Advances in Difference Equations
a SpringerOpen Journal

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Best proximity point results and application to a system of integro-differential equations

Anupam Das[1,2], Hemant Kumar Nashine[3*], Rabha W. Ibrahim[4] and Manuel De la Sen[5]

*Correspondence:
hemantkumarnashine@tdtu.edu.vn
[3]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract

In this work, we solve the system of integro-differential equations (in terms of Caputo–Fabrizio calculus) using the concepts of the best proximity pair (point) and measure of noncompactness. We first introduce the concept of cyclic (noncyclic) Θ-condensing operator for a pair of sets using the measure of noncompactness and then establish results on the best proximity pair (point) on Banach spaces and strictly Banach spaces. In addition, we have illustrated the considered system of integro-differential equations by three examples and discussed the stability, efficiency, and accuracy of solutions.

MSC: 47H10; 54H25

Keywords: Best proximity point, measure of noncompactness (MNC); Darbo fixed point theorem (DFPT); Fractional calculus; Fractional differential operator; Fractional integral equation

1 Introduction and preliminaries

In the last three decades, numerous physical problems have been presented employing the notion of fractional calculus. This theory becomes very attractive for researchers because of its flexibility, accuracy, and effectiveness in science. Moreover, it becomes a way of reformulation and reconstruction depending of the nature of the problem. Straightforward applications of fractional calculus can be observed in different areas. The critical differences among the arbitrary derivatives are their varied kernels, which can fit the structure of various applications. For example, the main variations between the Caputo fractional derivative, the Caputo–Fabrizio derivative [6], and others are that the Caputo calculus is expressed employing a power law, the Caputo–Fabrizio derivative is presented operating an exponential growth performance. Some of the recent work on fractional differential equations can be found in [2, 17] and the references cited therein.

One of the central problems in approximation theory is to determine points that minimize the distance to a given point or subset. Operator $\mathcal{T}$ on a nonempty subset $\mathcal{A}$ of a metric space $\mathcal{X}$ has a fixed point (FP) if $\mathcal{A} \cap \mathcal{T}(\mathcal{A}) \neq \emptyset$. If $\mathcal{T}$ is FP-free then we try to find $z \in \mathcal{A}$ such that z and $\mathcal{T}z$ have the smallest possible distance. The point z is a best approximant for $\mathcal{T}$. The best approximation has always attracted analysts because it carries enough potential to be extended especially with the functional analytic approach in nonlinear analysis. In the mid-20th century, it was found that the existence of a fixed point has its relevance in proving the existence of the best approximation. The best approximation is termed as an invariant approximation in the case of self-mappings. When $\mathcal{A}$ is mapped by $\mathcal{T}$ into $\mathcal{B} \subseteq \mathcal{X}$, the problem extends to that of finding a point which estimates the distance between these two subsets; this was handled by Ky Fan in 1969.

If $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of a normed linear space ($\mathbb{NLS}$, for short) $\mathcal{X}$ and if $\mathcal{T}: \mathcal{A} \to \mathcal{B}$ is a mapping, then the pair $(\vartheta^*, \mathcal{T}\vartheta^*)$ is called a best proximal pair of $\mathcal{T}$ if $\vartheta^* \in \mathcal{A}$ satisfies the condition $d(\vartheta^*, \mathcal{T}\vartheta^*) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. Define

$$\mathcal{A}_0 = \bigl\{w \in \mathcal{A} : \exists \varpi \in \mathcal{B}, \|w - \varpi\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})\bigr\},$$
$$\mathcal{B}_0 = \bigl\{\hat{z} \in \mathcal{B} : \exists \varpi \in \mathcal{A}, \|\varpi - \hat{z}\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})\bigr\}.$$

If $(\mathcal{A}, \mathcal{B})$ is a pair of nonempty, convex, and weakly compact subsets of $\mathcal{X}$, then the respective pair $(\mathcal{A}_0, \mathcal{B}_0)$ is of the same kind. If $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$, then the pair $(\mathcal{A}, \mathcal{B})$ is said to be proximinal.

A mapping $\mathcal{T}: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be cyclic if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$; it is called noncylic if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{A}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{B}$. Also $\mathcal{T}$ is said to be relatively nonexpansive if $\|\mathcal{T}p - \mathcal{T}q\| \leq \|p - q\|$ for all $p \in \mathcal{A}, q \in \mathcal{B}$. A point $z \in \mathcal{A} \cup \mathcal{B}$ satisfying $\|z - \mathcal{T}z\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$ is called a best proximity point (BPP, for short) of a cyclic mapping $\mathcal{T}$. If the mapping $\mathcal{T}$ is noncyclic, a pair $(q, p) \in (\mathcal{A}, \mathcal{B})$ is called a best proximity pair (BPPR, for short), if $q = \mathcal{T}q$, $p = \mathcal{T}p$ and $\|q - p\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. The previous notions were introduced by Eldred et al. in [10], where some related results were established.

The collection of all nonempty, closed, bounded, and convex sets (NBCC, for short) in an $\mathbb{NLS}$ $\mathcal{X}$ will be denoted by $\Lambda(\mathcal{X})$. A map $\mathcal{T}: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called compact if the pair $(\overline{\mathcal{T}(\mathcal{A})}, \overline{\mathcal{T}(\mathcal{B})})$ is compact. Gabeleh [11] proved the following results.

Theorem 1.1 ([11]) *Let $\mathcal{X}$ be a Banach space(BS) and let $\mathcal{T}: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a compact, relatively nonexpansive cyclic mapping with $\mathcal{A}_0 \neq \emptyset$. Then $\mathcal{T}$ has a BPP.*

A Banach space $\mathcal{X}$ is strictly convex if for $u, v, x \in \mathcal{X}$ and $\tau > 0$,

$$\bigl[\|u - x\| \leq \tau, \|v - x\| \leq \tau, u \neq v\bigr] \Longrightarrow \left\|\frac{u + v}{2} - x\right\| < \tau$$

holds. The L^p space $(1 < p < \infty)$ and Hilbert space are strictly convex Banach spaces.

Theorem 1.2 ([11]) *Let $\mathcal{X}$ be a strictly convex Banach space and let $\mathcal{T}: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive noncyclic mapping, where $\mathcal{A}, \mathcal{B} \in \Lambda(\mathcal{X})$. Then $\mathcal{T}$ has a BPPR.*

The MNC is coined by Kuratowski [19].

Let $(\mathcal{X}, \|\cdot\|)$ represents BS, and by $\bar{\theta}$ we denote zero element; $\mathcal{B}(\bar{\vartheta}, \bar{\zeta}) = \{\kappa \in \mathcal{X} : \|\kappa - \bar{\vartheta}\| \leq \bar{\zeta}\}$ and $\mathcal{B}_{\bar{\zeta}}$ denotes $\mathcal{B}(\bar{\theta}, \bar{\zeta})$. Also, $\mathcal{M}_{\mathcal{X}}$ represents the family of nonempty bounded subsets of $\mathcal{X}$ and $\mathcal{N}_{\mathcal{X}}$ its subfamily consisting of all relatively compact sets.

Definition 1.3 ([4, 5]) A function $\chi : \mathcal{M}_{\mathcal{X}} \to \mathbb{R}^+$ is called an MNC in $\mathcal{X}$ if $(\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{M}_{\mathcal{X}})$:
(1°) $\ker \chi := \{\mathcal{K}_1 \in \mathcal{M}_{\mathcal{X}} : \chi(\mathcal{K}_1) = 0\} \neq \emptyset$ and $\ker \chi \subset \mathcal{N}_{\mathcal{X}}$,
(2°) $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Longrightarrow \chi(\mathcal{K}_1) \leq \chi(\mathcal{K}_2)$,

(3°) $\chi(\overline{\mathcal{K}_1}) = \chi(\mathcal{K}_1)$,
(4°) $\chi(\overline{\operatorname{conv}\mathcal{K}_1}) = \chi(\mathcal{K}_1)$,
(5°) $\chi(\mu\mathcal{K}_1 + (1-\mu)\mathcal{K}_2) \leq \mu\chi(\mathcal{K}_1) + (1-\mu)\chi(\mathcal{K}_2)$ for $\mu \in [0,1]$,
(6°) $\chi(\mathcal{K}_1 \cup \mathcal{K}_2) = \max\{\chi(\mathcal{K}_1), \chi(\mathcal{K}_2)\}$,
(7°) If $\{\mathcal{K}_n\}_{n=1}^{\infty}$ is a sequence such that for all $n \in \mathbb{N}, \mathcal{K}_n \supseteq \mathcal{K}_{n+1}$, and $\mathcal{K}_n(\neq \phi)$ are closed sets in $\mathcal{M}_{\mathcal{X}}$, and if $\lim_{n\to\infty}\chi(\mathcal{K}_n) = 0$, then $\mathcal{K}_\infty = \bigcap_{n=1}^{\infty}\mathcal{K}_n(\neq\phi)$ is compact.

For some of the work in this direction, one can be referred in the papers [9, 16, 18, 20–23]. Very recently (see [12–15, 24]), the concept of a BP point has been associated with MNC, and some results have been obtained. In this paper, we discuss best proximity point results using MNC for cyclic and noncyclic Θ-condensing operator. Further we solve a system of IDEs (in terms of Caputo–Fabrizio calculus)

$$\mathcal{D}^{\mu}x(\varsigma) = \Upsilon_1\left(\varsigma, x(\varsigma), \int_{\tau_1}^{\tau_1+\tau_2}\phi_1\big(\varsigma,\hat{\tau},x(\hat{\tau})\big)\,d\hat{\tau}, \int_{\tau_1}^{\varsigma}\psi_1\big(\varsigma,\hat{\tau},x(\hat{\tau})\big)\,d\hat{\tau}\right), \quad x(\tau_1) = x_1,$$
$$\mathcal{D}^{\mu}y(\varsigma) = \Upsilon_2\left(\varsigma, y(\varsigma), \int_{\tau_1}^{\tau_1+\tau_2}\phi_2\big(\varsigma,\hat{\tau},y(\hat{\tau})\big)\,d\hat{\tau}, \int_{\tau_1}^{\varsigma}\psi_2\big(\varsigma,\hat{\tau},y(\hat{\tau})\big)\,d\hat{\tau}\right), \quad y(\tau_1) = y_1,$$

where $I = [\tau_1 - \tau_2, \tau_1 + \tau_2]$, $I_x = [x_1 - \epsilon, x_1 + \epsilon]$, $I_y = [y_1 - \epsilon, y_1 + \epsilon]$, and $I_\epsilon = [\tau_1 - \epsilon, \tau_1 + \epsilon]$, using the established result. Our results extend the works [12, 24].

2 Main results

Definition 2.1 ([7]) A continuous mapping $\varUpsilon : \mathbb{R}_+^2 \to \mathbb{R}$ is said to be a $\mathcal{C}$-class function if:
(1) $\varUpsilon(\zeta,\xi) \leq \zeta$,
(2) $\varUpsilon(\zeta,\xi) = \zeta$ gives either $\zeta = 0$ or $\xi = 0$, for all $\zeta,\xi \in \mathbb{R}_+$.

Definition 2.2 ([7]) A $\mathcal{C}$-class function is said to have the $\mathcal{C}_\varUpsilon$ property, if we can find a $\mathcal{C}_\varUpsilon \geq 0$ such that
(1) $\varUpsilon(\zeta,\xi) > \mathcal{C}_F \Rightarrow \zeta > \xi$,
(2) $\varUpsilon(\xi,\xi) \leq \mathcal{C}_\varUpsilon$, for all $\zeta,\xi \in \mathbb{R}_+$.

Definition 2.3 ([7]) Let $\Delta(\Theta_1, \mathcal{C}_\varUpsilon)$ be the family of extended $\mathcal{C}_\varUpsilon$-simulation functions $\Theta_1 : \mathbb{R}_+^2 \to \mathbb{R}$, satisfying
(Δ_1) $\Theta_1(\zeta,\xi) < \varUpsilon(\xi,\zeta)$ for all $\zeta,\xi > 0$, where $\varUpsilon \in \mathcal{C}$ has the $\mathcal{C}_\varUpsilon$ property;
(Δ_2) if $\{\zeta_n\},\{\xi_n\} \in \mathbb{R}_+$ are such that $\lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}\xi_n = \ell$, where $\ell \in \mathbb{R}_+$ and $\xi_n > \ell$, $n \in \mathbb{N}$, then $\limsup_{n\to\infty}\Theta_1(\zeta_n,\xi_n) < \mathcal{C}_\varUpsilon$;
(Δ_3) if $\{\zeta_n\} \in \mathbb{R}_+$ is such that $\lim_{n\to\infty}\zeta_n = \ell \in \mathbb{R}_+$, $\Theta_1(\zeta_n,\ell) \geq \mathcal{C}_F$ implies $\ell = 0$.

In this section, $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$ will be fixed convex subsets of a Banach space $\mathcal{X}$. We define a new notion of cyclic (noncyclic) Θ-condensing operators.

Definition 2.4 A cyclic (noncyclic) operator $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be Θ-condensing if there exist $\Theta \in \Delta(\Theta, \mathcal{C}_F)$ such that $\chi(\mathcal{T}\mathcal{K}_1), \chi(\mathcal{T}\mathcal{K}_2) > 0$ implies

$$\Theta\big(\chi(\mathcal{T}\mathcal{K}_1 \cup \mathcal{T}\mathcal{K}_2), \chi(\mathcal{K}_1 \cup \mathcal{K}_2)\big) \geq \mathcal{C}_F$$

for every proximinal and $\mathcal{T}$-invariant pair $\Lambda \ni (\mathcal{K}_1, \mathcal{K}_2) \subseteq (\mathcal{A}, \mathcal{B})$ with $\text{dist}(\mathcal{K}_1, \mathcal{K}_2) = \text{dist}(\mathcal{A}, \mathcal{B})$ where χ is the Kuratowski MNC.

Theorem 2.5 *Let $\mathcal{X}$ be a Banach space and $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive cyclic Θ-condensing operator. Then $\mathcal{T}$ has a best proximity point, provided $\mathcal{A}_0 \neq \emptyset$.*

Proof From $\mathcal{A}_0 \neq \emptyset$, $(\mathcal{A}_0, \mathcal{B}_0) \neq \emptyset$. Using the conditions on $\mathcal{T}$, it is easy to show that $(\mathcal{A}_0, \mathcal{B}_0)$ is convex, closed, $\mathcal{T}$-invariant, and proximinal pair. For $p \in \mathcal{A}_0$, there is a $q \in \mathcal{B}_0$ such that $\|p - q\| = \text{dist}(\mathcal{A}, \mathcal{B})$. Since $\mathcal{T}$ is relatively nonexpansive,

$$\|\mathcal{T}p - \mathcal{T}q\| \leq \|p - q\| = \text{dist}(\mathcal{A}, \mathcal{B}),$$

which gives $\mathcal{T}p \in \mathcal{B}_0$, that is, $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$. Likewise, $\mathcal{T}(\mathcal{B}_0) \subseteq \mathcal{A}_0$ and so $\mathcal{T}$ is cyclic on $\mathcal{A}_0 \cup \mathcal{B}_0$.

We start denoting $\mathcal{P}_0 = \mathcal{A}_0$ and $\mathcal{Q}_0 = \mathcal{B}_0$ and define a sequence pair $\{(\mathcal{P}_n, \mathcal{Q}_n)\}$ as $\mathcal{P}_n = \overline{\text{conv}}(\mathcal{T}(\mathcal{P}_{n-1}))$ and $\mathcal{Q}_n = \overline{\text{conv}}(\mathcal{T}(\mathcal{Q}_{n-1}))$, $n \geq 1$. We claim that $\mathcal{P}_{n+1} \subseteq \mathcal{Q}_n$ and $\mathcal{Q}_n \subseteq \mathcal{P}_{n-1}$ for all $n \in \mathbb{N}$. We have $\mathcal{Q}_1 = \overline{\text{conv}}(\mathcal{T}(\mathcal{Q}_0)) = \overline{\text{conv}}(\mathcal{T}\mathcal{B}_0) = \overline{\text{conv}}(\mathcal{A}_0) \subseteq \mathcal{A}_0 = \mathcal{G}_0$. Therefore,

$$\mathcal{T}(\mathcal{Q}_1) \subseteq \mathcal{T}(\mathcal{P}_0), \qquad \mathcal{Q}_2 = \overline{\text{conv}}\big(\mathcal{T}(\mathcal{Q}_1)\big) \subseteq \overline{\text{conv}}\big(\mathcal{T}(\mathcal{P}_0)\big) = \mathcal{P}_1.$$

Continuing this process, we get $\mathcal{Q}_n \subseteq \mathcal{P}_{n-1}$ by using induction. Similarly, $\mathcal{P}_{n+1} \subseteq \mathcal{Q}_n$ for all $n \in \mathbb{N}$. Thus $\mathcal{P}_{n+2} \subseteq \mathcal{Q}_{n+1} \subseteq \mathcal{P}_n \subseteq \mathcal{Q}_{n-1}$ for all $n \in \mathbb{N}$. Consequently, $\{(\mathcal{P}_{2n}, \mathcal{Q}_{2n})\} \neq \emptyset$ is a decreasing sequence of closed and convex pairs in $\mathcal{A}_0 \times \mathcal{B}_0$. Moreover,

$$\mathcal{T}(\mathcal{Q}_{2n}) \subseteq \mathcal{T}(\mathcal{P}_{2n-1}) \subseteq \overline{\text{conv}}\big(\mathcal{T}(\mathcal{P}_{2n-1})\big) = \mathcal{P}_{2n}$$

and

$$\mathcal{T}(\mathcal{P}_{2n}) \subseteq \mathcal{T}(\mathcal{Q}_{2n-1}) \subseteq \overline{\text{conv}}\big(\mathcal{T}(\mathcal{Q}_{2n-1})\big) = \mathcal{Q}_{2n}.$$

Thus the pair $(\mathcal{P}_{2n}, \mathcal{Q}_{2n})$ is $\mathcal{T}$-invariant for all $n \in \mathbb{N}$,

Next, if $(\nu, \vartheta) \in \mathcal{A}_0 \times \mathcal{B}_0$ is a proximinal pair then

$$\text{dist}(\mathcal{P}_{2n}, \mathcal{Q}_{2n}) \leq \big\|\mathcal{T}^{2n}\nu - \mathcal{T}^{2n}\vartheta\big\| \leq \|\nu - \vartheta\| = \text{dist}(\mathcal{A}, \mathcal{B}).$$

We next show, using mathematical induction, that the pair $(\mathcal{P}_n, \mathcal{Q}_n)$ is proximinal. Trivially, for $n = 0$, the pair $(\mathcal{P}_0, \mathcal{Q}_0)$ is proximinal. Suppose that $(\mathcal{P}_k, \mathcal{Q}_k)$ is proximinal. Let ζ be an arbitrary member in $\mathcal{P}_{k+1} = \overline{\text{conv}}(\mathcal{T}(\mathcal{P}_k))$. Then $\zeta = \sum_{\ell=1}^{m} \lambda_\ell \mathcal{T}(x_\ell)$ with $\vartheta_\ell \in \mathcal{P}_k$, $m \in \mathbb{N}$, $\lambda_\ell \geq 0$ and $\sum_{\ell=1}^{m} \lambda_\ell = 1$. Due to proximinality of the pair $(\mathcal{P}_k, \mathcal{Q}_k)$, there exists $\xi_\ell \in \mathcal{Q}_k$ for $1 \leq \ell \leq m$ such that $\|\xi_\ell - \xi_\ell\| = \text{dist}(\mathcal{P}_k, \mathcal{Q}_k) = \text{dist}(\mathcal{A}, \mathcal{B})$. Take $\xi = \sum_{\ell=1}^{m} \lambda_\ell \mathcal{T}(\xi_\ell)$. Then $\xi \in \overline{\text{conv}}(\mathcal{T}(\mathcal{Q}_k)) = \mathcal{Q}_{k+1}$ and

$$\|\zeta - \xi\| = \left\| \sum_{\ell=1}^{m} \lambda_\ell \mathcal{T}(x_\ell) - \sum_{\ell=1}^{m} \lambda_\ell \mathcal{T}(\xi_\ell) \right\| \leq \sum_{\ell=1}^{m} \lambda_\ell \|\zeta_\ell - \xi_\ell\| = \text{dist}(\mathcal{A}, \mathcal{B}).$$

Therefore the pair $(\mathcal{P}_{k+1}, \mathcal{Q}_{k+1})$ is proximinal and thus $(\mathcal{P}_n, \mathcal{Q}_n)$ is proximinal for all $n \in \mathbb{N}$.

Next, if $\max\{\chi(\mathcal{P}_{2n_0}), \chi(\mathcal{Q}_{2n_0})\} = 0$ for some natural number $n_0 \in \mathbb{N}$, then $\mathcal{T} : \mathcal{P}_{2n_0} \cup \mathcal{Q}_{2n_0} \to \mathcal{P}_{2n_0} \cup \mathcal{Q}_{2n_0}$ is compact, and the result follows from Theorem 1.1.

So we assume $\max\{\chi(\mathcal{P}_n), \chi(\mathcal{Q}_n)\} > 0$ for all $n \in \mathbb{N}$. As $\mathcal{P}_{2n+1} \subseteq \mathcal{T}(\mathcal{P}_{2n})$ and $\mathcal{Q}_{2n+1} \subseteq \mathcal{T}(\mathcal{Q}_{2n})$, we have

$$\begin{aligned}
&\Theta\big(\chi(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}), \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) \\
&\quad = \Theta\big(\max\big\{\chi(\mathcal{P}_{2n+1}), \chi(\mathcal{Q}_{2n+1})\big\}, \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) \\
&\quad = \Theta\big(\max\big\{\chi\big(\overline{\mathrm{conv}}\big(\mathcal{T}(\mathcal{P}_{2n})\big)\big), \chi\big(\overline{\mathrm{conv}}\big(\mathcal{T}(\mathcal{Q}_{2n})\big)\big)\big\}, \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) \\
&\quad = \Theta\big(\max\big\{\chi\big(\mathcal{T}(\mathcal{P}_{2n})\big), \chi\big(\mathcal{T}(\mathcal{Q}_{2n})\big)\big\}, \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) \\
&\quad = \Theta\big(\chi\big(\mathcal{T}(\mathcal{P}_{2n}) \cup \mathcal{T}(\mathcal{Q}_{2n})\big), \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) \\
&\quad \geq \mathcal{C}_F,
\end{aligned}$$

that is,

$$\begin{aligned}
&\mathcal{C}_F \leq \Theta\big(\chi(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}), \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) \\
&\quad \leq F(\chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}), \chi(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}), \\
&\chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) > \chi(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}),
\end{aligned}$$

for $n \in \mathbb{N}$, i.e., $\{\chi_n = \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\}$ is such that $\chi_n \geq \chi_{n+1} \geq 0$ and so we can find $\ell \geq 0$ such that $\chi_n \to \ell$ as $n \to \infty$. Let $\ell > 0$. Applying Definition 2.3 on sequences $\zeta_n = \chi(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1})$ and $\xi_n = \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})$, we get $\zeta_n, \xi_n \to \gamma$ and for $\xi_n > \ell$, we have

$$\limsup_{n\to\infty} \Theta\big(\chi(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}), \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) = \limsup_{n\to\infty} \chi(\zeta_n, \xi_n) < \mathcal{C}_F,$$

a contradiction. Thus, $\ell = 0$ and $\chi_n = \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) \to 0$ as $n \to \infty$. In other words, $\lim_{n\to\infty} \chi(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) = \max\{\lim_{n\to\infty} \chi(\mathcal{P}_{2n}), \lim_{n\to\infty} \chi(\mathcal{Q}_{2n})\} = 0$.

Now, let $\mathcal{P}_\infty = \bigcap_{n=0}^{\infty} \mathcal{P}_{2n}$ and $\mathcal{Q}_\infty = \bigcap_{n=0}^{\infty} \mathcal{Q}_{2n}$. Using property (7^0) of Definition 1.3, the pair $(\mathcal{P}_\infty, \mathcal{Q}_\infty) \neq \emptyset$ is convex, compact, and $\mathcal{T}$-invariant with $\mathrm{dist}(\mathcal{P}_\infty, \mathcal{Q}_\infty) = \mathrm{dist}(\mathcal{A}, \mathcal{B})$. Therefore, $\mathcal{T}$ admits a best proximity point. □

Theorem 2.6 *Let $\mathcal{X}$ be a strictly convex Banach space and a pair $(\mathcal{A}, \mathcal{B}) \in \Lambda$ in $\mathcal{X}$ such that $\mathcal{A}_0$ is nonempty. Let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive noncyclic $\mathcal{FG}$-contractive operator. Then $\mathcal{T}$ has a best proximity pair.*

Proof Following the proof of Theorem 2.5, we define a pair $(\mathcal{P}_n, \mathcal{Q}_n)$ as $\mathcal{P}_n = \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{P}_{n-1}))$ and $\mathcal{Q}_n = \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{Q}_{n-1}))$, $n \geq 1$ with $\mathcal{P}_0 = \mathcal{A}_0$ and $\mathcal{Q}_0 = \mathcal{B}_0$. We have $\mathcal{Q}_1 = \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{Q}_0)) = \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{B}_0)) \subseteq \mathcal{B}_0 = \mathcal{Q}_0$. Therefore, $\mathcal{T}(\mathcal{Q}_1) \subseteq \mathcal{T}(\mathcal{Q}_0)$. Thus $\mathcal{Q}_2 = \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{Q}_1)) \subseteq \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{Q}_0)) = \mathcal{Q}_1$. Continuing, we get $\mathcal{Q}_n \subseteq \mathcal{Q}_{n-1}$ by using induction. Likewise $\mathcal{P}_{n+1} \subseteq \mathcal{G}_n$ for all $n \in \mathbb{N}$. Hence $\{(\mathcal{P}_n, \mathcal{Q}_n)\}$ is a decreasing sequence of nonempty, closed, and convex pairs in $\mathcal{A}_0 \times \mathcal{B}_0$. Also, $\mathcal{T}(\mathcal{Q}_n) \subseteq \mathcal{T}(\mathcal{Q}_{n-1}) \subseteq \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{Q}_{n-1})) = \mathcal{Q}_n$ and $\mathcal{T}(\mathcal{P}_n) \subseteq \mathcal{T}(\mathcal{P}_{n-1}) \subseteq \overline{\mathrm{conv}}(\mathcal{T}(\mathcal{P}_{n-1})) = \mathcal{P}_n$. Therefore for all $n \in \mathbb{N}$, the pair $(\mathcal{P}_n, \mathcal{Q}_n)$ is $\mathcal{T}$-invariant. Following as in the proof of Theorem 2.5, we have $(\mathcal{P}_n, \mathcal{Q}_n)$ is a proximinal pair with $\mathrm{dist}(\mathcal{P}_n, \mathcal{Q}_n) = \mathrm{dist}(\mathcal{A}, \mathcal{B})$ for all $n \in \mathbb{N} \cup \{0\}$.

Next, if $\max\{\chi(\mathcal{P}_{n_0}),\chi(\mathcal{Q}_{n_0})\}=0$ for some $n_0\in\mathbb{N}$, then $\mathcal{T}:\mathcal{P}_{n_0}\cup\mathcal{Q}_{n_0}\to\mathcal{P}_{n_0}\cup\mathcal{Q}_{n_0}$ is compact. Then we can conclude by Theorem 1.2 when $\mathcal{T}$ is noncyclic relatively nonexpansive mapping.

Next, we assume that $\max\{\chi(\mathcal{P}_n),\chi(\mathcal{Q}_n)\}>0$ for all $n\in\mathbb{N}$. Since $\mathcal{P}_{n+1}\subseteq\mathcal{T}(\mathcal{P}_n)$ and $\mathcal{Q}_{n+1}\subseteq\mathcal{T}(\mathcal{Q}_n)$, we have

$$\begin{aligned}
&\Theta\bigl(\chi(\mathcal{P}_{2n+1}\cup\mathcal{Q}_{2n+1}),\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)\\
&\quad=\Theta\bigl(\max\bigl\{\chi(\mathcal{P}_{2n+1}),\chi(\mathcal{Q}_{2n+1})\bigr\},\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)\\
&\quad=\Theta\bigl(\max\bigl\{\chi\bigl(\overline{\operatorname{conv}}\bigl(\mathcal{T}(\mathcal{P}_{2n})\bigr)\bigr),\chi\bigl(\overline{\operatorname{conv}}\bigl(\mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr)\bigr\},\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)\\
&\quad=\Theta\bigl(\max\bigl\{\chi\bigl(\mathcal{T}(\mathcal{P}_{2n})\bigr),\chi\bigl(\mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr\},\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)\\
&\quad=\Theta\bigl(\chi\bigl(\mathcal{T}(\mathcal{P}_{2n})\cup\mathcal{T}(\mathcal{Q}_{2n})\bigr),\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)\\
&\quad\geq\mathcal{C}_F,
\end{aligned}$$

that is,

$$\begin{aligned}
&\mathcal{C}_F\leq\Theta\bigl(\chi(\mathcal{P}_{2n+1}\cup\mathcal{Q}_{2n+1}),\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)\\
&\qquad\leq F(\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n}),\chi(\mathcal{P}_{2n+1}\cup\mathcal{Q}_{2n+1}),\\
&\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})>\chi(\mathcal{P}_{2n+1}\cup\mathcal{Q}_{2n+1}),
\end{aligned}$$

for all $n\in\mathbb{N}$. The sequence $\{\chi_n=\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\}$ is such that $\chi_n\geq\chi_{n+1}\geq 0$, so we can find $\ell\geq 0$ satisfying $\chi_n\to\ell$ as $n\to\infty$. Let $\ell>0$. Applying Definition 2.3 on the sequences $\zeta_n=\chi(\mathcal{P}_{2n+1}\cup\mathcal{Q}_{2n+1})$ and $\xi_n=\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})$ gives $\zeta_n,\xi_n\to\gamma$ and $\xi_n>\ell$, so we have

$$\limsup_{n\to\infty}\Theta\bigl(\chi(\mathcal{P}_{2n+1}\cup\mathcal{Q}_{2n+1}),\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\bigr)=\limsup_{n\to\infty}\chi(\zeta_n,\xi_n)<\mathcal{C}_F,$$

a contradiction. Thus, $\ell=0$ and $\chi_n=\chi(\mathcal{P}_{2n}\cup\mathcal{Q}_{2n})\to 0$ as $n\to\infty$. That is, $\lim_{n\to\infty}\chi(\mathcal{P}_n\cup\mathcal{Q}_n)=\max\{\lim_{n\to\infty}\chi(\mathcal{P}_n),\lim_{n\to\infty}\chi(\mathcal{Q}_n)\}=0$. Now, let $\mathcal{P}_\infty=\bigcap_{n=0}^{\infty}\mathcal{P}_n$ and $\mathcal{Q}_\infty=\bigcap_{n=0}^{\infty}\mathcal{Q}_n$. Using property (7^0) of Definition 1.3, the pair $(\mathcal{P}_\infty,\mathcal{Q}_\infty)$ is nonempty, convex, compact, and $\mathcal{T}$-invariant with $\operatorname{dist}(\mathcal{P}_\infty,\mathcal{Q}_\infty)=\operatorname{dist}(\mathcal{A},\mathcal{B})$. Therefore, $\mathcal{T}$ has a best proximity point. □

3 Particular cases

Here we discuss some consequences of Theorems 2.5 and 2.6 with various condensing operators for different forms of Θ and $\mathcal{C}_F=0$ that include some existing results.

(i) (Patle, Patel, and Arab [24]) If $\mathcal{C}_F=0$, then for $(\mathcal{K}_1,\mathcal{K}_2)\subseteq(\mathcal{A},\mathcal{B})$,

$$0\leq\Theta\bigl(\chi(\mathcal{T}\mathcal{K}_1\cup\mathcal{T}\mathcal{K}_2),\chi(\mathcal{K}_1\cup\mathcal{K}_2)\bigr)\tag{1}$$

(ii) (Gabeleh and Markin [12]) If $\Theta(\zeta,\xi)=\lambda\xi-\zeta$ $(\lambda\in(0,1))$ in (1), then for $(\mathcal{K}_1,\mathcal{K}_2)\subseteq(\mathcal{A},\mathcal{B})$,

$$\chi(\mathcal{T}\mathcal{K}_1\cup\mathcal{T}\mathcal{K}_2)\leq\lambda\chi(\mathcal{K}_1\cup\mathcal{K}_2)).\tag{2}$$

Next, we deal with an application in terms of fractional calculus.

4 Applications to fractional calculus

Fractional differential/integral equations (FDE/FIE) have been extensively studied as an application of fixed point theory. In fact, to get the unique solution of an FDE, one has to apply Banach fixed point theorem or its variants. There are different types of FDEs in the literature but the FDEs in the Caputo sense are the easiest to solve. The main advantage of Caputo derivative is that the derivative of the constant function is 0, while most of the other fractional derivatives do not have such an important property. This property helps in initial value problems to apply fixed point theorems. In [1], the existence of solutions for some Atangana–Baleanu fractional differential equations in the Caputo sense have been discussed. Some other FDE related work can be seen in [2–8, 10–12, 15–18] and the references cited therein.

Definition 4.1 (Fractional differential operator) A fractional differential operator $\mathcal{D}^{\mu}$ is called fractional Caputo–Fabrizio derivative (CFD) of order $0 < \mu < 1$ of a function x if and only if $\mathcal{D}^{\mu}$ satisfies

$$\mathcal{D}^{\mu}x(\varsigma) = \frac{1}{1-\mu}\int_0^{\varsigma} x'(\tau)\exp\left(\frac{-\mu}{1-\mu}(t-\tau)\right)d\tau, \quad \varsigma \geq 0. \tag{3}$$

The fractional integral is introduced by

$$J^{\mu}x(\varsigma) = (1-\mu)x(\varsigma) + \mu\int_0^{\varsigma} x(\tau)\,d\tau. \tag{4}$$

In this section, we consider integro-differential system in terms of CFD as follows:

$$\begin{aligned}
&\mathcal{D}^{\mu}x(\varsigma) = \Upsilon_1\left(\varsigma, x(\varsigma), \int_{\tau_1}^{\tau_1+\tau_2}\phi_1\big(\varsigma,\tau,x(\tau)\big)\,d\tau, \int_{\tau_1}^{\varsigma}\psi_1\big(\varsigma,\tau,x(\tau)\big)\,d\tau\right),\\
&\quad x(\tau_1) = x_1,\\
&\mathcal{D}^{\mu}y(\varsigma) = \Upsilon_2\left(\varsigma, y(\varsigma), \int_{\tau_1}^{\tau_1+\tau_2}\phi_2\big(\varsigma,\tau,y(\tau)\big)\,d\tau, \int_{\tau_1}^{\varsigma}\psi_2\big(\varsigma,\tau,y(\tau)\big)\,d\tau\right),\\
&\quad y(\tau_1) = y_1,
\end{aligned} \tag{5}$$

where $I = [\tau_1 - \tau_2, \tau_1 + \tau_2]$, $I_x = [x_1 - \epsilon, x_1 + \epsilon]$, $I_y = [y_1 - \epsilon, y_1 + \epsilon]$, and $I_\epsilon = [\tau_1 - \epsilon, \tau_1 + \epsilon]$. We have the following assumptions:

- (A1) The given functions are continuous in $\mathbb{R}$ and such that $\phi_1 : I \times I \times I_x \to \mathbb{R}$, $\phi_2 : I \times I \times I_y \to \mathbb{R}$, $\Upsilon_1 : I_\epsilon \times I_x \times I_x \times I_x \to \mathbb{R}$, $\Upsilon_2 : I_\epsilon \times I_y \times I_y \times I_y \to \mathbb{R}$, and x, y belong to nonempty, bounded, closed, and convex sets $\mathsf{T}_1 \subset C(I_\epsilon, \mathbb{R})$ and $\mathsf{T}_2 \subset C(I_\epsilon, \mathbb{R})$, respectively.
- (A2) For the sup-norm, we suppose that $\|x_1 - y_1\| \leq \epsilon\|x - y\|$, $0 < \epsilon \leq 1$, so that $\operatorname{dist}(\mathsf{T}_1, \mathsf{T}_2) = \|x_1 - y_1\|$. Moreover, for all $x \in \mathsf{T}_1$ and $y \in \mathsf{T}_2$, we assume that there is a positive constant $\sigma > 0$ such that

$$\|\Upsilon_1 - \Upsilon_2\| \leq \sigma\big(\|x - y\| - \|x_1 - y_1\|\big).$$

(A3) For any I_x, I_y, there is a positive function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which is upper semicontinuous and satisfies $\varphi(\iota) < \iota$ and

$$\chi\big(\Upsilon_1(I_\epsilon \times I_x \times I_x \times I_x) \cup \Upsilon_2(I_\epsilon \times I_y \times I_y \times I_y)\big) < \frac{\varphi(x(I_x \cup I_y))}{\epsilon_\mu}, \quad \epsilon_\mu > 0.$$

Now, we want to find conditions for the existence of solution for the system (5). We define an operator $\mathcal{T} : \mathsf{T}_1 \cup \mathsf{T}_2 \to C(I_\epsilon, \mathbb{R})$ (the space of all continuous functions on the interval I_ϵ) as follows:

$$\mathcal{T}(\varsigma) := \begin{cases} y_1 + (1-\mu)\Upsilon_1 + \mu \int_{\tau_1}^{\varsigma} \Upsilon_1(\eta)\, d\eta & \text{if } x \in \mathsf{T}_1, \\ x_1 + (1-\mu)\Upsilon_2 + \mu \int_{\tau_2}^{\varsigma} \Upsilon_2(\eta)\, d\eta & \text{if } x \in \mathsf{T}_2. \end{cases} \tag{6}$$

Theorem 4.2 *Consider system* (5) *satisfying the hypotheses* (A1)–(A3). *It has a solution in* $C(I_\epsilon, \mathbb{R})$, *whenever*

$$(1-\mu)\sigma < \frac{1}{4}, \quad \sigma > 0, 0 < \mu < 1.$$

Proof Consider the operator $\mathcal{T}$. We claim that it is a cyclic operator. Let $x \in \mathsf{T}_1$. Then we obtain

$$\begin{aligned} \|\mathcal{T}x - y_1\| &= \left\|(1-\mu)\Upsilon_1 + \mu \int_{\tau_1}^{\varsigma} \Upsilon_1(\eta)\, d\eta\right\| \\ &\le (1-\mu)\|\Upsilon_1\| + \mu \int_{\tau_1}^{\varsigma} \big\|\Upsilon_1(\eta)\big\|\, d\eta \\ &\le S_1(\varsigma - \tau_1) \le S_1[1 + \mu\epsilon_1 - \mu] \\ &\le S_1[1 + \mu\bar{\epsilon} - \mu] := S_1\epsilon_\mu, \end{aligned}$$

where $S_1 := \sup(\Upsilon_1) = \|\Upsilon_1\|$. By letting $\epsilon_1 < \frac{\epsilon}{\max\{S_1,S_2\}} := \bar{\epsilon}$, where $S_2 := \sup(\Upsilon_2)$, we have

$$\|\mathcal{T}x - y_1\| < \epsilon_\mu, \quad \forall x \in \mathsf{T}_1.$$

Thus, $\mathcal{T}x \in \mathsf{T}_2$. In the same manner, we can show that, for $y \in \mathsf{T}_2$, this implies

$$\|\mathcal{T}y - x_1\| < \epsilon_\mu,$$

and hence $\mathcal{T}y \in \mathsf{T}_1$. We conclude that $\mathcal{T}$ is cyclic. The above conclusion shows that the set $\mathcal{T}(\mathsf{T}_1)$ is bounded in T_2, as well as the set $\mathcal{T}(\mathsf{T}_2)$ is bounded in T_1.

Note that $\omega \in \mathsf{T}_1 \cup \mathsf{T}_2$ indicates an solution of the system (5) if and only if $\operatorname{dist}(\mathsf{T}_1 \cup \mathsf{T}_2) = \|\omega - \mathcal{T}\omega\|$. Therefore, we proceed to prove such a conclusion. Now, we aim to show that $\mathcal{T}(\mathsf{T}_1)$ is equicontinuous in T_2. For ς and ς', we have

$$\begin{aligned} \big\|\mathcal{T}x(\varsigma) - \mathcal{T}x(\varsigma')\big\| &= \mu\left\|\int_{\tau_1}^{\varsigma} \Upsilon_1(\eta)\, d\eta - \int_{\tau_1}^{\varsigma'} \Upsilon_1(\eta)\, d\eta\right\| \\ &\le \mu\left|\int_{\varsigma}^{\varsigma'} \big\|\Upsilon_1(\eta)\big\|\, d\eta\right| \\ &\le \mu S_1\big|\varsigma - \varsigma'\big| \le \epsilon_\mu S_1, \end{aligned}$$

which indicates that $\mathcal{T}(\Upsilon_1)$ is equicontinuous in Υ_2. In a similar manner, we deduce that $\mathcal{T}(\Upsilon_2)$ is equicontinuous in Υ_1. Hence, by the Arzela–Ascoli theorem, we conclude that the pair (Υ_1, Υ_2) is relatively compact. Next, we show that $\mathcal{T}$ is relatively nonexpansive.

For $(x, y) \in (\Upsilon_1, \Upsilon_2)$, we have

$$\begin{aligned}
&\left\|\mathcal{T}x(\varsigma) - \mathcal{T}y(\varsigma)\right\| \\
&\quad = \left\|y_1 + (1-\mu)\Upsilon_1 + \mu\int_{\tau_1}^{\varsigma}\Upsilon_1(\eta)\,d\eta - x_1 - (1-\mu)\Upsilon_2 - \mu\int_{\tau_1}^{\varsigma}\Upsilon_2(\eta)\,d\eta\right\| \\
&\quad \le \|x-y\| + (1-\mu)\left\|\Upsilon_1(\varsigma) - \Upsilon_2(\varsigma)\right\| + \mu\left|\int_{\tau_1}^{\varsigma}\left\|\Upsilon_1(\eta) - \Upsilon_2(\eta)\right\|\,d\eta\right| \\
&\quad \le \epsilon\|x-y\| + (1-\mu)\sigma\left(\|x-y\| - \|x_1-y_1\|\right) + \mu\sigma\left(\|x-y\| - \|x_1-y_1\|\right)(\varsigma - \tau_1) \\
&\quad \le \epsilon\|x-y\| + (1-\mu)\sigma\left(\|x-y\| - \|x_1-y_1\|\right) + \mu\sigma\left(\|x-y\| - \|x_1-y_1\|\right)\epsilon_1 \\
&\quad \le \left(\left[\epsilon + (1-\mu)\sigma + \sigma\bar{\epsilon}\right] + \left[(1-\mu)\sigma + \sigma\bar{\epsilon}\right]\right)\|x-y\| \\
&\quad \le \left(\epsilon + 2\left[(1-\mu)\sigma + \sigma\epsilon_\mu\right]\right)\|x-y\|.
\end{aligned}$$

But ϵ is an arbitrary constant, thus when $\epsilon \to 0$, we have $\epsilon_\mu = 1-\mu$. Hence, we obtain the inequality

$$\left\|\mathcal{T}x(\varsigma) - \mathcal{T}y(\varsigma)\right\| \le 4(1-\mu)\sigma\|x-y\| < \|x-y\|.$$

This implies that $\mathcal{T}$ is relatively nonexpansive.

We proceed to show that $\mathcal{T}$ is χ-condensing. Assume that $(I_x, I_y) \subseteq (\Upsilon_1, \Upsilon_2)$ is nonempty, bounded, closed, and convex set such that

$$\begin{aligned}
&\chi\left(\mathcal{T}(I_x) \cup \mathcal{T}(I_y)\right) \\
&\quad = \max\left\{\left(\chi\mathcal{T}(I_x), \chi\mathcal{T}(I_y)\right)\right\} \\
&\quad = \max\left(\sup_x\{\chi\mathcal{T}x, x \in I_x\}, \sup_y\{\chi\mathcal{T}y, y \in I_y\}\right) \\
&\quad = \max\left(\sup_x\left\{\chi\left(y_1 + (1-\mu)\overline{\operatorname{conv}}\Upsilon_1 + \mu\epsilon_1\overline{\operatorname{conv}}\Upsilon_1\right)\right\},\right. \\
&\qquad \left.\sup_y\left\{\chi\left(x_1 + (1-\mu)\overline{\operatorname{conv}}\Upsilon_2 + \mu\epsilon_2\overline{\operatorname{conv}}\Upsilon_2\right)\right\}\right) \\
&\quad \le \max\left(\sup_x\left\{\chi\left(y_1 + (1-\mu)\overline{\operatorname{conv}}\Upsilon_1(I_\epsilon \times I_x \times I_x \times I_x)\right.\right.\right. \\
&\qquad \left.\left. + \mu\bar{\epsilon}\overline{\operatorname{conv}}\Upsilon_1(I_\epsilon \times I_x \times I_x \times I_x)\right)\right\}, \\
&\qquad \left.\sup_y\left\{\chi\left(x_1 + (1-\mu)\overline{\operatorname{conv}}\Upsilon_2(I_\epsilon \times I_y \times I_y \times I_y) + \mu\bar{\epsilon}\overline{\operatorname{conv}}\Upsilon_2(I_\epsilon \times I_y \times I_y \times I_y)\right)\right\}\right) \\
&\quad \le \epsilon_\mu\max\left(\left\{\chi\left(\Upsilon_1(I_\epsilon \times I_x \times I_x \times I_x)\right)\right\}, \left\{\chi\left(\Upsilon_2(I_\epsilon \times I_y \times I_y \times I_y)\right)\right\}\right) \\
&\quad \le \epsilon_\mu\chi\left(\Upsilon_1(I_\epsilon \times I_x \times I_x \times I_x) \cup \Upsilon_2(I_\epsilon \times I_y \times I_y \times I_y)\right) \\
&\quad < \epsilon_\mu\frac{\varphi(\chi(I_x \cup I_y))}{\epsilon_\mu} \\
&\quad = \varphi\left(\chi(I_x \cup I_y)\right).
\end{aligned}$$

Thus, we obtain

$$\varphi\big(\chi(I_x \cup I_y)\big) - \chi\big(\mathcal{T}(I_x) \cup \mathcal{T}(I_y)\big) \geq 0.$$

By putting $\Theta(\tau, \varsigma) := \varphi(\varsigma) - \tau$, then we arrive at

$$\Theta\big(\chi\big(\mathcal{T}(I_x) \cup \mathcal{T}(I_y)\big), \varphi\big(\chi(I_x \cup I_y)\big)\big) \geq 0.$$

Hence, necessary requirements of Theorem 2.5 are verified. Therefore, the operator $\mathcal{T}$ has a best proximity point, and thus system (5) has a solution.

This completes the proof. □

4.1 Numerical constructions

Here, we present some particular systems aiming to use Theorem 4.2. The main condition in this theorem is $(1 - \mu)\sigma < 1/4$. This inequality is very easy to check comparing with other existence theorems requiring (A1)–(A3). Theorem 4.2 indicates that the system obeying formula (5) has a solution. This type of solution is very important in dynamic and control systems. By this solution, one can study the stability, as well as oscillatory and other behaviors of the solution.

Example 4.3 Consider the system

$$\begin{aligned} &\mathcal{D}^{0.9}x(\varsigma) = x(\kappa_1 - \kappa_2 y), \quad x(0) = x_0, \\ &\mathcal{D}^{0.9}y(\varsigma) = y(\kappa_3 x - k_4), \quad y(0) = y_0. \end{aligned} \tag{7}$$

With the help on Mathematica 11.2, the solution is given by the integral

$$\begin{aligned} &\int_{\tau_0=0}^{x[\varsigma]1/\tau} \frac{1}{(1 + W(-(\kappa_2)/(\kappa_1)\exp((\kappa_3\tau - \text{constant})/\kappa_1)\tau^(- \kappa_4/\kappa_1)))}\, d\tau \\ &\quad \approx \kappa_1\varsigma + \text{constant}, \\ &y[\varsigma] = -\kappa_1/\kappa_2 W\big(-\kappa_2/\kappa_1 \exp\big(\big(\kappa_3 x[\varsigma] - \text{constant}\big)/\kappa_1\big)x[\varsigma]^{-\kappa_4/\kappa_1}\big), \end{aligned}$$

where W represents the product log-function. In order to apply Theorem 4.2, we take $(x_0, y_0) = (1, 1)$ and $\sigma = (\kappa_2\kappa_3 - \kappa_1\kappa_4)$, and then $\sigma(1 - \mu) < 0.25$. For instance, for $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (2, 1, 0.4, 0.1)$, we have $\sigma(1 - \mu) = 0.2 \cdot 0.1 = 0.02 < 0.25$; thus, by Theorem 4.2, the system has a solution that converges to a limit cycle. For another case, assume that $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (1, 1, 0.9, 0.8)$; then $\sigma(1 - \mu) = 0.1 \cdot 0.1 = 0.01 < 0.25$. According to Theorem 4.2, system (7) has a solution converging to a limit cycle. Similarly, for $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (1, 1, 1, 0.8)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (1, 0.9, 1, 0.5)$, Fig. 1 indicates the different cases depending on the value of σ.

Example 4.4 Consider the following system:

$$\begin{aligned} &\mathcal{D}^{0.5}x(\varsigma) = y, \quad x(0) = x_0, \\ &\mathcal{D}^{0.5}y(\varsigma) = -x + \sigma y, \quad y(0) = y_0, \end{aligned} \tag{8}$$

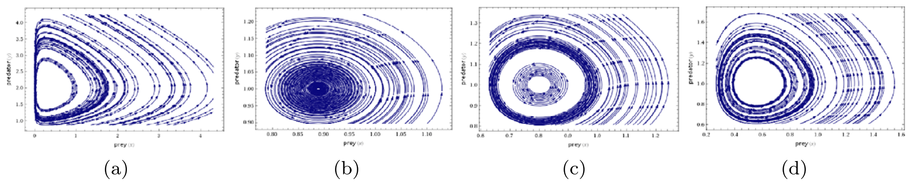

Figure 1 Solutions of (7) for different values of $\sigma(1-\mu)$. Fig. 1(**a**) and Fig. 1(**c**) are the solutions for $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (2, 1, 0.4, 0.1)$ and $(1, 1, 0.9, 0.8)$, respectively. Fig. 1(**b**) and Fig. 1(**d**) graphs indicate the solutions for $(1, 1, 1, 0.8)$ and $(1, 0.9, 1, 0.5)$, respectively. We see that the cyclic solution for these cases is cyclic

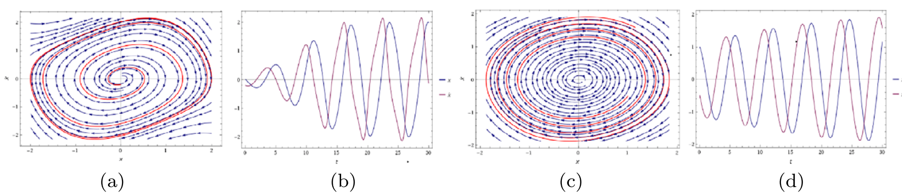

Figure 2 Solutions of (8) for different values of $\sigma(1-\mu)$. We see that the solution for these cases is cyclic

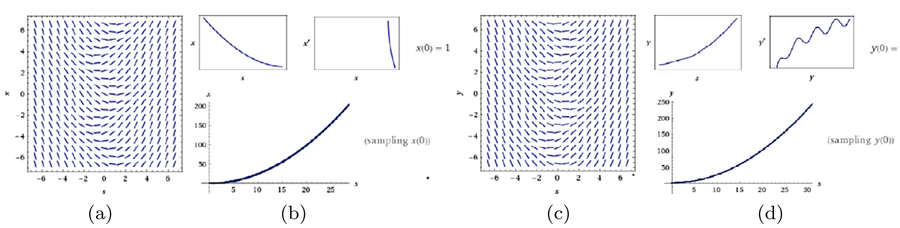

Figure 3 Solutions of (10) for the value of $\sigma(1-\mu) = 0.2$, indicating the sample solution family

where the value of σ satisfies $\sigma(1-\mu) < 0.25$. For instance, when $\sigma = 0.4$, we have $\sigma(1-\mu) = 0.2 < 0.25$ and then the system has a solution with the initial condition $(x_0, y_0) = (0, 0)$. Moreover, the solution is cyclic and its portrait indicates an unstable limit cycle (see Fig. 2(a)–(b)). When $\sigma = 0.1$, we have $\sigma(1-\mu) = 0.05 < 0.25$ and then the system has a solution with a portrait of an unstable limit cycle (see Fig. 2(c)–(d)).

Example 4.5 Consider the integro-differential system

$$\begin{aligned} \mathcal{D}^{0.6}x(\varsigma) &= \sigma y + \sigma \int \sin(x)\,dx, \quad x(0) = x_0, \\ \mathcal{D}^{0.6}y(\varsigma) &= (1-\sigma)x + \sigma \int \cos(y)\,dy, \quad y(0) = y_0, \end{aligned} \tag{9}$$

which is equivalent to

$$\begin{aligned} \mathcal{D}^{0.6}x(\varsigma) &= \sigma y - \sigma \cos(x) + c_1, \quad x(0) = x_0, \\ \mathcal{D}^{0.6}y(\varsigma) &= (1-\sigma)x + \sigma \sin(y) + c_2. \quad y(0) = y_0. \end{aligned} \tag{10}$$

For $\sigma = 0.5$, one has $\sigma(1-\mu) = 0.2 < 0.25$. Then (10) admits a solution where the constants $c_1 \le \sigma$ and $c_2 \le \sigma$ (see Fig. 3).

Acknowledgements
This research is funded by the Foundation for Science and Technology Development of Ton Duc Thang University (FOSTECT), website: http://fostect.tdtu.edu.vn, under Grant FOSTECT.2019.14. The authors are grateful to the Basque Government for the Grant IT1207-19. We are also grateful to the learned referees for useful suggestions which helped us improve the text in several places.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Cotton University, Panbazar, Guwahati, 781001, Assam, India. [2]Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh, 791112, Arunachal Pradesh, India. [3]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [4]Institute of Electrical and Electronics Engineers (IEEE): 94086547, Kuala Lumpur 59200, Malaysia. [5]Institute of Research and Development of Processes, University of the Basque Country, 48940 Leioa (Bizkaia), Spain.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 March 2021 Accepted: 28 August 2021 Published online: 10 September 2021

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