta(\flat)$ for all $\flat \in \mathcal{M} \cap \partial\Theta_2$ is satisfied.*

Then $\mathcal{L}$ has at least one fixed point in $\mathcal{M} \cap (\overline{\Theta_2} \backslash \Theta_1)$.

Let $\wp, \varrho, \varsigma$ be nonnegative continuous convex functionals on P and α, β be nonnegative continuous concave functionals on P, then for nonnegative real numbers h, a, b, d, and c, we define the following convex sets:

$$P(\wp, c) = \{\flat \in P : \wp(\flat) < c\},$$
$$P(\wp, \alpha, a, c) = \{\flat \in P : a \leq \alpha(\flat); \wp(\flat) \leq c\},$$
$$Q(\wp, \varrho, d, c) = \{\flat \in P : \varrho(\flat) \leq d; \wp(\flat) \leq c\},$$
$$P(\wp, \varsigma, \alpha, a, b, c) = \{\flat \in P : a \leq \alpha(\flat); \varsigma(\flat) \leq b, \wp(\flat) \leq c\},$$
$$Q(\wp, \varrho, \beta, h, d, c) = \{\flat \in P : h \leq \beta(\flat); \varrho(\flat) \leq d, \wp(\flat) \leq c\}.$$

Theorem 2.2 ([45]) *Let P be a cone in the real Banach space E. Suppose that α and β are nonnegative continuous concave functionals on P and $\wp$, ϱ, and ς are nonnegative continuous convex functions on P. Suppose that there exist positive numbers c and M with*

$$\alpha(\flat) \leq \varrho(\flat) \quad \text{and} \quad \|\flat\| \leq M\wp(\flat) \quad \text{for all } \flat \in \overline{P(\wp, c)}.$$

Suppose that $A : \overline{P(\wp, c)} \to \overline{P(\wp, c)}$ is a completely continuous operator, and there exist nonnegative numbers h, a, k, b with $0 < a < b$ such that:

$$(D1) \quad \{\flat \in P(\wp, \varsigma, \alpha, b, k, c) : \alpha(\flat) > b\} \neq \emptyset \quad \text{and}$$
$$\alpha(A\flat) > b \quad \text{for } \flat \in P(\wp, \varsigma, \alpha, b, k, c);$$

$$(D2)\quad \{\flat \in Q(\wp,\varrho,\beta,h,a,c) : \varrho(\flat) < a\} \neq \emptyset \quad \textit{and}$$

$$\varrho(A\flat) < a \quad \textit{for } \flat \in Q(\wp,\varrho,\beta,h,a,c);$$

$$(D3)\quad \alpha(A\flat) > b \quad \textit{for } \flat \in P(\wp,\alpha,b,c) \textit{ with } \varsigma(A\flat) > k;$$

$$(D4)\quad \varrho(A\flat) < a \quad \textit{for } \flat \in Q(\wp,\varrho,a,c) \textit{ with } \beta(A\flat) < h.$$

Then A has at least three fixed points $\flat_1, \flat_2, \flat_3 \in \overline{P(\wp,c)}$ *such that*

$$\varrho(\flat_1) < a, \qquad b < \alpha(\flat_2) \quad \textit{and} \quad a < \varrho(\flat_3) \quad \textit{with } \alpha(\flat_3) < b.$$

3 Green function and bounds

In this section, we construct the Green function for the homogeneous two-point boundary value problem

$$-{}^{H}\mathcal{D}_{1^+}^{\sigma_1}\text{ß}(t) = 0, \quad 1 < t < e, \tag{3}$$

$$\text{ß}(1) = \text{ß}'(1) = 0, \qquad \lambda_1\text{ß}(e) + \mu_1{}^{H}\mathcal{D}_{1^+}^{\delta_1}\text{ß}(e) = \psi_1. \tag{4}$$

Lemma 3.1 *Let* $\Lambda_1 = \lambda_1\Gamma(\sigma_1 - \delta_1) + \mu_1\Gamma(\sigma_1) > 0$. *If* $x \in [1,e]$, *then the Hadamard fractional differential order BVP*

$${}^{H}\mathcal{D}_{1^+}^{\sigma_1}\text{ß}(t) + x(t) = 0, \quad 1 < t < e, \tag{5}$$

subject to the two-point boundary conditions (4), *has a unique solution*

$$\text{ß}(t) = \int_1^e \mathcal{G}_1(t,s)x(s)\frac{ds}{s} + \frac{\psi_1\Gamma(\sigma_1 - \delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1},$$

where

$$\mathcal{G}_1(t,s) = \begin{cases} \mathcal{G}_{11}(t,s), & 1 \le t \le s \le e, \\ \mathcal{G}_{12}(t,s), & 1 \le s \le t \le e, \end{cases} \tag{6}$$

$$\mathcal{G}_{11}(t,s) = \frac{1}{\Lambda_1}\left[\mu_1(1-\log s)^{-\delta_1} + \frac{\lambda_1\Gamma(\sigma_1-\delta_1)}{\Gamma(\sigma_1)}\right](\log t)^{\sigma_1-1}(1-\log s)^{\sigma_1-1},$$

$$\mathcal{G}_{12}(t,s) = \frac{1}{\Lambda_1}\left[\mu_1(1-\log s)^{-\delta_1} + \frac{\lambda_1\Gamma(\sigma_1-\delta_1)}{\Gamma(\sigma_1)}\right](\log t)^{\sigma_1-1}(1-\log s)^{\sigma_1-1} - \frac{1}{\Gamma(\sigma_1)}\left(\log\frac{t}{s}\right)^{\sigma_1-1}.$$

Proof As argued in [43] the solution of Hadamard fractional order BVPs (5) and (4) can be written as the following equivalent integral equation:

$$\text{ß}(t) = c_1(\log t)^{\sigma_1-1} + c_2(\log t)^{\sigma_1-2} + c_3(\log t)^{\sigma_1-3} - \frac{1}{\Gamma(\sigma_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma_1-1}x(s)\frac{ds}{s}.$$

From $\text{ß}(1) = \text{ß}'(1) = 0$, we have $c_2 = c_3 = 0$. Furthermore, we can get

$${}^{H}\mathcal{D}_{1^+}^{\delta_1}\text{ß}(t) = c_1\frac{\Gamma(\sigma_1)}{\Gamma(\sigma_1-\delta_1)}(\log t)^{\sigma_1-\delta_1-1} - \frac{1}{\Gamma(\sigma_1-\delta_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma_1-\delta_1-1}x(s)\frac{ds}{s}.$$

From the boundary condition, $\lambda_1 ß(e) + \mu_1 {}^H\mathcal{D}_{1^+}^{\delta_1} ß(e) = \psi_1$, we obtain

$$c_1 = \frac{1}{\Lambda_1}\int_1^e \biggl[\mu_1(1-\log s)^{-\delta_1} + \frac{\lambda_1\Gamma(\sigma_1-\delta_1)}{\Gamma(\sigma_1)}\biggr](1-\log s)^{\sigma_1-1}x(s)\frac{ds}{s} + \frac{\psi_1\Gamma(\sigma_1-\delta_1)}{\Lambda_1}.$$

Hence, the unique solution of (5), (4) is

$$\begin{aligned} ß(t) &= \frac{1}{\Lambda_1}\int_1^e \biggl[\mu_1(1-\log s)^{-\delta_1} + \frac{\lambda_1\Gamma(\sigma_1-\delta_1)}{\Gamma(\sigma_1)}\biggr](1-\log s)^{\sigma_1-1}(\log t)^{\sigma_1-1}x(s)\frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\sigma_1)}\int_1^t \biggl(\log\frac{t}{s}\biggr)^{\sigma_1-1}x(s)\frac{ds}{s} + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1} \\ &= \int_1^t \biggl[\frac{1}{\Lambda_1}\biggl(\mu_1(1-\log s)^{-\delta_1} + \frac{\lambda_1\Gamma(\sigma_1-\delta_1)}{\Gamma(\sigma_1)}\biggr)(\log t)^{\sigma_1-1}(1-\log s)^{\sigma_1-1} \\ &\quad - \frac{1}{\Gamma(\sigma_1)}\biggl(\log\frac{t}{s}\biggr)^{\sigma_1-1}\biggr]x(s)\frac{ds}{s} \\ &\quad + \int_t^e \frac{1}{\Lambda_1}\biggl(\mu_1(1-\log s)^{-\delta_1} + \frac{\lambda_1\Gamma(\sigma_1-\delta_1)}{\Gamma(\sigma_1)}\biggr)(\log t)^{\sigma_1-1}(1-\log s)^{\sigma_1-1}x(s)\frac{ds}{s} \\ &\quad + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1} \\ &= \int_1^e \mathcal{G}_1(t,s)x(s)\frac{ds}{s} + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}. \end{aligned}$$

□

Lemma 3.2 *Let* $2 < \sigma_1 \leq 3$, $0 < \rho_1 \leq 1$, *and* $y \in C[1,e]$. *Then the Hadamard fractional order BVP*

$$\begin{cases} {}^H\mathcal{D}_{1^+}^{\rho_1}(\phi_{p_1}({}^H\mathcal{D}_{1^+}^{\sigma_1}ß(t))) + y(t) = 0, \quad t \in (1,e), \\ ß(1) = ß'(1) = 0, \qquad {}^H\mathcal{D}_{1^+}^{\sigma_1}ß(1) = 0, \qquad \lambda_1 ß(e) + \mu_1 {}^H\mathcal{D}_{1^+}^{\delta_1}ß(e) = \psi_1, \end{cases} \tag{7}$$

has a unique solution

$$ß(t) = \int_1^e \mathcal{G}_1(t,s)\phi_{q_1}\biggl(\int_1^s b_1\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_1-1}y(\kappa)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}. \tag{8}$$

For convenience, let $b_1 = \Gamma(\rho_1)^{-1}$.

Proof In fact, let $\phi = {}^H\mathcal{D}_{1^+}^{\sigma_1}ß$, $\Upsilon = \phi_{p_1}(\phi)$. Then the solution of the IVP

$$\begin{cases} {}^H\mathcal{D}_{1^+}^{\rho_1}\Upsilon(t) + y(t) = 0, \quad t \in (1,e), \\ \Upsilon(1) = 0. \end{cases} \tag{9}$$

By the Lemma 3.1, we can reduce IVP (9) to an equivalent integral equation

$$\Upsilon(t) = c_1(\log t)^{\rho_1-1} - I_{1^+}^{\rho_1}y(t), \quad t \in (1,e).$$

From the relation $\Upsilon(1) = 0$, we get $c_1 = 0$; and consequently

$$\Upsilon(t) = -I_{1^+}^{\rho_1}y(t), \quad t \in (1,e). \tag{10}$$

Noting that ${}^{H}\mathcal{D}_{1^+}^{\sigma_1}\text{ß} = \phi$, $\phi = \phi_{p_1}^{-1}(\Upsilon)$, we have from (10) that the solution of (7) satisfies

$$\begin{cases} {}^{H}\mathcal{D}_{1^+}^{\sigma_1}\text{ß} = \phi_{p_1}^{-1}(-I_{1^+}^{\rho_1}y(t)), & t \in (1,e), \\ \text{ß}(1) = \text{ß}'(1) = 0, & \lambda_1\text{ß}(e) + \mu_1 {}^{H}\mathcal{D}_{1^+}^{\delta_1}\text{ß}(e) = \psi_1. \end{cases} \tag{11}$$

By Lemma 3.1, the solution of (11) can be obtained as

$$\text{ß}(t) = -\int_1^e \mathcal{G}_1(t,s)\phi_{p_1}^{-1}\bigl(-I_{1^+}^{\rho_1}y(s)\bigr)\frac{ds}{s} + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}, \quad t \in (1,e),$$

since $y(s) \geq 0$, $s \in [1,e]$, we have

$$\phi_{p_1}^{-1}\bigl(-I_{1^+}^{\rho_1}y(s)\bigr) = -\phi_{q_1}\bigl(I_{1^+}^{\rho_1}y(s)\bigr), \quad s \in [1,e],$$

which implies that boundary value problem (7) has a unique solution

$$\begin{aligned} \text{ß}(t) = {} & \int_1^e \mathcal{G}_1(t,s)\phi_{q_1}\left(\int_1^s b_1\left(\log\frac{s}{\kappa}\right)^{\rho_1-1} y(\kappa)\frac{d\kappa}{\kappa}\right)\frac{ds}{s} \\ & + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}, \quad t \in [1,e]. \end{aligned}$$

□

Lemma 3.3 ([40]) *Assume that* (H2) *holds. Then the function* $\mathcal{G}_1(t,s)$ *given by* (6) *satisfies the following inequalities*:

(i) $\mathcal{G}_1(t,s) \geq 0$ *for all* $t,s \in [1,e]$,

(ii) $\mathcal{G}_1(t,s) \leq \mathcal{G}_1(e,s)$ *for all* $t,s \in [1,e]$,

(iii) $\mathcal{G}_1(t,s) \geq (\frac{1}{4})^{\sigma_1-1}\mathcal{G}_1(e,s)$ *for all* $t \in I$, $s \in (1,e)$, *where* $I = [e^{1/4}, e^{3/4}]$.

We can also formulate similar results to Lemmas 3.1–3.3 for the Hadamard fractional boundary value problems

$$ {}^{H}\mathcal{D}_{1^+}^{\rho_2}\bigl(\phi_{p_2}\bigl({}^{H}\mathcal{D}_{1^+}^{\sigma_2}\varpi(t)\bigr)\bigr) + z(t) = 0, \quad 1 < t < e, \tag{12}$$

$$\varpi(1) = \varpi'(1) = 0, \qquad {}^{H}\mathcal{D}_{1^+}^{\sigma_2}\varpi(1) = 0, \qquad \lambda_2\varpi(e) + \mu_2 {}^{H}\mathcal{D}_{1^+}^{\delta_2}\varpi(e) = \psi_2, \tag{13}$$

and

$$ {}^{H}\mathcal{D}_{1^+}^{\rho_3}\bigl(\phi_{p_3}\bigl({}^{H}\mathcal{D}_{1^+}^{\sigma_3}\omega(t)\bigr)\bigr) + \phi(t) = 0, \quad 1 < t < e, \tag{14}$$

$$\omega(1) = \omega'(1) = 0, \qquad {}^{H}\mathcal{D}_{1^+}^{\sigma_3}\omega(1) = 0, \qquad \lambda_3\omega(e) + \mu_3 {}^{H}\mathcal{D}_{1^+}^{\delta_3}\omega(e) = \psi_3, \tag{15}$$

where $\sigma_j, \rho_j, \delta_j \in \mathbb{R}$, $\sigma_j \in (2,3]$, $\rho_j \in (0,1]$, $\delta_j \in (1,2]$, $\lambda_j, \mu_j > 0$, ψ_j is a parameter for $j = 2,3$.

We denote by Λ_2, $\mathcal{G}_2$, $\mathcal{G}_{21}$, $\mathcal{G}_{22}$ and Λ_3, $\mathcal{G}_3$, $\mathcal{G}_{31}$, $\mathcal{G}_{32}$ the corresponding constants and Green functions for problem (12)–(13) and problem (14)–(15), respectively, defined in a similar manner as Λ_1, $\mathcal{G}_1$, $\mathcal{G}_{11}$, $\mathcal{G}_{12}$. More precisely, we have

$$\Lambda_2 = \lambda_2\Gamma(\sigma_2-\delta_2) + \mu_2\Gamma(\sigma_2),$$

$$\mathcal{G}_2(t,s) = \begin{cases} \mathcal{G}_{21}(t,s), & 1 \leq t \leq s \leq e, \\ \mathcal{G}_{22}(t,s), & 1 \leq s \leq t \leq e, \end{cases}$$

$$\begin{aligned}
\mathcal{G}_{21}(t,s) &= \frac{1}{\Lambda_2}\left[\mu_2(1-\log s)^{-\delta_2} + \frac{\lambda_2\Gamma(\sigma_2-\delta_2)}{\Gamma(\sigma_2)}\right](\log t)^{\sigma_2-1}(1-\log s)^{\sigma_2-1}, \\
\mathcal{G}_{22}(t,s) &= \frac{1}{\Lambda_2}\left[\mu_2(1-\log s)^{-\delta_2} + \frac{\lambda_2\Gamma(\sigma_2-\delta_2)}{\Gamma(\sigma_2)}\right](\log t)^{\sigma_2-1}(1-\log s)^{\sigma_2-1} \\
&\quad - \frac{1}{\Gamma(\sigma_2)}\left(\log\frac{t}{s}\right)^{\sigma_2-1},
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda_3 = \lambda_3\Gamma(\sigma_3-\delta_3) + \mu_3\Gamma(\sigma_3), \\
&\mathcal{G}_3(t,s) = \begin{cases} \mathcal{G}_{31}(t,s), & 1 \le t \le s \le e, \\ \mathcal{G}_{32}(t,s), & 1 \le s \le t \le e, \end{cases} \\
&\mathcal{G}_{31}(t,s) = \frac{1}{\Lambda_3}\left[\mu_3(1-\log s)^{-\delta_3} + \frac{\lambda_3\Gamma(\sigma_3-\delta_3)}{\Gamma(\sigma_3)}\right](\log t)^{\sigma_3-1}(1-\log s)^{\sigma_3-1}, \\
&\mathcal{G}_{32}(t,s) = \frac{1}{\Lambda_3}\left[\mu_3(1-\log s)^{-\delta_3} + \frac{\lambda_3\Gamma(\sigma_3-\delta_3)}{\Gamma(\sigma_3)}\right](\log t)^{\sigma_3-1}(1-\log s)^{\sigma_3-1} \\
&\qquad - \frac{1}{\Gamma(\sigma_3)}\left(\log\frac{t}{s}\right)^{\sigma_3-1}.
\end{aligned}$$

The inequalities from Lemma 3.3 for the functions $\mathcal{G}_2$, $\mathcal{G}_3$ are the following: $\mathcal{G}_2(t,s) \le \mathcal{G}_2(e,s)$, $\mathcal{G}_3(t,s) \le \mathcal{G}_3(e,s)$ for all $t,s \in [1,e]$ and $\mathcal{G}_2(t,s) \ge (\frac{1}{4})^{\sigma_2-1}\mathcal{G}_2(e,s)$, $\mathcal{G}_3(t,s) \ge (\frac{1}{4})^{\sigma_3-1} \times \mathcal{G}_3(e,s)$ for all $t \in I$, $s \in (1,e)$.

Remark Consider the following condition:

$$\mathcal{G}_i(t,s) \ge m\mathcal{G}_i(t,s) \quad \text{for all } t \in I, s \in (1,e), i = 1,2,3,$$

where $m = \min\{(\frac{1}{4})^{\sigma_1-1}, (\frac{1}{4})^{\sigma_2-1}, (\frac{1}{4})^{\sigma_3-1}\}$.

By using Green's functions $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ our problem (1)–(2) can be written equivalently as the following nonlinear system of integral equations:

$$\begin{cases}
ß(t) = \int_1^e \mathcal{G}_1(t,s)\phi_{q_1}(\int_1^s b_1(\log\frac{s}{\kappa})^{\rho_1-1}f_1(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa))\frac{d\kappa}{\kappa})\frac{ds}{s} \\
\qquad + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}, \\
\varpi(t) = \int_1^e \mathcal{G}_2(t,s)\phi_{q_2}(\int_1^s b_2(\log\frac{s}{\kappa})^{\rho_2-1}f_2(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa))\frac{d\kappa}{\kappa})\frac{ds}{s} \\
\qquad + \frac{\psi_2\Gamma(\sigma_2-\delta_2)(\log t)^{\sigma_2-1}}{\Lambda_2}, \\
\omega(t) = \int_1^e \mathcal{G}_3(t,s)\phi_{q_3}(\int_1^s b_3(\log\frac{s}{\kappa})^{\rho_3-1}f_3(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa))\frac{d\kappa}{\kappa})\frac{ds}{s} \\
\qquad + \frac{\psi_3\Gamma(\sigma_3-\delta_3)(\log t)^{\sigma_3-1}}{\Lambda_3}.
\end{cases}$$

We consider the Banach space $\mathcal{R} = \mathcal{T} \times \mathcal{T} \times \mathcal{T}$, where $\mathcal{T} = \{ß : ß \in C[1,e]\}$ equipped with the norm $\|(ß,\varpi,\omega)\|_{\mathcal{R}} = \|ß\| + \|\varpi\| + \|\omega\|$ for $(ß,\varpi,\omega) \in \mathcal{R}$, and the norm is defined as $\|ß\| = \max_{t\in[1,e]} |ß(t)|$. We define a cone $\mathcal{W} \subset \mathcal{R}$ by

$$\mathcal{W} = \Big\{(ß,\varpi,\omega) \in \mathcal{R}, ß(t) \geq 0, \varpi(t) \geq 0, \omega(t) \geq 0, \forall t \in [1,e] \text{ and } \min_{t\in I}\big[ß(t) + \varpi(t) + \omega(t)\big] \geq m\big\|(ß,\varpi,\omega)\big\|_{\mathcal{R}}\Big\},$$

where $I = [e^{1/4}, e^{3/4}]$, $m = \min\{(\frac{1}{4})^{\sigma_1-1}, (\frac{1}{4})^{\sigma_2-1}, (\frac{1}{4})^{\sigma_3-1}\}$.

We define now the operators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : \mathcal{W} \to \mathcal{T}$ and $\mathcal{L} : \mathcal{W} \to \mathcal{R}$ by

$$\mathcal{L}(ß,\varpi,\omega) = \big(\mathcal{L}_1(ß,\varpi,\omega), \mathcal{L}_2(ß,\varpi,\omega), \mathcal{L}_3(ß,\varpi,\omega)\big), \tag{16}$$

with

$$\begin{cases} \mathcal{L}_1(ß,\varpi,\omega)(t) = \int_1^e \mathcal{G}_1(t,s)\phi_{q_1}(\int_1^s b_1(\log\frac{s}{\kappa})^{\rho_1-1} f_1(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa))\frac{d\kappa}{\kappa})\frac{ds}{s} \\ \qquad + \frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}, \quad t \in [1,e], (ß,\varpi,\omega) \in \mathcal{W}, \\ \mathcal{L}_2(ß,\varpi,\omega)(t) = \int_1^e \mathcal{G}_2(t,s)\phi_{q_2}(\int_1^s b_2(\log\frac{s}{\kappa})^{\rho_2-1} f_2(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa))\frac{d\kappa}{\kappa})\frac{ds}{s} \\ \qquad + \frac{\psi_2\Gamma(\sigma_2-\delta_2)(\log t)^{\sigma_2-1}}{\Lambda_2}, \quad t \in [1,e], (ß,\varpi,\omega) \in \mathcal{W}, \\ \mathcal{L}_3(ß,\varpi,\omega)(t) = \int_1^e \mathcal{G}_3(t,s)\phi_{q_3}(\int_1^s b_3(\log\frac{s}{\kappa})^{\rho_3-1} f_3(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa))\frac{d\kappa}{\kappa})\frac{ds}{s} \\ \qquad + \frac{\psi_3\Gamma(\sigma_3-\delta_3)(\log t)^{\sigma_3-1}}{\Lambda_3}, \quad t \in [1,e], (ß,\varpi,\omega) \in \mathcal{W}. \end{cases}$$

Lemma 3.4 *If* (H1)–(H2) *hold, then* $\mathcal{L} : \mathcal{W} \to \mathcal{W}$ *is a completely continuous operator.*

Proof Let $(ß,\varpi,\omega) \in \mathcal{W}$ be an arbitrary element. Clearly, $\mathcal{L}_1(ß,\varpi,\omega) \geq 0$, $\mathcal{L}_2(ß,\varpi,\omega) \geq 0$ and $\mathcal{L}_3(ß,\varpi,\omega) \geq 0$ for $t \in [1,e]$. Also, for $(ß,\varpi,\omega) \in \mathcal{W}$,

$$\begin{aligned} &\big\|\mathcal{L}_1(ß,\varpi,\omega)\big\| \\ &\quad\leq \int_1^e \mathcal{G}_1(e,s)\phi_{q_1}\left(\int_1^s b_1\left(\log\frac{s}{\kappa}\right)^{\rho_1-1} f_1\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\right)\frac{ds}{s} \\ &\qquad + \frac{\psi_1\Gamma(\sigma_1-\delta_1)}{\Lambda_1}, \\ &\big\|\mathcal{L}_2(ß,\varpi,\omega)\big\| \\ &\quad\leq \int_1^e \mathcal{G}_2(e,s)\phi_{q_2}\left(\int_1^s b_2\left(\log\frac{s}{\kappa}\right)^{\rho_2-1} f_2\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\right)\frac{ds}{s} \\ &\qquad + \frac{\psi_2\Gamma(\sigma_2-\delta_2)}{\Lambda_2}, \\ &\big\|\mathcal{L}_3(ß,\varpi,\omega)\big\| \\ &\quad\leq \int_1^e \mathcal{G}_3(e,s)\phi_{q_3}\left(\int_1^s b_3\left(\log\frac{s}{\kappa}\right)^{\rho_3-1} f_3\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\right)\frac{ds}{s} \\ &\qquad + \frac{\psi_3\Gamma(\sigma_3-\delta_3)}{\Lambda_3}, \end{aligned}$$

and

$$\begin{aligned}
&\min_{t\in I}\mathcal{L}_1(\text{ß},\varpi,\omega)(t)\\
&\quad=\min_{t\in I}\biggl[\int_1^e\mathcal{G}_1(t,s)\phi_{q_1}\biggl(\int_1^s b_1\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_1-1}f_1\bigl(\kappa,\text{ß}(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_1\Gamma(\sigma_1-\delta_1)(\log t)^{\sigma_1-1}}{\Lambda_1}\biggr]\\
&\quad\ge\biggl(\frac{1}{4}\biggr)^{\sigma_1-1}\biggl[\int_1^e\mathcal{G}_1(e,s)\phi_{q_1}\biggl(\int_1^s b_1\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_1-1}f_1\bigl(\kappa,\text{ß}(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_1\Gamma(\sigma_1-\delta_1)}{\Lambda_1}\biggr]\\
&\quad\ge m\biggl[\int_1^e\mathcal{G}_1(e,s)\phi_{q_1}\biggl(\int_1^s b_1\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_1-1}f_1\bigl(\kappa,\text{ß}(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_1\Gamma(\sigma_1-\delta_1)}{\Lambda_1}\biggr]\\
&\quad\ge m\bigl\|\mathcal{L}_1(\text{ß},\varpi,\omega)\bigr\|.
\end{aligned}$$

Similarly, $\min_{t\in I}\mathcal{L}_2(\text{ß},\varpi,\omega)(t)\ge m\|\mathcal{L}_2(\text{ß},\varpi,\omega)\|$ and $\min_{t\in I}\mathcal{L}_3(\text{ß},\varpi,\omega)(t)\ge m\|\mathcal{L}_3(\text{ß},\varpi,\omega)\|$. Therefore

$$\begin{aligned}
&\min_{t\in I}\bigl\{\mathcal{L}_1(\text{ß},\varpi,\omega)(t)+\mathcal{L}_2(\text{ß},\varpi,\omega)(t)+\mathcal{L}_3(\text{ß},\varpi,\omega)(t)\bigr\}\\
&\quad\ge m\bigl\|\mathcal{L}_1(\text{ß},\varpi,\omega)\bigr\|+m\bigl\|\mathcal{L}_2(\text{ß},\varpi,\omega)\bigr\|+m\bigl\|\mathcal{L}_3(\text{ß},\varpi,\omega)\bigr\|\\
&\quad=m\bigl\|\mathcal{L}_1(\text{ß},\varpi,\omega),\mathcal{L}_2(\text{ß},\varpi,\omega),\mathcal{L}_3(\text{ß},\varpi,\omega)\bigr\|\\
&\quad=m\bigl\|\mathcal{L}(\text{ß},\varpi,\omega)\bigr\|.
\end{aligned}$$

Hence, we get $\mathcal{L}(\mathcal{W})\subset\mathcal{W}$. By the Arzela–Ascoli theorem, we see that $\mathcal{L}$ is a completely continuous operator from $\mathcal{W}$ to $\mathcal{W}$. □

4 Main results

For computational convenience, we denote

$$\begin{aligned}
A&=\max\biggl\{\frac{(1/4)^{\rho_1(q_1-1)}}{(\Gamma(\rho_1+1))^{q_1-1}}\int_{s\in I}\mathcal{G}_1(e,s)\frac{ds}{s},\frac{(1/4)^{\rho_2(q_2-1)}}{(\Gamma(\rho_2+1))^{q_2-1}}\int_{s\in I}\mathcal{G}_2(e,s)\frac{ds}{s},\\
&\quad\frac{(1/4)^{\rho_3(q_3-1)}}{(\Gamma(\rho_3+1))^{q_3-1}}\int_{s\in I}\mathcal{G}_3(e,s)\frac{ds}{s}\biggr\},\\
B&=\min\biggl\{\frac{1}{(\Gamma(\rho_1+1))^{q_1-1}}\int_1^e\mathcal{G}_1(e,s)(\log s)^{\rho_1(q_1-1)}\frac{ds}{s},\\
&\quad\frac{1}{(\Gamma(\rho_2+1))^{q_2-1}}\int_1^e\mathcal{G}_2(e,s)(\log s)^{\rho_2(q_2-1)}\frac{ds}{s},\\
&\quad\frac{1}{(\Gamma(\rho_3+1))^{q_3-1}}\int_1^e\mathcal{G}_3(e,s)(\log s)^{\rho_3(q_3-1)}\frac{ds}{s}\biggr\}.
\end{aligned}$$

Let us define two continuous functionals η and ξ on the cone $\mathcal{W}$ by

$$\eta(ß,\varpi,\omega)=\min_{t\in I}\{|ß|+|\varpi|+|\omega|\}\quad\text{and}$$
$$\xi(ß,\varpi,\omega)=\max_{t\in[1,e]}\{|ß|+|\varpi|+|\omega|\}=\|(ß,\varpi,\omega)\|_{\mathcal{R}}.$$

It is clear that $\eta(ß,\varpi,\omega)\leq\xi(ß,\varpi,\omega)$ for all $(ß,\varpi,\omega)\in\mathcal{W}$.

Theorem 4.1 *Assume that* (H_1)–(H_3) *hold, suppose that there exist positive real numbers* q *and* Q *with* $q<mQ$ *and* $0<\psi_i<\frac{q\Lambda_i}{\aleph_i\Gamma(\sigma_i-\delta_i)}\leq\frac{Q\Lambda_i}{\aleph_i\Gamma(\sigma_i-\delta_i)}$ *such that* f_i, $i=1,2,3$, *satisfy the following conditions*:

(F_1) $f_i(t,ß,\varpi,\omega)\geq\phi_{p_i}(\frac{q}{3mA})$ *for all* $t\in I$, $(ß,\varpi,\omega)\in[q,Q]$,

(F_2) $f_i(t,ß,\varpi,\omega)\leq\phi_{p_i}(\frac{Q}{\mho_i B})$ *for all* $t\in[1,e]$, $(ß,\varpi,\omega)\in[1,Q]$.

Then the system of Hadamard fractional order boundary value problem (1)–(2) *has at least one positive solution and nondecreasing solution* $(ß^\star,\varpi^\star,\omega^\star)$ *satisfying* $q\leq\eta(ß^\star,\varpi^\star,\omega^\star)$ *with* $\xi(ß^\star,\varpi^\star,\omega^\star)\leq Q$.

Proof Let $\Omega_1=\{(ß,\varpi,\omega):\eta(ß,\varpi,\omega)<q\}$ and $\Omega_2=\{(ß,\varpi,\omega):\xi(ß,\varpi,\omega)<Q\}$. It is easy to see that $0\in\Omega_1$ and Ω_1, Ω_2 are bounded open subsets of $\mathcal{T}$. Let $(ß,\varpi,\omega)\in\Omega_1$, we have

$$q>\eta(ß,\varpi,\omega)=\min_{t\in I}\{ß(t)+\varpi(t)+\omega(t)\}\geq m\{\|ß\|+\|\varpi\|+\|\omega\|\}=m\xi(ß,\varpi,\omega).$$

Thus $Q>\frac{q}{m}>\xi(ß,\varpi,\omega)$ i.e. $(ß,\varpi,\omega)\in\Omega_2$, so $\Omega_1\subseteq\Omega_2$.

Claim 1: If $(ß,\varpi,\omega)\in\mathcal{W}\cap\partial\Omega_1$, then $\eta(\mathcal{L}(ß,\varpi,\omega))\geq\eta(ß,\varpi,\omega)$. To see this, let $(ß,\varpi,\omega)\in\mathcal{W}\cap\partial\Omega_1$, then $Q=\xi(ß,\varpi,\omega)\geq(ß(s)+\varpi(s)+\omega(s))\geq\eta(ß,\varpi,\omega)=q$ for $s\in I$. It follows from (F_1) and Lemma 3.3 that

$$\begin{aligned}
&\eta\big(\mathcal{L}(ß,\varpi,\omega)(t)\big)\\
&=\min_{t\in I}\sum_{i=1}^{3}\Bigg[\int_1^e\mathcal{G}_i(t,s)\phi_{q_i}\Bigg(\int_1^s b_i\Big(\log\frac{s}{\kappa}\Big)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\Bigg]\\
&\geq\sum_{i=1}^{3}\Bigg[\int_{s\in I}m\mathcal{G}_i(e,s)\phi_{q_i}\Bigg(\int_1^s b_i\Big(\log\frac{s}{\kappa}\Big)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\Bigg]\\
&\geq\frac{1}{3}\frac{q}{mA}\int_{s\in I}m\mathcal{G}_1(e,s)\phi_{q_1}\bigg(\frac{(\log s)^{\rho_1}}{\Gamma(\rho_1+1)}\bigg)\frac{ds}{s}\\
&\quad+\frac{1}{3}\frac{q}{mA}\int_{s\in I}m\mathcal{G}_2(e,s)\phi_{q_2}\bigg(\frac{(\log s)^{\rho_2}}{\Gamma(\rho_2+1)}\bigg)\frac{ds}{s}\\
&\quad+\frac{1}{3}\frac{q}{mA}\int_{s\in I}m\mathcal{G}_3(e,s)\phi_{q_3}\bigg(\frac{(\log s)^{\rho_3}}{\Gamma(\rho_3+1)}\bigg)\frac{ds}{s}\\
&\geq\frac{1}{3}\frac{q}{A}\frac{(1/4)^{\rho_1(q_1-1)}}{(\Gamma(\rho_1+1))^{q_1-1}}\int_{s\in I}\mathcal{G}_1(e,s)\frac{ds}{s}+\frac{1}{3}\frac{q}{A}\frac{(1/4)^{\rho_2(q_2-1)}}{(\Gamma(\rho_2+1))^{q_2-1}}\int_{s\in I}\mathcal{G}_2(e,s)\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{3}\frac{q}{A}\frac{(1/4)^{\rho_3(q_3-1)}}{(\Gamma(\rho_3+1))^{q_3-1}}\int_{s\in I}\mathcal{G}_3(e,s)\frac{ds}{s} \\
&= \frac{q}{3}+\frac{q}{3}+\frac{q}{3} = q = \eta(ß,\varpi,\omega).
\end{aligned}
$$

Claim 2: If $(ß,\varpi,\omega)\in\mathcal{W}\cap\partial\Omega_2$, then $\xi(\mathcal{L}(ß,\varpi,\omega))\le\xi(ß,\varpi,\omega)$. To see this, let $(ß,\varpi,\omega)\in\mathcal{W}\cap\partial\Omega_2$, then $(ß(s)+\varpi(s)+\omega(s))\ge\xi(ß,\varpi,\omega)=Q$ for $s\in[1,e]$. It follows from (F_2) and Lemma 3.3 that

$$
\begin{aligned}
&\xi\bigl(\mathcal{L}(ß,\varpi,\omega)(t)\bigr) \\
&= \max_{t\in[1,e]}\sum_{i=1}^{3}\biggl[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1} f_i\bigl(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} \\
&\quad + \frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr] \\
&\le \sum_{i=1}^{3}\biggl[\int_1^e \mathcal{G}_i(e,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1} f_i\bigl(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} \\
&\quad + \frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\biggr] \\
&\le \frac{1}{\mho_1}\frac{Q}{B}\int_1^e \mathcal{G}_1(e,s)\phi_{q_1}\biggl(\frac{(\log s)^{\rho_1}}{\Gamma(\rho_1+1)}\biggr)\frac{ds}{s} + \frac{1}{\mho_2}\frac{Q}{B}\int_1^e \mathcal{G}_2(e,s)\phi_{q_2}\biggl(\frac{(\log s)^{\rho_2}}{\Gamma(\rho_2+1)}\biggr)\frac{ds}{s} \\
&\quad + \frac{1}{\mho_3}\frac{Q}{B}\int_1^e \mathcal{G}_3(e,s)\phi_{q_3}\biggl(\frac{(\log s)^{\rho_3}}{\Gamma(\rho_3+1)}\biggr)\frac{ds}{s} + \frac{Q}{\aleph_1}+\frac{Q}{\aleph_2}+\frac{Q}{\aleph_3} \\
&\le \frac{1}{\mho_1}\frac{Q}{B}\frac{1}{(\Gamma(\rho_1+1))^{q_1-1}}\int_1^e \mathcal{G}_1(e,s)(\log s)^{\rho_1(q_1-1)}\frac{ds}{s} \\
&\quad + \frac{1}{\mho_2}\frac{Q}{B}\frac{1}{(\Gamma(\rho_2+1))^{q_2-1}}\int_1^e \mathcal{G}_2(e,s)(\log s)^{\rho_2(q_2-1)}\frac{ds}{s} \\
&\quad + \frac{1}{\mho_3}\frac{Q}{B}\frac{1}{(\Gamma(\rho_3+1))^{q_3-1}}\int_1^e \mathcal{G}_3(e,s)(\log s)^{\rho_3(q_3-1)}\frac{ds}{s} + \frac{Q}{\aleph_1}+\frac{Q}{\aleph_2}+\frac{Q}{\aleph_3} \\
&= Q\biggl[\frac{1}{\mho_1}+\frac{1}{\mho_2}+\frac{1}{\mho_3}+\frac{1}{\aleph_1}+\frac{1}{\aleph_2}+\frac{1}{\aleph_3}\biggr] \le Q = \xi(ß,\varpi,\omega).
\end{aligned}
$$

Clearly, η satisfies Property 2.1(iii) and ξ satisfies Property 2.2(i). Therefore condition (i) of Theorem 2.1 is satisfied and hence $\mathcal{L}$ has at least one fixed point $(ß^\star,\varpi^\star,\omega^\star)\in\mathcal{W}\cap(\overline{\Omega_2}\backslash\Omega_1)$ i.e. the system of Hadamard fractional order boundary value problems (1)–(2) has at least one positive solution and nondecreasing solution $(ß^\star,\varpi^\star,\omega^\star)$ satisfying $q\le\eta(ß^\star,\varpi^\star,\omega^\star)$ with $\xi(ß^\star,\varpi^\star,\omega^\star)\le Q$. □

Theorem 4.2 *Assume that (H_1)–(H_3) hold, suppose that there exist positive real numbers q and Q with $q<Q$ and $0<\psi_i<\frac{q\Lambda_i}{\aleph_i\Gamma(\sigma_i-\delta_i)}\le\frac{Q\Lambda_i}{\aleph_i\Gamma(\sigma_i-\delta_i)}$ such that f_i, $i=1,2,3$, satisfy the following conditions*:

(F_3) $f_i(t,ß,\varpi,\omega)\le\phi_{p_i}(\frac{q}{\mho_i B})$ *for all* $t\in[1,e]$, $(ß,\varpi,\omega)\in[1,q]$,

(F_4) $f_i(t,ß,\varpi,\omega)\ge\phi_{p_i}(\frac{Q}{3mA})$ *for all* $t\in I$, $(ß,\varpi,\omega)\in[q,\frac{Q}{m}]$.

Then the system of Hadamard fractional order boundary value problems (1)–(2) has at least one positive solution and nondecreasing solution $(ß^\star,\varpi^\star,\omega^\star)$ satisfying $q\le\xi(ß^\star,\varpi^\star,\omega^\star)$ with $\eta(ß^\star,\varpi^\star,\omega^\star)\le Q$.

Proof Let $\Omega_3 = \{(ß,\varpi,\omega) : \xi(ß,\varpi,\omega) < q\}$ and $\Omega_4 = \{(ß,\varpi,\omega) : \eta(ß,\varpi,\omega) < Q\}$. We have $0 \in \Omega_3$ and $\Omega_3 \subseteq \Omega_4$ with Ω_3 and Ω_4 are bounded open subsets of $\mathcal{T}$.

Claim 1: If $(ß,\varpi,\omega) \in \mathcal{W} \cap \partial\Omega_3$, then $\xi(\mathcal{L}(ß,\varpi,\omega)) \leq \xi(ß,\varpi,\omega)$. To see this, let $(ß,\varpi,\omega) \in \mathcal{W} \cap \partial\Omega_3$, then $(ß(s) + \varpi(s) + \omega(s)) \leq \xi(ß,\varpi,\omega) = q$ for $s \in [1,e]$. It follows from (F_3) and Lemma 3.3 that

$$\begin{aligned}
&\xi\bigl(\mathcal{L}(ß,\varpi,\omega)(t)\bigr) \\
&= \max_{t\in[1,e]} \sum_{i=1}^{3} \biggl[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1} f_i\bigl(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} \\
&\quad + \frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr] \\
&\leq \sum_{i=1}^{3} \biggl[\int_1^e \mathcal{G}_i(e,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1} f_i\bigl(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} \\
&\quad + \frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\biggr] \\
&\leq \frac{1}{\mho_1}\frac{q}{B}\frac{1}{(\Gamma(\rho_1+1))^{q_1-1}}\int_1^e \mathcal{G}_1(e,s)(\log s)^{\rho_1(q_1-1)}\frac{ds}{s} \\
&\quad + \frac{1}{\mho_2}\frac{q}{B}\frac{1}{(\Gamma(\rho_2+1))^{q_2-1}}\int_1^e \mathcal{G}_2(e,s)(\log s)^{\rho_2(q_2-1)}\frac{ds}{s} \\
&\quad + \frac{1}{\mho_3}\frac{q}{B}\frac{1}{(\Gamma(\rho_3+1))^{q_3-1}}\int_1^e \mathcal{G}_3(e,s)(\log s)^{\rho_3(q_3-1)}\frac{ds}{s} + \frac{q}{\aleph_1} + \frac{q}{\aleph_2} + \frac{q}{\aleph_3} \\
&= q\biggl[\frac{1}{\mho_1} + \frac{1}{\mho_2} + \frac{1}{\mho_3} + \frac{1}{\aleph_1} + \frac{1}{\aleph_2} + \frac{1}{\aleph_3}\biggr] \leq q = \xi(ß,\varpi,\omega).
\end{aligned}$$

Claim 2: If $(ß,\varpi,\omega) \in \mathcal{W} \cap \partial\Omega_4$, then $\eta(\mathcal{L}(ß,\varpi,\omega)) \geq \eta(ß,\varpi,\omega)$. To see this, let $(ß,\varpi,\omega) \in \mathcal{W} \cap \partial\Omega_4$, then $\frac{Q}{m} = \frac{\eta(ß,\varpi,\omega)}{m} \geq \xi(ß,\varpi,\omega) \geq (ß(s) + \varpi(s) + \omega(s)) \geq \eta(ß,\varpi,\omega) = Q$ for $s \in I$. It follows from (F_4) and Lemma 3.3 that

$$\begin{aligned}
&\eta\bigl(\mathcal{L}(ß,\varpi,\omega)(t)\bigr) \\
&= \min_{t\in I} \sum_{i=1}^{3} \biggl[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1} f_i\bigl(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} \\
&\quad + \frac{\psi_i\Gamma(\sigma_i-\delta_i)(\ln t)^{\sigma_i-1}}{\Lambda_i}\biggr] \\
&\geq \sum_{i=1}^{3} \biggl[\int_{s\in I} m\mathcal{G}_i(e,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1} f_i\bigl(\kappa, ß(\kappa), \varpi(\kappa), \omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s} \\
&\quad + \frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\biggr] \\
&\geq \frac{1}{3}\frac{Q}{mA}\int_{s\in I} m\mathcal{G}_1(e,s)\phi_{q_1}\biggl(\frac{(\log s)^{\rho_1}}{\Gamma(\rho_1+1)}\biggr)\frac{ds}{s} + \frac{1}{3}\frac{Q}{mA}\int_{s\in I} m\mathcal{G}_2(e,s)\phi_{q_2}\biggl(\frac{(\log s)^{\rho_2}}{\Gamma(\rho_2+1)}\biggr)\frac{ds}{s} \\
&\quad + \frac{1}{3}\frac{Q}{mA}\int_{s\in I} m\mathcal{G}_3(e,s)\phi_{q_3}\biggl(\frac{(\log s)^{\rho_3}}{\Gamma(\rho_3+1)}\biggr)\frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{3}\frac{Q}{A}\frac{(1/4)^{\rho_1(q_1-1)}}{(\Gamma(\rho_1+1))^{q_1-1}}\int_{s\in I}\mathcal{G}_1(e,s)\frac{ds}{s}+\frac{1}{3}\frac{Q}{A}\frac{(1/4)^{\rho_2(q_2-1)}}{(\Gamma(\rho_2+1))^{q_2-1}}\int_{s\in I}\mathcal{G}_2(e,s)\frac{ds}{s}\\
&\quad+\frac{1}{3}\frac{Q}{A}\frac{(1/4)^{\rho_3(q_3-1)}}{(\Gamma(\rho_3+1))^{q_3-1}}\int_{s\in I}\mathcal{G}_3(e,s)\frac{ds}{s}\\
&=\frac{Q}{3}+\frac{Q}{3}+\frac{Q}{3}=Q=\eta(ß,\varpi,\omega).
\end{aligned}$$

Clearly, η satisfies Property 2.1(iii) and ξ satisfies Property 2.2(i). Therefore condition (ii) of Theorem 2.1 is satisfied, and hence $\mathcal{L}$ has at least one fixed point $(ß^\star,\varpi^\star,\omega^\star)\in\mathcal{W}\cap(\overline{\Omega_4}\backslash\Omega_3)$ i.e. the system of Hadamard fractional order boundary value problems (1)–(2) has at least one positive solution and nondecreasing solution $(ß^\star,\varpi^\star,\omega^\star)$ satisfying $q\leq\xi(ß^\star,\varpi^\star,\omega^\star)$ with $\eta(ß^\star,\varpi^\star,\omega^\star)\leq Q$. □

Theorem 4.3 *Assume that (H_1)–(H_3) hold, suppose that there exist nonnegative numbers a, b, and c such that $0<a<b<\frac{b}{m}\leq c$ and $0<\psi_i<\frac{a\Lambda_i}{\aleph_i\Gamma(\sigma_i-\delta_i)}\leq\frac{c\Lambda_i}{\aleph_i\Gamma(\sigma_i-\delta_i)}$ such that f_i, $i=1,2,3$, satisfy the following conditions*:

(F_5) *$f_i(t,ß,\varpi,\omega)<\phi_{p_i}(\frac{a}{\mho_i B})$ for all $t\in[1,e]$ and $(ß,\varpi,\omega)\in[ma,a]$,*
(F_6) *$f_i(t,ß,\varpi,\omega)>\phi_{p_i}(\frac{b}{3mA})$ for all $t\in I$ and $(ß,\varpi,\omega)\in[b,\frac{b}{m}]$,*
(F_7) *$f_i(t,ß,\varpi,\omega)<\phi_{p_i}(\frac{c}{\mho_i B})$ for all $t\in[1,e]$ and $(ß,\varpi,\omega)\in[0,c]$.*

Then the Hadamard fractional order BVP (1)–(2) has at least three positive solutions $(ß_1,\varpi_1,\omega_1)$, $(ß_2,\varpi_2,\omega_2)$, and $(ß_3,\varpi_3,\omega_3)$ such that $\varrho(ß_1,\varpi_1,\omega_1)<a$, $b<\alpha(ß_2,\varpi_2,\omega_2)$ and $a<\varrho(ß_3,\varpi_3,\omega_3)$ with $\alpha(ß_3,\varpi_3,\omega_3)<b$.

Proof Define the nonnegative continuous concave functionals α, β and the nonnegative continuous convex functionals $\wp$, ϱ, ς on $\mathcal{W}$:

$$\begin{aligned}
&\alpha(ß,\varpi,\omega)=\min_{t\in I}\{|ß|+|\varpi|+|\omega|\};\qquad \beta(ß,\varpi,\omega)=\min_{t\in I_1}\{|ß|+|\varpi|+|\omega|\};\\
&\wp(ß,\varpi,\omega)=\max_{t\in[1,e]}\{|ß|+|\varpi|+|\omega|\};\qquad \varrho(ß,\varpi,\omega)=\max_{t\in I_1}\{|ß|+|\varpi|+|\omega|\};\\
&\varsigma(ß,\varpi,\omega)=\max_{t\in I}\{|ß|+|\varpi|+|\omega|\};\quad \text{where } I_1=\left[e^{1/3},e^{2/3}\right].
\end{aligned}$$

For any $(ß,\varpi,\omega)\in\mathcal{W}$, we have

$$\begin{aligned}
&\alpha(ß,\varpi,\omega)=\min_{t\in I}\{|ß|+|\varpi|+|\omega|\}\leq\max_{t\in I_1}\{|ß|+|\varpi|+|\omega|\}=\varrho(ß,\varpi,\omega),\\
&\|(ß,\varpi,\omega)\|_{\mathcal{R}}\leq\frac{1}{m}\min_{t\in I}\{|ß|+|\varpi|+|\omega|\}\leq\frac{1}{m}\max_{t\in[1,e]}\{|ß|+|\varpi|+|\omega|\}=\frac{1}{m}\wp(ß,\varpi,\omega).
\end{aligned}$$

Thus, for each $(ß,\varpi,\omega)\in\mathcal{W}$, $\alpha(ß,\varpi,\omega)\leq\varrho(ß,\varpi,\omega)$ and $\|(ß,\varpi,\omega)\|_{\mathcal{R}}\leq\frac{1}{m}\wp(ß,\varpi,\omega)$. We show that $\mathcal{L}:\overline{\mathcal{W}(\wp,c)}\to\overline{\mathcal{W}(\wp,c)}$. Let $(ß,\varpi,\omega)\in\overline{\mathcal{W}(\wp,c)}$, then $0\leq|ß|+|\varpi|+|\omega|\leq c$. From condition (F_7) we obtain

$$\begin{aligned}
&\wp\big(\mathcal{L}(ß,\varpi,\omega)(t)\big)\\
&=\max_{t\in[1,e]}\sum_{i=1}^{3}\left[\int_1^e\mathcal{G}_i(t,s)\phi_{q_i}\left(\int_1^s b_i\left(\log\frac{s}{\kappa}\right)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\right)\frac{ds}{s}\right.\\
&\quad\left.+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{3}\left[\int_{1}^{e} \mathcal{G}_i(e,s)\phi_{q_i}\left(\int_{1}^{s} b_i\left(\log\frac{s}{\kappa}\right)^{\rho_i-1} f_i\bigl(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\right)\frac{ds}{s}\right.\\
&\quad\left.+\frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\right]\\
&< \frac{c}{\mho_1 B}\frac{1}{(\Gamma(\rho_1+1))^{q_1-1}}\int_{1}^{e}\mathcal{G}_1(e,s)(\log s)^{\rho_1(q_1-1)}\frac{ds}{s}\\
&\quad+\frac{c}{\mho_2 B}\frac{1}{(\Gamma(\rho_2+1))^{q_2-1}}\int_{1}^{e}\mathcal{G}_2(e,s)(\log s)^{\rho_2(q_2-1)}\frac{ds}{s}\\
&\quad+\frac{c}{\mho_3 B}\frac{1}{(\Gamma(\rho_3+1))^{q_3-1}}\int_{1}^{e}\mathcal{G}_3(e,s)(\log s)^{\rho_3(q_3-1)}\frac{ds}{s}+\frac{c}{\aleph_1}+\frac{c}{\aleph_2}+\frac{c}{\aleph_3}\\
&= c\left[\frac{1}{\mho_1}+\frac{1}{\mho_2}+\frac{1}{\mho_3}+\frac{1}{\aleph_1}+\frac{1}{\aleph_2}+\frac{1}{\aleph_3}\right]\leq c.
\end{aligned}
$$

Therefore $\mathcal{L}:\overline{\mathcal{W}(\wp,c)}\to\overline{\mathcal{W}(\wp,c)}$. Now conditions (F_5) and (F_6) of Theorem 2.2 are to be verified. It is obvious that

$$
\begin{aligned}
&\frac{mb+b}{3m}\in\left\{(ß,\varpi,\omega)\in\mathcal{W}\left(\wp,\varsigma,\alpha,b,\frac{b}{m},c\right);\alpha(ß,\varpi,\omega)>b\right\}\neq\emptyset\quad\text{and}\\
&\frac{ma+a}{3}\in\bigl\{(ß,\varpi,\omega)\in Q(\wp,\varrho,\alpha,\beta,ma,a,c);\varrho(ß,\varpi,\omega)<a\bigr\}\neq\emptyset.
\end{aligned}
$$

Next, let $(ß,\varpi,\omega)\in\mathcal{W}(\wp,\varsigma,\alpha,b,\frac{b}{m},c)$ (or) $(ß,\varpi,\omega)\in Q(\wp,\varrho,\alpha,\beta,ma,a,c)$. Then $b\leq|ß(t)|+|\varpi(t)|+|\omega(t)|\leq\frac{b}{m}$ and $ma\leq|ß(t)|+|\varpi(t)|+|\omega(t)|\leq a$. Now, we apply condition (F_6) to get

$$
\begin{aligned}
&\alpha\bigl(\mathcal{L}(ß,\varpi,\omega)(t)\bigr)\\
&\quad=\min_{t\in I}\sum_{i=1}^{3}\left[\int_{1}^{e}\mathcal{G}_i(t,s)\phi_{q_i}\left(\int_{1}^{s}b_i\left(\log\frac{s}{\kappa}\right)^{\rho_i-1}f_i\bigl(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\right)\frac{ds}{s}\right.\\
&\quad\quad\left.+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\right]\\
&\quad\geq\sum_{i=1}^{3}\left[\int_{s\in I}m\mathcal{G}_i(e,s)\phi_{q_i}\left(\int_{1}^{s}b_i\left(\log\frac{s}{\kappa}\right)^{\rho_i-1}f_i\bigl(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\bigr)\frac{d\kappa}{\kappa}\right)\frac{ds}{s}\right.\\
&\quad\quad\left.+\frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\right]\\
&\quad>\frac{1}{3}\frac{b}{mA}\int_{s\in I}m\mathcal{G}_1(e,s)\phi_{q_1}\left(\frac{(\log s)^{\rho_1}}{\Gamma(\rho_1+1)}\right)\frac{ds}{s}+\frac{1}{3}\frac{b}{mA}\int_{s\in I}m\mathcal{G}_2(e,s)\phi_{q_2}\left(\frac{(\log s)^{\rho_2}}{\Gamma(\rho_2+1)}\right)\frac{ds}{s}\\
&\quad\quad+\frac{1}{3}\frac{b}{mA}\int_{s\in I}m\mathcal{G}_3(e,s)\phi_{q_3}\left(\frac{(\log s)^{\rho_3}}{\Gamma(\rho_3+1)}\right)\frac{ds}{s}\\
&\quad=\frac{1}{3}\frac{b}{A}\frac{(1/4)^{\rho_1(q_1-1)}}{(\Gamma(\rho_1+1))^{q_1-1}}\int_{s\in I}\mathcal{G}_1(e,s)\frac{ds}{s}+\frac{1}{3}\frac{b}{A}\frac{(1/4)^{\rho_2(q_2-1)}}{(\Gamma(\rho_2+1))^{q_2-1}}\int_{s\in I}\mathcal{G}_2(e,s)\frac{ds}{s}\\
&\quad\quad+\frac{1}{3}\frac{b}{A}\frac{(1/4)^{\rho_3(q_3-1)}}{(\Gamma(\rho_3+1))^{q_3-1}}\int_{s\in I}\mathcal{G}_3(e,s)\frac{ds}{s}\\
&\quad=\frac{b}{3}+\frac{b}{3}+\frac{b}{3}=b.
\end{aligned}
$$

Clearly, by condition ($D1$), we have

$$\begin{aligned}
&\varrho\big(\mathcal{L}(ß,\varpi,\omega)(t)\big)\\
&= \max_{t\in[1,e]}\sum_{i=1}^{3}\Bigg[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\Bigg(\int_1^s b_i\bigg(\log\frac{s}{\kappa}\bigg)^{\rho_i-1} f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\Bigg]\\
&\le \sum_{i=1}^{3}\Bigg[\int_1^e \mathcal{G}_i(e,s)\phi_{q_i}\Bigg(\int_1^s b_i\bigg(\log\frac{s}{\kappa}\bigg)^{\rho_i-1} f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)}{\Lambda_i}\Bigg]\\
&< \frac{1}{\mho_1}\frac{a}{B}\frac{1}{(\Gamma(\rho_1+1))^{q_1-1}}\int_1^e \mathcal{G}_1(e,s)(\log s)^{\rho_1(q_1-1)}\frac{ds}{s}\\
&\quad+\frac{1}{\mho_2}\frac{a}{B}\frac{1}{(\Gamma(\rho_2+1))^{q_2-1}}\int_1^e \mathcal{G}_2(e,s)(\log s)^{\rho_2(q_2-1)}\frac{ds}{s}\\
&\quad+\frac{1}{\mho_3}\frac{a}{B}\frac{1}{(\Gamma(\rho_3+1))^{q_3-1}}\int_1^e \mathcal{G}_3(e,s)(\log s)^{\rho_3(q_3-1)}\frac{ds}{s}+\frac{a}{\aleph_1}+\frac{a}{\aleph_2}+\frac{a}{\aleph_3}\\
&= a\bigg[\frac{1}{\mho_1}+\frac{1}{\mho_2}+\frac{1}{\mho_3}+\frac{1}{\aleph_1}+\frac{1}{\aleph_2}+\frac{1}{\aleph_3}\bigg]\le a.
\end{aligned}$$

To see that ($D2$) is satisfied, let $(ß,\varpi,\omega)\in\mathcal{W}(\wp,\alpha,b,c)$ with $\varsigma(\mathcal{L}(ß,\varpi,\omega)(t))>\frac{b}{m}$, we have

$$\begin{aligned}
&\alpha\big(\mathcal{L}(ß,\varpi,\omega)(t)\big)\\
&= \min_{t\in I}\sum_{i=1}^{3}\Bigg[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\Bigg(\int_1^s b_i\bigg(\log\frac{s}{\kappa}\bigg)^{\rho_i-1} f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\Bigg]\\
&\ge \sum_{i=1}^{3}\Bigg[\int_1^e m\mathcal{G}_i(e,s)\phi_{q_i}\Bigg(\int_1^s b_i\bigg(\log\frac{s}{\kappa}\bigg)^{\rho_i-1} f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\Bigg]\\
&\ge m\max_{t\in[1,e]}\sum_{i=1}^{3}\Bigg[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\Bigg(\int_1^s b_i\bigg(\log\frac{s}{\kappa}\bigg)^{\rho_i-1} f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\Bigg]\\
&\ge m\max_{t\in I}\sum_{i=1}^{3}\Bigg[\int_1^e \mathcal{G}_i(t,s)\phi_{q_i}\Bigg(\int_1^s b_i\bigg(\log\frac{s}{\kappa}\bigg)^{\rho_i-1} f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\Bigg)\frac{ds}{s}\\
&\quad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\Bigg]\\
&= m\varsigma\big(\mathcal{L}(ß,\varpi,\omega)\big)(t)>b.
\end{aligned}$$

Finally, it is shown that (*D*4) holds. Let $(ß,\varpi,\omega)\in\mathcal{L}(\wp,\varrho,a,c)$ with $\beta(\mathcal{L}(ß,\varpi,\omega))<ma$. Then we have

$$\begin{aligned}
&\varrho\big(\mathcal{L}(ß,\varpi,\omega)(t)\big)\\
&\quad=\max_{t\in I_1}\sum_{i=1}^{3}\biggl[\int_1^e\mathcal{G}_i(t,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr]\\
&\quad\le\max_{t\in[1,e]}\sum_{i=1}^{3}\biggl[\int_1^e\mathcal{G}_i(e,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr]\\
&\quad=\frac{1}{m}\sum_{i=1}^{3}\biggl[m\int_1^e\mathcal{G}_i(e,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr]\\
&\quad\le\frac{1}{m}\min_{t\in I}\sum_{i=1}^{3}\biggl[\int_1^e\mathcal{G}_i(t,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr]\\
&\quad\le\frac{1}{m}\min_{t\in I_1}\sum_{i=1}^{3}\biggl[\int_1^e\mathcal{G}_i(t,s)\phi_{q_i}\biggl(\int_1^s b_i\biggl(\log\frac{s}{\kappa}\biggr)^{\rho_i-1}f_i\big(\kappa,ß(\kappa),\varpi(\kappa),\omega(\kappa)\big)\frac{d\kappa}{\kappa}\biggr)\frac{ds}{s}\\
&\qquad+\frac{\psi_i\Gamma(\sigma_i-\delta_i)(\log t)^{\sigma_i-1}}{\Lambda_i}\biggr]\\
&\quad=\frac{1}{m}\beta\big(\mathcal{L}(ß,\varpi,\omega)\big)(t)<a.
\end{aligned}$$

It has been proved that all the conditions of Theorem 2.2 are fulfilled. Therefore, the Hadamard fractional order BVP (1)–(2) has at least three positive solutions $(ß_1,\varpi_1,\omega_1)$, $(ß_2,\varpi_2,\omega_2)$, and $(ß_3,\varpi_3,\omega_3)$ such that $\varrho(ß_1,\varpi_1,\omega_1)<a$, $b<\alpha(ß_2,\varpi_2,\omega_2)$ and $a<\varrho(ß_3,\varpi_3,\omega_3)$ with $\alpha(ß_3,\varpi_3,\omega_3)<b$. □

5 Examples

Let $\sigma_1=\frac{5}{2}$, $\sigma_2=\frac{7}{3}$, $\sigma_3=\frac{9}{4}$, $\rho_1=\frac{1}{2}$, $\rho_2=\frac{1}{3}$, $\rho_3=\frac{1}{4}$, $\delta_1=\delta_2=\delta_3=\frac{3}{2}$, $\mu_1=\mu_2=\mu_3=8$, and $\lambda_1=\lambda_2=\lambda_3=3$. Let $p_1=p_2=p_3=2$, $q_1=q_2=q_3=2$, $\phi_{p_i}(s)=s$, $\phi_{q_i}(s)=s$, $i=1,2,3$.

Consider the following system of Hadamard fractional differential equations:

$$\begin{cases}-{}^H\mathcal{D}_{1^+}^{1/2}(\phi_{p_1}({}^H\mathcal{D}_{1^+}^{5/2}ß(t)))=f_1(t,ß(t),\varpi(t),\omega(t)), & 1<t<e,\\ -{}^H\mathcal{D}_{1^+}^{1/3}(\phi_{p_2}({}^H\mathcal{D}_{1^+}^{7/3}\varpi(t)))=f_2(t,ß(t),\varpi(t),\omega(t)), & 1<t<e,\\ -{}^H\mathcal{D}_{1^+}^{1/4}(\phi_{p_3}({}^H\mathcal{D}_{1^+}^{9/4}\omega(t)))=f_3(t,ß(t),\varpi(t),\omega(t)), & 1<t<e,\end{cases}\tag{17}$$

with the boundary conditions

$$\begin{cases} ß(1) = ß'(1) = 0, & {}^{H}\mathcal{D}_{1^+}^{5/2} ß(1) = 0, \quad 3ß(e) + 8^{H}\mathcal{D}_{1^+}^{3/2} ß(e) = \psi_1, \\ \varpi(1) = \varpi'(1) = 0, & {}^{H}\mathcal{D}_{1^+}^{7/3} \varpi(1) = 0, \quad 3\varpi(e) + 8^{H}\mathcal{D}_{1^+}^{3/2} \varpi(e) = \psi_2, \\ \omega(1) = \omega'(1) = 0, & {}^{H}\mathcal{D}_{1^+}^{9/4} \omega(1) = 0, \quad 3\omega(e) + 8^{H}\mathcal{D}_{1^+}^{3/2} \omega(e) = \psi_3, \end{cases} \tag{18}$$

where ψ_1, ψ_2, ψ_3 are parameters. We have $m = 0.125$, $\Lambda_1 = 13.63472 > 0$, $\Lambda_2 = 12.90201 > 0$, $\Lambda_3 = 12.74027 > 0$, so assumption (H2) is satisfied. Besides we deduce

$$\mathcal{G}_1(t,s) = \frac{1}{13.63472} \begin{cases} (8(1-\log s)^{-1.5} + 2.256758)(1-\log s)^{2.5-1}, & 1 \le t \le s \le e, \\ (8(1-\log s)^{-1.5} + 2.256758)(1-\log s)^{2.5-1} \\ \quad - \frac{1}{2.5}(1-\log s)^{2.5-1}, & 1 \le s \le t \le e, \end{cases}$$

$$\mathcal{G}_2(t,s) = \frac{1}{12.90201} \begin{cases} (8(1-\log s)^{-1.5} + 2.858519)(1-\log s)^{2.33-1}, & 1 \le t \le s \le e, \\ (8(1-\log s)^{-1.5} + 2.858519)(1-\log s)^{2.33-1} \\ \quad - \frac{1}{2.33}(1-\log s)^{2.33-1}, & 1 \le s \le t \le e, \end{cases}$$

$$\mathcal{G}_3(t,s) = \frac{1}{12.74027} \begin{cases} (8(1-\log s)^{-1.5} + 3.244696)(1-\log s)^{2.25-1}, & 1 \le t \le s \le e, \\ (8(1-\log s)^{-1.5} + 3.244696)(1-\log s)^{2.25-1} \\ \quad - \frac{1}{2.25}(1-\log s)^{2.25-1}, & 1 \le s \le t \le e. \end{cases}$$

Then we obtain

$$\begin{aligned} \int_{e^{1/4}}^{e^{3/4}} \mathcal{G}_1(e,s)\frac{ds}{s} &= \frac{1}{13.63472} \int_{e^{1/4}}^{e^{3/4}} \biggl(\bigl[8(1-\log s)^{-1.5} + 2.256758 \bigr] (1-\log s)^{1.5} \\ &\quad - \frac{1}{2.5}(1-\log s)^{1.5} \biggr) \frac{ds}{s} \\ &\approx 0.232374. \end{aligned}$$

Similarly, $\int_{e^{1/4}}^{e^{3/4}} \mathcal{G}_2(e,s)\frac{ds}{s} \approx 0.323496$ and $\int_{e^{1/4}}^{e^{3/4}} \mathcal{G}_3(e,s)\frac{ds}{s} \approx 0.338443$.

$$\begin{aligned} A &= \max\biggl\{ \frac{(1/4)^{\rho_1(q_1-1)}}{(\Gamma(\rho_1+1))^{q_1-1}} \int_{e^{1/4}}^{e^{3/4}} \mathcal{G}_1(e,s)\frac{ds}{s}, \frac{(1/4)^{\rho_2(q_2-1)}}{(\Gamma(\rho_2+1))^{q_2-1}} \int_{e^{1/4}}^{e^{3/4}} \mathcal{G}_2(e,s)\frac{ds}{s}, \\ &\quad \frac{(1/4)^{\rho_3(q_3-1)}}{(\Gamma(\rho_3+1))^{q_3-1}} \int_{e^{1/4}}^{e^{3/4}} \mathcal{G}_3(e,s)\frac{ds}{s} \biggr\} \\ &= \max\{0.131103, 0.228309, 0.264027\} \approx 0.264027, \\ B &= \min\biggl\{ \frac{1}{(\Gamma(\rho_1+1))^{q_1-1}} \int_1^e \mathcal{G}_1(e,s)(\log s)^{\rho_1(q_1-1)}\frac{ds}{s}, \\ &\quad \frac{1}{(\Gamma(\rho_2+1))^{q_2-1}} \int_1^e \mathcal{G}_2(e,s)(\log s)^{\rho_2(q_2-1)}\frac{ds}{s}, \\ &\quad \frac{1}{(\Gamma(\rho_3+1))^{q_3-1}} \int_1^e \mathcal{G}_3(e,s)(\log s)^{\rho_3(q_3-1)}\frac{ds}{s} \biggr\} \\ &= \min\{0.29447, 0.177359, 0.141563\} \approx 0.141563. \end{aligned}$$

Example 5.1 We consider the functions

$$f_1(t,ß,\varpi,\omega) = \frac{13(t-1)}{15} + \frac{11}{17}e^{-(ß+\varpi+\omega)} + 25,$$
$$f_2(t,ß,\varpi,\omega) = \frac{15}{17}(\log t) + e^{-(ß+\varpi+\omega)} + 21,$$
$$f_3(t,ß,\varpi,\omega) = 5(\log t) + \frac{15}{17}\log(ß+\varpi+\omega) + 22.$$

If we choose $q = 2$, $Q = 100$ and $\frac{1}{\mho_1} = \frac{1}{\mho_2} = \frac{1}{\mho_3} = \frac{1}{\aleph_1} = \frac{1}{\aleph_2} = \frac{1}{\aleph_3} = \frac{1}{8}$, then $q < mQ$ and f_i $(i = 1,2,3)$ fulfill the following conditions:

(F_1) $f_i(t,ß,\varpi,\omega) \geq 20.19993 = \phi_{p_i}(\frac{q}{3mA})$, for $t \in [e^{1/4}, e^{3/4}]$, $(ß,\varpi,\omega) \in [2,100]$,

(F_2) $f_i(t,ß,\varpi,\omega) \leq 88.30011 = \phi_{p_i}(\frac{Q}{\mho_i B})$for $t \in [1,e]$, $(ß,\varpi,\omega) \in [1,100]$.

Consequently, all presumptions in Theorem 4.1 are agreeable. Thus, for $\psi_1 \leq 170.434$, $\psi_2 \leq 161.275125$, $\psi_3 \leq 159.253375$, the system of (17)–(18) has at least one positive solution.

Example 5.2 We consider the functions

$$f_1(t,ß,\varpi,\omega) = \begin{cases} \frac{1}{9}t(ß+\varpi+\omega) + \frac{27}{29}(\log t), & 0 < ß+\varpi+\omega \leq 10, \\ (\log t)(ß+\varpi+\omega) + \frac{29}{7}(ß+\varpi+\omega) + 61, & 10 < ß+\varpi+\omega \leq 80, \\ (\log t) + \frac{7t}{19}(e^{-(ß+\varpi+\omega)} + 55) + \frac{13}{17}, & 80 < ß+\varpi+\omega \leq 90, \end{cases}$$

$$f_2(t,ß,\varpi,\omega) = \begin{cases} \frac{17}{9}t + \frac{1}{6}(ß+\varpi+\omega), & 0 < ß+\varpi+\omega \leq 10, \\ (\log t)(ß+\varpi+\omega) + \frac{17}{19}t + 3(ß+\varpi+\omega) + 69, & 10 < ß+\varpi+\omega \leq 80, \\ (\log t) + \frac{17}{19}(t+4) + 10e^{-(ß+\varpi+\omega)} + 73 + \frac{13}{17}, & 80 < ß+\varpi+\omega \leq 90, \end{cases}$$

$$f_3(t,ß,\varpi,\omega) = \begin{cases} \frac{2}{3}\log t + \frac{3}{5}(t+1) + e^{-(ß+\varpi+\omega)}, & 0 < ß+\varpi+\omega \leq 10, \\ (\log t)(ß+\varpi+\omega) + \frac{3}{4}(\log t) & \\ \quad + 7(ß+\varpi+\omega) + 24t, & 10 < ß+\varpi+\omega \leq 80, \\ (\log t) + \frac{1}{2}(\log t) + \frac{3}{2}\sin(ß+\varpi+\omega) + 28t, & 80 < ß+\varpi+\omega \leq 90. \end{cases}$$

Choosing $a = 5$, $b = 10$, $c = 90$, evidently, $0 < a < b < \frac{b}{m} \leq c$ and $\frac{1}{\mho_1} = \frac{1}{\mho_2} = \frac{1}{\mho_3} = \frac{1}{\aleph_1} = \frac{1}{\aleph_2} = \frac{1}{\aleph_3} = \frac{1}{8}$ and f_i $(i = 1,2,3)$ fulfill the following conditions:

(F_5) $f_i(t,ß,\varpi,\omega) < 4.415005 = \phi_{p_i}(\frac{a}{\mho_i B})$ for $t \in [1,e]$ and $|ß| + |\varpi| + |\omega| \in [0.625, 5]$,

(F_6) $f_i(t,ß,\varpi,\omega) > 100.9996 = \phi_{p_i}(\frac{b}{3mA})$ for $t \in [e^{1/4}, e^{3/4}]$ and $|ß| + |\varpi| + |\omega| \in [10, 80]$,

(F_7) $f_i(t,ß,\varpi,\omega) < 79.4701 = \phi_{p_i}(\frac{c}{\mho_i B})$ for $t \in [1,e]$ and $|ß| + |\varpi| + |\omega| \in [0, 90]$.

Thus, all the conditions of Theorem 4.3 are fulfilled. Hence, for $\psi_1 \leq 153.3906$, $\psi_2 \leq 145.14762$, $\psi_3 \leq 143.32803$, the system of (17)–(18) has at least three positive solutions.

6 Conclusion

In this study, we are pleased to investigate the multiplicity of positive solutions for the system of three Hadamard fractional two-point boundary value problems with parameters and (p_1,p_2,p_3)-Laplacian operators by using the cone expansion and compression of functional type and five functional fixed point theorems for cones in ordered Banach spaces respectively.

Acknowledgements
We are indebted to the most respected Professor K. Rajendra Prasad, and our heartfelt sincere thanks go to the referees for their valuable suggestions and comments.

Funding
Not applicable.

Abbreviations
Not applicable.

Availability of data and materials
Data sharing not applicable to this article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Authors' information
Abdullah Ali H. Ahmadini, Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia. email: aahmadini@jazanu.edu.sa

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 March 2021 Accepted: 16 September 2021 Published online: 02 October 2021

References

1. Ge, F.D., Chen, Y.Q., Kou, C.H., Podlubny, I.: On the regional controllability of the sub-diffusion process with Caputo fractional derivative. Fract. Calc. Appl. Anal. **19**, 1262–1281 (2016)
2. Petráš, I.: Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. Springer, Berlin (2011)
3. Garra, R., Giusti, A., Mainardi, F., Pagnini, G.: Fractional relaxation with time-varying coefficient. Fract. Calc. Appl. Anal. **17**(2), 424–439 (2014)
4. Wang, T., Wang, G., Yang, X.: On a Hadamard-type fractional turbulent flow model with deviating arguments in a porous medium. Nonlinear Anal., Model. Control **2017**(22), 765–784 (2017)
5. Wang, G., Ren, X., Zhang, L., Ahmad, B.: Explicit iteration and unique positive solution for a Caputo-Hadamard fractional turbulent flow model. IEEE Access **2019**(7), 109833–109839 (2019)
6. Saxena, R.K., Garra, R., Orsingher, E.: Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives. Integral Transforms Spec. Funct. **27**(1), 30–42 (2016)
7. Garra, R., Orsingher, E., Polito, F.: A note on Hadamard fractional differential equations with varying coefficients and their applications in probability. Mathematics **6**(1), 4 (2018)
8. Arafa, A.A.M., Rida, S.Z., Khalil, M.: Fractional modeling dynamics of HIV and $\mathrm{CD}4^{+}$ T-cells during primary infection. Nonlinear Biomed. Phys. **6**(1), 1–7 (2012)
9. Ma, Q., Wang, J., Wang, R., Ke, X.: Study on some qualitative properties for solutions of a certain two-dimensional fractional differential system with Hadamard derivative. Appl. Math. Lett. **36**, 7–13 (2014)
10. Jesus, I.S., Machado, J.A.T., Cunha, J.B.: Fractional electrical impedances in botanical elements. J. Vib. Control **14**, 1389–1402 (2008)
11. Henderson, J., Luca, R.: Boundary Value Problems for Systems of Differential, Difference and Fractional Equations: Positive Solutions. Elsevier, Amsterdam (2016)
12. Henderson, J., Luca, R., Tudorache, A.: Existence and nonexistence of positive solutions for coupled Riemann–Liouville fractional boundary value problems. Discrete Dyn. Nat. Soc. **2016**, Article ID 2823971 (2016)
13. Henderson, J., Luca, R.: Systems of Riemann Liouville fractional equations with multi-point boundary conditions. Appl. Math. Comput. **309**, 303–323 (2017)
14. Luca, R.: Positive solutions for a system of Riemann Liouville fractional differential equations with multi-point fractional boundary conditions. Bound. Value Probl. **2017**, 102 (2017)
15. Ahmad, B., Luca, R.: Existence of solutions for a system of fractional differential equations with coupled nonlocal boundary conditions. Fract. Calc. Appl. Anal. **21**(2), 423–441 (2018)
16. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. Chaos Solitons Fractals **83**, 234–241 (2016)
17. Rao, S.N.: Multiplicity of positive solutions for fractional differential equation with *p*-Laplacian boundary value problem. Int. J. Differ. Equ. **2016**, Article ID 6906049 (2016)
18. Rao, S.N., Zico, M.M.: Positive solutions for a coupled system of nonlinear semipositone fractional boundary value problems. Int. J. Differ. Equ. **2019**, Article ID 2893857 (2019)
19. Li, D., Liu, Y., Wang, C.: Multiple positive solutions for fractional three-point boundary value problem with *p*-Laplacian operator. Math. Probl. Eng. **2020**, Article ID 2327580 (2020)
20. Chai, G.: Positive solutions for boundary value problem of fractional differential equation with *p*-Laplacian operator. Bound. Value Probl. **2012**, 18 (2012)
21. Luca, R.: On a system of fractional boundary value problems with *p*-Laplacian operator. Dyn. Syst. Appl. **28**(3), 691–713 (2019)

22. Xu, J., O'Regan, D.: Positive solutions for a fractional p-Laplacian boundary value problem. Filomat **31**(6), 1549–1558 (2017)
23. Tian, Y., Bai, Z., Sun, S.: Positive solutions for a boundary value problem of fractional differential equation with p-Laplacian operator. Adv. Differ. Equ. **2019**, 349 (2019)
24. Dong, X., Bai, Z., Zhang, S.: Positive solutions to boundary value problems of p-Laplacian with fractional derivative. Bound. Value Probl. **2017**, 5 (2017)
25. Hao, X., Wang, H., Liu, L., Cui, Y.: Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator. Bound. Value Probl. **2017**, 182 (2017)
26. Rao, S.N.: Multiple positive solutions for coupled system of p-Laplacian fractional order three-point boundary value problems. Rocky Mt. J. Math. **49**(2), 609–626 (2019)
27. Luca, R.: Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions. Nonlinear Anal., Model. Control **23**(5), 771–801 (2018)
28. Wang, Y.: Multiple positive solutions for mixed fractional differential system with p-Laplacian operators. Bound. Value Probl. **2019**(1), 144 (2019)
29. Lv, Z.W., Liu, J., Xu, J.: Multiple positive solutions for a Caputo fractional p-Laplacian boundary value problems. Complexity **2020**, Article ID 6723791 (2020)
30. Kong, L., Piao, D., Wang, L.: Positive solutions for third boundary value problems with p-Laplacian. Results Math. **55**, 111–128 (2009)
31. Han, Z., Lu, H., Sun, S., Yang, D.: Positive solutions to boundary value problems of p-Laplacian fractional differential equations with a parameter in the boundary. Electron. J. Differ. Equ. **2012**, 213 (2012)
32. Zhang, H., Li, Y., Xu, J.: Positive solutions for a system of fractional integral boundary value problems involving Hadamard-type fractional derivatives. Complexity **2019**, Article ID 2671539 (2019)
33. Ding, Y., Jiang, J., O'Regan, D., Xu, J.: Positive solutions for a system of Hadamard-type fractional differential equations with semipositone nonlinearities. Complexity **2020**, Article ID 9742418 (2020)
34. Jiang, J., O'Regan, D., Xu, J., Fu, Z.: Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions. J. Inequal. Appl. **2019**, 204 (2019)
35. Zhang, K., Wang, J., Ma, W.: Solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations. J. Funct. Spaces **2018**, Article ID 2193234 (2018)
36. Rao, S.N., Msmali, A.H., Singh, M., Ahmadini, A.H.: Existence and uniqueness for a system of Caputo–Hadamard fractional differential equations with multi-point boundary conditions. J. Funct. Spaces **2020**, Article ID 8821471 (2020)
37. Jiang, J., O'Regan, D., Xu, J., Cui, Y.: Positive solutions for a Hadamard fractional p-Laplacian three-point boundary value problem. Mathematics **7**(5), 439 (2019)
38. Xu, J., Jiang, J., O'Regan, D.: Positive solutions for a class of p-Laplacian Hadamard fractional order three point boundary value problems. Mathematics **8**(3), 308 (2020)
39. Alesemi, M.: Solvability for a class of nonlinear Hadamard fractional differential equations with parameters. Bound. Value Probl. **2019**, 101 (2019)
40. Rao, S.N., Singh, M., Meetei, M.Z.: Multiplicity of positive solutions for Hadamard fractional differential equations with p-Laplacian operator. Bound. Value Probl. **43**(1), 1–25 (2020)
41. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
42. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
43. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
44. Avery, R.I., Henderson, J., O'Regan, D.: Functional compression expansion fixed point theorem. Electron. J. Differ. Equ. **2008**, 22 (2008)
45. Avery, R.I.: A generalization of the Leggett–Williams fixed point theorem. Math. Sci. Res. Hot-Line **3**, 9–14 (1999)