Kim *Advances in Difference Equations* (2021) 2021:426
https://doi.org/10.1186/s13662-021-03581-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Calculation of vectorial derivatives for functions of a quaternion variable and their properties

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Abstract

Various attempts have been made in defining the derivative of a quaternionic function due to the noncommutativity of the product over quaternions. We observe that the difference in the left and right operations caused by the noncommutativity of the quaternion product is determined by the vector part of the quaternion. In this paper, we propose a corresponding derivative to replace the derivative of a quaternion-valued function of a quaternionic variable using the component terms of a quaternion. Further, the analogous constant, product, and quotient rules for the proposed calculations are given. Application of the proposed derivatives is provided to compute the derivatives of elementary functions. Several illustrations are also presented.

MSC: Primary 32A99; secondary 32A10; 30G35; 26B05

Keywords: Quaternion; Derivatives; Differential operator; Elementary quaternionic functions

1 Introduction

The quaternion algebra was introduced by Hamilton [13] in 1843. Further, according to Frobenius' theorem in [14], every finite-dimensional associative division algebra over $\mathbb{R}$ (the real numbers) is isomorphic to $\mathbb{H}$ (the quaternions). Later, studies and results on the algebraic properties and functions of the quaternions were compiled in [7, 25]. There have been many attempts to extend the typical theory of derivatives of complex-valued functions to that for a function over the quaternion field. The theory of holomorphic (regular) complex-valued functions of a complex variable is established based on the limit definition of a derivative. Similarly, a study has been conducted to deal with the limit definition of the holomorphy (regularity) over the field of quaternions. Buff [6] attempted to define an analytic quaternion-valued function of a quaternion variable, by using the structure of a quaternion and the properties of the units. For a quaternionic function, there are two versions of the limit definition of a derivative due to the noncommutative multiplication of the quaternions. For a function f of a quaternion variable q of the form $aq + b$, where a

and b are quaternion constants, the limit

$$\lim_{h\to 0}\{f(q+h)-f(q)\}h^{-1}$$

exists; thus, the right-derivative of f exists. Similarly, for a function f of the form $qa+b$, where a and b are quaternion constants, the limit

$$\lim_{h\to 0}h^{-1}\{f(q+h)-f(q)\}$$

exists; thus, the left-derivative of f exists. However, [24] has showed that these limit definitions do not apply to functions other than those mentioned above, namely $aq+b$ and $qa+b$. Bantsuri [4] has proposed that the conditions of differentiability on the right are almost equivalent to the two-sided differentiability and the existence of a strong gradient. Gentili and Struppa [12] gave definitions of regularity for functions of a quaternionic variable and developed representations of the Cullen-regularity of quaternion analysis. Kim and Shon [17] have proposed ternary numbers, modified to have the product that can be closed in ternary numbers, and have defined the hyperholomophicity of a ternary function. Kim [15] has provided the hyperholomorphy and properties of functions of split-quaternionic variables which are expressed in hyperbolic coordinates, using Cullen's form. In addition to studying quaternion functions of the real variables, Kim [16] has examined the properties of functions for special quaternion variables such as dual quaternions and split quaternions.

Using the limit definition of the holomorphy for a complex function, Loomann [20] and Menchoff [22] proved that any complex-valued continuous function satisfies the Cauchy–Riemann condition in a complex domain. To extend the theory of holomorphic functions over the complex field to holomorphic functions over the quaternion field, a quaternionic analog of the Cauchy–Riemann equation was introduced by Fueter [10], in 1935. Fueter [11] proposed two quaternionic gradient operators as follows: for a quaternion q,

$$\frac{\partial^r}{\partial q}=\frac{\partial}{\partial x_0}+\frac{\partial}{\partial x_1}i+\frac{\partial}{\partial x_2}j+\frac{\partial}{\partial x_3}k$$

and

$$\frac{\partial^l}{\partial q}=\frac{\partial}{\partial x_0}+i\frac{\partial}{\partial x_1}+j\frac{\partial}{\partial x_2}+k\frac{\partial}{\partial x_3}.$$

These operators give rise to the definition of regularity of a quaternion function $f(q)$ as follows:

A function $f(q)$ is said to be right-(resp., left-)regular if f satisfies $\frac{\partial^r f}{\partial q}=0$ (resp., $\frac{\partial^l f}{\partial q}=0$). However, this definition of regularity does not apply to general polynomial functions and functions multiplied by regular functions. As such, functions satisfying the definition of holomorphy (regularity) are limited. However, if the condition of holomorphy (regularity) is not satisfied, a formula for derivatives that can be applied to the quaternions is required. For example, in optimization, the objective function of a quaternion variable is not holomorphic (regular), but there are cases where a derivative is needed. The derivative might be required to minimize, maximize, or approximate the values of quaternion functions.

Also, this is of interest in the study of derivatives and their properties, such as proposing the derivation and expansion of differential operators and integral formulas corresponding to special functions and functions of several variables (see [1–3]).

Since the vectorial derivatives of a quaternion function use both the inner and outer product of the vector used in the calculus, it is easy to understand and perform the calculation required for obtaining the derivative. The aim of employing the vectorial derivative is to interpret the quaternion derivative expression so as to overcome the noncommutative product of the quaternions and define the specific scalar functions in consideration of the characteristics of the basis of the quaternions causing the noncommutativity. Through the analysis and examples, the proposed derivatives expand the applicable range of derivatives in a general, complete, and intuitive way.

This paper proposes the corresponding derivative of a quaternion-valued function of a quaternionic variable using the scalar and vector part, denoted by S_p and V_p, respectively, of a quaternion p. Section 2 defines the composition of quaternions and examines the algebraic properties of quaternions expressed as scalar and vector parts. In addition, the function defined by the composition of the scalar and vector parts is presented, and the differential operator to be applied to these functions is defined. Section 3 defines the operation of a new derivative that will be called the vectorial derivative of a quaternionic function. We investigate how the rules found in the existing complex analysis (such as the analogous constant, product, and quotient rules given by the proposed derivative operation) are applied and extended to the vectorial derivative. In Sect. 4, we apply the proposed derivative to some elementary functions of a quaternion variable, and examine the properties of the considered derivative. Furthermore, it is confirmed through the figures that the remainder terms of each elementary function derived from the definition of the vectorial derivative are approximated to 0 except for some specific set. Finally, in Sect. 5, we present conclusions on this paper, and propose areas where we expect to utilize the differential operation proposed in this paper.

2 Preliminaries

The set of quaternions is an associative, but not commutative algebra. The set is denoted by $\mathbb{H}$ and defined as

$$\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3 | x_r \in \mathbb{R}\ (r = 0, 1, 2, 3)\},$$

where $1, i, j, k$ are the imaginary units satisfying

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik. \end{aligned}$$

Any quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is also written as

$$q = S_q + V_q, \tag{2.1}$$

known as the vectorial form of quaternions, where S_q is called the scalar part of q, defined as $S_q = x_0$, and V_q is called the vector part of q, defined as $V_q = ix_1 + jx_2 + kx_3$. A quaternion with $S_q = 0$ is called a pure quaternion. Given two quaternions $p = S_p + V_p$ and $q = S_q + V_q$, their product is given by

$$pq = S_pS_q + S_pV_q + V_pS_q + V_pV_q,$$

where $V_pV_q = -V_p \cdot V_q + V_p \times V_q$ and the symbols $\cdot$ and $\times$ denote the standard scalar product (or dot product) and vector product (or cross product), respectively. The quaternion product is not commutative due to the presence of the vector product V_p and V_q, i.e., $pq \neq qp$. Since it is convenient to perform the calculus of the noncommutative product of quaternions, we express quaternions in their vectorial form.

The conjugate of a quaternion $q = S_q + V_q$, denoted by q^*, is defined as $q^* = S_q - V_q$ and satisfies $(pq)^* = q^*p^*$. The modulus of a quaternion $q = S_q + V_q$ is defined as

$$|q| = \sqrt{qq^*} = \sqrt{S_q^2 + |V_q|^2}$$

and satisfies the property $|pq| = |p||q|$. If $|p| = 1$, then p is called a unit quaternion. The inverse of $q = S_q + V_q$ ($\neq 0$) is given as $q^{-1} = \frac{q^*}{|q|^2}$ and satisfies $(pq)^{-1} = q^{-1}p^{-1}$.

A pure quaternion q satisfies the following:

$$q^* = -q \quad \text{and} \quad q^2 = -|q|^2.$$

So, a pure unit quaternion satisfies the conditions $|q| = 1$ and $q^2 = -1$. For example, the imaginary units i, j, k in $\mathbb{H}$ are pure unit quaternions.

A function $f : \mathbb{H} \to \mathbb{H}$ is said to be a quaternion function of a quaternion variable if f is defined as $f(q) = f_0 + if_1 + jf_2 + kf_3$, where $f_r = f_r(x_0, x_1, x_2, x_3)$ $(r = 0, 1, 2, 3)$ are real-valued functions. Compactly, a quaternion function f is written as $f(q) = S_f + V_f$, where $S_f = f_0$ and $V_f = if_1 + jf_2 + kf_3$.

Definition 2.1 ([8]) A quaternion function $f(q)$ is said to be real-differentiable if each component function f_r $(r = 0, 1, 2, 3)$ of $f(q)$ is differentiable as a function of real variables x_0, x_1, x_2, and x_3.

Motivated by Cauchy–Riemann–Fueter equation, the vectorial differential operator D of a quaternion function is given as

$$D = \frac{1}{2}(S_D - V_D),$$

where

$$S_D = \frac{\partial}{\partial x_0} \quad \text{and} \quad V_D = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

If the operator is calculated for a quaternion function f, we obtain

$$\begin{aligned} Df &= \frac{1}{2}(S_D - V_D)(S_f + V_f) = \frac{1}{2}(S_DS_f + S_DV_f - V_DS_f - V_DV_f) \\ &= \frac{1}{2}(S_DS_f + S_DV_f - V_DS_f + V_D \cdot V_f - V_D \times V_f) \end{aligned}$$

and

$$fD = \frac{1}{2}(S_f + V_f)(S_D - V_D) = \frac{1}{2}(S_f S_D - S_f V_D + V_f S_D - V_f V_D)$$
$$= \frac{1}{2}(S_D S_f + S_D V_f - V_D S_f + V_D \cdot V_f + V_D \times V_f),$$

where

$$S_D S_f = S_f S_D = \frac{\partial f_0}{\partial x_0}, \qquad S_D V_f = V_f S_D = i\frac{\partial f_1}{\partial x_0} + j\frac{\partial f_2}{\partial x_0} + k\frac{\partial f_3}{\partial x_0},$$
$$V_D S_f = S_f V_D = i\frac{\partial f_0}{\partial x_1} + j\frac{\partial f_0}{\partial x_2} + k\frac{\partial f_0}{\partial x_3},$$
$$V_D \cdot V_f = V_f \cdot V_D = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3},$$
$$V_D \times V_f = i\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) + j\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) + k\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right),$$

These results are related as

$$Df = fD - V_D \times V_f.$$

Moreover, if the nth order derivative is introduced, it is expressed by

$$D^n f = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} S_D^{n-k}(-1)^k V_D^k,$$

where

$$S_D = \frac{\partial^k}{\partial x_0^k}, V_D^k = \begin{cases} (-1)^{\frac{k}{2}}|V_D|^k & k \text{ is even;} \\ (-1)^{\frac{k-1}{2}} V_D |V_D|^{k-1} & k \text{ is odd} \end{cases}$$

and when k is even,

$$|V_D|^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)^{k/2}.$$

3 The notion of vectorial derivatives for a quaternion function

In this section, we define the derivative, called the vectorial derivative, of a quaternionic function and give some corresponding rules that can be derived for the vectorial differentiation.

Definition 3.1 Let $f : \mathbb{H} \to \mathbb{H}$ be real-differentiable. The left vectorial derivative of a quaternion function $f(q) = S_f + V_f$ with respect to q is defined as

$$Df = \frac{1}{2}(S_D S_f + S_D V_f - V_D S_f + V_D \cdot V_f - V_D \times V_f)$$

and the right vectorial derivative of f is defined as

$$fD = \frac{1}{2}(S_D S_f + S_D V_f - V_D S_f + V_D \cdot V_f + V_D \times V_f).$$

The left and right vectorial derivatives differ only in the operation of $V_D \times V_f$. So, in the following description, the formulas and properties are derived using the definition of the left vectorial derivative.

Proposition 3.2 (Constant rule) *Let $f : \mathbb{H} \to \mathbb{H}$ be real-differentiable. For $\alpha \in \mathbb{H}$, the left vectorial derivative of the function αf is given as*

$$D(\alpha f) = \alpha(Df) + (V_\alpha \times V_D)f,$$

where

$$V_\alpha \times V_D = i\left(\alpha_2 \frac{\partial}{\partial x_3} - \alpha_3 \frac{\partial}{\partial x_2}\right) + j\left(\alpha_3 \frac{\partial}{\partial x_1} - \alpha_1 \frac{\partial}{\partial x_3}\right) + k\left(\alpha_1 \frac{\partial}{\partial x_2} - \alpha_2 \frac{\partial}{\partial x_1}\right)$$

and

$$(V_\alpha \times V_D)f = V_\alpha \times V_D S_f - (V_\alpha \times V_D) \cdot V_f + (V_\alpha \times V_D) \times V_f.$$

Proof For a quaternion α, the function αf is expressed by

$$\alpha f = S_\alpha S_f + S_\alpha V_f + V_\alpha S_f + V_\alpha V_f,$$

where $V_\alpha V_f = -V_\alpha \cdot V_f + V_\alpha \times V_f$. Since $S_D = \frac{\partial}{\partial x_0}$, we have

$$S_D(\alpha f) = \alpha(S_D f).$$

Consider the expression of $V_D(\alpha f)$. The formula for $V_D(\alpha f)$ can be written as

$$\begin{aligned} V_D(\alpha f) &= V_D(S_\alpha S_f + S_\alpha V_f + V_\alpha S_f + V_\alpha V_f) \\ &= S_\alpha(V_D S_f) + S_\alpha(V_D V_f) + V_D(V_\alpha S_f) + V_D(V_\alpha V_f), \end{aligned}$$

where $V_D(V_\alpha S_f)$ and $V_D(V_\alpha V_f)$ are expressed as follows:

$$\begin{aligned} V_D(V_\alpha S_f) &= -V_D \cdot (V_\alpha S_f) + V_D \times (V_\alpha S_f) \\ &= -V_\alpha \cdot V_D S_f - V_\alpha \times V_D S_f \\ &= V_\alpha(V_D S_f) - 2V_\alpha \times V_D S_f \end{aligned}$$

and

$$\begin{aligned} V_D(V_\alpha V_f) &= (V_D V_\alpha)V_f = (-V_D \cdot V_\alpha + V_D \times V_\alpha)V_f \\ &= (-V_D \cdot V_\alpha - V_\alpha \times V_D)V_f \end{aligned}$$

$$= V_\alpha(V_D V_f) - 2(V_\alpha \times V_D)V_f,$$

respectively. Hence,

$$\begin{aligned}V_D(\alpha f) &= S_\alpha(V_D S_f) + S_\alpha(V_D V_f) + V_\alpha(V_D S_f)\\ &\quad - 2V_\alpha \times V_D S_f + V_\alpha(V_D V_f) - 2(V_\alpha \times V_D)V_f.\end{aligned}$$

Since

$$\begin{aligned}\alpha(Df) &= (S_\alpha + V_\alpha)(S_D S_f + S_D V_f - V_D S_f - V_D V_f)\\ &= S_\alpha(S_D S_f) + S_\alpha(S_D V_f) - S_\alpha(V_D S_f) - S_\alpha(V_D V_f)\\ &\quad + V_\alpha(S_D S_f) + V_\alpha(S_D V_f) - V_\alpha(V_D S_f) - V_\alpha(V_D V_f),\end{aligned}$$

we can compare with $D(\alpha f)$ and then

$$D(\alpha f) = \alpha(Df) + V_\alpha(V_D S_f) + V_\alpha(V_D V_f) = \alpha(Df) + (V_\alpha \times V_D)f,$$

where $(V_\alpha \times V_D)f$

$$\begin{aligned}&= -\left(\alpha_2\frac{\partial f_1}{\partial x_3} - \alpha_3\frac{\partial f_1}{\partial x_2}\right) - \left(\alpha_3\frac{\partial f_2}{\partial x_1} - \alpha_1\frac{\partial f_2}{\partial x_3}\right) - \left(\alpha_1\frac{\partial f_3}{\partial x_2} - \alpha_2\frac{\partial f_3}{\partial x_1}\right)\\ &\quad + i\left\{\left(\alpha_2\frac{\partial f_0}{\partial x_3} - \alpha_3\frac{\partial f_0}{\partial x_2}\right) + \left(\alpha_3\frac{\partial f_3}{\partial x_1} - \alpha_1\frac{\partial f_3}{\partial x_3}\right) - \left(\alpha_1\frac{\partial f_2}{\partial x_2} - \alpha_2\frac{\partial f_2}{\partial x_1}\right)\right\}\\ &\quad + j\left\{\left(\alpha_3\frac{\partial f_0}{\partial x_1} - \alpha_1\frac{\partial f_0}{\partial x_3}\right) + \left(\alpha_1\frac{\partial f_1}{\partial x_2} - \alpha_2\frac{\partial f_1}{\partial x_1}\right) - \left(\alpha_2\frac{\partial f_3}{\partial x_3} - \alpha_3\frac{\partial f_3}{\partial x_2}\right)\right\}\\ &\quad + k\left\{\left(\alpha_1\frac{\partial f_0}{\partial x_2} - \alpha_2\frac{\partial f_0}{\partial x_1}\right) + \left(\alpha_2\frac{\partial f_2}{\partial x_3} - \alpha_3\frac{\partial f_2}{\partial x_2}\right) - \left(\alpha_3\frac{\partial f_1}{\partial x_1} - \alpha_1\frac{\partial f_1}{\partial x_3}\right)\right\}.\end{aligned}$$

□

Proposition 3.3 (Product rule) *Let $f, g : \mathbb{H} \to \mathbb{H}$ be real-differentiable. The left vectorial derivative of the function fg is given as*

$$D(fg) = (Df)g + f(Dg) + (V_f \times V_D)g,$$

where

$$V_f \times V_D = i\left(f_2\frac{\partial}{\partial x_3} - f_3\frac{\partial}{\partial x_2}\right) + j\left(f_3\frac{\partial}{\partial x_1} - f_1\frac{\partial}{\partial x_3}\right) + k\left(f_1\frac{\partial}{\partial x_2} - f_2\frac{\partial}{\partial x_1}\right)$$

and

$$\begin{aligned}(V_f \times V_D)g &= (V_f \times V_D)S_g + (V_f \times V_D)V_g\\ &= f_3\frac{\partial g_1}{\partial x_2} - f_2\frac{\partial g_1}{\partial x_3} + f_1\frac{\partial g_2}{\partial x_3} - f_3\frac{\partial g_2}{\partial x_1} + f_2\frac{\partial g_3}{\partial x_1} - f_1\frac{\partial g_3}{\partial x_2}\\ &\quad + i\left(f_2\frac{\partial g_0}{\partial x_3} - f_3\frac{\partial g_0}{\partial x_2} + f_3\frac{\partial g_3}{\partial x_1} - f_1\frac{\partial g_3}{\partial x_3} - f_1\frac{\partial g_2}{\partial x_2} + f_2\frac{\partial g_2}{\partial x_1}\right)\end{aligned}$$

$$+ j\left(f_3\frac{\partial g_0}{\partial x_1} - f_1\frac{\partial g_0}{\partial x_3} + f_1\frac{\partial g_1}{\partial x_2} - f_2\frac{\partial g_1}{\partial x_1} - f_2\frac{\partial g_3}{\partial x_3} + f_3\frac{\partial g_3}{\partial x_2}\right)$$
$$+ k\left(f_1\frac{\partial g_0}{\partial x_2} - f_2\frac{\partial g_0}{\partial x_1} + f_2\frac{\partial g_2}{\partial x_3} - f_3\frac{\partial g_2}{\partial x_2} - f_3\frac{\partial g_1}{\partial x_1} + f_1\frac{\partial g_1}{\partial x_3}\right).$$

Proof Since we have

$$S_D(fg) = (S_D f)g + f(S_D g),$$

we consider

$$\begin{aligned} V_D(fg) &= V_D(S_f S_g + S_f V_g + V_f S_g + V_f V_g) \\ &= V_D(S_f S_g) + V_D(S_f V_g) + V_D(V_f S_g) + V_D(V_f V_g) \\ &= (V_D S_f)S_g + S_f(V_D S_g) + (V_D S_f)V_g + S_f(V_D V_g) \\ &\quad + V_D(V_f S_g) + V_D(V_f V_g). \end{aligned}$$

In particular, since

$$V_D(V_f S_g) = (V_D V_f)S_g + (V_f V_D - 2V_f \times V_D)S_g$$

and

$$V_D(V_f V_g) = (V_D V_f)V_g + (V_f V_D - 2V_f \times V_D)V_g,$$

we have

$$V_D(fg) = (V_D f)g + f(V_D g) - 2(V_f \times V_D)g.$$

So, we obtain

$$\begin{aligned} D(fg) &= \frac{1}{2}\big(S_D(fg) - V_D(fg)\big) \\ &= \frac{1}{2}\big\{(S_D f)g + f(S_D g) - (V_D f)g - f(V_D g) + 2(V_f \times V_D)g\big\} \\ &= (Df)g + f(Dg) + (V_f \times V_D)g. \end{aligned}$$ □

If f is a real-valued function of a quaternion variable, we have

$$D(fg) = (Df)g + f(Dg).$$

Proposition 3.4 (Quotient rule) *Let $f, g : \mathbb{H} \to \mathbb{H}$ be real-differentiable. If $g \neq 0$, then the left vectorial derivative of the function g^{-1} is given as*

$$Dg^{-1} = \big(D|g|^{-2}\big)g^* + |g|^{-2}\big(Dg^*\big). \tag{3.1}$$

In equation (3.1), g^{-1} *is written as*

$$g^{-1} = |g|^{-2}g^* = \left(S_g^2 + |V_g|^2\right)^{-1}(S_g - V_g)$$

and $|g|^{-2}$ *is expressed by*

$$|g|^{-2} = \left(S_g^2 + |V_g|^2\right)^{-1}.$$

Furthermore, the left vectorial derivative of the function fg^{-1} (*resp.*, $g^{-1}f$) *is given as*

$$D\left(fg^{-1}\right) = D\left(|g|^{-2}fg^*\right) = \left(D|g|^{-2}\right)\left(fg^*\right) + |g|^{-2}\left(D\left(fg^*\right)\right),$$

resp.,

$$D\left(g^{-1}f\right) = D\left(|g|^{-2}g^*f\right) = \left(D|g|^{-2}\right)\left(g^*f\right) + |g|^{-2}\left(D\left(g^*f\right)\right).$$

4 Examples of the left vectorial derivative

We look at the elementary functions as examples of the quaternion function of a quaternion variable and illustrate our findings with figures. A study was conducted on the definition and properties of elementary functions of quaternion variables (see [5, 9, 19, 23]).

Proposition 4.1 (Power function) *Let* $f : \mathbb{H} \to \mathbb{H}$ *be the power function defined as* $f(p) = p^n$, *where* n *is any nonnegative integer. Then, the left vectorial derivative of the function* $f(p) = S_f + V_f$ *with respect to* p *is given as*

$$f'(p) = D\left(p^n\right) = np^{n-1} + \mathcal{O}_n(p),$$

where the scalar function $\mathcal{O}_n(p)$ *is*

$$\mathcal{O}_n(p) = \sum_{k=0}^{m} \binom{n}{2k+1}(-1)^k S_p^{n-2k-1}|V_p|^{2k} \quad (n \geq 2m+1).$$

Proof First, we find the vectorial expression for the power function p^n. We note the expression in which the power of the vector part of p is calculated on the basis of the noncommutativity property of the product for quaternions. If a similar operation as in

$$V_p = V_p, \qquad V_p^2 = -|V_p|^2, \qquad V_p^3 = -V_p|V_p|^2$$

is continuously performed, the following expression is obtained:

$$V_p^n = \begin{cases} (-1)^{\frac{n}{2}}|V_p|^n & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}}V_p|V_p|^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Hence, we obtain

$$p^n = \sum_{k=0}^{n} \binom{n}{k} S_p^{n-k} V_p^k.$$

So, the left vectorial derivative is obtained as

$$\begin{aligned} D(p^n) &= \frac{1}{2}(S_D - V_D)p^n \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (n-k) S_p^{n-k-1} V_p^k + \sum_{k=1}^{n} \binom{n}{k} S_p^{n-k} \left(V_D V_p^k\right), \end{aligned}$$

where

$$V_D V_p^k = \begin{cases} (-1)^{\frac{k}{2}} k |V_p|^{k-2} V_p & \text{if } k \text{ is even;} \\ (-1)^{\frac{k+1}{2}} (k+2) |V_p|^{k-1} & \text{if } k \text{ is odd.} \end{cases}$$

Furthermore,

$$S_D p^n = n p^{n-1}, \qquad V_D p^n = -n p^{n-1} - 2 \sum_{k=0}^{m} \binom{n}{2k+1} S_p^{n-2k-1} V_p^{2k},$$

Thus, we obtain

$$f'(p) = D(p^n) = n p^{n-1} + \sum_{k=0}^{m} \binom{n}{2k+1} S_p^{n-2k-1} V_p^{2k}.$$

□

Proposition 4.2 (Exponential function) *Let $f : \mathbb{H} \to \mathbb{H}$ be the exponential function defined as $f(p) = \exp(p)$. Then, the left vectorial derivative of f with respect to p is given as*

$$f'(p) = D(\exp(p)) = \exp(p) + \frac{\exp(S_p) \sin(|V_p|)}{|V_p|}.$$

Proof The function $\exp(p)$ is expressed as

$$\exp(p) = \exp(S_p)\left(\cos|V_p| + \frac{V_p}{|V_p|} \sin|V_p| \right).$$

So, to obtain the left vectorial derivatives of $\exp(p)$, we calculate

$$\begin{aligned} S_D \exp(p) &= S_D \left(\exp(S_p) \left(\cos|V_p| + \frac{V_p}{|V_p|} \sin|V_p| \right) \right) \\ &= \left(\frac{\partial}{\partial x_0} \exp(S_p) \right) \left(\cos|V_p| + \frac{V_p}{|V_p|} \sin|V_p| \right) \\ &= \exp(p). \end{aligned}$$

Also,

$$\begin{aligned} V_D \exp(p) = {} & \exp(S_p) \left(i \frac{\partial}{\partial x_1} \cos|V_p| + j \frac{\partial}{\partial x_2} \cos|V_p| + k \frac{\partial}{\partial x_3} \cos|V_p| \right) \\ & + \exp(S_p) \left(-V_D \cdot \frac{V_p}{|V_p|} \sin|V_p| + V_D \times \frac{V_p}{|V_p|} \sin|V_p| \right), \end{aligned}$$

where

$$\begin{aligned}
&\frac{\partial}{\partial x_r}\cos|V_p| = -\frac{x_r}{|V_p|}\sin|V_p| \quad (r=1,2,3),\\
&-V_D\cdot\frac{V_p}{|V_p|}\sin|V_p|\\
&\quad= -\frac{\partial}{\partial x_1}\left(\frac{x_1}{|V_p|}\sin|V_p|\right) - \frac{\partial}{\partial x_2}\left(\frac{x_2}{|V_p|}\sin|V_p|\right) - \frac{\partial}{\partial x_3}\left(\frac{x_3}{|V_p|}\sin|V_p|\right)\\
&\quad= -\frac{1}{|V_p|^2}\left(|V_p|\sin|V_p| + |V_p|^2\cos|V_p| - |V_p|\sin|V_p|\right)\\
&\quad= -\cos|V_p| - 2\frac{\sin(|V_p|)}{|V_p|}
\end{aligned}$$

and

$$\begin{aligned}
V_D\times\frac{V_p}{|V_p|}\sin|V_p| = {}& i\left\{\frac{\partial}{\partial x_2}\left(\frac{x_3}{|V_p|}\sin|V_p|\right) - \frac{\partial}{\partial x_3}\left(\frac{x_2}{|V_p|}\sin|V_p|\right)\right\}\\
&+ j\left\{\frac{\partial}{\partial x_3}\left(\frac{x_1}{|V_p|}\sin|V_p|\right) - \frac{\partial}{\partial x_1}\left(\frac{x_3}{|V_p|}\sin|V_p|\right)\right\}\\
&+ k\left\{\frac{\partial}{\partial x_1}\left(\frac{x_2}{|V_p|}\sin|V_p|\right) - \frac{\partial}{\partial x_2}\left(\frac{x_1}{|V_p|}\sin|V_p|\right)\right\})\\
={}& 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
V_D\exp(p) &= -\exp(S_p)\frac{V_p}{|V_p|}\sin|V_p| - \exp(S_p)\cos|V_p|\\
&= -\exp(S_p)\left(\cos|V_p| + \frac{V_p}{|V_p|}\sin|V_p| + 2\frac{\sin(|V_p|)}{|V_p|}\right)\\
&= -\exp(p) - 2\frac{\exp(S_p)\sin(|V_p|)}{|V_p|}.
\end{aligned}$$

Thus, the formula is obtained as

$$D\exp(p) = \frac{1}{2}(S_D - V_D)\exp(p) = \exp(p) + \frac{\exp(S_p)\sin(|V_p|)}{|V_p|}. \qquad \square$$

In particular, the remainder term

$$\frac{\exp(S_p)\sin(|V_p|)}{|V_p|}$$

of the vectorial derivative of an exponential function is a scalar function. In the illustration of the remainder term of the vectorial derivative of $\exp(p)$, S_p and $|V_p|$ are represented by real variables x and $y > 0$, respectively, as shown in Figs. 1 and 2. Further, excluding the specific area in Figs. 1 and 2, if $|V_p|$ is treated as a sufficiently large number compared to S_p, $D\exp(p)$ can be used as the derivative of $\exp(p)$.

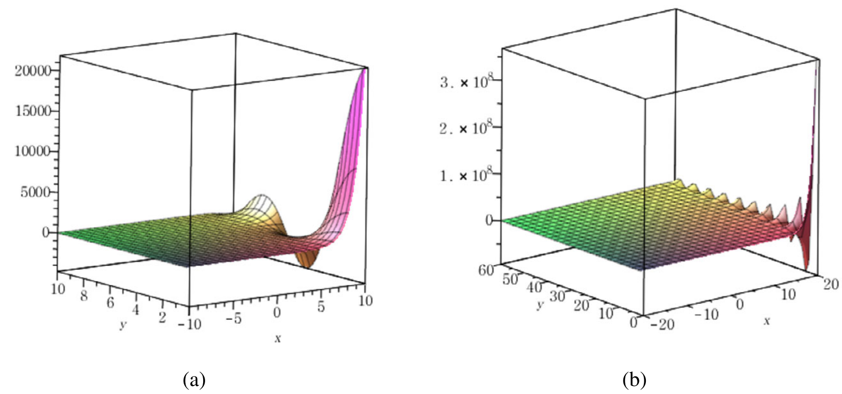


Figure 1 Illustration of the remainder term of $D\exp(p)$ according to the range of $x(=S_p)$ and $y(=|V_p|)$ using Maple 20 'Plot3D' function: (**a**) output of Plot3D of $\frac{\exp(x)\sin(y)}{y}$ with $x = -10 \cdots 10$, $y = 0 \cdots 10$; (**b**) output of Plot3D of $\frac{\exp(x)\sin(y)}{y}$ with $x = -20 \cdots 20$, $y = 0 \cdots 60$

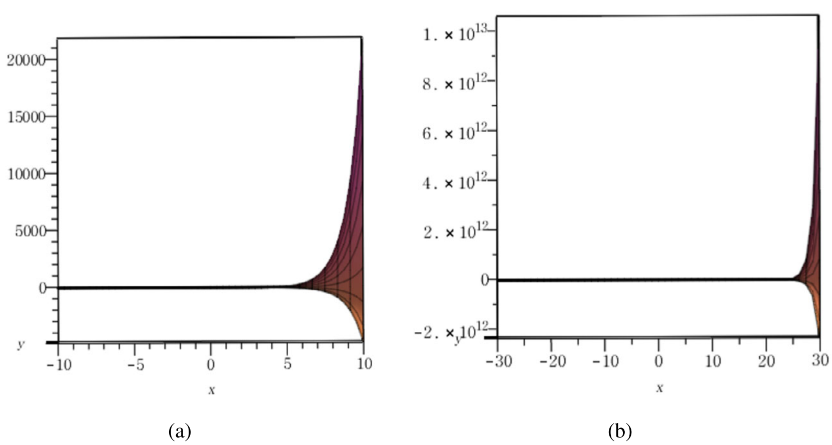


Figure 2 Figures observed in the direction perpendicular to the xz plane of each graph in Fig. 1. Depending on the range of y, it can be checked which points x will be excluded: (**a**) graph observed perpendicular to the xz plane with $x = -10 \cdots 10$, $y = 0 \cdots 10$; (**b**) graph observed perpendicular to the xz plane with $x = -30 \cdots 30$, $y = 0 \cdots 10$

We consider a corresponding logarithm of $\exp(p)$. In [5], the logarithm of quaternions is given as

$$\log p = \ln|p| + \frac{V_p}{|V_p|}\arg(p), \quad |V_p| \neq 0,$$

where $\arg(p) = \tan^{-1}(\frac{|V_p|}{x_0}) + 2n\pi$, $n \in \mathbb{Z}$, for some real number α, such that $\alpha < \arg(p) < \alpha + 2\pi$.

Proposition 4.3 (Logarithm function) *Let $f : \mathbb{H} \to \mathbb{H}$ be the logarithm function defined as $f(p) = \log(p)$. Then, the left vectorial derivative of f with respect to p is given as*

$$f'(p) = D\bigl(\log(p)\bigr) = p^{-1} + \frac{\tan^{-1}(\frac{|V_p|}{x_0})}{|V_p|}$$

Proof By the definition of vectorial derivative of $\log(p)$, we have

$$D\log(p) = \frac{1}{2}(S_D - V_D)\log(p).$$

The calculation of the vectorial derivative of $\log(p)$ is as follows:

$$\begin{aligned} S_D \log(p) &= \frac{1}{|p|}\frac{x_0}{|p|} + \frac{V_p}{|V_p|}\frac{-|V_p|}{|p|^2} \\ &= \frac{x_0 - V_p}{|p|^2} = \frac{p^*}{|p|^2} \end{aligned}$$

and

$$\begin{aligned} V_D \log(p) = V_D \ln|p| &- \left(V_D \cdot \frac{V_p}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right)\right) \\ &+ V_D \times \frac{V_p}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right). \end{aligned}$$

In particular, each term is calculated as

$$\begin{aligned} &V_D \ln|p| = \frac{V_p}{|p|^2}, \\ &V_D \cdot \frac{V_p}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right) = 2\frac{1}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right) + \frac{x_0}{|p|^2} \end{aligned}$$

and

$$V_D \times \frac{V_p}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right) = 0.$$

Hence, we have

$$V_D \log(p) = \frac{V_p}{|p|^2} - 2\frac{1}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right) - \frac{x_0}{|p|^2}.$$

Thus, the vectorial derivative of the logarithm function is given by

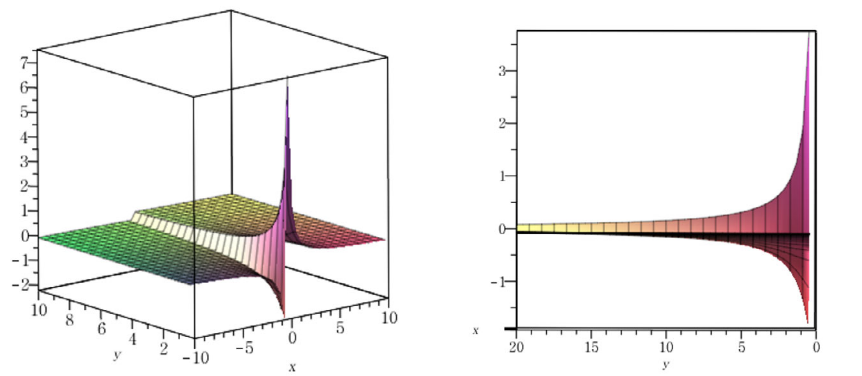
$$\begin{aligned} D\log(p) &= \frac{1}{2}\left(\frac{p^*}{|p|^2} - \frac{V_p}{|p|^2} + 2\frac{1}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{x_0}\right) + \frac{x_0}{|p|^2}\right) \\ &= \frac{p^*}{|p|^2} + \frac{1}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{S_p}\right). \end{aligned}$$

□

The remainder term

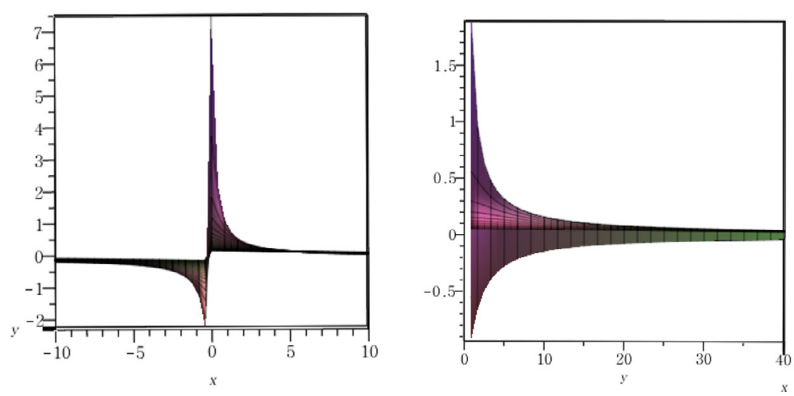
$$\frac{1}{|V_p|}\tan^{-1}\left(\frac{|V_p|}{S_p}\right)$$

of the vectorial derivative of the logarithm function is a scalar function. The illustration of the remainder term of the vectorial derivative of a logarithm function is expressed in several figures. In the remainder term of the vectorial derivative of $\log(p)$, S_p and $|V_p|$ are



(a) (b)

Figure 3 Figures showing what values of the remainder term of $D\log(p)$ have according to the size of $x = S_p$ and $y = |V_p|$ using Maple 20 performance. It can be seen that as x and y approach 0, the value of the remainder term increases infinitely. (**a**) Output of Maple 20 'Plot3D' performance of $\frac{1}{y}\tan^{-1}\frac{y}{x}$ with $x = -10 \cdots 10$, $y = 0 \cdots 10$. (**b**) Graph observed perpendicular to the yz plane with $x = -20 \cdots 20$, $y = 0 \cdots 20$



(a) (b)

Figure 4 Side views where we can observe the remainder term of $D\log(p)$ according to the respective sizes of x and y using Maple 20 performance. (**a**) Graph observed perpendicular to the xz plane with $x = -10 \cdots 10$, $y = 0 \cdots 10$. As the absolute value of x increases, the remainder term approaches zero. (**b**) Graph observed perpendicular to the yz plane with $x = -20 \cdots 20$, $y = 0 \cdots 10$. As the value of y increases, the remainder term gets closer to zero

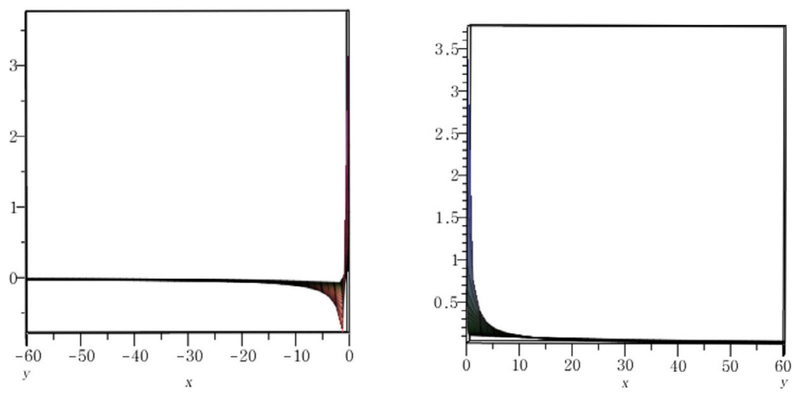
represented by real variables x and $y > 0$, respectively. Except for a specific area that both S_p and $|V_p|$ are close to 0, in each of Figs. 3–5, $D\log(p)$ can be used approximately as the derivative of $\log(p)$.

We consider $\sin(p)$ and $\cos(p)$. In [5], we have

$$\exp(p) = \exp(S_p)\left(\cos\bigl(|V_p|\bigr) + \frac{V_p}{|V_p|}\sin\bigl(|V_p|\bigr)\right)$$

and, if $S_p = 0$,

$$\exp\left(\frac{V_p}{|V_p|}|V_p|\right) = \cos\bigl(|V_p|\bigr) + \frac{V_p}{|V_p|}\sin\bigl(|V_p|\bigr).$$



(a) (b)

Figure 5 Side views that can observe the effect of the size of x on the remainder term over a certain range of y using Maple 20 performance. (**a**) Graph observed perpendicular to the xz plane with $x = -60 \cdots 0$, $y = 0 \cdots 20$. As the size of x gets smaller than −10, the remainder term gets closer to zero. (**b**) Graph observed perpendicular to the xz plane with $x = 0 \cdots 60$, $y = 0 \cdots 20$. For a certain range of y, as the magnitude of x becomes greater than 10, the remainder term approaches zero

From the expression in [5], we can write

$$\exp\left(\frac{V_p}{|V_p|}p\right) = \cos(p) + \frac{V_p}{|V_p|}\sin(p) \quad \text{and} \quad \exp\left(\frac{-V_p}{|V_p|}p\right) = \cos(p) - \frac{V_p}{|V_p|}\sin(p).$$

Hence, we get

$$\cos(p) = \frac{1}{2}\left(\exp\left(\frac{V_p}{|V_p|}p\right) + \exp\left(\frac{-V_p}{|V_p|}p\right)\right)$$

and

$$\sin(p) = \frac{1}{2}\frac{-V_p}{|V_p|}\left(\exp\left(\frac{V_p}{|V_p|}p\right) - \exp\left(\frac{-V_p}{|V_p|}p\right)\right).$$

Proposition 4.4 (Trigonometric functions) *Let* $\cos, \sin : \mathbb{H} \to \mathbb{H}$ *be the sine and cosine functions defined as* $\cos(p)$ *and* $\sin(p)$, *respectively. Then, the left vectorial derivatives of* $\cos(p)$ *and* $\sin(p)$ *with respect to p are given as*

$$\cos'(p) = D\big(\cos(p)\big) = -\sin(p) - \frac{\sin(S_p)\sinh(|V_p|)}{|V_p|}$$

and

$$\sin'(p) = D\big(\sin(p)\big) = \cos(p) + \frac{\cos(S_p)\cosh(|V_p|)}{|V_p|}.$$

Proof The function $\cos(p)$ is expressed by

$$\cos(p) = \frac{1}{2}\left(\exp\left(\frac{V_p}{|V_p|}p\right) + \exp\left(\frac{-V_p}{|V_p|}p\right)\right).$$

So, to obtain the left vectorial derivative of $\cos(p)$, we calculate

$$\begin{aligned}
S_D \cos(p) &= \frac{1}{2} S_D\left(\exp\left(\frac{V_p}{|V_p|}p\right) + \exp\left(\frac{-V_p}{|V_p|}p\right)\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x_0}\left(\exp(-|V_p|)\exp\left(\frac{V_p}{|V_p|}S_p\right) + \exp(|V_p|)\exp\left(\frac{-V_p}{|V_p|}S_p\right)\right) \\
&= \frac{1}{2}\left(\frac{V_p}{|V_p|}\exp(-|V_p|)\exp\left(\frac{V_p}{|V_p|}S_p\right) + \frac{-V_p}{|V_p|}\exp(|V_p|)\exp\left(\frac{-V_p}{|V_p|}S_p\right)\right) \\
&= \frac{1}{2}\frac{V_p}{|V_p|}\left(\exp\left(\frac{V_p}{|V_p|}p\right) - \exp\left(\frac{-V_p}{|V_p|}p\right)\right) \\
&= -\sin(p).
\end{aligned}$$

Also,

$$V_D \cos(p) = \frac{1}{2}(V_D\left(\exp(-|V_p|)\exp\left(\frac{V_p}{|V_p|}S_p\right) + V_D\exp(|V_p|)\exp\left(\frac{-V_p}{|V_p|}S_p\right)\right), \tag{4.1}$$

and the first term of equation (4.1) is

$$\begin{aligned}
&V_D(\exp(-|V_p|)\exp\left(\frac{V_p}{|V_p|}S_p\right) \\
&\quad = V_D\left(\exp(-|V_p|)\left(\cos x_0 + \frac{V_p}{|V_p|}\sin x_0\right)\right) \\
&\quad = (V_D\exp(-|V_p|)\cos x_0 + \left(V_D\frac{\exp(-|V_p|)V_p}{|V_p|}\right)\sin x_0,
\end{aligned}$$

where

$$\begin{aligned}
V_D\exp(-|V_p|)\cos x_0 &= \left(i\frac{\partial}{\partial x_1}\exp(-|V_p|) + j\frac{\partial}{\partial x_2}\exp(-|V_p|)\right. \\
&\qquad \left. + k\frac{\partial}{\partial x_3}\exp(-|V_p|)\right)\cos x_0 \\
&= \frac{-V_p}{|V_p|}\exp(-|V_p|)\cos x_0
\end{aligned}$$

and

$$\begin{aligned}
&V_D\frac{\exp(-|V_p|)V_p}{|V_p|}\sin x_0 \\
&\quad = -V_D\cdot\frac{\exp(-|V_p|)V_p}{|V_p|} + V_D\times\frac{\exp(-|V_p|)V_p}{|V_p|} \\
&\quad = -\left(\frac{\partial}{\partial x_1}\exp(-|V_p|)\frac{x_1}{|V_p|} + \frac{\partial}{\partial x_2}\exp(-|V_p|)\frac{x_2}{|V_p|} + \frac{\partial}{\partial x_3}\exp(-|V_p|)\frac{x_3}{|V_p|}\right)\sin x_0 \\
&\quad = \exp(-|V_p|)\sin x_0 - 2\frac{\exp(-|V_p|)\sin x_0}{|V_p|}.
\end{aligned}$$

So,

$$
\begin{aligned}
&V_D\left(\exp\bigl(-|V_p|\bigr)\exp\left(\frac{V_p}{|V_p|}S_p\right)\right.\\
&\quad = \frac{-V_p}{|V_p|}\exp\bigl(-|V_p|\bigr)\left(\cos(x_0)+\frac{V_p}{|V_p|}\sin(x_0)\right)-2\frac{\exp(-|V_p|)\sin x_0}{|V_p|}.
\end{aligned}
$$

And the second term of equation (4.1) is

$$
\begin{aligned}
&V_D\exp\bigl(|V_p|\bigr)\exp\left(\frac{-V_p}{|V_p|}S_p\right)\\
&\quad = V_D\left(\exp\bigl(|V_p|\bigr)\left(\cos x_0-\frac{V_p}{|V_p|}\sin x_0\right)\right)\\
&\quad = \frac{V_p}{|V_p|}\exp\bigl(|V_p|\bigr)\cos x_0+\exp\bigl(|V_p|\bigr)\sin x_0+2\frac{\exp(|V_p|)\sin x_0}{|V_p|}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V_D\cos(p) &= \frac{1}{2}\frac{-V_p}{|V_p|}\left(\exp\left(\frac{V_p}{|V_p|}p\right)-\exp\left(\frac{-V_p}{|V_p|}p\right)\right)-\frac{\exp(-|V_p|)\sin x_0}{|V_p|}\\
&\quad +\frac{\exp(|V_p|)\sin x_0}{|V_p|}\\
&= \sin(p)-\frac{\exp(-|V_p|)\sin x_0}{|V_p|}+\frac{\exp(|V_p|)\sin x_0}{|V_p|}.
\end{aligned}
$$

Thus, the formula is obtained as

$$
\begin{aligned}
D\cos(p) &= -\sin(p)+\frac{1}{2}\left(\frac{\exp(-|V_p|)}{|V_p|}\sin x_0-\frac{\exp(|V_p|)}{|V_p|}\sin x_0\right)\\
&= -\sin(p)-\frac{\sin(S_p)\sinh(|V_p|)}{|V_p|}.
\end{aligned}
$$

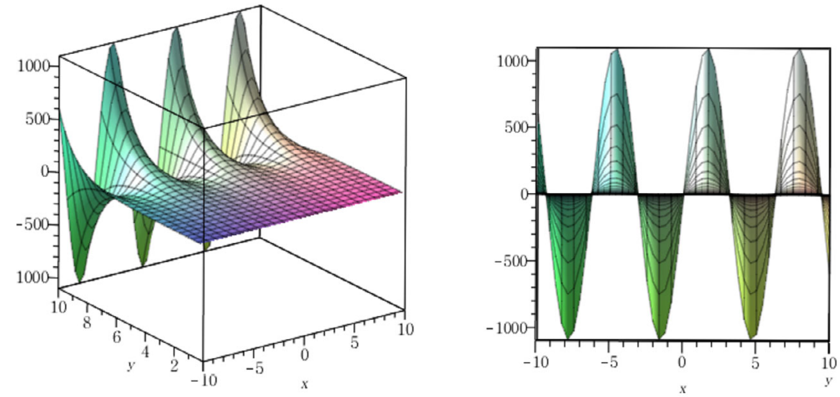
Similarly as for $D\cos(p)$, the left vectorial derivative $D\sin(p)$ of $\sin(p)$ is obtained as follows:

$$
\begin{aligned}
D\sin(p) &= \cos(p)+\frac{1}{2}\left(\frac{\exp(-|V_p|)}{|V_p|}\cos x_0+\frac{\exp(|V_p|)}{|V_p|}\cos x_0\right)\\
&= \cos(p)+\frac{\cos(S_p)\cosh(|V_p|)}{|V_p|}.
\end{aligned}
$$
□

The remainder term

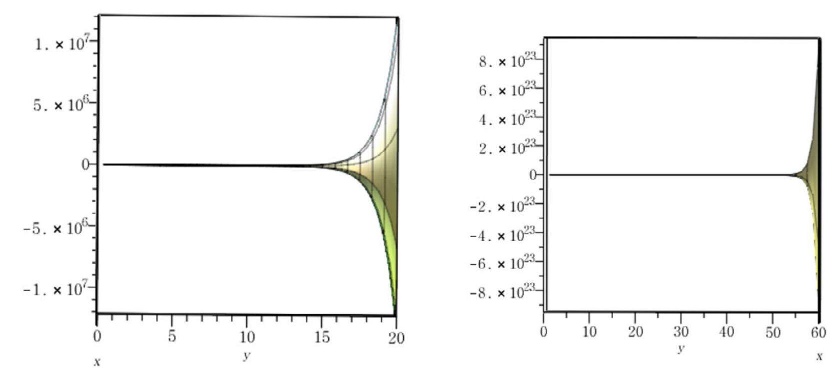
$$
\frac{\sin(S_p)\sinh(|V_p|)}{|V_p|}
$$

of the vectorial derivative of the cosine function is a scalar function. In the illustration of the remainder term of the vectorial derivative of the cosine function, S_p and $|V_p|$ are represented by x and y, respectively, as shown in Figs. 6 and 7. Observe that $D\cos(p)$ can be used as a derivative of $\cos(p)$ in the region where the remainder term is close to 0 by



(a) (b)

Figure 6 Figures showing the size of the remainder term of $D\cos(p)$ as the size of $x = S_p$ and $y = |V_p|$ changes using Maple 20 performance. (**a**) Output of Maple 20 Plot3D of $\frac{1}{y}\sin(x)\sinh(y)$ with $x = -10\cdots10$, $y = 0\cdots10$. (**b**) Graph observed perpendicular to the xz plane with $x = -10\cdots10$, $y = 0\cdots10$



(a) (b)

Figure 7 Side views showing the size of the remainder term of $D\cos(p)$ that is output according to the size of a positive real y in a certain range of x using Maple 20 performance. (**a**) Graph observed perpendicular to the yz plane with $x = -10\cdots10$, $y = 0\cdots20$. For x within $-10 \le x \le 10$, the remainder term becomes 0 in a region where the size of y is less than about 15. (**b**) Graph observed perpendicular to the yz plane with $x = -20\cdots20$, $y = 0\cdots60$. For x where $-20 \le x \le 20$, the remainder term is zero, excluding the area where y is larger than 55

considering appropriate ranges of x and y and their size. In particular, as the size of y decreases, the remainder term approaches zero.

Further, the remainder term of $D\sin(p)$ is represented by figures similar to those of $D\cos(p)$, and the remainder term of $D\sin(p)$ also approaches 0 as the size of $|V_p|$ decreases for a certain range of S_p. Therefore, if the size of $|V_p|$ is sufficiently small in each interval for S_p, $D\sin(p)$ can be used as the derivative of $\sin(p)$. The remainder term

$$\frac{\cos(S_p)\cosh(|V_p|)}{|V_p|}$$

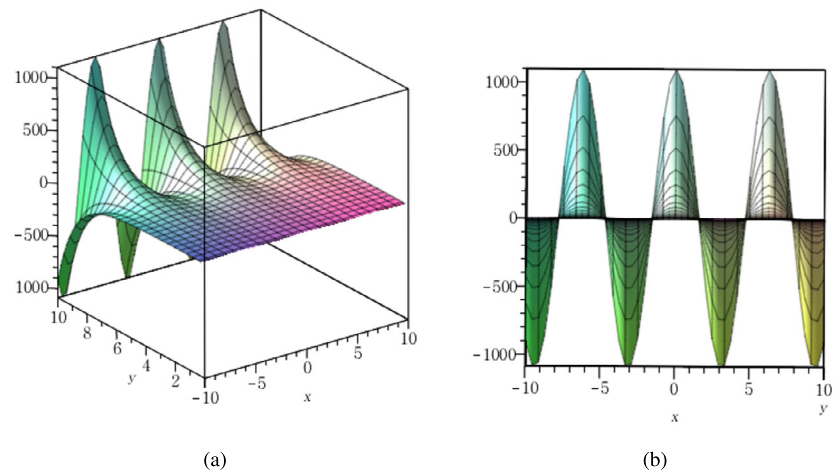


Figure 8 Figures showing the size of the remainder term of $D\sin(p)$ according to the range of $x = S_p$ and $y = |V_p|$ using Maple 20 performance. (**a**) Output of Maple 20 Plot3d of $\frac{1}{y}\cos(x)\cosh(y)$ with $x = -10\cdots 10, y = 0\cdots 10$. For x within a certain interval, as the size of y decreases, the remainder term is approximated to zero. (**b**) Graph observed perpendicular to the xz plane with $x = -10\cdots 10, y = 0\cdots 10$

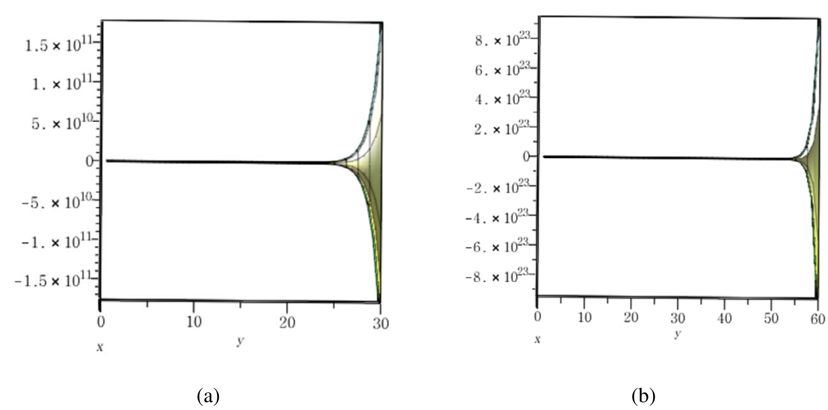


Figure 9 Side views that can observe the effect of the size of x on the remainder term over a certain range of y using Maple 20 performance. (**a**) Graph observed perpendicular to the yz plane with $x = -20\cdots 20, y = 0\cdots 30$. For x within $-20 \leq x \leq 20$, as the size of y is less than about 15, the remainder term is approximated to zero. (**b**) Graph observed perpendicular to the yz plane with $x = -20\cdots 20, y = 0\cdots 60$. Except for areas where y is larger than about 55, the remainder term of $D\sin(p)$ is closed to zero

of the vectorial derivative of the sine function is a scalar function. In the illustration of the remainder term of the vectorial derivative of the sine function, S_p and $|V_p|$ are represented by x and y, respectively, as shown in Figs. 8 and 9.

5 Conclusion

This paper has presented a novel notion of a quaternion derivative, called the vectorial derivative. Since the noncommutative product rule applies to quaternions, various results about derivatives are needed. Although there are some differences between the calculation methods and the properties of differentiation in complex analysis, this paper introduces the newly defined derivative calculation method using the Fueter operator and examines the computational properties according to this definition.

The vectorial derivative calculation method flexibly interprets the existence conditions for the derivative of a general nonlinear function of a quaternion variable. Furthermore, this paper has shown that the defined derivative of the quaternion function is simplified by presenting the properties of operations such as the product and quotient of the quaternion functions, applied to the vectorial derivative calculation. Unlike quaternion function derivatives which require certain existence assumptions, the vectorial derivative calculation method is general; it can be used for either analytic or nonanalytic function of a quaternion variable.

Because quaternions provide more efficient modeling of rotations and transformations than real vectors, they are utilized in physics and engineering applications. For example, since the general motion of a rigid body is a combination of translation and rotation, it is possible to interpret the motion of a rigid body over time using the corresponding quaternion structure of such transformation. In addition, the motion of a rigid body can be formulated for relativity using two quaternionic operators, and a differential operation for space–time intervals in the special theory of relativity can be defined. In electrodynamics, the Lorenz–Gauge condition can be reached by using the quaternionic derivative to express the electron velocity as the potential of the quaternion structure. In quantum physics, the Dirac special wave equation can be dealt with by combining the square magnitude of energy and the wave function of the quaternion system. (Actual formulas and symbols to which the quaternary structure is applied can be found in [18, 21].) From the algebraic characteristics and analytic properties caused by the noncommutativity of the quaternion operations, the definition of the derivative for applications requires the establishment of some restrictions and strong conditions. The vectorial derivative can be used to extend the range of applications that require derivatives. Our work is intended to help in setting conditions. Furthermore, algorithms for quaternions can be developed by extending the range that can be set by usual calculation methods for the real- and complex-valued optimization algorithms.

Acknowledgements
This work was supported by the Dongguk University Research Fund 2019.

Funding
This work was supported by the Dongguk University Research Fund 2019 and the National Research Foundation of Korea (NRF) (2021R1F1A1063356).

Availability of data and materials
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author confirms sole responsibility for the following: study conception and design, data collection, analysis and interpretation of results, and manuscript preparation. Author read and approved the final manuscript.

Publisher's Note

Received: 20 April 2021 Accepted: 2 September 2021 Published online: 23 September 2021

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