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A nonlinear fractional Rayleigh–Stokes equation under nonlocal integral conditions

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Abstract

In this paper, we study the fractional nonlinear Rayleigh–Stokes equation under nonlocal integral conditions, and the existence and uniqueness of the mild solution to our problem are considered. The ill-posedness of the mild solution to the problem recovering the initial value is also investigated. To tackle the ill-posedness, a regularized solution is constructed by the Fourier truncation method, and the convergence rate to the exact solution of this method is demonstrated.

Keywords: Fractional Rayleigh–Stokes equation; Ill-posed problem; Regularization; Existence; Uniqueness; Convergence estimation

1 Introduction

Most of fluids in the real world, such as in food products (mayonnaise, mustard, chocolate, ketchup, butter, cheese, yogurt, etc.), in natural substances (honey, magma, lava, gums, etc.), in biology (blood, semen, synovia, mucus, etc.), in industry (paint, glue, lubricant, ink, molten polymer, etc.), in cosmetics (soap solution, toothpaste, cream, silicone, nail polish, etc.) are treated as non-Newtonian fluids. Therefore, the study on non-Newtonian fluids is a substantial subject in science and industrial applications. As the Rayleigh–Stokes problem for an edge, the first problem of Stokes for a non-Newtonian fluid flow past an impulsively started flat plate has received much attention because of its practical importance [1, 2]. For a second grade fluid, the equation of motion is of higher order than the Navier–Stokes equation, because it exhibits all properties of viscoelastic fluids.

Recently, fractional calculus has encountered much success in the description of constitutive relations of viscoelastic fluids. The starting point of the fractional derivative model of a viscoelastic fluid is usually a classical differential equation which is modified by replacing the time derivative of an integer order with a fractional calculus operator. This generalization allows one to define precisely noninteger order integrals or derivatives.

Moreover, there has currently been a considerable increase in examining fractional partial differential equations (FPDEs). We can list several current remarkable research studies; for instance, Abdolrazaghi and Razani [3], Behboudi *et al.* [4], Ding and Neito [5], Agarwal *et al.* [6], Baleanu [7], Adiguzel *et al.* [8–10], Afshari *et al.* [11], Alqahtani *et al.* [12], Karapinar *et al.* [13], Abdeljawad *et al.* [14], Baitiche *et al.* [15], Ardjouni [16] and the references therein.

In this research study, we focus on the Rayleigh–Stokes problem as follows:

$$\begin{cases} \partial_t u - \Delta u - \mu \partial_t^\alpha \Delta u = F(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

associated with the nonlocal integral condition

$$\xi_1 u(x,0) + \xi_2 \int_0^T \nu(t) u(x,s)\, ds = g(x), \quad x \in \Omega, \tag{1.2}$$

where $\xi_1, \xi_2 > 0$ and $\xi_1^2 + \xi_2^2 > 0$. Here, $u(x,t)$ is fluid velocity, μ is viscosity, F is a source function, Δ is the Laplacian, $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a smooth domain with the boundary $\partial\Omega$, and $T > 0$ is a given time, g is the final data in $L^2(\Omega)$, $\partial_t = \partial/\partial t$, and ∂_t^α is the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ defined in [17, 18]:

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \left(\int_0^t \frac{u(x,s)}{(t-s)^\alpha}\, ds \right),$$

where $\Gamma(.)$ is the gamma function.

The forward problems for equation (1.1) have been examined in a plenty of studies. For instance, Zierep and Fetecau [19] discussed the energetic balance in the Rayleigh–Stokes problem for a Maxwell fluid for several initial and/or boundary conditions. Fetecau and Zierep [20] found the exact solutions both for the Stokes' problem and for the Rayleigh–Stokes problem within the context of the fluids of second grade. Bazhlekova *et al.* [21] presented an introduction about an analysis of the Rayleigh–Stokes problem for a generalized second-grade fluid. The exact solution of the Rayleigh–Stokes problem for a generalized second grade fluid in a porous half-space with a heated flat plate was considered by Xue *et al.* [22]. Exact solutions of the Rayleigh–Stokes equation in the case of homogeneous initial and boundary conditions was considered by Zhao and Yang [23]. The backward problems have currently been studied by many mathematicians. Ngoc *et al.* [24], for instance, pondered the inverse problem for the nonlinear fractional Rayleigh–Stokes equations. Equation (1.1) associated with Gaussian random noise was examined by Triet *et al.* [25], and so forth.

To the best of our knowledge, there are still very few studies on the Rayleigh–Stokes equation accompanied with nonlocal integral conditions. In comparison with the initial condition $u(x,0) = f(x)$ or the final condition $u(x,T) = g(x)$, the nonlocal conditions are of more significant complexity. Furthermore, it is emphasized that we cannot apply Parseval's equality to obtain stable estimates in the L^p space. To tackle this limitation, we need to develop additional techniques and Sobolev embedding in our study.

This manuscript is structured as follows. An introduction of preliminary results is described in Sect. 2; the regularity of the mild solution to the problem in a linear case is illustrated in Sect. 3. In Sect. 4, the problem recovering the initial value and the convergence of the regularized solution are detailed. The regularity of the mild solution to the problem in the nonlinear case is mentioned in Sect. 5. Eventually, the conclusion is presented in Sect. 6.

2 Preliminary results

We recall the spectral problem as follows:

$$\begin{cases} \Delta\phi_n(x) = -\lambda\phi_n(x), & x \in \Omega, \\ \phi_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admitting the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$ (see [26]). The corresponding eigenfunctions $\phi_n \in H_0^1(\Omega)$. For all $s \ge 0$, the operator A^s (here $A = -\Delta$) also possesses the following representation:

$$A^s h = \sum_{n=1}^{\infty} (h, \phi_n)\lambda_n^s \phi_n, \quad h \in D\bigl(A^s\bigr) = \left\{ h \in L^2(\Omega) : \sum_{n=1}^{\infty} \bigl|(h,\phi_n)\bigr|^2 \lambda_n^{2s} < \infty \right\}.$$

Consider on $D(A^s)$ the norm (noting that $\lambda_1 > 0$)

$$\|h\|_{D(A^s)} = \left(\sum_{n=1}^{\infty} \bigl|(h,\phi_n)\bigr|^2 \lambda_n^{2s} \right)^{\frac{1}{2}}, \quad h \in D\bigl(A^s\bigr).$$

By duality, we can set $D(A^{-s}) = (D(A^s))^*$ by identifying $(L^2(\Omega))^* = L^2(\Omega)$ and using the so-called Gelfand triple (see [27]). Then $D(A^{-s})$ is a Hilbert space with the following norm:

$$\|h\|_{D(A^{-s})} = \left(\sum_{n=1}^{\infty} \bigl|\langle h,\phi_j\rangle\bigr|^2 \lambda_n^{-2s} \right)^{\frac{1}{2}},$$

wherein $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $D(A^{-s})$ and $D(A)$. For any $p \ge 0$, we define the space

$$\mathcal{H}^p(\Omega) = \left\{ v \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^p \bigl|\bigl\langle v(x), \phi_n(x)\bigr\rangle\bigr|^2 < +\infty \right\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega)$, then $\mathcal{H}(\Omega)$ is a Hilbert space with the norm

$$\|v\|_{\mathcal{H}^p(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^p \bigl|\bigl\langle v(x), \phi_n(x)\bigr\rangle\bigr|^2 \right)^{\frac{1}{2}}.$$

Suppose that problem (1.1) has a solution u which admits the form $u(x,t) = \sum_{n=1}^{\infty} \langle u(x,t), \phi_n(x)\rangle \phi_n(x)$. From [21, 24], we deduce that the solution of problem (1.1) with the initial condition $u(x,0) = u_0(x)$ is given by

$$\bigl\langle u(x,t), \phi_n(x)\bigr\rangle = \mathbf{S}_{n,\alpha}(t)\bigl\langle u_0(x), \phi_n(x)\bigr\rangle + \int_0^t \mathbf{S}_{n,\alpha}(t-s)\bigl\langle F\bigl(u(x,s)\bigr), \phi_n(x)\bigr\rangle\,ds,$$

where $\mathbf{S}_{n,\alpha}(t)$ is given by

$$\mathbf{S}_{n,\alpha}(t) = \int_0^{\infty} e^{-zt}\mathcal{K}(n,\alpha,z)\,dz, \tag{2.1}$$

$$\mathcal{K}(n,\alpha,z) = \frac{\mu}{\pi}\frac{\lambda_j \sin(\alpha\pi)z^\alpha}{(-z+\lambda_n\mu z^\alpha \cos\alpha\pi + \lambda_j)^2 + (\lambda_n\mu z^\alpha \sin\alpha\pi)^2}, \tag{2.2}$$

and its Laplace transform is given by $\mathcal{L}(\mathbf{S}_{n,\alpha}(t)) = \frac{1}{t+\mu\lambda_n t^\alpha+\lambda_n}$. This implies that

$$u(x,t) = \sum_{n=1}^{\infty}\left[\mathbf{S}_{n,\alpha}(t)\langle u_0(x),\phi_n(x)\rangle + \int_0^t \mathbf{S}_{n,\alpha}(t-s)\langle F\big(u(x,s)\big),\phi_n(x)\rangle\,ds\right]\phi_n(x). \tag{2.3}$$

We can easily see that

$$u_n(t) = \mathbf{S}_{n,\alpha}(t)u_n(0) + \int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds, \tag{2.4}$$

where $F_n(u(s)) = \langle F(u(\cdot,s)),\phi(x)\rangle$.

From the equation

$$\xi_1 u(x,0) + \xi_2\int_0^T \nu(t)u(x,s)\,ds = g(x), \tag{2.5}$$

we have that

$$\xi_1\sum_1^\infty u_n(0)\phi_n(x) + \xi_2\int_0^T \nu(t)\left(\sum_1^\infty u_n(t)\phi_n(x)\right)dt = \sum_1^\infty g_n\phi_n(x), \tag{2.6}$$

where $g_n = \langle g(\cdot),\phi(x)\rangle$.

Thanks to the uniqueness of the Fourier expansion of a function in L^2 space, we get

$$\begin{aligned} g_n &= \xi_1 u_n(0) + \xi_2\int_0^T \nu(t)u_n(t)\,dt \\ &= \xi_1 u_n(0) + \xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)u_n(0)\,dt \\ &\quad + \xi_2\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\,dt. \end{aligned} \tag{2.7}$$

By a straightforward computation, we obtain

$$\begin{aligned} u_n(0) &= \frac{g_n - \xi_2\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n(u(s))\,ds\,dt}{\xi_1 + \xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \\ &= \frac{1}{\xi_1 + \xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\left(g_n - \xi_2\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\,dt\right). \end{aligned} \tag{2.8}$$

Conjoining (2.4) and (2.8), we have

$$\begin{aligned} u_n(t) &= \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\left(g_n - \xi_2\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\,dt\right) \\ &\quad + \int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds. \end{aligned} \tag{2.9}$$

As a result, we have the formula or the mild solution of problem (1.1)–(1.2)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \\ &\quad\times \left(g_n - \xi_2 \int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds\,dt \right) \phi_n(x) \\ &\quad+ \sum_{n=1}^{\infty} \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds \phi(x). \end{aligned} \tag{2.10}$$

We posit the terms $\mathcal{A}_{j=1,2,3}(x,t)$ as follows:

$$\begin{aligned} \mathcal{A}_1(x,t) &= \sum_{n=1}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} g_n \phi_n(x), \\ \mathcal{A}_2(x,t) &= \sum_{n=1}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \left(\xi_2 \int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds\,dt \right) \phi_n(x), \\ \mathcal{A}_3(x,t) &= \sum_{n=1}^{\infty} \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds \phi(x). \end{aligned}$$

Then we have

$$u(x,t) = \mathcal{A}_1(x,t) - \mathcal{A}_2(x,t) + \mathcal{A}_3(x,t). \tag{2.11}$$

Lemma 2.1 *If* $\alpha \in (0,1)$, *we have that the subsequent estimations hold*

$$\frac{C_2(\mu,\alpha,t)}{\lambda_n} \le \mathbf{S}_{n,\alpha}(t) \le \frac{C_1(\mu,\alpha)}{1+\lambda_n t^{1-\alpha}}, \quad 0 \le t \le T, \tag{2.12}$$

where

$$C_1(\mu,\alpha) = \frac{\Gamma(1-\alpha)}{\mu\pi \sin(\alpha\pi)} + 1, \qquad C_2(\mu,\alpha,t) = \frac{\mu \sin(\alpha\pi) e^{-t}}{3\pi(\alpha+1)(\mu^2+1+\frac{1}{\lambda_1^2})}.$$

Proof The detailed interpretation can be found in [24]. We briefly present some main steps of the proof. We have

$$\mathbf{S}_{n,\alpha}(t) = \frac{\mu}{\pi} \int_0^{\infty} \frac{e^{-zt}\lambda_n \sin(\alpha\pi) z^{\alpha}}{(-z+\lambda_n\mu z^{\alpha}\cos(\alpha\pi))^2 + (\lambda_n \mu z^{\alpha} \sin(\alpha\pi))^2}\,dz. \tag{2.13}$$

In terms of the left-hand side of inequality (2.12), we have

$$\mathbf{S}_{n,\alpha}(t) \le \frac{1}{\mu\pi \sin(\alpha\pi) t^{1-\alpha}\lambda_n} \int_0^{\infty} e^{-z} z^{-\alpha}\,dz = \frac{\Gamma(1-\alpha)}{\mu\pi \sin(\alpha\pi) t^{1-\alpha}\lambda_n}. \tag{2.14}$$

It is worth noting that

$$\int_0^{\infty} e^{-z} z^{-\alpha}\,dz = \Gamma(1-\alpha).$$

With regard to the right-hand side of inequality (2.12), we have

$$\begin{aligned}\mathbf{S}_{n,\alpha}(t) &\geq \frac{\mu\sin(\alpha\pi)}{3\pi\lambda_n}\int_0^\infty \frac{e^{-zt}z^\alpha}{\mu^2 z^{2\alpha}+1+\frac{z^2}{\lambda_1^2}}\,dz\\ &\geq \frac{\mu\sin(\alpha\pi)}{3\pi\lambda_n}\frac{e^{-t}}{\mu^2+1+\frac{1}{\lambda_1^2}}\frac{1}{\alpha+1}.\end{aligned}\tag{2.15}$$

□

For the rest of paper, we give the definition of a mild solution to problem (1.1)–(1.2) in the subsequent notion.

Definition 2.1 The function u is called a mild solution of problem (1.1)–(1.2) if
(i) u belongs to the $L^m(0,T;L^2(\Omega))$ space;
(ii) u satisfies equality (2.9).

3 The regularity of the mild solution to problem (1.1)–(1.2) in the linear case

In this section, we consider the regularity of the mild solution of problem (1.1)–(1.2) under the condition that the source term is linear.

Theorem 3.1 *Suppose that there exist M, N such that $Nt^{-\gamma}\leq \nu(t)\leq Mt^{-\theta}$ for $\gamma,\theta<\frac{1}{2}$.*
(i) Let $1/2<\alpha<1$ and $s>0$. If $\xi_1>0$, $\xi_2>0$ and $g\in D(A^s)$, $F\in L^2(0,T;D(A^{s-1}))$, we rest assured that

$$\|u(\cdot,t)\|_{D(A^s)}\leq \frac{C_1(\mu,\alpha)}{\xi_1}\|g\|_{D(A^s)}+M_2\|F\|_{L^2(0,T;D(A^{s-1}))}+M_3\|F\|_{L^2(0,T;D(A^{s-1}))}.\tag{3.1}$$

(ii) Let $1/2<\alpha<1$ and $s>0$. If $\xi_1=0$, $\xi_2>0$ and $g\in D(A^s)$, $F\in L^2(0,T;D(A^{s-1}))$, then we can conclude that

$$\|u(\cdot,t)\|_{D(A^s)}\leq M_4\|g\|_{D(A^s)}+M_5\|F\|^2_{L^2(0,T;D(A^{s-1}))}+M_3\|F\|_{L^2(0,T;D(A^{s-1}))}.\tag{3.2}$$

Proof We will estimate the terms $\mathcal{A}_1(x,t)$, $\mathcal{A}_2(x,t)$, and $\mathcal{A}_3(x,t)$ in two cases: $\xi_1,\xi_2>0$ and $\xi_1=0$, $\xi_2>0$.

Part i: In the case of $\xi_1,\xi_2>0$. Thanks to Parseval's equality, we have

$$\begin{aligned}\|\mathcal{A}_1(\cdot,t)\|^2_{D(A^s)} &= \sum_{n=1}^\infty\left(\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1+\xi_2\int_0^T\nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\right)^2\lambda_n^{2s}|g_n|^2\\ &\leq \frac{C_1^2(\mu,\alpha)}{\xi_1^2}\sum_1^\infty\lambda_n^{2s}|g_n|^2\\ &= \frac{C_1^2(\mu,\alpha)}{\xi_1^2}\|g\|^2_{D(A^s)}.\end{aligned}\tag{3.3}$$

This implies

$$\|\mathcal{A}_1(\cdot,t)\|_{D(A^s)}\leq \frac{C_1(\mu,\alpha)}{\xi_1}\|g\|_{D(A^s)}.\tag{3.4}$$

Similarly, the term $\|\mathcal{A}_2(\cdot,t)\|_D(A^2)$ can be assessed as follows:

$$\begin{aligned}
&\left\|\mathcal{A}_2(\cdot,t)\right\|^2_{D(A^s)}\\
&\quad= \sum_{n=1}^{\infty}\left(\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1+\xi_2\int_0^T v(t)\mathbf{S}_{n,\alpha}(t)\,dt}\right)^2\lambda^{2s}\left(\xi_2\int_0^T v(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\,dt\right)^2\\
&\quad\le \sum_{n=1}^{\infty}\lambda_n^{2s-2}\frac{\xi_2^2}{\xi_1^2}\left(\int_0^T v^2(t)\,dt\right)\left(\int_0^T\left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\right)^2 dt\right).
\end{aligned}\tag{3.5}$$

Note that $\theta < \frac{1}{2}$, we have

$$\int_0^T v^2(t)\,dt \le \int_0^T M^2 t^{-2\theta} = M_\theta.\tag{3.6}$$

Put another way, the integral $\int_0^T v^2(t)\,dt$ is convergent

$$\begin{aligned}
&\int_0^T\left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\right)^2 dt\\
&\quad\le \int_0^T\left(\int_0^t \mathbf{S}^2_{n,\alpha}(t-s)\,ds\right)\left(\int_0^t \left|F\big(u(s)\big)\right|^2 ds\right)dt\\
&\quad\le \frac{C_1^2(\mu,\alpha)}{\lambda_n^2}\int_0^T\left(\int_0^t (t-s)^{2\alpha-2}\,ds\right)\left(\int_0^t \left|F\big(u(s)\big)\right|^2 ds\right)dt\\
&\quad\le \frac{C_1^2(\mu,\alpha)}{\lambda_n^2}\frac{T^{2\alpha}}{2\alpha-1}\left(\int_0^T \left|F\big(u(s)\big)\right|^2 ds\right).
\end{aligned}\tag{3.7}$$

Conjoining (3.5), (3.6), and (3.7), we arrive at

$$\begin{aligned}
\left\|\mathcal{A}_2(\cdot,t)\right\|^2_{D(A^s)} &\le M_\theta C_1^2(\mu,\alpha)\frac{\xi_2^2}{\xi_1^2}\frac{T^{2\alpha}}{2\alpha-1}\int_0^T\sum_{n=1}^{\infty}\lambda_n^{2s-4}\left|F\big(u(s)\big)\right|^2 ds\\
&\le M_\theta C_1^2(\mu,\alpha)\frac{\xi_2^2}{\xi_1^2}\frac{T^{2\alpha}}{2\alpha-1}\|F\|^2_{L^2(0,T;D(A^{s-2}))}.
\end{aligned}\tag{3.8}$$

In other words, we have

$$\left\|\mathcal{A}_2(\cdot,t)\right\|_{D(A^s)} \le \sqrt{M_\theta C_1^2(\mu,\alpha)\frac{\xi_2^2}{\xi_1^2}\frac{T^{2\alpha}}{2\alpha-1}}\|F\|_{L^2(0,T;D(A^{s-2}))} = M_2\|F\|_{L^2(0,T;D(A^{s-2}))}.\tag{3.9}$$

Using Parseval's equality again, we can estimate the term $\mathcal{A}_3(\cdot,t)$ as follows:

$$\begin{aligned}
\left\|\mathcal{A}_3(\cdot,t)\right\|^2_{D(A^S)} &= \sum_{n=1}^{\infty}\lambda_n^{2s}\left(\int_0^t \mathbf{S}_n(\alpha,t-s)F_n\big(u(s)\big)\,ds\right)^2\\
&\le C_1^2(\mu,\alpha)\sum_{n=1}^{\infty}\lambda_n^{2s-2}\left(\int_0^t (t-s)^{2\alpha-2}\,ds\right)\left(\int_0^t \left|F\big(u(s)\big)\right|^2 ds\right)dt\\
&\le C_1^2(\mu,\alpha)\frac{T^{2\alpha}}{2\alpha-2}\left(\int_0^T \lambda_n^{2s-2}\left|F\big(u(s)\big)\right|^2 ds\right)
\end{aligned}$$

$$\le C_1^2(\mu,\alpha)\frac{T^{2\alpha}}{2\alpha-2}\|F\|^2_{L^2(0,T;D(A^{s-1}))}. \tag{3.10}$$

The latter estimation allows us to deduce that

$$\left\|\mathcal{A}_3(\cdot,t)\right\|_{D(A^S)} \le \sqrt{C_1^2(\mu,\alpha)\frac{T^{2\alpha}}{2\alpha-2}}\|F\|_{L^2(0,T;D(A^{s-2}))} = M_3\|F\|_{L^2(0,T;D(A^{s-1}))}. \tag{3.11}$$

Combining (3.4), (3.9), and (3.11) helps us rest assured that

$$\left\|u(\cdot,t)\right\|_{D(A^s)} \le \frac{C_1(\mu,\alpha)}{\xi_1}\|g\|_{D(A^s)} + M_2\|F\|_{L^2(0,T;D(A^{s-2}))} + M_3\|F\|_{L^2(0,T;D(A^{s-1}))}. \tag{3.12}$$

Part ii: In the case of $\xi_1 = 0$, $\xi_2 > 0$.

Paserval's equality gives us

$$\left\|\mathcal{A}_1(\cdot,t)\right\|^2_{D(A^s)} = \sum_{n=1}^{\infty}\left(\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\right)^2\lambda_n^{2s}|g_n|^2. \tag{3.13}$$

It should be noted that

$$\mathbf{S}_{n,\alpha}(t) \ge \frac{C_2(\mu,\alpha,t)}{\lambda_n}$$

and

$$\nu(t) \ge Nt^{-\gamma}.$$

We get

$$\begin{aligned}\left\|\mathcal{A}_1(\cdot,t)\right\|^2_{D(A^s)} &\le \frac{-2\gamma+2}{\xi_2^2N^2T^{-2\gamma+2}}\frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}}\sum_{n=1}^{\infty}\lambda_n^{2s}|g_n|^2\\ &\le \frac{-2\gamma+2}{\xi_2^2N^2T^{-2\gamma+2}}\frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}}\|g\|^2_{D(A^s)}.\end{aligned} \tag{3.14}$$

Therefore, we immediately obtain that

$$\left\|\mathcal{A}_1(\cdot,t)\right\|_{D(A^s)} \le \frac{-\gamma+1}{\xi_2NT^{-\gamma+1}}\frac{C_1(\mu,\alpha)}{C_2(\mu,\alpha,t)}\frac{1}{t^{1-\alpha}}\|g\|_{D(A^s)} = M_4\|g\|_{D(A^s)}. \tag{3.15}$$

In terms of $\mathcal{A}_2(\cdot,t)$, by computations similar to above, we have

$$\begin{aligned}&\left\|\mathcal{A}_2(\cdot,t)\right\|^2_{D(A^s)}\\ &\quad= \sum_{n=1}^{\infty}\left(\frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\right)^2\lambda_n^{2s}\left(\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\,dt\right)^2\\ &\quad\le \frac{-2\gamma+2}{N^2T^{-2\gamma+2}}\frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}}\\ &\qquad\times\sum_{n=1}^{\infty}\lambda_n^{2s}\left(\int_0^T \nu^2(t)\,dt\right)\left(\int_0^T\left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)F_n\big(u(s)\big)\,ds\right)^2dt\right)\end{aligned}$$

$$\begin{aligned}
&\leq \frac{M_\theta(-2\gamma+2)}{N^2 T^{-2\gamma+2}} \frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{T^{2\alpha-2}}{t^{2-2\alpha}(2\alpha-1)} \sum_{n=1}^{\infty} \lambda_n^{2s-2} \left(\int_0^T \left| F\big(u(s)\big) \right|^2 ds \right) \\
&\leq M_5^2 \|F\|^2_{L^2(0,T;D(A^{s-1}))}.
\end{aligned} \tag{3.16}$$

This implies

$$\left\| \mathcal{A}_2(\cdot,t) \right\|_{D(A^s)} \leq M_5 \|F\|^2_{L^2(0,T;D(A^{s-1}))}. \tag{3.17}$$

Connecting (3.15), (3.16), and (3.17) allows us to come to a conclusion that

$$\left\| u(\cdot,t) \right\|_{D(A^s)} \leq M_4 \|g\|_{D(A^s)} + M_5 \|F\|^2_{L^2(0,T;D(A^{s-1}))} + M_3 \|F\|_{L^2(0,T;D(A^{s-1}))}. \tag{3.18}$$

□

4 The problem recovering the initial value

In this section, we are inclined to deliberate on the following problem:

$$\begin{cases} \partial_t u - \Delta u - \mu \partial_t^\alpha \Delta u = F(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \\ \xi_2 \int_0^T \nu(t) u(x,s)\, ds = g(x), & x \in \Omega. \end{cases} \tag{4.1}$$

Our primary aim in this part is to recover the initial data $u(x,0) = V(x)$. To achieve this goal, we firstly go to prove the point that the problem is ill-posed in $L^2(0,T)$. For convenience of readers, we presume that $F = 0$.

From the formula of the mild solution (2.10), we have that

$$V(x) = u(x,0) = \sum_{n=0}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2 \int_0^T \nu(t) \mathbf{S}_{n,\alpha}(t)\, dt} g_n \phi_n(x). \tag{4.2}$$

Theorem 4.1 *In the sense of Hadamard, problem (4.1) is ill-posed in the space $L^2(0,T)$ with reference to the case of $t = 0$.*

Proof We ponder on the linear operator $\mathcal{K} : L^2(D) \to L^2(D)$ as follows:

$$\begin{aligned}
\mathcal{K} V(x) &= \sum_{n=0}^{\infty} \frac{\xi_2 \int_0^T \nu(t) \mathbf{S}_{n,\alpha}(t)\, dt}{\mathbf{S}_{n,\alpha}(t)} \langle V(x), \phi_n(x) \rangle \phi_n(x) \\
&= \int_D m(x,\tau) V(\tau)\, d\tau,
\end{aligned} \tag{4.3}$$

where

$$m(x,\tau) = \sum_{n=0}^{\infty} \frac{\xi_2 \int_0^T \nu(t) \mathbf{S}_{n,\alpha}(t)\, dt}{\mathbf{S}_{n,\alpha}(t)} \phi_n(x) \phi_n(\tau).$$

It is worth mentioning that $m(x,\tau) = m(\tau,x)$. Therefore, the operator $\mathcal{K}$ is self-adjoint and the compactness of $\mathcal{K}$ is presented as follows.

We define the finite rank operator $\mathcal{K}_L$

$$\mathcal{K}_L V = \sum_{n=1}^{L} \frac{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}{\mathbf{S}_{n,\alpha}(t)} \langle V(x), \phi_n(x)\rangle \phi_n(x). \tag{4.4}$$

From (4.3) and (4.4), we have

$$\begin{aligned} \|\mathcal{K}_L V - \mathcal{K}V\|^2_{L^2(D)} &= \sum_{n=L+1}^{\infty} \left(\frac{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}{\mathbf{S}_{n,\alpha}(t)} \right)^2 \langle V(x), \phi_n(x)\rangle^2 \\ &\le \frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} t^{2-2\alpha} \frac{M^2 T^{-2\theta+2}}{(-\theta+1)^2} \sum_{n=L+1}^{\infty} \langle V(x), \phi_n(x)\rangle^2. \end{aligned} \tag{4.5}$$

In that

$$\|\mathcal{K}_L V - \mathcal{K}V\|_{L^2(D)} \le \frac{C_1(\mu,\alpha)}{C_2(\mu,\alpha,t)} t^{1-\alpha} \frac{M T^{-\theta+1}}{(-\theta+1)} \|V\|_{L^2(D)}. \tag{4.6}$$

This allows us to conclude that $\mathcal{K}$ is a compact operator. In conjunction with (4.3), we have

$$\mathcal{K}V(x) = g(x). \tag{4.7}$$

Combining the latter conclusion and using Kirsch [28], we deduce that the problem of recovering the initial value V from (4.7) is ill-posed. To ensure mathematical clarity, we provide an example as follows. If we choose the input data $g^j(x) = \frac{\phi_j(x)}{\lambda_j^{1/2}}$, the L^2 norm of g^j is

$$\|g^j\|_{L^2(D)} = \frac{1}{\lambda_j^{1/2}} \to 0 \quad \text{when } j \to +\infty, \tag{4.8}$$

and the initial data regarding g^j is

$$\begin{aligned} V^j(x) &= \sum_{n=1}^{\infty} \frac{\mathbf{S}_{j,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{j,\alpha}(t)\,dt} \langle g^j(x), \phi_j(x)\rangle \phi_j(x) \\ &= \frac{\mathbf{S}_{j,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{j,\alpha}(t)\,dt} \frac{\phi_j(x)}{\lambda_j^{1/2}}. \end{aligned} \tag{4.9}$$

In the next step, we assess the initial data V^j in respect of L^2 norm

$$\begin{aligned} \|V^j\|_{L^2(D)} &= \left\| \frac{\mathbf{S}_{j,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{j,\alpha}(t)\,dt} \frac{\phi_j(x)}{\lambda_j^{1/2}} \right\|_{L^2(D)} \\ &\ge \frac{\frac{C_2(\mu,\alpha,t)}{\lambda_1}}{\xi_2 M_{1,\beta} \frac{C_1(\mu,\alpha)}{1+\lambda_j t^{1-\alpha}}} \frac{1}{\lambda_j^{1/2}} \\ &\ge \frac{C_2(\mu,\alpha,t) t^{1-\alpha}}{\xi_2 M_{1,\beta} \lambda_1 C_1(\mu,\alpha)} \lambda_j^{1/2} \to +\infty \quad \text{when } j \to +\infty. \end{aligned} \tag{4.10}$$

Conjoining (4.8) with (4.10), we arrive at the conclusion that the solution of problem (4.1) is instable. □

With the aim of putting forward the next theory, we give an assumption as follows. Presume that $g^\epsilon \in L^p(D)$ and $F^\epsilon \in L^\infty(0,T;L^p(D))$ are noisy data, and presume

$$\|g - g^\epsilon\|_{L^p(D)} + \|F - F^\epsilon\|_{L^\infty(0,T;L^p(D))} \le \epsilon. \tag{4.11}$$

Using the Fourier truncation method, we provide a construction of regularized solution to problem (4.1) as follows:

$$\begin{aligned} V^\epsilon(x) &= \sum_{n=1}^{K} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \langle g^\epsilon(x), \phi_n(x)\rangle \phi_n(x), \\ &\quad + \sum_{n=1}^{K} \frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \left(\int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n^\epsilon\big(u(s)\big)\,ds\,dt \right) \phi_n(x). \end{aligned} \tag{4.12}$$

Theorem 4.2 *Let (g^ϵ, F^ϵ) satisfy (4.11), and let b, δ be such that*

$$-\frac{d}{4} < b \le \min\left(0, \frac{(q-2)d}{4q}\right), \quad 0 \le \delta < \frac{d}{4}. \tag{4.13}$$

Suppose that there exists β such that

$$V \in D\big(A^\beta\big).$$

Choose $K = K(\epsilon)$ such that

$$\lim_{\epsilon \to 0} \lambda_K^{\delta-b}\epsilon = 0, \qquad \lim_{\epsilon \to 0} \lambda_k = +\infty. \tag{4.14}$$

Then we have that the following estimation holds:

$$\|V - V^\epsilon\|_{L^{\frac{2d}{d-4\delta}}(\Omega)} \le C\|V - V^\epsilon\|_{D(A^\delta)} \le \lambda_K^{\delta-b}\epsilon + \lambda_K^{\delta-b}\epsilon + \lambda_{K+1}^{-\gamma}\|V\|_{D(A^{\delta+\beta})}. \tag{4.15}$$

Remark 4.1 On account of $\lambda_K \sim K^{2/d}$, we need to choose K such that

$$K^{\frac{2}{d}(\delta-b)}\epsilon \to 0, \quad \text{when } \epsilon \to 0.$$

In fact, we choose $K = \epsilon^{(r-1)d/2(\delta-b)}$ for $0 < r < 1$, and then the error is of order

$$\max\left(\left(\epsilon^{(r-1)d/2(\eta-b)} + 1\right)^{\frac{-2\delta}{d}}, \epsilon^r\right).$$

Proof To begin with, we define the function $W^\epsilon(x)$ in the following way:

$$W^\epsilon(x) = \sum_{n=1}^{K} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \langle g(x), \phi_n(x)\rangle \phi_n(x),$$

$$+\sum_{n=1}^{K} \frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \left(\int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds\,dt \right) \phi_n(x). \tag{4.16}$$

As the next step, we ponder the term $\|V^\epsilon - V\|_{D(A^\delta)}$ and present an estimation of it with regard to the $D(A^\delta)$ norm for $\delta > 0$. Thanks to the triangle inequality, we obtain

$$\|V^\delta - V\|_{D(A^\eta)} \le \|W^\delta - V^\delta\|_{D(A^\eta)} + \|W^\delta - V\|_{D(A^\eta)}. \tag{4.17}$$

In terms of $\|W^\delta - V^\delta\|_{D(A^\eta)}$, we have

$$\begin{aligned}
&\|W^\epsilon - V^\epsilon\|^2_{D(A^\delta)} \\
&\quad \le 2\sum_{n=1}^{K} \left| \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \right|^2 \lambda_n^{2\delta} \langle g^\epsilon - g, \phi_n\rangle^2 \\
&\qquad + \sum_{n=1}^{K} \left| \frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \right|^2 \lambda_n^{2\delta} \\
&\qquad \times \left(\int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n^\epsilon\big(u(s)\big) - F_n\big(u(s)\big)\big)\,ds\,dt \right)^2.
\end{aligned} \tag{4.18}$$

From the Sobolev embedding $L^q(\Omega) \hookrightarrow D(A^b)$ for $-\frac{d}{4} \le b < 0$ and $q \ge \frac{2d}{d-4b}$, it should be noted that there exists a positive constant $C(b,q)$ such that

$$\|g^\epsilon - g\|_{D(A^b)} \le C(b,q)\|g^\epsilon - g\|_{L^q(D)} \le C(b,q)\epsilon. \tag{4.19}$$

We denote the two terms $\mathcal{B}_1$ and $\mathcal{B}_2$ as follows:

$$\begin{aligned}
\mathcal{B}_1 &= \sum_{n=1}^{K} \left| \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \right| \langle g^\epsilon - g, \phi_n\rangle \phi_n(x), \\
\mathcal{B}_2 &= \sum_{n=1}^{K} \left| \frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \right. \\
&\quad \times \left| \left(\int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n^\epsilon\big(u(s)\big) - F_n\big(u(s)\big)\big)\,ds\,dt \right) \phi_n(x). \right.
\end{aligned} \tag{4.20}$$

It is worth noting that $\delta > b$, the term $\mathcal{B}_1$ can be assessed in the following way:

$$\begin{aligned}
\|\mathcal{B}_1\|_{D(A^\delta)} &= \sqrt{\sum_{n=1}^{K} \left| \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \right|^2 \lambda_n^{2\delta-2b} \lambda_n^{2b} \langle g^\epsilon - g, \phi_n\rangle^2} \\
&\le \frac{-\gamma+1}{\xi_2 N T^{-\gamma+1}} \frac{C_1(\mu,\alpha)}{C_2(\mu,\alpha,t)} \frac{1}{t^{1-\alpha}} \lambda_K^{\delta-b} \sqrt{\sum_{n=1}^{K} \lambda_n^{2b} \langle g^\epsilon - g, \phi_n\rangle^2} \\
&\le \frac{-\gamma+1}{\xi_2 N T^{-\gamma+1}} \frac{C_1(\mu,\alpha)}{C_2(\mu,\alpha,t)} \frac{1}{t^{1-\alpha}} \lambda_K^{\delta-b} \|g^\epsilon - g\|_{D(A^b)} \\
&\le \frac{-\gamma+1}{\xi_2 N T^{-\gamma+1}} \frac{C_1(\mu,\alpha)}{C_2(\mu,\alpha,t)} \frac{1}{t^{1-\alpha}} C(b,q) \lambda_K^{\delta-b} \epsilon = M_6 \lambda_K^{\delta-b} \epsilon.
\end{aligned} \tag{4.21}$$

Similarly, the term $\mathcal{B}_2$ can be estimated as follows:

$$\begin{aligned}
\|\mathcal{B}_2\|^2_{D(A^\delta)} &= \sum_{n=1}^{K}\left|\frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\right|^2 \lambda_n^{2\delta} \\
&\quad \times \left(\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t)\big(F_n^\epsilon\big(u(s)\big) - F_n\big(u(s)\big)\big)\,ds\,dt\right)^2 \\
&\le \frac{-2\gamma+2}{N^2 T^{-2\gamma+2}}\frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}} \\
&\quad \times \sum_{n=1}^{K}\lambda_n^{2\delta-2b}\lambda_n^{2b}\left(\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n^\epsilon\big(u(s)\big) - F_n\big(u(s)\big)\big)\,ds\,dt\right)^2 \\
&\le \frac{-2\gamma+2}{N^2 T^{-2\gamma+2}}\frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}}\big\|F^\epsilon - F\big\|_{L^\infty(0,T;D(A^b))} \\
&\quad \times \sum_{n=1}^{K}\lambda_n^{2\delta-2b}\left(\int_0^T \nu(t)\int_0^t \mathbf{S}_{n,\alpha}(t-s)\,ds\,dt\right)^2 \\
&\le \frac{-2\gamma+2}{N^2 T^{-2\gamma+1}}\frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}}\big\|F^\epsilon - F\big\|_{L^\infty(0,T;D(A^b))}\sum_{n=1}^{K}\lambda_n^{2\delta-2b}\left(\int_0^T \nu(t)\,dt\right)^2 \\
&\le \frac{-2\gamma+2}{N^2 T^{-2\gamma+1}}\frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)}\frac{1}{t^{2-2\alpha}}\|\nu\|^2_{L^2(0,T)}\lambda_K^{2\delta-2b-2}\big\|F^\delta - F\big\|_{L^\infty(0,T;D(A^b))} \\
&\le M_7^2\lambda_K^{2\delta-2b}\epsilon^2.
\end{aligned} \tag{4.22}$$

The latter estimation and the previous one allow us to have that

$$\big\|W^\epsilon - V^\epsilon\big\|_{D(A^\delta)} \le \|\mathcal{B}_1\|_{D(A^\delta)} + \|\mathcal{B}_2\|_{D(A^\delta)} \le M_6\lambda_K^{\delta-b}\epsilon + M_7\lambda_K^{\delta-b}\epsilon. \tag{4.23}$$

Regarding the term $\|W^\epsilon - V\|_{D(A^\delta)}$, we have

$$\big\|W^\epsilon - V\big\|_{D(A^\delta)} = \sqrt{\sum_{n=K+1}^{\infty}\lambda_n^{2\delta}V_n^2} = \sqrt{\sum_{n=K+1}^{\infty}\lambda_n^{-2\beta}\lambda_n^{2\delta+2\beta}V_{n^2}} \le \lambda_{K+1}^{-\beta}\|V\|_{D(A^{\delta+\beta})}. \tag{4.24}$$

Conjoining (4.23) and (4.24), we come to the conclusion that

$$\begin{aligned}
\big\|V - V^\epsilon\big\|_{D(A^\delta)} &\le \big\|W^\epsilon - V^\epsilon\big\|_{D(A^\delta)} + \big\|W^\epsilon - V\big\|_{D(A^\delta)} \\
&\le M_6\lambda_K^{\delta-b}\epsilon + M_7\lambda_K^{\delta-b}\epsilon + \lambda_{K+1}^{-\beta}\|V\|_{D(A^{\delta+\beta})}.
\end{aligned} \tag{4.25}$$

The Sobolev embedding $D(A^\delta) \hookrightarrow L^{\frac{2d}{d-4\delta}}(\Omega)$ allows us to rest assured that

$$\big\|V - V^\epsilon\big\|_{L^{\frac{2d}{d-4\delta}}(\Omega)} \le C\big\|V - V^\epsilon\big\|_{D(A^\delta)} \le \lambda_K^{\delta-b}\epsilon + \lambda_K^{\delta-b}\epsilon + \lambda_{K+1}^{-\beta}\|V\|_{D(A^{\delta+\beta})}. \tag{4.26}$$

□

5 The regularity of the mild solution to problem (1.1)–(1.2) in the nonlinear case

In this section, we concentrate on examining the subsequent nonlinear problem

$$\begin{cases} \partial_t u - \Delta u - \mu \partial_t^\alpha \Delta u = F(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \\ \xi_1 u(x,0) + \xi_2 \int_0^T \nu(t) u(x,s)\, ds = g(x), & x \in \Omega. \end{cases} \tag{5.1}$$

Theorem 5.1 *Suppose that there exists $Nt^{-\gamma} \le \nu(t) \le Mt^{-\theta}$ for $\theta < 1/2$.*

(i) Let $1/2 < \alpha < 1$, and presume $0 < s < 1$. If $\xi_1, \xi_2 > 0$, g belongs to $D(A^s)$, and F is a global Lipschitz source function satisfying

$$\|F(u) - F(v)\|_{L^2(D)} \le K_F \|u - v\|_{L^2(D)} \tag{5.2}$$

for T is small enough, then problem (5.1) has a unique solution

$$u \in L^\infty(0,T; D(A^s)).$$

Furthermore, we have

$$\|u\|_{L^\infty(0,T;D(A^s))} \le \frac{C_1^2(\mu,\alpha)}{\xi_1^2(1 - M_9\sqrt{T} - M_{10}T^{\alpha-1})} \|g\|^2_{D(A^s)}.$$

(ii) Let $1/2 < \alpha < 1$, and presume $0 < s < 1$. If $\xi_1 = 0$, $\xi_2 > 0$, $g \in D(A^{s+\alpha})$, and F is a global Lipschitz source function satisfying

$$\|F(u) - F(v)\|_{D(A^\alpha)} \le K_F \|u - v\|_{D(A^\alpha)} \tag{5.3}$$

for T is small enough, then problem (5.1) has a unique solution

$$u \in L^\infty(0,T; D(A^s)).$$

Moreover, we have

$$\|u\|_{L^\infty(0,T;D(A^s))} \le \frac{M_4}{1 - M_{11}\sqrt{T} - M_{10}T^{\alpha-1}} \|g\|^2_{D(A^s)}.$$

Proof Part i. Using (2.11) in Sect. 2, we obtain

$$u(x,t) = \mathcal{A}_1(x,t) - \mathcal{A}_2(u)(x,t) + \mathcal{A}_3(u)(x,t). \tag{5.4}$$

For convenience of calculations in the next steps, we posit

$$\mathcal{A}_1(x,t) = \sum_{n=1}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\, dt} g_n \phi_n(x),$$

$$\mathcal{A}_2(u)(x,t) = \sum_{n=1}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\, dt}$$

$$
\begin{aligned}
&\times \left(\xi_2 \int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\bigl(s, u(s)\bigr)\, ds\, dt \right) \phi_n(x), \\
\mathcal{A}_3(u)(x,t) &= \sum_{n=1}^{\infty} \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\bigl(s, u(s)\bigr)\, ds \phi(x),
\end{aligned}
$$

where

$$
F_n\bigl(x, u(s)\bigr) = \bigl\langle F\bigl(x, s, u(s)\bigr), \phi_n(x)\bigr\rangle. \tag{5.5}
$$

With the purpose of proving the point that nonlinear equation (5.4) has a unique solution in $L^\infty(0,T;D(A^s))$, we posit the following mapping:

$$
\begin{aligned}
&\mathcal{I} : L^\infty\bigl(0,T;D\bigl(A^s\bigr)\bigr) \to L^\infty\bigl(0,T;D\bigl(A^s\bigr)\bigr), \\
&\mathcal{I}w = \mathcal{A}_1(x,t) - \mathcal{A}_2(w)(x,t) + \mathcal{A}_3(w)(x,t).
\end{aligned} \tag{5.6}
$$

Using the triangle inequality again, we have

$$
\begin{aligned}
\|\mathcal{I}w_1 - \mathcal{I}w_2\|_{L^\infty(0,T;D(A^s))} &\le \bigl\| \mathcal{A}_2(w_1) - \mathcal{A}_2(w_2)\bigr\|_{L^\infty(0,T;D(A^s))} \\
&\quad + \bigl\| \mathcal{A}_3(w_1) - \mathcal{A}_3(w_2)\bigr\|_{L^\infty(0,T;D(A^s))}.
\end{aligned} \tag{5.7}
$$

We will show that the mapping $\mathcal{I}$ is a contraction mapping in $L^\infty(0,T;D(A^s))$. Now, we give estimations of the terms $\|\mathcal{A}_2(w_1) - \mathcal{A}_2(w_2)\|_{L^\infty(0,T;D(A^s))}$ and $\|\mathcal{A}_3(w_1) - \mathcal{A}_3(w_2)\|_{L^\infty(0,T;D(A^s))}$ as follows.

Step 1: In terms of the term $\|\mathcal{A}_2(w_1) - \mathcal{A}_2(w_2)\|_{L^\infty(0,T;D(A^s))}$.

Firstly, we have

$$
\begin{aligned}
&\bigl\| \mathcal{A}_2(w_1)(\cdot,t) - \mathcal{A}_2(w_2)(\cdot,t)\bigr\|^2_{D(A^s)} \\
&\quad = \sum_{n=1}^{\infty} \left(\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t) \mathbf{S}_{n,\alpha}(t)\, dt} \right)^2 \lambda_n^{2s} \\
&\quad\quad \times \left(\xi_2 \int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) \bigl(F_n\bigl(s, w_1(s)\bigr) - F_n\bigl(s, w_2(s)\bigr)\bigr)\, ds\, dt \right)^2.
\end{aligned} \tag{5.8}
$$

It is noteworthy that $\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2 \int_0^T \nu(t) \mathbf{S}_{n,\alpha}(t)\, dt} \le \frac{C_1(\mu,\alpha)}{\xi_1}$, we obtain

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} \frac{\xi_2^2 C_1^2(\mu,\alpha)}{\xi_1^2} \lambda_n^{2s} \int_0^T \nu^2(t)\, dt \\
&\quad \times \int_0^T \left(\int_0^t \mathbf{S}_{n,\alpha}(t-s) \bigl(F_n\bigl(s, w_1(s)\bigr) - F_n\bigl(s, w_2(s)\bigr)\bigr)\, ds \right)^2 dt \\
&\le \sum_{n=1}^{\infty} \frac{\xi_2^2 C_1^2(\mu,\alpha)}{\xi_1^2} \lambda_n^{2s} M_\theta \\
&\quad \times \int_0^T \left(\int_0^t \mathbf{S}_{n,\alpha}(t-s) \bigl(F_n\bigl(s, w_1(s)\bigr) - F_n\bigl(s, w_2(s)\bigr)\bigr)\, ds \right)^2 dt.
\end{aligned} \tag{5.9}
$$

Thanks to the inequality $\mathbf{S}_{n,\alpha}(t) \le \frac{C_1(\mu,\alpha)}{1+\lambda_n(t-s)^{1-\alpha}}$ (see Lemma (2.1)), we get

$$\begin{aligned}
&\int_0^T \left(\int_0^t \mathbf{S}_{n,\alpha}(t)\big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)\,ds \right)^2 dt \\
&\quad \le \int_0^T \left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)\,ds \right) \left(\int_0^t \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2 ds \right) dt \\
&\quad \le \frac{C_1^2(\mu,\alpha)T^{2\alpha}}{2\alpha-1}\lambda_n^{-2} \int_0^T \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2 ds. \qquad (5.10)
\end{aligned}$$

On account of $s < 1$, we have that

$$\begin{aligned}
&\big\|\mathcal{A}_2(w_1)(\cdot,t) - \mathcal{A}_2(w_2)(\cdot,t)\big\|^2_{D(A^s)} \\
&\quad \le \sum_{n=1}^{\infty} \frac{\xi_2^2 C_1^4(\mu,\alpha)T^{2\alpha}M_\theta}{\xi_1^2(2\alpha-1)}\lambda_n^{2s-2} \int_0^T \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2 ds \\
&\quad \le M_8 \int_0^T \sum_{n=1}^{\infty} \lambda_n^{2s-2}\big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2 ds \\
&\quad \le M_8 \lambda_1^{2s-2} \int_0^T \big\|F\big(s,w_1(s)\big) - F\big(s,w_2(s)\big)\big\|^2_{L^2(D)} ds. \qquad (5.11)
\end{aligned}$$

Using the global Lipschitz property of F, we arrive at

$$\big\|\mathcal{A}_2(w_1)(\cdot,t) - \mathcal{A}_2(w_2)(\cdot,t)\big\|^2_{D(A^s)} \le M_8\lambda_1^{2s-2}K_F^2 \int_0^T \big\|w_1(s) - w_2(s)\big\|^2_{L^2(D)} ds. \qquad (5.12)$$

The Sobolev embedding $D(A^s) \hookrightarrow L^2(D)$ allows us to deduce that

$$\big\|\mathcal{A}_2(w_1)(\cdot,t) - \mathcal{A}_2(w_2)(\cdot,t)\big\|^2_{D(A^s)} \le M_8\lambda_1^{2s-2}K_F^2 C_s^2 \int_0^T \big\|w_1(s) - w_2(s)\big\|^2_{D(A^s)} ds. \qquad (5.13)$$

This implies that, for any t, $0 \le t \le T$, we have

$$\big\|\mathcal{A}_2(w_1)(\cdot,t) - \mathcal{A}_2(w_2)(\cdot,t)\big\|^2_{D(A^s)} \le M_8\lambda_1^{2s-2}K_F^2 C_s^2 T\|w_1 - w_2\|^2_{L^\infty(0,T;D(A^s))}. \qquad (5.14)$$

In other words,

$$\big\|\mathcal{A}_2(w_1)(\cdot,t) - \mathcal{A}_2(w_2)(\cdot,t)\big\|_{D(A^s)} \le M_9\sqrt{T}\|w_1 - w_2\|_{L^\infty(0,T;D(A^s))}. \qquad (5.15)$$

We can see that the right-hand side of (5.14) is independent of t, as a result we reach the conclusion that

$$\big\|\mathcal{A}_2(w_1) - \mathcal{A}_2(w_2)\big\|_{L^\infty(0,T;D(A^s))} \le M_9\sqrt{T}\|w_1 - w_2\|_{L^\infty(0,T;D(A^s))}. \qquad (5.16)$$

Step 2: In terms of the term $\|\mathcal{A}_3(w_1)(\cdot,t) - \mathcal{A}_3(w_2)(\cdot,t)\|_{L^\infty(0,T;D(A^s))}$.

Thanks to Parseval's equality and Holder's inequality, we can assess the term $\|\mathcal{A}_3(w_1)(\cdot,t) - \mathcal{A}_3(w_2)(\cdot,t)\|_{L^\infty(0,T;D(A^s))}$ in the following way:

$$\begin{aligned}
&\big\|\mathcal{A}_3(w_1)(\cdot,t) - \mathcal{A}_3(w_2)(\cdot,t)\big\|_{D(A^s)} \\
&\quad = \sum_{n=1}^{\infty} \lambda_n^{2s} \left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)\,ds \right)^2 \\
&\quad \le \sum_{n=1}^{\infty} \lambda_n^{2s} \int_0^t \mathbf{S}_{n,\alpha}(t-s)\,ds \int_0^t \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2\,ds \\
&\quad \le C_1^2(\mu,\alpha) \sum_{n=1}^{\infty} \lambda_n^{2s-2} \int_0^t (t-s)^{2\alpha-2}\,ds \int_0^t \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2\,ds.
\end{aligned} \tag{5.17}$$

It is worth noting that $\alpha > 1/2$, we get

$$\begin{aligned}
&\big\|\mathcal{A}_3(w_1)(\cdot,t) - \mathcal{A}_3(w_2)(\cdot,t)\big\|_{D(A^s)} \\
&\quad \le C_1^2(\mu,\alpha)\frac{T^{2\alpha-1}}{2\alpha-1} \sum_{n=1}^{\infty} \lambda_n^{2s-2} \int_0^T \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2\,ds \\
&\quad \le C_1^2(\mu,\alpha)\frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \sum_{n=1}^{\infty} \lambda_n^{2s-2} \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2\,ds.
\end{aligned} \tag{5.18}$$

We can estimate the term $\int_0^T \sum_{n=1}^\infty \lambda_n^{2s-2}(F_n(s,w_1(s)) - F_n(s,w_2(s)))^2\,ds$ as follows:

$$\begin{aligned}
&\int_0^T \sum_{n=1}^{\infty} \lambda_n^{2s-2} \big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)^2\,ds \\
&\quad \le \lambda_1^{2s-2} \int_0^T \big\|F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big\|_{L^2(D)}^2\,ds \\
&\quad \le \lambda_1^{2s-2} K_F^2 \int_0^T \big\|w_1(s)-, w_2(s)\,ds\big\|_{L^2(D)}^2\,ds \\
&\quad \le \lambda_1^{2s-2} K_F^2 T \|w_1 - w_2\|_{L^\infty(0,T;D(A^s))}^2,
\end{aligned} \tag{5.19}$$

where we used the global Lipschitz property of F function and note that $s < 1$.

Conjoining the latter estimate and the previous one, we claim that

$$\big\|\mathcal{A}_3(w_1)(\cdot,t) - \mathcal{A}_3(w_2)(\cdot,t)\big\|_{D(A^s)}^2 \le \frac{T^{2\alpha-2}}{2\alpha-1}\lambda_1^{2s-2}K_F^2\|w_1 - w_2\|_{L^\infty(0,T;D(A^s))}^2.$$

Put another way,

$$\big\|\mathcal{A}_3(w_1)(\cdot,t) - \mathcal{A}_3(w_2)(\cdot,t)\big\|_{D(A^s)} \le M_{10}T^{\alpha-1}\|w_1 - w_2\|_{L^\infty(0,T;D(A^s))},$$

this implies that

$$\big\|\mathcal{A}_3(w_1) - \mathcal{A}_3(w_2)\big\|_{L^\infty(0,T;D(A^s))} \le M_{10}T^{\alpha-1}\|w_1 - w_2\|_{L^\infty(0,T;D(A^s))}. \tag{5.20}$$

Combining (5.7), (5.16), and (5.20), we have that

$$\|\mathcal{I}w_1 - \mathcal{I}w_2\|_{L^\infty(0,T;D(A^s))} \le \left(M_9\sqrt{T} + M_{10}T^{\alpha-1}\right)\|w_1 - w_2\|_{L^\infty(0,T;D(A^s))}. \tag{5.21}$$

Suppose that T is small enough such that

$$M_9\sqrt{T} + M_{10}T^{\alpha-1} < 1,$$

we can claim that $\mathcal{I}$ is a contraction mapping in $L^\infty(0,T;D(A^s)$.

Furthermore, it should be noted that if $w_1 = 0$, then

$$\mathcal{I}w_1(x,t) = \mathcal{A}_1(x,t). \tag{5.22}$$

This implies

$$\begin{aligned}\left\|\mathcal{I}w_1(\cdot,t)\right\|^2_{D(A^s)} &= \sum_{n=1}^{\infty}\left(\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_1 + \xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}\right)^2 \lambda_n^{2s}|g_n|^2 \\ &\le \frac{C_1^2(\mu,\alpha)}{\xi_1^2}\sum_1^{\infty}\lambda_n^{2s}|g_n|^2 \\ &= \frac{C_1^2(\mu,\alpha)}{\xi_1^2}\|g\|^2_{D(A^s)}.\end{aligned} \tag{5.23}$$

The latter estimation helps us assert that

$$\mathcal{I}w_1 \in L^\infty\left(0,T;D\left(A^s\right)\right). \tag{5.24}$$

On the basis of the Banach fixed point theorem, we claim that problem (5.1) has a unique solution belonging to the $L^\infty(0,T;D(A^s))$ space. In addition, if we take $w = 0$, we have that

$$\begin{aligned}\|u\|_{L^\infty(0,T;D(A^s))} &\le \|\mathcal{I}u - \mathcal{I}w\|_{L^\infty(0,T;D(A^s))} + \|\mathcal{I}w\|_{L^\infty(0,T;D(A^s))} \\ &\le \left(M_9\sqrt{T} + M_{10}T^{\alpha-1}\right)\|u\|_{L^\infty(0,T;D(A^s))} + \frac{C_1^2(\mu,\alpha)}{\xi_1^2}\|g\|^2_{D(A^s)}.\end{aligned} \tag{5.25}$$

It is worth noting that $M_9\sqrt{T} + M_{10}T^{\alpha-1} < 1$, we get

$$\|u\|_{L^\infty(0,T;D(A^s))} \le \frac{C_1^2(\mu,\alpha)}{\xi_1^2(1 - M_9\sqrt{T} - M_{10}T^{\alpha-1})}\|g\|^2_{D(A^s)}. \tag{5.26}$$

We break the back of the proof of Part i.

Part ii: From (2.11) in Sect. 2, we get

$$u(x,t) = \mathcal{B}_1(x,t) + \mathcal{B}_2(u)(x,t) + \mathcal{B}_3(u)(x,t), \tag{5.27}$$

where $\mathcal{B}_j$ ($j = 1,2,3$) are defined as follows:

$$\mathcal{B}_1(x,t) = \sum_{n=1}^{\infty}\frac{\mathbf{S}_{n,\alpha}(t)}{\xi_2\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt}g_n\phi_n(x),$$

$$\mathcal{B}_2(u)(x,t) = \sum_{n=1}^{\infty} \frac{\mathbf{S}_{n,\alpha}(t)}{\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t)\,dt} \left(\int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds\,dt \right) \phi_n(x),$$
$$\mathcal{B}_3(u)(x,t) = \sum_{n=1}^{\infty} \int_0^t \mathbf{S}_{n,\alpha}(t-s) F_n\big(u(s)\big)\,ds\phi(x),$$

and

$$F_n\big(x,u(s)\big) = \big\langle F\big(x,s,u(s)\big), \phi_n(x)\big\rangle.$$

Firstly, we posit the following function:

$$\begin{aligned} &\mathcal{J} : L^{\infty}\big(0,T;D\big(A^s\big)\big) \to L^{\infty}\big(0,T;D\big(A^s\big)\big), \\ &\mathcal{J}w = \mathcal{B}_1(x,t) + \mathcal{B}_2(w)(x,t) + \mathcal{B}_3(w)(x,t). \end{aligned} \tag{5.28}$$

Similar to Part i, we go to prove the point that $\mathcal{J}$ is a contraction mapping in $L^{\infty}(0,t;D(A^s))$. In view of the triangle inequality, we have

$$\begin{aligned} \|\mathcal{J}w_1 - \mathcal{J}w_2\|_{L^{\infty}(0,T;D(A^s))} &\le \big\|\mathcal{B}_2(w_1) - \mathcal{B}_2(w_2)\big\|_{L^{\infty}(0,T;D(A^s))} \\ &\quad + \big\|\mathcal{B}_3(w_1) - \mathcal{B}_3(w_2)\big\|_{L^{\infty}(0,T;D(A^s))}. \end{aligned} \tag{5.29}$$

The term $\|\mathcal{B}_2(w_1) - \mathcal{B}_2(w_2)\|_{L^{\infty}(0,T;D(A^s))}$ can be estimated in the following way:

$$\begin{aligned} &\big\|\mathcal{B}_2(w_1)(\cdot,t) - \mathcal{B}_2(w_2)(\cdot,t)\big\|^2_{D(A^s)} \\ &\quad= \sum_{n=1}^{\infty} \frac{\mathbf{S}^2_{n,\alpha}(t)}{(\int_0^T \nu(t)\mathbf{S}_{n,\alpha}(t))^2} \lambda_n^{2s} \\ &\quad\times \left(\int_0^T \nu(t) \int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)\,ds\,dt \right)^2. \end{aligned} \tag{5.30}$$

It is important to note that $\mathbf{S}_{n,\alpha}(t) \ge \frac{C_2(\mu,\alpha,t)}{\lambda_n}$ and $\nu(t) \ge Nt^{-\gamma}$, we get

$$\begin{aligned} &\big\|\mathcal{B}_2(w_1)(\cdot,t) - \mathcal{B}_2(w_2)(\cdot,t)\big\|^2_{D(A^s)} \\ &\quad\le \frac{-2\gamma+2}{N^2 T^{-2\gamma+2}} \frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{1}{t^{2-2\alpha}} \sum_{n=1}^{\infty} \lambda_n^{2s} \\ &\quad\times \left(\int_0^T \nu^2(t)\,dt \right) \left(\int_0^T \left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)\,ds \right)^2 dt \right) \\ &\quad\le \frac{M_\theta^2(-2\gamma+2)}{N^2 T^{-2\gamma+2}} \frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{1}{t^{2-2\alpha}} \\ &\quad\times \sum_{n=1}^{\infty} \lambda_n^{2s} \left(\int_0^T \left(\int_0^t \mathbf{S}_{n,\alpha}(t-s)\big(F_n\big(s,w_1(s)\big) - F_n\big(s,w_2(s)\big)\big)\,ds \right)^2 dt \right) \\ &\quad\le \frac{M_\theta^2(-2\gamma+2)}{N^2 T^{-2\gamma+2}} \frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{T^{2\alpha-1}}{t^{2-2\alpha}(2\alpha-1)} \lambda_1^{-2} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{n=1}^{\infty} \lambda_n^{2s} \int_0^T \big(F_n\big(s, w_1(s)\big) - F_n\big(s, w_2(s)\big)\big)^2\, ds \\
&\leq \frac{M_\theta^2(-2\gamma+2)}{N^2 T^{-2\gamma+2}} \frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{T^{2\alpha-1}}{t^{2-2\alpha}(2\alpha-1)} \lambda_1^{-2} \\
&\times \int_0^T \sum_{n=1}^{\infty} \lambda_n^{2s} \big(F_n\big(s, w_1(s)\big) - F_n\big(s, w_2(s)\big)\big)^2\, ds \\
&\leq \frac{M_\theta^2(-2\gamma+2)}{N^2 T^{-2\gamma+2}} \frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{T^{2\alpha-1}}{t^{2-2\alpha}(2\alpha-1)} \lambda_1^{-2} \\
&\times \int_0^T \big\| F_n\big(s, w_1(s)\big) - F_n\big(s, w_2(s)\big)\big\|_{D(A^s)}^2\, ds \\
&\leq \frac{M_\theta^2(-2\gamma+2)}{N^2 T^{-2\gamma+2}} \frac{C_1^4(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{T^{2\alpha-1}}{t^{2-2\alpha}(2\alpha-1)} \lambda_1^{-2} K_F^2 \int_0^T \big\| w_1(s) - w_2(s)\big\|_{D(A^s)}^2\, ds \\
&\leq M_{11}^2 T \big\| w_1(s) - w_2(s)\big\|_{L^\infty(0,T;D(A^s))}^2.
\end{aligned}
\tag{5.31}
$$

In that,

$$
\big\| \mathcal{B}_2(w_1) - \mathcal{B}_2(w_2)\big\|_{L^\infty(0,T;D(A^s))} \leq M_{11}\sqrt{T} \big\| w_1(s) - w_2(s)\big\|_{L^\infty(0,T;D(A^s))}^2. \tag{5.32}
$$

With regard to the term $\|\mathcal{B}_3(w_1) - \mathcal{B}_3(w_2)\|_{L^\infty(0,T;D(A^s))}$, by a similar calculation in Part i, we immediately obtain

$$
\big\| \mathcal{B}_3(w_1) - \mathcal{B}_3(w_2)\big\|_{L^\infty(0,T;D(A^s))} \leq M_{10} T^{\alpha-1} \| w_1 - w_2\|_{L^\infty(0,T;D(A^s))}. \tag{5.33}
$$

Conjoining (5.29), (5.32), and (5.33), we have

$$
\| \mathcal{J} w_1 - \mathcal{J} w_2 \|_{L^\infty(0,T;D(A^s))} \leq \big(M_{11}\sqrt{T} + M_{10} T^{\alpha-1}\big) \| w_1 - w_2\|_{L^\infty(0,T;D(A^s))}. \tag{5.34}
$$

Suppose that there exists T small enough such that

$$
M_{11}\sqrt{T} + M_{10} T^{\alpha-1} < 1, \tag{5.35}
$$

we arrive at the conclusion that $\mathcal{J}$ is a contraction mapping in the space $L^\infty(0,T;D(A^s))$. Moreover, it is noteworthy that if $w_1 = 0$, then

$$
\mathcal{J} w_1(x,t) = \mathcal{B}_1(x,t).
$$

In conjunction with (3.4), we have

$$
\big\| \mathcal{J} w_1(\cdot,t)\big\|_{L^\infty(0,T;D(A^s))}^2 \leq \frac{-2\gamma+2}{\xi_2^2 N^2 T^{-2\gamma+2}} \frac{C_1^2(\mu,\alpha)}{C_2^2(\mu,\alpha,t)} \frac{1}{t^{2-2\alpha}} \|g\|_{D(A^s)}^2.
$$

In other words,

$$
\big\| \mathcal{J} w_1(\cdot,t)\big\|_{L^\infty(0,T;D(A^s))}^2 \leq M_4 \|g\|_{D(A^s)}^2.
$$

The latter estimation allows us to deduce that if $w_1(x,t) \in L^\infty(0,T;D(A^s))$, then

$$\mathcal{J} w_1(x,t) \in L^\infty\big(0,T;D\big(A^s\big)\big).$$

Thanks to the Banach fixed point theorem, we come to the conclusion that $\mathcal{J} w = w$ has a unique solution $w \in L^\infty(0,T;D(A^s))$. Furthermore, in a similar way to Part i, we take $w = 0$, and we immediately get

$$\begin{aligned}\|u\|_{L^\infty(0,T;D(A^s))} &\leq \|\mathcal{I}u - \mathcal{I}w\|_{L^\infty(0,T;D(A^s))} + \|\mathcal{I}w\|_{L^\infty(0,T;D(A^s))} \\ &\leq \big(M_{11}\sqrt{T} + M_{10}T^{\alpha-1}\big)\|u\|_{L^\infty(0,T;D(A^s))} + M_4\|g\|^2_{D(A^s)}.\end{aligned}$$

Put another way,

$$\|u\|_{L^\infty(0,T;D(A^s))} \leq \frac{M_4}{1 - M_{11}\sqrt{T} - M_{10}T^{\alpha-1}}\|g\|^2_{D(A^s)}. \tag{5.36}$$

We get the proof out of the way. □

6 Conclusion

In this paper, we examined the fractional nonlinear Rayleigh–Stokes equation under nonlocal integral conditions. The existence and uniqueness are considered using the Fourier truncation method. The convergence rate between the obtained solution and the regularized solution is demonstrated.

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References

1. Tan, W.C., Masuoka, T.: Stokes' first problem for a second grade fluid in a porous halfspace with heated boundary. Int. J. Non-Linear Mech. **40**(4), 515–522 (2005)
2. Tan, W.C., Masuoka, T.: Stokes' first problem for an Oldroyd-B fluid in a porous halfspace. Phys. Fluids **17**(2), 023101 (2005)
3. Abdolrazaghi, F., Razani, A.: A unique weak solution for a kind of coupled system of fractional Schrodinger equations. Opusc. Math. **40**(3), 313–322 (2020)
4. Behboudi, F., Razani, A., Oveisiha, M.: Existence of a mountain pass solution for a nonlocal fractional (p,q)-Laplacian problem. Bound. Value Probl. **2020**, 149 (2020)
5. Ding, X.L., Nieto, J.J.: Analytical solutions for multi-term time-space fractional partial differential equations with nonlocal damping terms. Fract. Calc. Appl. Anal. **21**, 312–335 (2018)
6. Agarwal, P., Nieto, J.J., Luo, M.J.: Extended Riemann–Liouville type fractional derivative operator with applications. Open Math. **15**(1), 1667–1681 (2017)
7. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus: Models and Numerical Methods. World Scientific, Singapore (2016)
8. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652
9. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. **20**, 2 (2021)
10. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **115**, 155 (2021). https://doi.org/10.1007/s13398-021-01095-3
11. Afshari, H., Kalantari, S., Karapinar, E.: Solution of fractional differential equations via coupled fixed point. Electron. J. Differ. Equ. **2015**, 286 (2015)
12. Alqahtani, B., Aydi, H., Karapınar, E., Rakočević, V.: A solution for Volterra fractional integral equations by hybrid contractions. Mathematics **7**, 694 (2019)
13. Karapınar, E., Fulga, A., Rashid, M., Shahid, L., Aydi, H.: Large contractions on quasi-metric spaces with an application to nonlinear fractional differential-equations. Mathematics **7**, 444 (2019)
14. Abdeljawad, T., Agarwal, R.P., Karapınar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Symmetry **11**, 686 (2019)
15. Baitiche, Z., Derbazi, C., Benchohra, M.: phi-Caputo fractional differential equations with multi-point boundary conditions by topological degree theory. Results Nonlinear Anal. **3**(4), 167–178 (2020)
16. Ardjouni, A.: Asymptotic stability in Caputo–Hadamard fractional dynamic equations. Results Nonlinear Anal. **4**(2), 77–86 (2021)
17. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Application of Fractional Differential Equations. North–Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
18. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1990)
19. Zierep, J., Fetecau, C.: Energetic balance for the Rayleigh–Stokes problem of a Maxwell fluid. Int. J. Eng. Sci. **45**(2–8), 617–627 (2007)
20. Fetecau, C., Zierep, J.: On a class of exact solutions of the equations of motion of a second grade fluid. Acta Mech. **150**(1–2), 135–138 (2001)
21. Bazhlekova, E., Jin, B., Lazarov, R., Zhou, Z.: An analysis of the Rayleigh–Stokes problem for a generalized second-grade fluid. Numer. Math. **131**, 1–31 (2015)
22. Xue, C., Nie, J.: Exact solutions of the Rayleigh–Stokes problem for a heated generalized second grade fluid in a porous half-space. Appl. Math. Model. **33**(1), 524–531 (2009)
23. Zhao, C., Yang, C.: Exact solutions for electro-osmotic flow of viscoelastic fluids in rectangular microchannels. Appl. Math. Comput. **211**(2), 502–509 (2009)
24. Ngoc, T.B., Luc, N.H., Au, V.V., Tuan, N.H., Zhou, Y.: Existence and regularity of inverse problem for the nonlinear fractional Rayleigh–Stokes equations. Math. Methods Appl. Sci. (2020) https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.6162
25. Triet, N.A., Hoan, L.V.C., Luc, N.H., Tuan, N.H., Thinh, N.V.: Identification of source term for the Rayleigh–Stokes problem with Gaussian random noise. Math. Methods Appl. Sci. **41**(14), 5593–5601 (2018)
26. Dehghan, M.: A computational study of the one-dimensional parabolic equation subject to nonclassical boundary specifications. Numer. Methods Partial Differ. Equ. **22**(1), 220–257 (2006)
27. Arendt, W., ter Elst, A.F.M., Warma, M.: Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator. Commun. Partial Differ. Equ. **43**(1), 1–24 (2018)
28. Kirsch, A.: An Introduction to the Mathematical Theory of Inverse Problems, 2nd edn. Applied Mathematical Sciences, vol. 120. Springer, New-York (2011)