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Analytical properties of type 2 degenerate poly-Bernoulli polynomials associated with their applications

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Abstract

Recently, Kim *et al.* (Adv. Differ. Equ. 2020:168, 2020) considered the poly-Bernoulli numbers and polynomials resulting from the moderated version of degenerate polyexponential functions. In this paper, we investigate the degenerate type 2 poly-Bernoulli numbers and polynomials which are derived from the moderated version of degenerate polyexponential functions. Our degenerate type 2 degenerate poly-Bernoulli numbers and polynomials are different from those of Kim *et al.* (Adv. Differ. Equ. 2020:168, 2020) and Kim and Kim (Russ. J. Math. Phys. 26(1):40–49, 2019). Utilizing the properties of moderated degenerate poly-exponential function, we explore some properties of our type 2 degenerate poly-Bernoulli numbers and polynomials. From our investigation, we derive some explicit expressions for type 2 degenerate poly-Bernoulli numbers and polynomials. In addition, we also scrutinize type 2 degenerate unipoly-Bernoulli polynomials related to an arithmetic function and investigate some identities for those polynomials. In particular, we consider certain new explicit expressions and relations of type 2 degenerate unipoly-Bernoulli polynomials and numbers related to special numbers and polynomials. Further, some related beautiful zeros and graphical representations are displayed with the help of *Mathematica*.

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1 Introduction

Carlitz [1, 2], Kim and Kim [14, 19, 21–23], Kim *et al.* [26, 28, 29, 31], Jang *et al.* [7, 8], Muhiuddin *et al.* [37–39], Khan *et al.* [10–13], Sharma *et al.* [41–43] introduced and studied various degenerate versions of special polynomials and numbers like degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, degenerate Fubini polynomials, and degenerate Stirling numbers of the first and second kinds.

For $\kappa \in \mathbb{Z}$, the modified degenerate polyexponential function [30] is specified by Kim and Kim to be

$$\mathrm{Ei}_{\kappa,\lambda}(\omega) = \sum_{\upsilon=1}^{\infty} \frac{(1)_{\upsilon,\lambda}\omega^{\upsilon}}{(\upsilon-1)!\upsilon^{\kappa}}, \quad \bigl(|\omega| < 1\bigr). \tag{1.1}$$

Note that

$$\mathrm{Ei}_{1,\lambda}(\omega) = \sum_{\upsilon=1}^{\infty} \frac{(1)_{\upsilon,\lambda}\omega^{\upsilon}}{\upsilon!} = e_{\lambda}(\omega) - 1. \tag{1.2}$$

The degenerate polyexponential functions were introduced by Kim and Kim as

$$e_{\lambda}(\omega,\delta|\kappa) = \sum_{\upsilon=0}^{\infty} \frac{(1)_{\upsilon,\lambda}\omega^{\upsilon}}{\upsilon!(\upsilon+\delta)^{\kappa}} \quad (\text{see [21]}), \tag{1.3}$$

where $\delta \in \mathbb{C}$ and $\kappa \in \mathbb{N} \cup \{0\}$ with $\Re(\delta) > 0$.

From (1.3), we readily get

$$e_{\lambda}(\omega,\delta|0) = e_{\lambda}(\omega), \qquad e_{\xi}(\omega,1|1) = \frac{1}{\omega}\frac{1}{1+\xi}\bigl(e_{\lambda}(\omega) - 1\bigr) + \frac{\lambda}{1+\lambda}e_{\lambda}(\omega). \tag{1.4}$$

We note here that

$$\lim_{\lambda\to 0} e_{\lambda}(\omega,1|1) = \frac{1}{\omega}\bigl(e^{\omega} - 1\bigr) = e(\omega,1|1),$$

which was defined by Hardy (see [5, 6]).

The degenerate poly-Genocchi polynomials [30] were considered by Kim and Kim and given as

$$\frac{2\mathrm{Ei}_{\kappa,\lambda}(\log_{\lambda}(1+z))}{e_{\lambda}(z)+1}e_{\lambda}^{\omega}(z) = \sum_{\upsilon=0}^{\infty} G_{\upsilon,\lambda}^{(\kappa)}(\omega)\frac{z^{\upsilon}}{\upsilon!} \quad (\kappa \in \mathbb{Z}). \tag{1.5}$$

When $\omega = 0$, $G_{\upsilon,\lambda}^{(\kappa)} = G_{\upsilon,\lambda}^{(\kappa)}(0)$ are called degenerate poly-Genocchi numbers.

From (1.1) and (1.5), we see that $G_{\upsilon,\lambda}^{(1)}(\omega) = G_{\upsilon,\lambda}(\omega)$, $(\upsilon \geq 0)$ which are called degenerate Genocchi polynomials.

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as follows (see [14–16, 18–34]):

$$e_{\lambda}^{\omega}(z) = (1+\lambda z)^{\frac{\omega}{\lambda}}, \qquad e_{\lambda}(z) := e_{\lambda}^{1}(z) = (1+\lambda z)^{\frac{1}{\lambda}} \tag{1.6}$$

and

$$e_{\lambda}^{\omega}(z) = \sum_{\upsilon=0}^{\infty}(\omega)_{\upsilon,\lambda}\frac{z^{\upsilon}}{\upsilon!}, \tag{1.7}$$

where $(\omega)_{0,\lambda} = 1$, $(\omega)_{\upsilon,\lambda} = \omega(\omega-\lambda)(\omega-2\lambda)\cdots(\omega-(\upsilon-1)\lambda)$, $(\upsilon \geq 1)$.

The degenerate Bernoulli polynomials considered by Carlitz [1, 2] are given by

$$\frac{z}{e_\lambda(z)-1}e_\lambda^\omega(z)=\sum_{\upsilon=0}^{\infty}\beta_{\upsilon,\lambda}(\omega)\frac{z^\upsilon}{\upsilon!}. \tag{1.8}$$

When $\omega=0$, $\beta_{\upsilon,\lambda}=\beta_{\upsilon,\lambda}(0)$ denotes degenerate Bernoulli numbers.

In 2019, Jang and Kim [7] introduced type 2 degenerate Bernoulli polynomials as follows:

$$\frac{z}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}e_\lambda^\omega(z)=\sum_{\upsilon=0}^{\infty}B_{\upsilon,\lambda}(\omega)\frac{z^\upsilon}{\upsilon!}. \tag{1.9}$$

When $\omega=0$, $B_{\upsilon,\lambda}=B_{\upsilon,\lambda}(0)$ are type 2 degenerate Bernoulli numbers.

We note that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{z}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}e_\lambda^\omega(z)&=\sum_{\upsilon=0}^{\infty}\lim_{\lambda\to 0}B_{\upsilon,\lambda}(\omega)\frac{z^\upsilon}{\upsilon!}\\&=\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}e^{\omega z}=\sum_{\upsilon=0}^{\infty}B_\upsilon(\omega)\frac{z^\upsilon}{\upsilon!},\end{aligned} \tag{1.10}$$

are called type 2 Bernoulli polynomials $B_\upsilon(\omega)$, $(\upsilon\ge 0)$.

The degenerate form of central Bell polynomials is given as (see [20])

$$e^{\omega(e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z))}=\sum_{\upsilon=0}^{\infty}Bel_{\upsilon,\lambda}^{(c)}(\omega)\frac{z^\upsilon}{\upsilon!}. \tag{1.11}$$

For $\omega=1$, $Bel_{\upsilon,\lambda}^{(c)}=Bel_{\upsilon,\lambda}^{(c)}(1)$ denotes degenerate central Bell numbers.

For $\lambda\in\mathbb{R}$, Kim and Kim [15] defined the degenerate version of the logarithm function denoted by $\log_\lambda(1+t)$ as follows:

$$\log_\lambda(1+z)=\sum_{\upsilon=1}^{\infty}\lambda^{\upsilon-1}(1)_{\upsilon,1/\lambda}\frac{z^\upsilon}{\upsilon!}, \tag{1.12}$$

being the inverse of the degenerate version of the exponential function $e_\lambda(z)$ as has been shown below:

$$e_\lambda\big(\log_\lambda(z)\big)=\log_\lambda\big(e_\lambda(z)\big)=z.$$

It is noteworthy to mention that

$$\lim_{\lambda\to 0}\log_\lambda(1+z)=\sum_{\upsilon=1}^{\infty}(-1)^{\upsilon-1}\frac{z^\upsilon}{\upsilon!}=\log(1+z).$$

The degenerate Daehee polynomials of order r are defined by (see [28, 41])

$$\left(\frac{\log_\lambda(1+z)}{z}\right)^r(1+z)^\omega=\sum_{\upsilon=0}^{\infty}D_{\upsilon,\lambda}^{(r)}(\omega)\frac{z^\upsilon}{\upsilon!}. \tag{1.13}$$

If $\omega = 0$, $D^{(r)}_{\upsilon,\lambda} = D^{(r)}_{\upsilon,\lambda}(0)$, $(\upsilon \geq 0)$ denotes degenerate Daehee numbers of order r.
For $r = 1$, (1.13) reduces to

$$\frac{\log_{\lambda}(1+z)}{z}(1+z)^{\omega} = \sum_{\upsilon=0}^{\infty} D_{\upsilon,\lambda}(\omega)\frac{z^{\upsilon}}{\upsilon!}. \tag{1.14}$$

If $\omega = 0$, $D_{\upsilon,\lambda} = D_{\upsilon,\lambda}(0)$, $(\upsilon \geq 0)$ denotes degenerate Daehee numbers.
The degenerate Bernoulli polynomials of the second kind are specified by

$$\frac{z}{\log_{\lambda}(1+z)}(1+z)^{\omega} = \sum_{\upsilon=0}^{\infty} b_{\upsilon,\lambda}(\omega)\frac{z^{\upsilon}}{\upsilon!} \quad (\text{see [18]}). \tag{1.15}$$

When $\omega = 0$, $b_{\upsilon,\lambda} = b_{\upsilon,\lambda}(0)$ $(\upsilon \geq 0)$ denotes degenerate Bernoulli numbers of the second kind.

The degenerate form of first kind Stirling numbers is defined by

$$\frac{1}{\kappa!}\bigl(\log_{\lambda}(1+z)\bigr)^{\kappa} = \sum_{\upsilon=\kappa}^{\infty} S_{1,\lambda}(\upsilon,\kappa)\frac{z^{\upsilon}}{\upsilon!} \quad (\kappa \geq 0), \ (\text{see [21, 30]}). \tag{1.16}$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(\upsilon,\kappa) = S_1(\upsilon,\kappa)$, where $S_1(\upsilon,\kappa)$ are first kind Stirling numbers given by

$$\frac{1}{\kappa!}\bigl(\log(1+z)\bigr)^{\kappa} = \sum_{\upsilon=\kappa}^{\infty} S_{1}(\upsilon,\kappa)\frac{z^{\upsilon}}{\upsilon!} \quad (\kappa \geq 0), \ (\text{see [1–14, 17, 21, 26, 28]}).$$

The degenerate form of second kind Stirling numbers is defined by

$$\frac{1}{\kappa!}\bigl(e_{\lambda}(z)-1\bigr)^{\kappa} = \sum_{\upsilon=\kappa}^{\infty} S_{2,\lambda}(\upsilon,\kappa)\frac{z^{\upsilon}}{\upsilon!} \quad (\text{see [17]}). \tag{1.17}$$

Here, $\lim_{\lambda\to 0} S_{2,\lambda}(\upsilon,\kappa) = S_2(\upsilon,\kappa)$, where $S_2(\upsilon,\kappa)$ are second kind Stirling numbers given by

$$\frac{1}{\kappa!}\bigl(e^{z}-1\bigr)^{\kappa} = \sum_{\upsilon=\kappa}^{\infty} S_{2}(\upsilon,\kappa)\frac{z^{\upsilon}}{\upsilon!} \quad (\text{see [1–43]}). \tag{1.18}$$

The degenerate form of second kind central factorial polynomials [3] is given as

$$\frac{1}{\kappa!}\bigl(e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)\bigr)^{\kappa} e_{\lambda}^{\omega}(z) = \sum_{j=\kappa}^{\infty} T_{2,\lambda}(j,\kappa\,|\,\omega)\frac{z^{j}}{j!} \quad (\kappa \geq 0). \tag{1.19}$$

If $\omega = 0$, $T_{2,\lambda}(j,\kappa) = T_{2,\lambda}(j,\kappa\,|\,0)$ denotes second kind degenerate central factorial numbers.

This article aims to present type 2 degenerate poly-Bernoulli numbers and polynomials arising from moderated degenerate polyexponential functions. Certain explicit expressions for these numbers and polynomials are derived. Also, we introduce type 2 degenerate unipoly-Bernoulli numbers and polynomials by utilizing unipoly functions and show some basic properties of them.

2 Type 2 degenerate poly-Bernoulli polynomials and numbers

For $\kappa \in \mathbb{Z}$, and utilizing the modified degenerate polyexponential functions, we consider type 2 degenerate poly-Bernoulli polynomials as

$$\frac{\mathrm{Ei}_{\kappa,\lambda}(\log_\lambda(1+z))}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}e_\lambda^{\omega}(z)=\sum_{\upsilon=0}^{\infty}\beta_{\upsilon,\lambda}^{(\kappa)}(\omega)\frac{z^\upsilon}{\upsilon!}. \tag{2.1}$$

If $\omega=0$, $\beta_{\upsilon,\lambda}^{(\kappa)}=\beta_{\upsilon,\lambda}^{(\kappa)}(0)$ denotes type 2 degenerate poly-Bernoulli numbers.

For $\kappa=1$, by using (2.1) and (1.2), we see that

$$\frac{\mathrm{Ei}_{1,\lambda}(\log_\lambda(1+z))}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}e_\lambda^{\omega}(z)=\frac{z}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}e_\lambda^{\omega}(z)=\sum_{\upsilon=0}^{\infty}B_{\upsilon,\lambda}(\omega)\frac{z^\upsilon}{\upsilon!}\quad(\upsilon\geq 0), \tag{2.2}$$

where $B_{\upsilon,\lambda}(\omega)$ are called type 2 degenerate Bernoulli polynomials (see [3]).

First, we can write that

$$\begin{aligned}\mathrm{Ei}_{\kappa,\lambda}\big(\log_\lambda(1+z)\big)&=\sum_{i=1}^{\infty}\frac{(1)_{i,\lambda}(\log_\lambda(1+z))^i}{(i-1)!i^\kappa}\\&=\sum_{i=1}^{\infty}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}\frac{(\log_\lambda(1+z))^i}{i!}\\&=\sum_{i=1}^{\infty}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}\sum_{\upsilon=i}^{\infty}S_{1,\lambda}(\upsilon,i)\frac{z^\upsilon}{\upsilon!}\\&=\sum_{\upsilon=1}^{\infty}\left(\sum_{i=1}^{\upsilon}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}S_{1,\lambda}(\upsilon,i)\right)\frac{z^\upsilon}{\upsilon!}.\end{aligned} \tag{2.3}$$

By (2.3), we see that (2.1) is equal to

$$\begin{aligned}\frac{\mathrm{Ei}_{\kappa,\lambda}(\log_\lambda(1+z))}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}&=\frac{1}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}\sum_{r=1}^{\infty}\left(\sum_{i=1}^{r}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}S_{1,\lambda}(r,i)\right)\frac{z^r}{r!}\\&=\frac{z}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}\sum_{r=0}^{\infty}\left(\sum_{i=1}^{r+1}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}\frac{S_{1,\lambda}(r+1,i)}{r+1}\right)\frac{z^r}{r!}\\&=\sum_{\upsilon=0}^{\infty}B_{\upsilon,\lambda}\frac{z^\upsilon}{\upsilon!}\sum_{r=0}^{\infty}\left(\sum_{i=1}^{r+1}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}\frac{S_{1,\lambda}(r+1,i)}{l+1}\right)\frac{z^r}{r!}\\&=\sum_{\upsilon=0}^{\infty}\left(\sum_{r=0}^{\upsilon}\binom{\upsilon}{r}\sum_{i=1}^{r+1}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}\frac{S_{1,\lambda}(r+1,i)}{r+1}B_{\upsilon-l,\lambda}\right)\frac{z^\upsilon}{\upsilon!}.\end{aligned} \tag{2.4}$$

Therefore, by (2.1) and (2.4), we arrive at the following theorem.

Theorem 2.1 *For $\kappa\in\mathbb{Z}$ and $\upsilon\geq 0$, we have*

$$\beta_{\upsilon,\lambda}^{(\kappa)}=\sum_{r=0}^{\upsilon}\binom{\upsilon}{r}\sum_{i=1}^{r+1}\frac{(1)_{i,\lambda}}{i^{\kappa-1}}\frac{S_{1,\lambda}(r+1,i)}{l+1}B_{\upsilon-r,\lambda}.$$

Corollary 2.1 *Putting $k = 1$ in Theorem 2.1 yields*

$$\beta_{\upsilon,\lambda} = \sum_{r=0}^{\upsilon}\binom{\upsilon}{r}\sum_{i=1}^{r+1}\frac{S_{1,\lambda}(r+1,i)}{r+1}(1)_{i,\lambda}B_{\upsilon-r,\lambda}.$$

Remark 2.1 Letting λ to 0 in Theorem 2.1 leads to

$$\beta_{\upsilon}^{(\kappa)} = \sum_{r=0}^{\upsilon}\binom{\upsilon}{r}\sum_{i=1}^{r+1}\frac{S_1(r+1,i)}{i^{\kappa-1}r+1}B_{\upsilon-r} \quad (\upsilon \geq 0).$$

Using (1.1), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{Ei}_{\kappa,\lambda}\bigl(\log_\lambda(1+z)\bigr) &= \frac{\mathrm{d}}{\mathrm{dz}}\sum_{\upsilon=1}^{\infty}\frac{(1)_{\upsilon,\lambda}(\log_\lambda(1+z))^{\upsilon}}{(\upsilon-1)!\upsilon^{\kappa}} \\ &= \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\sum_{\upsilon=1}^{\infty}\frac{(1)_{\upsilon,\lambda}(\log_\lambda(1+z))^{\upsilon}}{(\upsilon-1)!\upsilon^{\kappa-1}} \\ &= \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\mathrm{Ei}_{\kappa-1,\lambda}\bigl(\log_\lambda(1+z)\bigr).\end{aligned} \tag{2.5}$$

By (2.5), for $\kappa \geq 2$, we have

$$\mathrm{Ei}_{\kappa,\lambda}\bigl(\log_\lambda(1+z)\bigr) = \int_0^z \underbrace{\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\int_0^z \cdots \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}}_{(\kappa-2)\text{-times}}\int_0^z \frac{z(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\,dz\cdots dz. \tag{2.6}$$

Then, from (2.1) and (2.6), we have

$$\begin{aligned}\sum_{\upsilon=0}^{\infty}\beta_{\upsilon,\lambda}^{(\kappa)}\frac{z^{\upsilon}}{\upsilon!} &= \frac{1}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}\mathrm{Ei}_{\kappa,\lambda}\bigl(\log_\lambda(1+z)\bigr) \\ &= \frac{1}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}\int_0^z \underbrace{\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\int_0^z \cdots \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}}_{(\kappa-2)\text{-times}}\int_0^z \frac{z(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\,dz\cdots dz \\ &= \frac{z}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}\sum_{i=0}^{\infty}\sum_{i_1+\cdots+i_{\kappa-1}=i}\binom{i}{i_1+\cdots+i_{\kappa-1}} \\ &\quad\times \frac{b_{i_1,\lambda}(\lambda-1)}{i_1+1}\frac{b_{i_2,\lambda}(\lambda-1)}{i_1+i_2+1}\cdots\frac{b_{i_{\kappa-1},\lambda}(\lambda-1)}{i_1+\cdots+i_{\kappa-1}+1}\frac{z^i}{i!} \\ &= \sum_{\upsilon=0}^{\infty}\sum_{i=0}^{\upsilon}\binom{\upsilon}{i}\sum_{i_1+\cdots+i_{\kappa-1}=i}\binom{i}{i_1+\cdots+i_{\kappa-1}} \\ &\quad\times \frac{b_{i_1,\lambda}(\lambda-1)}{i_1+1}\frac{b_{i_2,\lambda}(\lambda-1)}{i_1+i_2+1}\cdots\frac{b_{i_{\kappa-1},\lambda}(\lambda-1)}{i_1+\cdots+i_{\kappa-1}+1}B_{\upsilon-i,\lambda}\frac{z^{\upsilon}}{\upsilon!}.\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we arrive at the following theorem.

Theorem 2.2 *For $\upsilon \geq 0$, then*

$$\beta_{\upsilon,\lambda}^{(\kappa)} = \sum_{i=0}^{\upsilon} \binom{\upsilon}{i} \sum_{i_1+\cdots+i_{\kappa-1}=i} \binom{i}{i_1+\cdots+i_{\kappa-1}} \\ \times \frac{b_{i_1,\lambda}(\lambda-1)}{i_1+1} \frac{b_{i_2,\lambda}(\lambda-1)}{i_1+i_2+1} \cdots \frac{b_{i_{\kappa-1},\lambda}(\lambda-1)}{i_1+\cdots+i_{\kappa-1}+1} B_{\upsilon-i,\lambda}. \tag{2.8}$$

By making use of (1.7) and (2.1), we note that

$$\begin{aligned} \sum_{\upsilon=0}^{\infty} \beta_{\upsilon,\lambda}^{(\kappa)}(\omega) \frac{z^{\upsilon}}{\upsilon!} &= \frac{\mathrm{Ei}_{\kappa,\lambda}(\log_{\lambda}(1+z))}{e_{\lambda}^{\frac{1}{2}}(z) - e_{\lambda}^{-\frac{1}{2}}(z)} e_{\lambda}^{\omega}(z) \\ &= \sum_{r=0}^{\infty} \beta_{r,\lambda}^{(\kappa)} \frac{z^r}{r!} \sum_{\upsilon=0}^{\infty} (\omega)_{\upsilon,\lambda} \frac{z^{\upsilon}}{\upsilon!} \\ &= \sum_{\upsilon=0}^{\infty} \left(\sum_{r=0}^{\upsilon} \binom{\upsilon}{r} \beta_{r,\lambda}^{(\kappa)}(\omega)_{\upsilon-r,\lambda} \right) \frac{z^{\upsilon}}{\upsilon!}, \end{aligned} \tag{2.9}$$

which on comparing the coefficients on both sides of the above equation yields the following theorem.

Theorem 2.3 *For $\upsilon \geq 0$, we have*

$$\beta_{\upsilon,\lambda}^{(\kappa)}(\omega) = \sum_{r=0}^{\upsilon} \binom{\upsilon}{r} \beta_{r,\lambda}^{(\kappa)}(\omega)_{\upsilon-r,\lambda}.$$

Let $\kappa \geq 1$ be an integer. For $s \in \mathbb{C}$, the function $\eta_{\kappa,\lambda}(s)$ can be defined as

$$\begin{aligned} \eta_{\kappa,\lambda}(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{z^{s-1}}{e_{\lambda}^{\frac{1}{2}}(z) - e_{\lambda}^{-\frac{1}{2}}(z)} \mathrm{Ei}_{\kappa,\lambda}\bigl(\log_{\lambda}(1+z)\bigr)\, dz \\ &= \frac{1}{\Gamma(s)} \int_0^{1} \frac{z^{s-1}}{e_{\lambda}^{\frac{1}{2}}(z) - e_{\lambda}^{-\frac{1}{2}}(z)} \mathrm{Ei}_{\kappa,\lambda}\bigl(\log_{\lambda}(1+z)\bigr)\, dz \\ &\quad + \frac{1}{\Gamma(s)} \int_1^{\infty} \frac{z^{s-1}}{e_{\lambda}^{\frac{1}{2}}(z) - e_{\lambda}^{-\frac{1}{2}}(z)} \mathrm{Ei}_{\kappa,\lambda}\bigl(\log_{\lambda}(1+z)\bigr)\, dz. \end{aligned} \tag{2.10}$$

Here, for any $s \in \mathbb{C}$, the second integral converges absolutely; hence, the second term on the r.h.s. vanishes at nonpositive integers, i.e.,

$$\lim_{s \to -n} \left| \frac{1}{\Gamma(s)} \int_1^{\infty} \frac{z^{s-1}}{e_{\lambda}^{\frac{1}{2}}(z) - e_{\lambda}^{-\frac{1}{2}}(z)} \mathrm{Ei}_{\kappa,\lambda}\bigl(\log_{\lambda}(1+z)\bigr)\, dz \right| \leq \frac{1}{\Gamma(-n)} N = 0. \tag{2.11}$$

On the other hand, for $\Re(s) > 0$, we can write the first integral in (2.10) as

$$\frac{1}{\Gamma(s)} \sum_{r=0}^{\infty} \frac{\beta_{r,\lambda}^{(\kappa)}}{r!} \frac{1}{s+r},$$

which defines an entire function of s. Therefore, we may say that $\eta_{\kappa,\lambda}(s)$ can be continued to an entire function of s.

Further, from (2.10) and (2.11), we get

$$\begin{aligned}
\eta_{\kappa,\lambda}(-n) &= \lim_{s\to -n}\frac{1}{\Gamma(s)}\int_0^1 \frac{z^{s-1}}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)}\operatorname{Ei}_{\kappa,\lambda}\bigl(\log_\lambda(1+z)\bigr)\,dz \\
&= \lim_{s\to -n}\frac{1}{\Gamma(s)}\int_0^1 z^{s-1}\sum_{r=0}^{\infty}\frac{B_{r,\lambda}^{(\kappa)}z^r}{r!}\,dz = \lim_{s\to -n}\frac{1}{\Gamma(s)}\sum_{r=0}^{\infty}\frac{\beta_{r,\lambda}^{(\kappa)}}{s+r}\frac{1}{r!} \\
&= \cdots + 0 + \cdots + 0 + \lim_{s\to -n}\frac{1}{\Gamma(s)}\frac{1}{s+n}\frac{\beta_{n,\lambda}^{(\kappa)}}{n!} + 0 + 0 + \cdots \\
&= \lim_{s\to -n}\frac{\left(\frac{\Gamma(1-s)\sin\pi s}{\pi}\right)}{s+n}\frac{\beta_{n,\lambda}^{(\kappa)}}{n!} = \Gamma(1+n)\cos(\pi n)\frac{\beta_{n,\lambda}^{(\kappa)}}{n!} \\
&= (-1)^n\beta_{n,\lambda}^{(\kappa)}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we arrive at the following theorem.

Theorem 2.4 *Let $\kappa \geq 1$ and $n \in \mathbb{N}\cup\{0\}$, $s \in \mathbb{C}$. Then*

$$\eta_{\kappa,\lambda}(-n) = (-1)^n\beta_{n,\lambda}^{(\kappa)}.$$

By making use of (1.2), we note that

$$\begin{aligned}
\operatorname{Ei}_{1,\lambda}\bigl(\log_\lambda(1+z)\bigr) &= \sum_{i=1}^{\infty}\frac{(1)_{i,\lambda}(\log_\lambda(1+z))^i}{i!} \\
&= \sum_{i=1}^{\infty}\frac{(1)_{i,\lambda}(\log_\lambda(1+z))^i}{i!} \\
&= \sum_{i=1}^{\infty}\sum_{\upsilon=i}^{\infty}(1)_{i,\lambda}S_{1,\lambda}(\upsilon,i)\frac{z^\upsilon}{\upsilon!} \\
&= \sum_{\upsilon=1}^{\infty}\left(\sum_{i=1}^{\upsilon}(1)_{i,\lambda}S_{1,\lambda}(\upsilon,i)\right)\frac{z^\upsilon}{\upsilon!}.
\end{aligned} \tag{2.13}$$

On the other hand,

$$\operatorname{Ei}_{1,\lambda}\bigl(\log_\lambda(1+z)\bigr) = z. \tag{2.14}$$

Therefore, by (2.13) and (2.14), we arrive at the following theorem.

Theorem 2.5 *For $\upsilon \in \mathbb{N}$, then*

$$\sum_{i=1}^{\upsilon}(1)_{i,\lambda}S_{1,\lambda}(\upsilon,i) = \delta_{\upsilon,1},$$

where $\delta_{\upsilon,\kappa}$ is the Kronecker delta.

From (2.1), we note that

$$\frac{\mathrm{Ei}_{\kappa,\lambda}(\log_\lambda(1+z))}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)} = \sum_{\upsilon=0}^{\infty}\beta_{\upsilon,\lambda}^{(\kappa)}\frac{z^\upsilon}{\upsilon!}. \tag{2.15}$$

On replacing z with $e_\lambda(z)-1$ in (2.15), we get

$$\begin{aligned}
&\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(\kappa)}\frac{(e_\lambda(z)-1)^m}{m!}\\
&\quad= \frac{\mathrm{Ei}_{\kappa,\lambda}(z)}{e_\lambda^{\frac{1}{2}}(e_\lambda(z)-1)-e_\lambda^{-\frac{1}{2}}(e_\lambda(z)-1)}\\
&\quad= \frac{e_\lambda(z)-1}{e_\lambda^{\frac{1}{2}}(e_\lambda(z)-1)-e_\lambda^{-\frac{1}{2}}(e_\lambda(z)-1)}\frac{z}{e_\lambda(z)-1}\frac{\mathrm{Ei}_{\kappa,\lambda}(z)}{z}\\
&\quad= \sum_{i=0}^{\infty}B_{i,\lambda}\frac{1}{i!}(e_\lambda(z)-1)^i\sum_{j=0}^{\infty}B_{j,\lambda}\frac{z^j}{j!}\sum_{\upsilon=0}^{\infty}\frac{(1)_{\upsilon+1,\lambda}z^\upsilon}{\upsilon!(\upsilon+1)^\kappa}\\
&\quad= \sum_{i=0}^{\infty}B_{i,\lambda}\sum_{l=i}^{\infty}S_{2,\lambda}(l,i)\frac{z^l}{l!}\sum_{j=0}^{\infty}B_{j,\lambda}\frac{z^j}{j!}\sum_{\upsilon=0}^{\infty}\frac{(1)_{\upsilon+1,\lambda}z^\upsilon}{\upsilon!(\upsilon+1)^\kappa}\\
&\quad= \sum_{l=0}^{\infty}\sum_{i=0}^{l}B_{i,\lambda}S_{2,\lambda}(l,i)\frac{z^l}{l!}\sum_{j=0}^{\infty}B_{j,\lambda}\frac{z^j}{j!}\sum_{\upsilon=0}^{\infty}\frac{(1)_{\upsilon+1,\lambda}z^\upsilon}{\upsilon!(\upsilon+1)^\kappa}\\
&\quad= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\sum_{i=0}^{l}\binom{j}{l}B_{i,\lambda}S_{2,\lambda}(l,i)B_{j-l,\lambda}\right)\frac{z^j}{j!}\times\sum_{\upsilon=0}^{\infty}\frac{(1)_{\upsilon+1,\lambda}z^\upsilon}{\upsilon!(\upsilon+1)^\kappa}\\
&\quad= \sum_{\upsilon=0}^{\infty}\left(\sum_{j=0}^{\upsilon}\sum_{l=0}^{j}\sum_{i=0}^{l}\binom{j}{l}\binom{\upsilon}{j}B_{i,\lambda}S_{2,\lambda}(l,i)B_{j-l,\lambda}\frac{(1)_{\upsilon-j+1,\lambda}}{(\upsilon-j+1)^\kappa}\right)\frac{z^\upsilon}{\upsilon!}.
\end{aligned} \tag{2.16}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(\kappa)}\frac{(e_\lambda(z)-1)^m}{m!} &= \sum_{m=0}^{\infty}\beta_{m,\lambda}^{(\kappa)}\sum_{\upsilon=m}^{\infty}S_{2,\lambda}(\upsilon,m)\frac{z^\upsilon}{\upsilon!}\\
&= \sum_{\upsilon=0}^{\infty}\left(\sum_{m=0}^{\upsilon}\beta_{m,\lambda}^{(\kappa)}S_{2,\lambda}(\upsilon,m)\right)\frac{z^\upsilon}{\upsilon!}.
\end{aligned} \tag{2.17}$$

Therefore, by (2.16) and (2.17), we arrive at the following theorem.

Theorem 2.6 *For $\upsilon\in\mathbb{N}$, we have*

$$\sum_{m=0}^{\upsilon}\beta_{m,\lambda}^{(\kappa)}S_{2,\lambda}(\upsilon,m) = \sum_{j=0}^{\upsilon}\sum_{l=0}^{j}\sum_{i=0}^{l}\binom{j}{l}\binom{\upsilon}{j}B_{i,\lambda}S_{2,\lambda}(l,i)B_{j-l,\lambda}\frac{(1)_{\upsilon-j+1,\lambda}}{(\upsilon-j+1)^\kappa}.$$

The higher-order type 2 degenerate Bernoulli polynomials of order $r \in \mathbb{N}$ are defined by (see [7])

$$\left(\frac{z}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)}\right)^r e_\lambda^\omega(z) = \sum_{\upsilon=0}^{\infty} B_{\upsilon,\lambda}^{(r)}(\omega)\frac{z^\upsilon}{\upsilon!}. \tag{2.18}$$

For $\omega = 0$, $B_{\upsilon,\lambda}^{(r)}(0) = B_{\upsilon,\lambda}^{(r)}$ denotes higher-order type 2 degenerate Bernoulli numbers of order $r \in \mathbb{N}$.

From (1.18) and (2.18), we get

$$\binom{\upsilon+\kappa}{\kappa} B_{\upsilon,\lambda}^{(-\kappa)} = T_{2,\lambda}(\upsilon+\kappa,\kappa),$$

where υ, κ are nonnegative positive integers.

Replacing z with $\log(1+z)$ in (2.18), we get

$$\begin{aligned}\sum_{m=0}^{\infty} B_{m,\lambda}^{(r)}\frac{(\log(1+z))^m}{m!} &= \left(\frac{\log(1+z)}{(1+\lambda\log(1+z))^{\frac{1}{2\lambda}} - (1+\lambda\log(1+z))^{-\frac{1}{2\lambda}}}\right)^r \\ &= \sum_{\upsilon=0}^{\infty} D_{\upsilon,\lambda}^{*(r)}\frac{z^\upsilon}{\upsilon!}. \end{aligned} \tag{2.19}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty} B_{m,\lambda}^{(r)}\frac{(\log(1+z))^m}{m!} &= \sum_{m=0}^{\infty} B_{m,\lambda}^{(r)}\sum_{\upsilon=m}^{\infty} S_1(\upsilon,m)\frac{z^\upsilon}{\upsilon!} \\ &= \sum_{\upsilon=0}^{\infty}\left(\sum_{m=0}^{\upsilon} B_{m,\lambda}^{(r)} S_1(\upsilon,m)\right)\frac{z^\upsilon}{\upsilon!}. \end{aligned} \tag{2.20}$$

Therefore, by (2.19) and (2.20), we arrive at the following theorem.

Theorem 2.7 *For $\upsilon \geq 0$, we have*

$$D_{\upsilon,\lambda}^{*(r)} = \sum_{m=0}^{\upsilon} B_{m,\lambda}^{(r)} S_1(\upsilon,m).$$

Now, we introduce type 2 degenerate central poly-Bell polynomials as

$$\mathrm{Ei}_{\kappa,\lambda}\big(\omega\big(e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)\big)\big) = \sum_{\upsilon=1}^{\infty} Bel_{\upsilon,\lambda}^{(\kappa,c)}(\omega)\frac{z^\upsilon}{\upsilon!}, \qquad Bel_{\upsilon,\lambda}^{(\kappa,c)}(\omega) = 0 \quad (\kappa \in \mathbb{Z}). \tag{2.21}$$

When $\omega = 1$, $Bel_{\upsilon,\lambda}^{(\kappa,c)} = Bel_{\upsilon,\lambda}^{(\kappa,c)}(1)$ are called type 2 degenerate poly-Bell numbers.

From (2.21), we note that

$$\mathrm{Ei}_{\kappa,\lambda}\big(\omega\big(e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)\big)\big) = \sum_{m=1}^{\infty}\frac{\omega^m(e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z))^m}{(m-1)!n^\kappa}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{\omega^m (e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z))^m}{(m-1)! m^\kappa} \\
&= \sum_{m=1}^{\infty} \frac{\omega^m}{m^{\kappa-1}} \sum_{\upsilon=m}^{\infty} T_{2,\lambda}(\upsilon, m) \frac{z^\upsilon}{\upsilon!} \\
&= \sum_{\upsilon=1}^{\infty} \left(\sum_{m=1}^{\upsilon} \frac{\omega^m}{m^{\kappa-1}} T_{2,\lambda}(\upsilon, m) \right) \frac{z^\upsilon}{\upsilon!}.
\end{aligned}
\tag{2.22}
$$

Thus, by (2.21) and (2.22), we obtain the following theorem.

Theorem 2.8 *For $\kappa \in \mathbb{Z}$ and $\upsilon \in \mathbb{N}$, we have*

$$
Bel_{\upsilon,\lambda}^{(\kappa,c)}(\omega) = \sum_{m=1}^{\upsilon} \frac{\omega^m}{m^{\kappa-1}} T_{2,\lambda}(\upsilon, m).
$$

3 Type 2 degenerate unipoly-Bernoulli polynomials and numbers

The unipoly function attached to polynomials $p(\omega)$ was defined by Kim and Kim [14] as

$$
u_\kappa(\omega|p) = \sum_{\upsilon=1}^{\infty} \frac{p(\upsilon)}{\upsilon^\kappa} \omega^\upsilon \quad (\kappa \in \mathbb{Z}),
\tag{3.1}
$$

where p denotes any arithmetic real or complex-valued function defined on $\mathbb{N}$.

Moreover,

$$
u_\kappa(\omega|1) = \sum_{\upsilon=1}^{\infty} \frac{\omega^\upsilon}{\upsilon^\kappa} = \mathrm{Li}_\kappa(\omega), \quad \text{(see [9])}
\tag{3.2}
$$

is the ordinary polylogarithm function.

Dolgy and Khan [4] introduced the degenerate unipoly function attached to polynomials $p(\omega)$ as follows:

$$
u_{\kappa,\lambda}(\omega|p) = \sum_{i=1}^{\infty} p(i) \frac{(1)_{i,\lambda} \omega^i}{i^\kappa}.
\tag{3.3}
$$

It is worthy to see that

$$
u_{\kappa,\lambda}\left(\omega \middle| \frac{1}{\Gamma}\right) = \mathrm{Ei}_{\kappa,\lambda}(\omega), \quad \text{(see [30])}
\tag{3.4}
$$

is the moderated degenerate polyexponential function.

Now, we define type 2 degenerate unipoly-Bernoulli polynomials which are given by the generating function as follows:

$$
\frac{u_{\kappa,\lambda}(\log_\lambda(1+z)|p)}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)} e_\lambda^\omega(z) = \sum_{r=0}^{\infty} \beta_{r,\lambda,p}^{(\kappa)}(\omega) \frac{z^r}{r!}.
\tag{3.5}
$$

For $\omega = 0$, $\beta_{r,\lambda,p}^{(\kappa)} = \beta_{r,\lambda,p}^{(\kappa)}(0)$ denotes type 2 degenerate unipoly-Bernoulli numbers attached to p.

By (3.5), we see that

$$\begin{aligned}\sum_{\upsilon=0}^{\infty}\beta^{(\kappa)}_{\upsilon,\lambda,\frac{1}{\Gamma}}\frac{z^{\upsilon}}{\upsilon!}&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}u_{\kappa,\lambda}\left(\log_{\lambda}(1+z)\Big|\frac{1}{\Gamma}\right)\\&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}\sum_{r=1}^{\infty}\frac{(1)_{r,\lambda}(\log_{\lambda}(1+z))^{r}}{r^{\kappa}(r-1)!}\\&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}u_{\kappa,\lambda}\big(\log_{\lambda}(1+z)\big)=\sum_{\upsilon=0}^{\infty}\beta^{(\kappa)}_{\upsilon,\lambda}\frac{z^{\upsilon}}{\upsilon!}.\end{aligned}\tag{3.6}$$

Thus, by (3.6), we have

$$\beta^{(\kappa)}_{\upsilon,\lambda,\frac{1}{\Gamma}}=\beta^{(\kappa)}_{\upsilon,\lambda}.\tag{3.7}$$

By making use of (3.5), we see that

$$\begin{aligned}\sum_{\upsilon=0}^{\infty}\beta^{(\kappa)}_{\upsilon,\lambda,p}\frac{z^{\upsilon}}{\upsilon!}&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}u_{\kappa,\lambda}\big(\log_{\lambda}(1+z)\big)|p)\\&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}\sum_{r=1}^{\infty}\frac{p(r)(1)_{r,\lambda}}{r^{\kappa}}\big(\log_{\lambda}(1+z)\big)^{r}\\&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}\sum_{r=1}^{\infty}\frac{p(r)(1)_{r,\lambda}}{r^{\kappa}}\big(\log_{\lambda}(1+z)\big)^{r}\frac{r!}{r!}\\&=\frac{1}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}\sum_{r=1}^{\infty}\frac{p(r)(1)_{r,\lambda}r!}{r^{\kappa}}\sum_{i=r}^{\infty}S_{1,\lambda}(i,r)\frac{z^{i}}{i!}\\&=\frac{z}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}\sum_{i=0}^{\infty}\sum_{r=1}^{i+1}\frac{p(r)(1)_{r,\lambda}r!}{r^{\kappa}(i+1)}S_{1,\lambda}(i+1,r)\frac{z^{i}}{i!}\\&=\left(\sum_{j=0}^{\infty}B_{j,\lambda}\frac{z^{j}}{j!}\right)\left(\sum_{i=0}^{\infty}\sum_{r=1}^{i+1}\frac{p(r)(1)_{r,\lambda}r!}{r^{\kappa}(i+1)}S_{1,\lambda}(i+1,r)\frac{z^{i}}{i!}\right)\\&=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\sum_{r=1}^{i+1}\binom{j}{i}\frac{p(r)(1)_{r,\lambda}r!}{r^{\kappa}(i+1)}S_{1,\lambda}(i+1,r)B_{j-i,\lambda}\right)\frac{z^{j}}{j!},\end{aligned}\tag{3.8}$$

which yields the following theorem.

Theorem 3.1 *Let j be a nonnegative integer and $\kappa\in\mathbb{Z}$. Then*

$$\beta^{(\kappa)}_{j,\lambda,p}=\sum_{i=0}^{j}\sum_{r=1}^{i+1}\binom{j}{i}\frac{p(r)(1)_{r,\lambda}r!}{r^{\kappa}(i+1)}S_{1,\lambda}(i+1,r)B_{j-i,\lambda}.$$

Recalling from (3.5), we have

$$\sum_{j=0}^{\infty}\beta^{(\kappa)}_{j,\lambda,p}(\omega)\frac{z^{j}}{j!}=\frac{u_{\kappa,\lambda}(\log_{\lambda}(1+z))|p)}{e_{\lambda}^{\frac{1}{2}}(z)-e_{\lambda}^{-\frac{1}{2}}(z)}e_{\lambda}^{\omega}(z)$$

$$
\begin{aligned}
&= \frac{u_{\kappa,\lambda}(\log_\lambda(1+z))|p)}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)} \sum_{s=0}^{\infty}(\omega)_{s,\lambda}\frac{z^s}{s!} \\
&= \sum_{j=0}^{\infty}\beta_{j,\lambda,p}^{(\kappa)}\frac{z^j}{j!}\sum_{s=0}^{\infty}(\omega)_{s,\lambda}\frac{z^s}{s!} \\
&= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\binom{j}{s}\beta_{j-s,\lambda,p}^{(\kappa)}(\omega)_{s,\lambda}\right)\frac{z^j}{j!}.
\end{aligned} \tag{3.9}
$$

By Eq. (3.9), we get the following theorem.

Theorem 3.2 *Let j be a nonnegative integer and $\kappa \in \mathbb{Z}$. Then*

$$
\beta_{j,\lambda,p}^{(\kappa)}(\omega) = \sum_{s=0}^{j}\binom{j}{s}\beta_{j-s,\lambda,p}^{(\kappa)}(\omega)_{s,\lambda}.
$$

From (3.5), we have

$$
\begin{aligned}
\sum_{j=0}^{\infty}\beta_{j,\lambda,p}^{(\kappa)}(\omega)\frac{z^j}{j!} &= \frac{1}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)}u_{\kappa,\lambda}\big(\log_\lambda(1+z)|p\big)\big(e_\lambda(z)-1+1\big)^{\omega} \\
&= \frac{u_{\kappa,\lambda}(\log_\lambda(1+z)|p)}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)}\sum_{i=0}^{\infty}(\omega)_i\frac{(e_\lambda(z)-1)^i}{i!} \\
&= \sum_{j=0}^{\infty}\beta_{j,\lambda,p}^{(\kappa)}\frac{z^j}{j!}\sum_{i=0}^{\infty}(\omega)_i\sum_{q=i}^{\infty}S_{2,\lambda}(q,i)\frac{z^q}{q!} \\
&= \sum_{j=0}^{\infty}\beta_{j,\lambda,p}^{(\kappa)}\frac{z^j}{j!}\sum_{q=0}^{\infty}\sum_{i=0}^{q}(\omega)_i S_{2,\lambda}(q,i)\frac{z^q}{q!} \\
&= \sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\sum_{i=0}^{q}\binom{j}{q}(\omega)_i S_{2,\lambda}(q,i)\beta_{j-q,\lambda,p}^{(\kappa)}\right)\frac{z^j}{j!},
\end{aligned} \tag{3.10}
$$

which yields the following theorem.

Theorem 3.3 *Let j be a nonnegative integer and $\kappa \in \mathbb{Z}$. Then*

$$
\beta_{j,\lambda,p}^{(\kappa)}(\omega) = \sum_{q=0}^{j}\sum_{i=0}^{q}\binom{j}{q}(\omega)_i S_{2,\lambda}(q,i)\beta_{j-q,\lambda,p}^{(\kappa)}.
$$

Using (3.5), we have

$$
\begin{aligned}
\sum_{\upsilon=0}^{\infty}\beta_{\upsilon,\lambda,p}^{(\kappa)}\frac{z^\upsilon}{\upsilon!} &= \frac{1}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)}u_{\kappa,\lambda}\big(\log_\lambda(1+z)|p\big) \\
&= \frac{1}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)}\sum_{m=1}^{\infty}\frac{p(m)(1)_{m,\lambda}}{m^\kappa}\big(\log_\lambda(1+z)\big)^m \\
&= \frac{1}{e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{-\frac{1}{2}}(z)}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}}{(m+1)^\kappa}\big(\log_\lambda(1+z)\big)^{m+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\log_\lambda(1+z)}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)} \sum_{m=0}^{\infty} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^\kappa} \frac{(\log_\lambda(1+z))^m}{m!} \\
&= \frac{\log_\lambda(1+z)}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)} \sum_{m=0}^{\infty} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^\kappa} \sum_{l=m}^{\infty} S_{1,\lambda}(l,m)\frac{z^l}{l!} \\
&= \frac{\log_\lambda(1+z)}{z} \frac{z}{e_\lambda^{\frac{1}{2}}(z)-e_\lambda^{-\frac{1}{2}}(z)} \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^\kappa} S_{1,\lambda}(l,m)\frac{z^l}{l!} \\
&= \sum_{\upsilon=0}^{\infty} D_{\upsilon,\lambda}\frac{z^\upsilon}{\upsilon!} \sum_{j=0}^{\infty} B_{j,\lambda}\frac{z^j}{j!} \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^\kappa} S_{1,\lambda}(l,m)\frac{z^l}{l!} \\
&= \sum_{\upsilon=0}^{\infty}\sum_{j=0}^{\upsilon} \binom{\upsilon}{j} D_{\upsilon-j,\lambda}B_{j,\lambda}\frac{z^\upsilon}{\upsilon!} \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^\kappa} S_{1,\lambda}(l,m)\frac{z^l}{l!} \\
&= \sum_{\upsilon=0}^{\infty}\Biggl(\sum_{l=0}^{\upsilon}\sum_{m=0}^{l}\sum_{j=0}^{\upsilon-l} \binom{\upsilon-l}{j}\binom{\upsilon}{l} \\
&\quad \times D_{\upsilon-j-l,\lambda}B_{j,\lambda}\frac{p(m+1)(1)_{m+1,\xi} m!}{(m+1)^\kappa} S_{1,\lambda}(l,m)\Biggr)\frac{z^\upsilon}{\upsilon!}.
\end{aligned}
\tag{3.11}
$$

Therefore, by (3.5) and (3.11), we obtain the following theorem.

Theorem 3.4 *For $\upsilon \geq 0$ and $\kappa \in \mathbb{Z}$, we have*

$$
\beta_{\upsilon,\lambda,p}^{(\kappa)} = \sum_{l=0}^{\upsilon}\sum_{m=0}^{l}\sum_{j=0}^{\upsilon-l} \binom{\upsilon-l}{j}\binom{\upsilon}{l} D_{\upsilon-j-l,\lambda}B_{j,\lambda}\frac{p(m+1)(1)_{m+1,\lambda} m!}{(m+1)^\kappa} S_{1,\lambda}(l,m).
$$

4 Numerical computations

In this section, some numerical computations are done to calculate certain zeros of type 2 degenerate poly-Bernoulli polynomials and present certain interesting graphical representations. The first five members of $\beta_{\upsilon,\lambda}^{(\kappa)}(\omega)$ are obtained and given as follows:

$$
\begin{aligned}
\beta_{0,\lambda}^{(\kappa)}(\omega) &= -\frac{1}{\log 3}, \\
\beta_{1,\lambda}^{(\kappa)}(\omega) &= \frac{1}{24(\log 3)^2} + \frac{1}{3\log 3} - \frac{\omega}{\log 3}, \\
\beta_{2,\lambda}^{(\kappa)}(\omega) &= -\frac{2}{729(\log 3)^3} - \frac{5}{72(\log 3)^2} + \frac{\omega}{12(\log 3)^2} - \frac{43}{108\log 3} + \frac{\omega}{\log 3} - \frac{\omega^2}{\log 3}, \\
\beta_{3,\lambda}^{(\kappa)}(\omega) &= \frac{8}{729(\log 3)^3} - \frac{2\omega}{243(\log 3)^3} + \frac{151}{864(\log 3)^2} - \frac{\omega}{4(\log 3)^2} + \frac{\omega^2}{8(\log 3)^2} + \frac{26}{27\log 3} \\
&\quad - \frac{7\omega}{4\log 3} + \frac{2\omega^2}{\log 3} - \frac{\omega^3}{\log 3},
\end{aligned}
$$

$$\beta_{4,\lambda}^{(\kappa)}(\omega) = -\frac{313}{6561(\log 3)^3} + \frac{4\omega}{81(\log 3)^3} - \frac{4\omega^2}{243(\log 3)^3} - \frac{805}{1296(\log 3)^2} + \frac{7\omega}{8(\log 3)^2} - \frac{7\omega^2}{12(\log 3)^2} + \frac{\omega^3}{6(\log 3)^2} - \frac{63{,}667}{19{,}440\log 3} + \frac{31\omega}{6\log 3} - \frac{89\omega^2}{18\log 3} + \frac{10\omega^3}{3\log 3} - \frac{\omega^4}{\log 3}.$$

Next, we present some graphs showing the behavior of $\beta_{\upsilon,\lambda}^{(\kappa)}(\omega)$ for $\kappa = 4$ and $\lambda = \frac{1}{3}$; these graphs are given in Fig. 1.

The approximate solutions of $\beta_{20,\lambda}^{(\kappa)}(\omega) = 0$ when $\kappa = 4$ and $\lambda = \frac{1}{3}$ are calculated and displayed in Fig. 2.

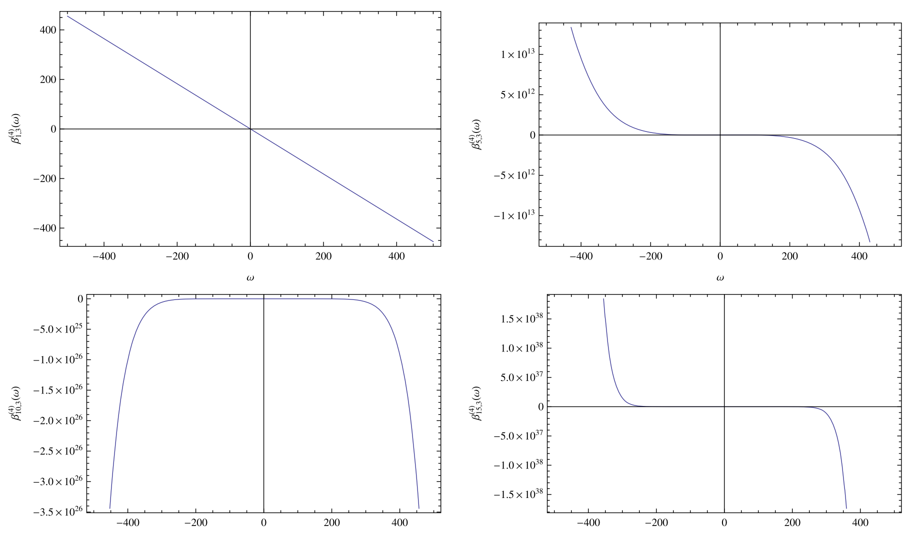

Figure 1 Graphs of $\beta_{\upsilon,\lambda}^{(\kappa)}(\omega)$

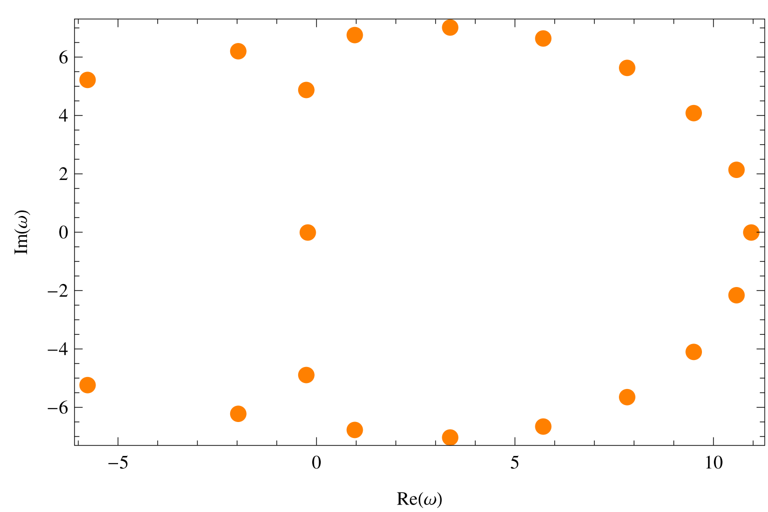

Figure 2 Zeros of $\beta_{20,\lambda}^{(\kappa)}(\omega)$

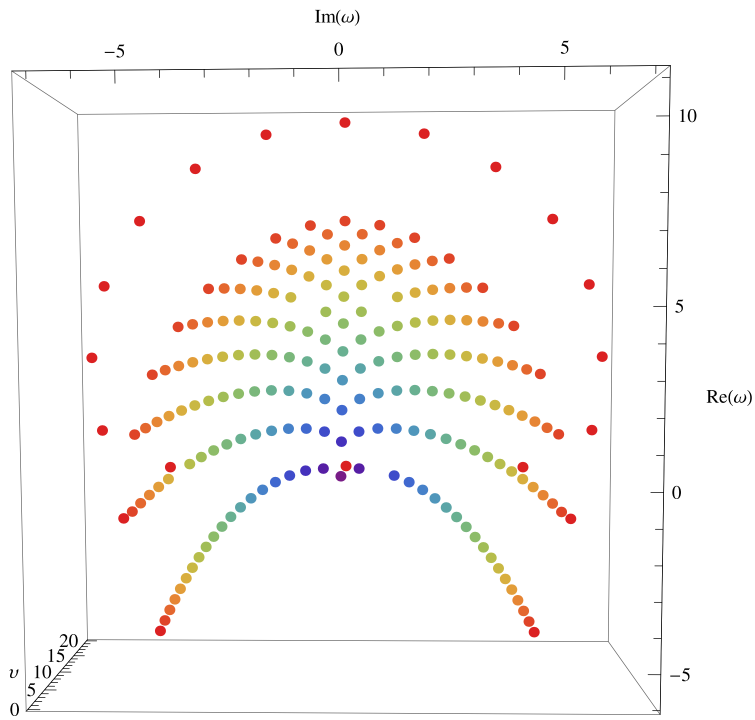

Figure 3 Stacking structure of zeros $\beta_{\upsilon,\lambda}^{(\kappa)}(\omega)$

Further, we calculate the approximate zeros of $\beta_{\upsilon,\lambda}^{(\kappa)}(\omega) = 0$ for $\kappa = 4$, $\lambda = \frac{1}{3}$ and $n = 1, 2, \ldots, 20$, and show the stacking structure of these zeros in Fig. 3.

5 Conclusions

In this paper, we have studied and introduced degenerate versions of type 2 Bernoulli numbers and polynomials and derived some properties of these polynomials. We have given some relationships between higher-order Bernoulli polynomials, degenerate type 2 Bernoulli polynomials, degenerate central Bell polynomials, degenerate Stirling numbers of the first and second kind, degenerate central factorials numbers. Besides, we have introduced degenerate type 2 unipoly-Bernoulli polynomials by using degenerate unipoly polynomials and derived some identities of these polynomials. We have derived some relationship between degenerate type 2 Bernoulli polynomials and degenerate Daehee polynomials.

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