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Improved criteria for oscillation of noncanonical neutral differential equations of even order

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Abstract

In this work, we aim at studying the asymptotic and oscillatory behavior of even-order neutral delay noncanonical differential equations. To the best of our knowledge, most of the related previous works are concerned only with neutral equations in the canonical case. Our new oscillation criteria essentially improve, simplify, and complement related results in the literature, especially those from a paper by Li and Rogovchenko (Abstr. Appl. Anal. 2014:395368, 2014). Some examples are presented that illustrate the importance of the new criteria.

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1 Introduction

Neutral delay differential equations (NDDEs) have many interesting applications in various branches of applied science. It is well known that the modeling of many natural and technological phenomena can be carried out using differential equations, often of a higher order (see [1, 2]). The study of half-linear/Emden–Fowler differential equations with deviating arguments has numerous applications in physics and engineering (e.g., half-linear/Emden–Fowler differential equations arise in the study of p -Laplace equations, porous medium problems, chemotaxis models, and so forth); see, e.g., the papers [3, 4] for more details, the papers [5–7] for the oscillation of half-linear differential equations, and the papers [3, 8–10] for the oscillation and asymptotic behavior of half-linear/Emden–Fowler differential equations with different neutral coefficients.

In this paper, we consider the oscillation and asymptotic behavior of even-order half-linear/Emden–Fowler NDDE of the form

$$\left(r \cdot \left((u + p \cdot (u \circ \tau))^{(m-1)\alpha} \right)' \right) + q(l)u^\beta(\sigma(l)) = 0, \quad (1.1)$$

where $l \geq l_0$, $m \geq 4$ is an even integer, α, β are ratios of odd positive integers, r is a known differentiable real-valued function, while p, τ, q and σ are known continuous real-valued

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functions on $[l_0, \infty)$. We also assume that $r(l) > 0, r'(l) \geq 0, p(l) \in [0, p_0], p_0$ is a constant, $q(l) \geq 0, q \neq 0$ on any half-line $[L, \infty)$ for all $L \geq l_0, \tau(l) \leq l, \sigma(l) \leq l, \lim_{l \rightarrow \infty} \tau(l) = \infty,$ and $\lim_{l \rightarrow \infty} \sigma(l) = \infty.$

To facilitate the analysis and presentation of results, we will define the function $v := u + p \cdot (u \circ \tau).$ A solution $u(l)$ of (1.1) different from zero means that $u(l)$ is a continuous real-valued function on $[l_0, \infty)$ such that $v \in C^{m-1}([l_0, \infty)), r \cdot (v^{(m-1)})^\alpha \in C^1([l_0, \infty)),$ and which satisfies the equation in (1.1).

A solution $u(l)$ of (1.1) is called *oscillatory* if it is neither positive nor negative and presents arbitrarily large zeros on $[l_0, \infty)$; otherwise, it is called *nonoscillatory*.

Although there are many works that have dealt with the oscillation of solutions of m -order neutral differential equations, as far as we know, most of them are concerned only with the canonical operator, that is, when $r(l)$ verifies that

$$\int_{l_0}^l r^{-1/\alpha}(\kappa) \, d\kappa \rightarrow \infty \quad \text{as } l \rightarrow \infty. \tag{1.2}$$

On the other hand, in the noncanonical case, when

$$\int_{l_0}^\infty r^{-1/\alpha}(\kappa) \, d\kappa < \infty, \tag{1.3}$$

the studied equations have the so-called Kneser’s solutions. The sign of one of such solutions differs from the sign of its first derivative, that is, $u(l)u'(l) < 0.$ Moreover, in case of even-order differential equations, the assumption (1.2) has been commonly used in the literature to ensure that any possible positive solution u satisfies $u > (1 - p)v,$ which does not generally hold in the case of (1.3). This results in the difficulty of studying the case when $u(l)u'(l) < 0,$ using the usual techniques (see [11–13]).

From 1969 until recently, the asymptotic behavior of a DDE of the form

$$(r(l)(u^{(m-1)}(l))^\alpha)' + q(l)u^\beta(\sigma(l)) = 0, \tag{1.4}$$

with the canonical condition in (1.2), has attracted the interest of several authors (see [14–17]). Nonetheless, in 2003 Agarwal et al. [18] obtained a criterion for the existence of a bounded solution of (1.4) under the noncanonical condition (1.3). Later on, Baculikova et al. [19], Li and Rogovchenko [7], and Zhang et al. [20–22] discussed the asymptotic and oscillatory behavior of (1.4) under the condition (1.3). Very recently, Moaaz and Muhib [23] improved and complemented the results in [19–21].

The authors in [24–29] were interested in studying and developing the oscillation theory of even-order neutral equations of the form

$$(u + p \cdot (u \circ \tau))^{(m)}(l) + q(l)u(\sigma(l)) = 0.$$

To see other oscillation criteria of more general neutral differential equations considering the canonical operator, one can see the references [30, 31].

Li and Rogovchenko [32] obtained some results on the oscillatory and asymptotic behavior of the solutions of (1.1) under the condition (1.3). For the reader’s convenience, we present the following result which appeared in [32].

Theorem 1.1 *Let $m \geq 4$ be even and $0 < \alpha = \beta \leq 1$. Assume that $0 \leq p(l) \leq p_0 < \infty$ for some constant p_0 , $\tau' \geq \tau_* > 0$, and $\tau \circ \sigma = \sigma \circ \tau$, and there exist three functions $\eta_1, \eta_2, \eta_3 \in C([l_0, \infty), \mathbb{R})$ such that*

$$\eta_1(l) \leq \sigma(l) \leq \eta_2(l), \quad \eta_1(l) \leq \tau(l) \leq l < \eta_2(l), \quad \eta_3(l) \geq \sigma(l), \quad \eta_3(l) > l,$$

and

$$\lim_{l \rightarrow \infty} \eta_1(l) = \infty.$$

Suppose also that

$$\frac{\tau_*(\tau_* + p_0^\beta)^{-1}}{((m-1)!)^\beta} \liminf_{l \rightarrow \infty} \int_{\tau^{-1}(\eta_1(l))}^l Q(s) \left(\frac{(\eta_1(s))^{m-1}}{(r(\eta_1(s)))^{1/\beta}} \right)^\beta ds > \frac{1}{e},$$

$$\frac{\tau_*(\tau_* + p_0^\beta)^{-1}}{((m-2)!)^\beta} \liminf_{l \rightarrow \infty} \int_l^{\eta_2(l)} (Q(s)(\sigma^{m-2}(s))^\beta (\delta_0(\eta_2(s)))^\beta) ds > \frac{1}{e},$$

and

$$\frac{\tau_*(\tau_* + p_0^\beta)^{-1}}{((m-3)!)^\beta} \liminf_{l \rightarrow \infty} \int_l^{\eta_3(l)} \left(Q(s) \left(\int_{\eta_3(s)}^\infty ((\eta - \eta_3(s))^{m-3} \delta_0(\eta)) d\eta \right)^\beta \right) ds > \frac{1}{e},$$

where $Q(l) = \min\{q(l), q(\tau(l))\}$. Then (1.1) is oscillatory.

In this work, we establish new oscillation criteria for neutral delay differential equations of even order. Unlike most of the previous related works, we are interested in studying the behavior of the solutions of equation (1.1) in the noncanonical case. As far as we know, the unique work related with oscillations in the noncanonical case of (1.1) is [32]. However, in [32], there is no detailed guideline about how to choose the functions $\eta_i, i = 1, 2, 3$, fulfilling the forcing conditions, an intriguing issue is how to build up oscillation criteria without requiring the presence of the obscure functions η_i . Here, we will address this topic and introduce some new oscillation criteria. Some examples are provided to illustrate the new results.

The following lemmas are needed in the proofs of our main results.

Lemma 1.1 ([33, Lemma 2.2.3]) *Let $f \in C^m([l_0, \infty), (0, \infty))$, $f^{(m)}$ be of fixed sign, $f^{(m)} \equiv 0$ on a subray of $[l_0, \infty)$, and $\lim_{l \rightarrow \infty} f(l) \neq 0$. Assume that there is an $l_1 \in [l_0, \infty)$ such that $f^{(m-1)}f^{(m)} \leq 0$ for $l \in [l_1, \infty)$. Then, there is an $l_\lambda \in [l_1, \infty)$ such that*

$$f(l) \geq \frac{\lambda}{(m-1)!} l^{m-1} |f^{(m-1)}(l)|,$$

for $\lambda \in (0, 1)$ and $l \in [l_\lambda, \infty)$.

Lemma 1.2 ([34]) *Assume that $B \geq 0, A > 0, \vartheta \geq 0$, and $\mu > 0$. Then, we have that*

$$B\vartheta - A\vartheta^{(\mu+1)/\mu} \leq \frac{\mu^\mu}{(\mu+1)^{\mu+1}} \frac{B^{\mu+1}}{A^\mu}.$$

Lemma 1.3 ([35, Lemma 1.1]) *Assume that $f(l) \in C^m([l_0, \infty), (0, \infty))$ and $f^{(m)}(l)$ is eventually of one sign for all large l . Then, there exists a nonnegative integer $h \leq m$, with $m + h$ even for $f^{(m)}(l) \geq 0$, or $m + h$ odd for $f^{(m)}(l) \leq 0$, such that*

$$h > 0 \text{ yields } f^{(k)}(l) > 0 \text{ for } k = 0, 1, \dots, h - 1,$$

and

$$h \leq m - 1 \text{ yields } (-1)^{h+k} f^{(k)}(l) > 0 \text{ for } k = h, h + 1, \dots, m - 1,$$

eventually.

2 Main results

In order to facilitate the calculation, let us define the following:

$$\delta_0(l) := \int_l^\infty r^{-1/\alpha}(\kappa) \, d\kappa,$$

$$\eta(l) := \begin{cases} c_1^{\beta-\alpha} & \text{if } \alpha \geq \beta, \\ c_2 \delta_0^{\beta-\alpha}(l) & \text{if } \alpha < \beta, \end{cases}$$

and

$$\mu(l) := \begin{cases} c_3^{\beta-\alpha} & \text{if } \alpha \geq \beta, \\ \left(\frac{c_4}{(m-3)!} \int_l^\infty (\varrho - l)^{m-3} \delta_0(\varrho) \, d\varrho\right)^{\beta-\alpha} & \text{if } \alpha < \beta, \end{cases}$$

where c_1, c_2, c_3 , and c_4 are any positive constants.

Lemma 2.1 *Assume that $u(l) \in C([l_0, \infty), (0, \infty))$ is a solution of (1.1). Then $v(l) > 0$, $(r(l)(v^{(m-1)}(l))^\alpha)' \leq 0$, and one of the following cases holds, for $l \in [l_1, \infty)$, $l_1 \geq l_0$:*

- (A) $v'(l), v^{(m-1)}(l)$ are positive and $v^{(m)}(l)$ is negative;
- (B) $v'(l), v^{(m-2)}(l)$ are positive and $v^{(m-1)}(l)$ is negative;
- (C) $(-1)^k v^{(k)}(l)$ are positive for all $k = 1, 2, \dots, m - 1$.

Proof Assume that u is an eventually positive solution of (1.1). Then, there exists $l_1 \geq l_0$ such that $u(l), u(\tau(l))$, and $u(\sigma(l))$ are positive for all $l \geq l_1$. Hence, we see that $v(l) > 0$ for $l \geq l_1$. It follows from (1.1) that $(r(l)(v^{(m-1)}(l))^\alpha)' \leq 0$. Now, using Lemma 1.3 with m even, we readily get the cases (A)–(C). □

Lemma 2.2 *Assume that $u(l) \in C([l_0, \infty), (0, \infty))$ is a solution of (1.1) and that $v(l)$ satisfies (B) in Lemma 2.1. Then $(v^{(m-2)}(l))^{\beta-\alpha} \geq \eta(l)$, eventually.*

Proof Assume that u is an eventually positive solution of (1.1) and that v satisfies (B) for $l \geq l_1$. Let us consider different possibilities.

If we assume firstly that $\alpha = \beta$, then $(v^{(m-2)}(l))^{\beta-\alpha} = 1$, and the result follows trivially.

Now, consider that $\alpha > \beta$. Since $v^{(m-2)}(l)$ is a nonincreasing positive function, there is an $m_1 > 0$ such that $v^{(m-2)}(l) \leq m_1$, which implies that

$$(v^{(m-2)}(l))^{\beta-\alpha} \geq m_1^{\beta-\alpha},$$

and thus the result holds taking $c_1 = m_1$.

Finally, suppose that $\alpha < \beta$.

Using the decreasingness property of $r(v^{(m-1)})^\alpha$, we obtain, for $l \geq l_1$,

$$r(l)(v^{(m-1)}(l))^\alpha \leq r(l_1)(v^{(m-1)}(l_1))^\alpha = -m_2 < 0,$$

from which

$$(r^{1/\alpha} v^{(m-1)})(l) \leq -m_2^{1/\alpha}. \tag{2.1}$$

Multiplying (2.1) by $r^{-1/\alpha}(l)$ and integrating it on $[l, L]$, we get

$$v^{(m-2)}(L) \leq v^{(m-2)}(l) - \int_l^L \frac{m_2^{1/\alpha}}{r^{1/\alpha}(\vartheta)} d\vartheta.$$

Letting $L \rightarrow \infty$, we get

$$0 \leq v^{(m-2)}(l) - m_2^{1/\alpha} \delta_0(l),$$

that is,

$$v^{(m-2)}(l) \geq m_2^{1/\alpha} \delta_0(l).$$

Thus, we see that

$$(v^{(m-2)}(l))^{\beta-\alpha} \geq m_2^{(\beta-\alpha)/\alpha} \delta_0^{\beta-\alpha}(l) = c_2 \delta_0^{\beta-\alpha}(l).$$

Therefore,

$$(v^{(m-2)}(l))^{\beta-\alpha} \geq \eta(l).$$

The proof is complete. □

Lemma 2.3 *Assume that $u(l) \in C([l_0, \infty), (0, \infty))$ is a solution of (1.1) and v satisfies condition (C) in Lemma 2.1. Then $v^{\beta-\alpha}(l) \geq \mu(l)$, eventually.*

Proof Assuming the hypothesis of the statement in the case $\alpha = \beta$, the result follows readily, as $(v)^{\beta-\alpha} = 1$.

Next, we assume that $\alpha > \beta$. Since v is a nonincreasing positive function, there are $M_3 > 0$ and $l_2 \geq l_1$ such that $v \leq M_3$, for every $l \geq l_2$, and hence

$$(v)^{\beta-\alpha} \geq M_3^{\beta-\alpha} = k_1.$$

Finally, we suppose that $\alpha < \beta$. Using the decreasingness property of $r(v^{(m-1)})^\alpha$, we obtain for $l \geq l_1$ that

$$(r(v^{(m-1)})^\alpha)(l) \leq (r(v^{(m-1)})^\alpha)(l_1) = -M_4 < 0,$$

which yields

$$(r^{1/\alpha} v^{(m-1)})(l) \leq -M_4^{1/\alpha}. \tag{2.2}$$

Multiplying (2.2) by $r^{-1/\alpha}(l)$ and integrating it on $[l, L]$, we get

$$v^{(m-2)}(L) \leq v^{(m-2)}(l) - \int_l^L \frac{M_4^{1/\alpha}}{r^{1/\alpha}(\vartheta)} d\vartheta.$$

Letting $L \rightarrow \infty$ and using (C), we obtain

$$0 \leq v^{(m-2)}(l) - M_4^{1/\alpha} \delta_0(l). \tag{2.3}$$

Integrating (2.3) $(m - 2)$ times from l to ∞ , we successively arrive at

$$\begin{aligned} v^{(m-3)}(l) &\leq -M_4^{1/\alpha} \int_l^\infty \delta_0(\varrho) d\varrho, \\ -v^{(m-4)}(l) &\leq -M_4^{1/\alpha} \int_l^\infty \left(\int_s^\infty \delta_0(\varrho) d\varrho \right) ds = -\frac{M_4^{1/\alpha}}{1!} \int_l^\infty (\varrho - l) \delta_0(\varrho) d\varrho, \end{aligned}$$

and finally, we get

$$v(l) \geq \frac{M_4^{1/\alpha}}{(m - 3)!} \int_l^\infty (\varrho - l)^{m-3} \delta_0(\varrho) d\varrho.$$

Therefore, taking $c_4 = M_4^{1/\alpha}$, we have that

$$(v(l))^{\beta-\alpha} \geq \mu(l).$$

This completes the proof. □

Lemma 2.4 *Assume that $u(l) \in C([l_0, \infty), (0, \infty))$ is a solution of (1.1) and v satisfies condition (C) in Lemma 2.1. If*

$$\int_{l_0}^\infty \left(\int_l^\infty (v - l)^{m-3} \left(\frac{1}{r(v)} \int_{l_1}^v q(s) ds \right)^{1/\alpha} dv \right) dl = \infty, \tag{2.4}$$

then $\lim_{l \rightarrow \infty} u(l) = 0$.

Proof Suppose that $u(l) \in C([l_0, \infty), (0, \infty))$ is a solution of (1.1) and v satisfies condition (C) in Lemma 2.1. Let us denote $\lim_{l \rightarrow \infty} v(l) = D$. We claim that $D = 0$. Indeed, for the

sake of contradiction, suppose that $D > 0$, and so for all $\varepsilon > 0$ there exists $l_1 \geq l_0$ such that $u(\sigma(l)) \geq D$ for $l \geq l_1$. Integrating (1.1) on $[l_1, l]$, we get

$$r(v^{(m-1)})^\alpha(l) = r(v^{(m-1)})^\alpha(l_1) - \int_{l_1}^l q(s)u^\beta(\sigma(s)) \, ds \leq -D^\beta \int_{l_1}^l q(s) \, ds,$$

that is,

$$v^{(m-1)}(l) < -D^{\beta/\alpha} \left(\frac{1}{r(l)} \int_{l_1}^l q(s) \, ds \right)^{1/\alpha}. \tag{2.5}$$

Integrating (2.5) twice on $[l, \infty)$, we successively have that

$$-v^{(m-2)}(l) < -D^{\beta/\alpha} \int_l^\infty \left(\frac{1}{r(v)} \int_{l_1}^v q(s) \, ds \right)^{1/\alpha} \, dv$$

and finally,

$$\begin{aligned} v^{(m-3)}(l) &< -D^{\beta/\alpha} \int_l^\infty \int_s^\infty \left(\frac{1}{r(v)} \int_{l_1}^v q(s) \, ds \right)^{1/\alpha} \, dv \, ds \\ &= -D^{\beta/\alpha} \int_l^\infty (v-l) \left(\frac{1}{r(v)} \int_{l_1}^v q(s) \, ds \right)^{1/\alpha} \, dv. \end{aligned}$$

Similarly, integrating the above inequality $(m - 4)$ times on $[l, \infty)$, we obtain

$$v'(l) < -D^{\beta/\alpha} \int_l^\infty (v-l)^{m-3} \left(\frac{1}{r(v)} \int_{l_1}^v q(s) \, ds \right)^{1/\alpha} \, dv.$$

Integrating this inequality on $[l_1, \infty)$, we find

$$v(l_1) > D^{\beta/\alpha} \int_{l_1}^\infty \left(\int_l^\infty (v-l)^{m-3} \left(\frac{1}{r(v)} \int_{l_1}^v q(s) \, ds \right)^{1/\alpha} \, dv \right) \, dl,$$

which is a contradiction to (2.4). Thus, $D = 0$; moreover, the inequality $u \leq v$ implies $\lim_{l \rightarrow \infty} u(l) = 0$. The proof is complete. \square

Theorem 2.1 *Let $0 \leq p(l) < 1$, and let us assume that (2.4) holds. If the first-order DDE*

$$y'(l) + q(l) \left(\frac{\lambda_0(1 - p(\sigma(l)))\sigma^{m-1}(l)}{(m-1)!r^{1/\alpha}(\sigma(l))} \right)^\beta y^{\beta/\alpha}(\sigma(l)) = 0 \tag{2.6}$$

is oscillatory for some constant $\lambda_0 \in (0, 1)$ and

$$\limsup_{l \rightarrow \infty} \int_{l_0}^l \left(\eta(s)q(s)(1 - p(\sigma(s)))^\beta \left(\frac{\lambda_1\sigma^{m-2}(s)}{(m-2)!} \right)^\beta \delta_0^\alpha(s) - \frac{\alpha^{\alpha+1}r^{-1/\alpha}(s)}{(\alpha+1)^{\alpha+1}\delta_0(s)} \right) \, ds = \infty \tag{2.7}$$

holds for some constant $\lambda_1 \in (0, 1)$, then every solution of (1.1) is either oscillatory or converges to zero as $l \rightarrow \infty$.

Proof Suppose on the contrary that there is a nonoscillatory solution u of (1.1). Then, we can assume $u(l)$, $u(\tau(l))$, and $u(\sigma(l))$ are positive for $l \geq l_1 \geq l_0$. It follows from Lemma 2.1 that there are three possible cases for the behavior of v and its derivatives.

First, suppose that case (A) holds. From the definition of v , we see that

$$u(l) = v(l) - p(l)u(\tau(l)) \geq (1 - p(l))v(l), \tag{2.8}$$

which, together with (1.1), gives

$$(r(v^{(m-1)})^\alpha)'(l) \leq -q(l)(1 - p(\sigma(l)))^\beta v^\beta(\sigma(l)). \tag{2.9}$$

From Lemma 1.1, we have

$$v(l) \geq \frac{\lambda l^{m-1}}{(m-1)!} v^{(m-1)}(l), \tag{2.10}$$

for every $\lambda \in (0, 1)$. From (2.10) and (2.9), we obtain

$$(r(v^{(m-1)})^\alpha)'(l) + q(l)(1 - p(\sigma(l)))^\beta \left(\frac{\lambda \sigma^{m-1}(l)}{(m-1)!}\right)^\beta (v^{(m-1)}(\sigma(l)))^\beta \leq 0.$$

Letting $y(l) = r(l)(v^{(m-1)}(l))^\alpha$, clearly, $y(l)$ is a positive solution of the first-order delay differential inequality

$$y'(l) + q(l) \left(\frac{\lambda(1 - p(\sigma(l)))\sigma^{m-1}(l)}{(m-1)!r^{1/\alpha}(\sigma(l))}\right)^\beta y^{\beta/\alpha}(\sigma(l)) \leq 0. \tag{2.11}$$

It follows from [36, Theorem 1] that the corresponding differential equation (2.6) also has a positive solution for all $\lambda_0 \in (0, 1)$, which is a contradiction.

Next, consider that case (B) holds. We define the function Φ by

$$\Phi := \frac{r(v^{(m-1)})^\alpha}{(v^{(m-2)})^\alpha}. \tag{2.12}$$

Then $\Phi(l) < 0$ for $l \geq l_1$. Noting that $(r(v^{(m-1)})^\alpha)' \leq 0$, we have

$$r^{1/\alpha}(s)v^{(m-1)}(s) \leq r^{1/\alpha}(l)v^{(m-1)}(l), \quad s \geq l \geq l_1. \tag{2.13}$$

Multiplying (2.13) by $r^{-1/\alpha}(s)$ and integrating it on $[l, \infty)$, we obtain

$$0 \leq v^{(m-2)}(l) + r^{1/\alpha}(l)v^{(m-1)}(l)\delta_0(l),$$

that is,

$$-\frac{r^{1/\alpha}(l)v^{(m-1)}(l)\delta_0(l)}{v^{(m-2)}(l)} \leq 1,$$

which in view of (2.12) may be written as

$$-\Phi(l)\delta_0^\alpha(l) \leq 1. \tag{2.14}$$

Differentiating (2.12), we have

$$\Phi'(l) = \frac{(r(v^{(m-1)})^\alpha)'(l)}{(v^{(m-2)}(l))^\alpha} - \frac{\alpha r(l)(v^{(m-1)}(l))^{\alpha+1}}{(v^{(m-2)}(l))^{\alpha+1}},$$

which, in view of (1.1) and (2.12), becomes

$$\Phi'(l) = -\frac{q(l)u^\beta(\sigma(l))}{(v^{(m-2)}(l))^\alpha} - \frac{\alpha\Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \tag{2.15}$$

Taking into account the fact that $v'(l) > 0$ and the definition of $v(l)$, we get that (2.8) holds. Hence, (2.15) becomes

$$\Phi'(l) \leq -\frac{q(l)(1-p(\sigma(l)))^\beta v^\beta(\sigma(l))}{(v^{(m-2)}(l))^\alpha} - \frac{\alpha\Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \tag{2.16}$$

From Lemma 1.1, we find

$$v(l) \geq \frac{\lambda l^{m-2}}{(m-2)!} v^{(m-2)}(l),$$

for all sufficiently large l and for every $\lambda \in (0, 1)$. Then, (2.16) becomes

$$\begin{aligned} \Phi'(l) &\leq -q(l)(1-p(\sigma(l)))^\beta \left(\frac{\lambda\sigma^{m-2}(l)}{(m-2)!}\right)^\beta (v^{(m-2)}(\sigma(l)))^{\beta-\alpha} \frac{(v^{(m-2)}(\sigma(l)))^\alpha}{(v^{(m-2)}(l))^\alpha} \\ &\quad - \frac{\alpha\Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \end{aligned}$$

Since $l \geq \sigma(l)$ and $v^{(m-2)}(l)$ is decreasing, in view of the definition of $\eta(l)$ and Lemma 2.2, we have that

$$\Phi'(l) \leq -\eta(l)q(l)(1-p(\sigma(l)))^\beta \left(\frac{\lambda\sigma^{m-2}(l)}{(m-2)!}\right)^\beta - \frac{\alpha\Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \tag{2.17}$$

Multiplying (2.17) by $\delta_0^\alpha(l)$ and integrating it on $[l_1, l]$, we get

$$\begin{aligned} 0 &\geq \delta_0^\alpha(l)\Phi(l) - \delta_0^\alpha(l_1)\Phi(l_1) + \int_{l_1}^l \frac{\alpha\delta_0^{\alpha-1}(s)}{r^{1/\alpha}(s)} \Phi(s) ds + \int_{l_1}^l \frac{\alpha\delta_0^\alpha(s)}{r^{1/\alpha}(s)} \Phi^{(\alpha+1)/\alpha}(s) ds \\ &\quad + \int_{l_1}^l \eta(s)q(s)(1-p(\sigma(s)))^\beta \left(\frac{\lambda\sigma^{m-2}(s)}{(m-2)!}\right)^\beta \delta_0^\alpha(s) ds. \end{aligned}$$

Setting $A = \delta_0^\alpha(s)/r^{1/\alpha}(s)$, $B = \delta_0^{\alpha-1}(s)/r^{1/\alpha}(s)$, and $\vartheta = -\Phi(s)$, and using Lemma 1.2, we get

$$\int_{l_1}^l \left(\eta(s)q(s)(1-p(\sigma(s)))^\beta \left(\frac{\lambda\sigma^{m-2}(s)}{(m-2)!}\right)^\beta \delta_0^\alpha(s) - \frac{\alpha^{\alpha+1}r^{-1/\alpha}(s)}{(\alpha+1)^{\alpha+1}\delta_0(s)} \right) ds \leq \frac{\Phi(l_1)}{\delta_0^{-\alpha}(l_1)} + 1,$$

due to (2.14), which contradicts (2.7).

Finally, suppose that (C) holds. From Lemma 2.4, we see that $\lim_{l \rightarrow \infty} u(l) = 0$, which is a contradiction.

The proof of the theorem is complete. □

Remark Combining Theorem 2.1 and the results reported in the papers [37, 38] for equation (2.6), one can obtain various oscillation criteria for equation (1.1) in the case where $\alpha = \beta$.

Theorem 2.2 *Let us assume that the first-order DDE (2.6) is oscillatory for some $\lambda_0 \in (0, 1)$ and that (2.7) holds for some $\lambda_1 \in (0, 1)$. If*

$$\tau \circ \sigma = \sigma \circ \tau, \quad \tau'(l) \geq \tau_0 > 0, \quad \sigma(l) \leq \tau(l),$$

and

$$\limsup_{l \rightarrow \infty} \left(\mu(l) \delta_{m-2}^\alpha(l) \int_{l_0}^l Q(\varrho) \, d\varrho \right) > \kappa \left(1 + \frac{p_0^\beta}{\tau_0} \right), \tag{2.18}$$

where

$$\delta_{k+1}(l) := \int_l^\infty \delta_k(\varrho) \, d\varrho \quad \text{for } k = 0, 1, \dots, m-3,$$

$$Q(l) := \min\{q(l), q(\tau(l))\},$$

and $\kappa = 1$ if $\beta \in (0, 1]$; otherwise, $\kappa = 2^{\beta-1}$, then every solution of (1.1) is oscillatory.

Proof We argue by contradiction. Assume to the contrary that there is a nonoscillatory solution u of (1.1). Then, we can assume $u(l), u(\tau(l)),$ and $u(\sigma(l))$ are positive for $l \geq l_1 \geq l_0$. It follows from Lemma 2.1 that there are three possible cases for the behavior of v and its derivatives.

The proofs of the cases in which (A) or (B) is fulfilled are similar to those of Theorem 2.1. Suppose that (C) holds. Since $(r(l)(v^{(m-1)}(l))^\alpha)' \leq 0$, we have that

$$r(s)(v^{(m-1)}(s))^\alpha - r(l)(v^{(m-1)}(l))^\alpha \leq 0 \quad \text{for all } s \geq l,$$

or

$$v^{(m-1)}(s) \leq r^{1/\alpha}(l)v^{(m-1)}(l) \frac{1}{r^{1/\alpha}(s)}.$$

Integrating this inequality from l to ∞ and using the fact that $v^{(m-2)}$ is a positive decreasing function, we arrive at

$$-v^{(m-2)}(l) \leq r^{1/\alpha}(l)v^{(m-1)}(l) \int_l^\infty \frac{1}{r^{1/\alpha}(\varrho)} \, d\varrho = r^{1/\alpha}(l)v^{(m-1)}(l)\delta_0(l).$$

Taking into account the behavior of the derivatives of $v(l)$ and integrating the last inequality $(m - 2)$ times from l to ∞ , we obtain

$$(-1)^{k+1}v^{(k)}(l) \leq r^{1/\alpha}(l)v^{(m-1)}(l)\delta_{m-k-2}(l), \tag{2.19}$$

for $k = 0, 1, \dots, m - 3$. On the other hand, from (1.1) we have

$$u^\beta(\sigma(l)) = -\frac{1}{q(l)}(r(l)(v^{(m-1)}(l))^\alpha)', \tag{2.20}$$

and taking into account that $\tau'(l) \geq \tau_0 > 0$, we get

$$\begin{aligned}
 u^\beta(\sigma(\tau(l))) &= -\frac{1}{\tau'(l)q(\tau(l))} (r(\tau(l))(v^{(m-1)}(\tau(l)))^\alpha)' \\
 &\leq -\frac{1}{\tau_0 q(\tau(l))} (r(\tau(l))(v^{(m-1)}(\tau(l)))^\alpha)'.
 \end{aligned}
 \tag{2.21}$$

From (2.20) and (2.21), after using [39, Lemma 1], we find that

$$\begin{aligned}
 v^\beta(\sigma(l)) &= (u(\sigma(l)) + p(\sigma(l))u(\tau(\sigma(l))))^\beta \\
 &\leq \kappa (u^\beta(\sigma(l)) + p_0^\beta u^\beta(\sigma(\tau(l)))) \\
 &\leq -\frac{\kappa}{q(l)} (r(l)(v^{(m-1)}(l))^\alpha)' - \frac{\kappa p_0^\beta}{\tau_0 q(\tau(l))} (r(\tau(l))(v^{(m-1)}(\tau(l)))^\alpha)' \\
 &\leq -\frac{\kappa}{Q(l)} \left(r(l)(v^{(m-1)}(l))^\alpha + \frac{p_0^\beta}{\tau_0} r(\tau(l))(v^{(m-1)}(\tau(l)))^\alpha \right)',
 \end{aligned}$$

or

$$\left(r(l)(v^{(m-1)}(l))^\alpha + \frac{p_0^\beta}{\tau_0} r(\tau(l))(v^{(m-1)}(\tau(l)))^\alpha \right)'(l) \leq -\frac{1}{\kappa} Q(l)v^\beta(\sigma(l)).$$

Integrating this inequality from l_1 to l , we obtain

$$\begin{aligned}
 &r(l)(v^{(m-1)}(l))^\alpha + \frac{p_0^\beta}{\tau_0} r(\tau(l))(v^{(m-1)}(\tau(l)))^\alpha \\
 &\leq r(l_1)(v^{(m-1)}(l_1))^\alpha + \frac{p_0^\beta}{\tau_0} r(\tau(l_1))(v^{(m-1)}(\tau(l_1)))^\alpha - \frac{1}{\kappa} \int_{l_1}^l Q(\varrho)v^\beta(\sigma(\varrho)) \, d\varrho \\
 &\leq -\frac{1}{\kappa} v^\beta(\sigma(l)) \int_{l_1}^l Q(\varrho) \, d\varrho.
 \end{aligned}$$

Since $(r(l)v^{(m-1)}(l))' \leq 0$ and $\tau(l) \leq l$, we arrive at

$$\begin{aligned}
 \left(1 + \frac{p_0^\beta}{\tau_0} \right) r(l)(v^{(m-1)}(l))^\alpha &\leq -\frac{1}{\kappa} v^\beta(\sigma(l)) \int_{l_1}^l Q(\varrho) \, d\varrho \\
 &\leq -\frac{1}{\kappa} v^\alpha(l)v^{\beta-\alpha}(l) \int_{l_1}^l Q(\varrho) \, d\varrho,
 \end{aligned}$$

which, in view of Lemma 2.3, gives

$$\left(1 + \frac{p_0^\beta}{\tau_0} \right) r(l)(v^{(m-1)}(l))^\alpha \leq -\frac{1}{\kappa} \mu(l)v^\alpha(l) \int_{l_1}^l Q(\varrho) \, d\varrho.
 \tag{2.22}$$

Finally, from the inequality in (2.19) for $k = 0$ and (2.22), we have that

$$\left(1 + \frac{p_0^\beta}{\tau_0} \right) \geq \frac{1}{\kappa} \mu(l)\delta_{m-2}^\alpha(l) \int_{l_1}^l Q(\varrho) \, d\varrho,$$

which is a contradiction to (2.18). This completes the proof. □

3 Some applications

Example 3.1 Consider the NDDE

$$(l^4(u(l) + p_0u(al))''')' + q_0u(bl) = 0, \tag{3.1}$$

where $a, b \in (0, 1)$ and $q_0 > 0$. Then, we note that

$$\begin{aligned} \alpha = \beta = 1, \quad m = 4, \quad r(l) = l^4, \quad p(l) = p_0, \\ \tau(l) = al, \quad q(l) = q_0, \quad \text{and} \quad \sigma(l) = bl. \end{aligned}$$

Therefore, it is easy to verify that

$$\delta_0(l) = \frac{1}{3l^3}, \quad \delta_1(l) = \frac{1}{6l^2}, \quad \text{and} \quad \delta_2(l) = \frac{1}{6l}.$$

Next, to apply Theorem 2.1, we must first check that conditions (2.4), (2.6), and (2.7) are fulfilled. A simple calculus shows that the integral in (2.4) is divergent. After replacing and simplifying, (2.6) becomes

$$y'(l) + q_0 \frac{\lambda_0(1-p_0)}{6b} \frac{1}{l} y(bl) = 0. \tag{3.2}$$

Applying a well-known oscillation result [40, Theorem 2.1.1] to the first-order DDE in (3.2), we obtain immediately that it is oscillatory if

$$\liminf_{l \rightarrow \infty} \int_{bl}^l q_0 \frac{\lambda_0(1-p_0)}{6b} \frac{1}{s} ds > \frac{1}{e},$$

that is,

$$q_0 \ln \frac{1}{b} > \frac{3!b}{(1-p_0)e}. \tag{3.3}$$

Now, we note that (2.7) reduces to

$$\limsup_{l \rightarrow \infty} \int_{l_0}^l \left(q_0(1-p_0) \frac{\lambda_1 b^2}{6} - \frac{3}{4} \right) \frac{1}{s} ds = \infty,$$

which is satisfied if

$$q_0 > \frac{18}{4(1-p_0)b^2}. \tag{3.4}$$

Thus, if conditions (3.3) and (3.4) hold, then every solution of (3.1) is oscillatory or tends to zero. Moreover, if $q_0 = \kappa b^{-\kappa} (\kappa + 1)(\kappa - 1)(2 - \kappa)(p_0 a^\kappa + 1)$ with $\kappa \in (-1, 0)$, then it is easy to verify that $u(l) = l^\kappa$ is a nonoscillatory solution of (3.1) and tends to zero as $l \rightarrow \infty$.

On the other hand, to apply Theorem 2.2, we see that the condition (2.18) becomes

$$\limsup_{l \rightarrow \infty} \left(\frac{1}{6l} \int_{l_0}^l q_0 d\varrho \right) > \left(1 + \frac{p_0}{a} \right),$$

and so

$$q_0 > 6 \left(1 + \frac{p_0}{a} \right). \tag{3.5}$$

Thus, if conditions (3.3), (3.4), and (3.5) hold, then every solution of (3.1) is oscillatory.

Example 3.2 Consider the NDDE

$$\left(e^{3l} \left(\left(u(l) + \left(1 - \frac{1}{l^2} \right) u(l-a) \right)''' \right)^3 \right)' + q_0 e^{3l} u^3(l-b) = 0, \tag{3.6}$$

where $l \geq 1, 0 < a < b$, and $q_0 > 0$. Then, we note that $\alpha = \beta = 3, m = 4$,

$$r(l) = e^{3l}, \quad p(l) = 1 - 1/l^2, \quad \tau(l) = l - a, \quad q(l) = q_0 e^{3l}, \quad \text{and} \quad \sigma(l) = l - b.$$

Therefore, it is easy to verify that

$$\delta_i(l) = e^{-l} \quad \text{for } i = 0, 1, 2.$$

A simple calculus shows that in this case the integral in (2.4) diverges. After replacing and simplifying, (2.6) becomes

$$y'(l) + q_0 e^{3l} \left(\lambda_0 \frac{(l-b)^3}{3! e^{l-b}} \right) y(l-b) = 0. \tag{3.7}$$

Applying a well-known oscillation result [40, Theorem 2.1.1], we see that (3.7) is oscillatory. Moreover, (2.7) reduces to

$$\limsup_{l \rightarrow \infty} \int_{l_0}^l \left(q_0 \frac{\lambda_1^3}{2^3} - \left(\frac{3}{4} \right)^4 \right) ds = \infty,$$

which is satisfied if $q_0 > 81/32$. Thus, every solution of (3.6) is oscillatory or tends to zero if $q_0 > 81/32$.

On the other hand, to apply Theorem 2.2, we see that the condition (2.18) becomes

$$\limsup_{l \rightarrow \infty} e^{-3l} \int_{l_0}^l q_0 e^{3(s-a)} ds > 2^3,$$

that is, $q_0 > 24e^{3a}$. Thus, every solution of (3.6) is oscillatory if $q_0 > \max\{24e^{3a}, 81/32\}$.

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