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On the perturbations of maps obeying Shannon–Whittaker–Kotel'nikov's theorem generalization

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Abstract
Let $f:\mathbb{R}\to\mathbb{R}$ be a map and $\tau\in\mathbb{R}^+$. The map f obeys the Shannon–Whittaker–Kotel'nikov theorem generalization (SWKTG) if $f(t)=\lim_{n\to\infty}(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}(\frac{k}{\tau})\,\mathrm{sinc}(\tau t-k))^n$ for every $t\in\mathbb{R}$. The aim of the present paper is to characterize the perturbations of the map f that obeys SWKTG. Our results enlarge the catalog of maps that can be recomposed using SWKTG. We underline that maps obeying SWKTG play a central role in applications to chemistry and signal theory between other fields.

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1 Introduction and description of the main results

In signal theory one of most well-known results is the so-called Shannon–Whittaker–Kotel'nikov's theorem (see for instance Refs. [10, 13] or [15]) acting over band-limited maps of $L^2(\mathbb{R})$ (i.e., for Paley–Wiener signals), and using the normalized cardinal sinus map $\mathrm{sinc}(t)$ given by

$$\mathrm{sinc}(t)=\begin{cases}1 & \text{if } t=0,\\ \frac{\sin(\pi t)}{\pi t} & \text{if } t\neq 0,\end{cases}$$

as key stone.

At least in the field of signal theory, also Middleton's sampling theorem is known for band step functions (see [12]). This result, to the best of our knowledge, was one of the first extensions of the classical sampling theorem which only works for band-limited maps; see [14]. After this starting point many different modifications and generalizations of this theorem appeared in the literature trying to obtain approximations of non-band-limited signals (see for instance [5] or [8]). Good surveys on these generalizations are [6] or [15].

Inspired by the results of [7] and [9] there have appeared several papers trying to obtain approximations of non-band-limited signals by using band-limited ones by means of the increasing of the band size. See for instance [1, 2] and [3] where an asymptotic generalization of the classical Shannon–Whittaker–Kotel'nikov theorem is stated. This work has had a very deep impact in chemistry because of their possible applications to recompose functions which models different chemical procedures; see for instance [11] or [16].

Theorem 1 (SWKTG) *Let $f : \mathbb{R} \to \mathbb{R}$ be a map and $\tau \in \mathbb{R}^+$. We say that f satisfies the Shannon–Whittaker–Kotel'nikov theorem generalization (SWKTG) for τ if*

$$f(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}} \left(\frac{k}{\tau} \right) \operatorname{sinc}(\tau t - k) \right)^n,$$

for every $t \in \mathbb{R}$ where the convergence of the series is considered in terms of the Cauchy principal value.

In [3] it was proved that, if $\boldsymbol{\lambda} = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a bounded sequence of positive real numbers obeying the property $\sum_{k\in\mathbb{Z},k\neq 0} |\frac{\log \lambda_k}{k}| < \infty$, then the function $\sigma_\lambda(t) := \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)}$ obeys SWKTG and it can be recomposed by

$$\sigma_\lambda(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t-k) \right)^n$$

for every $t \in \mathbb{R}$. In [4] the algebraic structure of the set of sequences of positive real numbers obeying $\sum_{k\in\mathbb{Z},k\neq 0} |\frac{\log \lambda_k}{k}| < \infty$ were characterized.

The aim of this paper is to study the problem of how to modify or perturb a map obeying SWKTG in such a way that the resultant map again can be recomposed in the form of SWKTG.

In a first approach it can be considered as natural to define a modification of f over $\{f(k)\}_{k\in\mathbb{Z}}$ by $\{f(k) + \varepsilon_k\}_{k\in\mathbb{Z}}$, with $\varepsilon_k > -f(k)$ for every $k \in \mathbb{Z}$. These changes are linked to a perturbed map in the form

$$\widetilde{f}(k) = \varepsilon_k + f(k) \quad \text{with } k \in \mathbb{Z},$$

but it is easy to note that a small variation in a unique point generates very different resultant maps, so this way of perturbing is not stable in the sense that we need (i.e., at least to have a chance of the resultant map can satisfies the SWKTG). Therefore, the way of perturbing a map in the sampling points that we have chosen is via products, i.e.,

$$\widetilde{f}(k) = \lambda_k f(k) \quad \text{with } k \in \mathbb{Z}.$$

Moreover, we shall consider $\lambda_k > 0$ for every $k \in \mathbb{Z}$ to avoid zeros in the new map. Note that the number of zeros in the resultant map will play a key role in the annulation of non-boundedness of such map essential to apply SWKTG; see Remark 5 for more details.

Thus, the modifications that we are going to consider will be given by sequences of values that we shall denote by $\boldsymbol{\lambda} = \{\lambda_k\}_{k\in\mathbb{Z}}$ with $\lambda_k > 0$ for every $k \in \mathbb{Z}$, and change the value of the map in the sampling values.

In short, our main objective is from a map f obeying SWKTG and a perturbation sequence $\boldsymbol{\lambda}$, to analyze if there exists a map

$$F_{\boldsymbol{\lambda}}(t) = \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right)^n. \tag{1}$$

In such a case the map $F_{\boldsymbol{\lambda}}$ obeys SWKTG and it is derived from f by a perturbation.

We shall study perturbations in a finite number of points on the one hand and on an infinite number of points in the second hand. As a consequence of our results it is stated that given two random maps obeying SWKTG one always can be derived from a proper perturbation of the other one and conversely. Thus, we shall center our attention to the case of the perturbed map being of the form

$$F_{\boldsymbol{\lambda}} = f\sigma_{\boldsymbol{\lambda}}. \tag{2}$$

In general this is not true but has the advantage that in the case f is analytic then $F_{\boldsymbol{\lambda}}$ is analytic too.

The structure of the paper is as follows: in Sect. 2 we shall introduce the essential notation and analyze the perturbations in a finite number of points (i.e., perturbations which modified the original function in a finite number of points) making the resultant map obeying too SWKTG. Section 3 is devoted to the introduction of different type of perturbations that will play essential role in the case of perturbing the original map in an infinite number of points. In Sect. 4 we shall study what type of maps obey SWKTG derived from perturbations of the form (2) in infinitely many points.

2 Perturbations in a finite number of points

Definition 2 Given $f:\mathbb{R}\to\mathbb{R}$ obeying SWKTG, $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ and $n\in\mathbb{N}$, we shall denote

$$F_{\boldsymbol{\lambda}}(t,n) := \sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k),$$

$$f(t,n) := \sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k).$$

Remark 3 A key point in the sequel will be the fact that if f obeys SWKTG then

$$\lim_{n\to\infty} f(t,n) = 1 \tag{3}$$

and for every $t\in\mathbb{R}$ there exists $n_0\in\mathbb{N}$ such that for every $n\ge n_0$ we have $f(t,n) > \delta > 0$.

In this section we shall work with sequences $\boldsymbol{\lambda} = \{\lambda_k\}_{k\in\mathbb{Z}}$ with $\lambda_k = 1$ except in a finite number of indices. Then, if we call

$$Q = \{k\in\mathbb{Z}; \lambda_k \neq 1\} = \{q_1,\dots,q_m\},$$

we find that there exists

$$\sigma_{\boldsymbol{\lambda}}(t) = \prod_{k\in\mathbb{Z}} {\lambda_k}^{\operatorname{sinc}(t-k)} = \prod_{i=1}^{m} {\lambda_{q_i}}^{\operatorname{sinc}(t-q_i)}.$$

Our main result in this section is the following.

Theorem 4 *Let $f:\mathbb{R}\to\mathbb{R}$ be a map obeying SWKTG, $Q=\{q_1,\dots,q_m\}\subset\mathbb{Z}$ a finite set and $\{\lambda_{q_1},\dots,\lambda_{q_m}\}\subset\mathbb{R}^+$. Let $\boldsymbol{\lambda}=\{\lambda_k\}_{k\in\mathbb{Z}}$ be the sequence given by*

$$\lambda_k=\begin{cases}1, & \text{if } k\notin Q,\\ \lambda_{q_i}, & \text{if } k=q_i\in Q;\end{cases}$$

and let $F_{\boldsymbol{\lambda}}$ be the function given in (1). Then

$$F_{\boldsymbol{\lambda}}(t)=f(t)\prod_{i=1}^{m}\lambda_{q_i}{}^{\operatorname{sinc}(t-q_i)}=f(t)\sigma_{\boldsymbol{\lambda}}(t)$$

and it obeys SWKTG.

Proof Note that $F_{\boldsymbol{\lambda}}(t)$ is a perturbation of $f(t)$ in the previously quoted sense because if $k\in\mathbb{Z}$

$$F_{\boldsymbol{\lambda}}(k)=\begin{cases}f(k), & \text{if } k\notin Q,\\ \lambda_{q_i}f(q_i), & \text{if } k=q_i, i=1,\dots,m.\end{cases}\tag{4}$$

Thus, taking account that, by (3), $f(t,n)\neq 0$ from some $n_0(t)$ we can state that

$$\begin{aligned}F_{\boldsymbol{\lambda}}(t,n)&=\sum_{\substack{k\in\mathbb{Z}\\ k\notin Q}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)+\sum_{i=1}^{m}\lambda_{q_i}^{\frac{1}{n}}f^{\frac{1}{n}}(q_i)\operatorname{sinc}(t-q_i)\\ &=f(t,n)+\sum_{i=1}^{m}\bigl(\lambda_{q_i}^{\frac{1}{n}}-1\bigr)f^{\frac{1}{n}}(q_i)\operatorname{sinc}(t-q_i)\\ &=f(t,n)\left(1+\sum_{i=1}^{m}\frac{\lambda_{q_i}^{\frac{1}{n}}-1}{f(t,n)}f^{\frac{1}{n}}(q_i)\operatorname{sinc}(t-q_i)\right).\end{aligned}$$

Therefore, if we define

$$H(t,n)=\sum_{i=1}^{m}\frac{\lambda_{q_i}^{\frac{1}{n}}-1}{f(t,n)}f^{\frac{1}{n}}(q_i)\operatorname{sinc}(t-q_i),$$

we can say that

$$\bigl(F_{\boldsymbol{\lambda}}(t,n)\bigr)^n=\bigl(f(t,n)\bigr)^n\bigl(1+H(t,n)\bigr)^n.\tag{5}$$

By (3) we have

$$\lim_{n\to\infty}H(t,n)=0.$$

Using the equivalents of limits and again (3)

$$\lim_{n\to\infty}\log\bigl(1+H(t,n)\bigr)^n=\lim_{n\to\infty}n\log\bigl(1+H(t,n)\bigr)=\lim_{n\to\infty}nH(t,n)$$

$$= \lim_{n\to\infty} n \sum_{i=1}^{m} \frac{\lambda_{q_i}^{\frac{1}{n}} - 1}{f(t,n)} f^{\frac{1}{n}}(q_i)\,\mathrm{sinc}(t-q_i)$$
$$= \sum_{i=1}^{m} \log \lambda_{q_i}\,\mathrm{sinc}(t-q_i) = \log \prod_{i=1}^{m} \lambda_{q_i}{}^{\mathrm{sinc}(t-q_i)},$$

and therefore

$$\lim_{n\to\infty} \bigl(1 + H(t,n)\bigr)^n = \prod_{i=1}^{m} \lambda_{q_i}{}^{\mathrm{sinc}(t-q_i)}.$$

Since f obeys SWKTG we have

$$\lim_{n\to\infty} \bigl(f(t,n)\bigr)^n = f(t).$$

Taking the limit when n tends to infinity in Eq. (5), we have

$$\lim_{n\to\infty} \bigl(F_\lambda(t,n)\bigr)^n = f(t) \prod_{i=1}^{m} \lambda_{q_i}{}^{\mathrm{sinc}(t-q_i)} = f(t)\sigma_\lambda(t),$$

ending the proof. □

Remark 5 Note that the proof of the previous theorem does not work if λ_{q_i} is 0 for some $i = 1,\dots,m$, because one is forbidden to use the equivalence $\log(1+x) \sim x$ when $x \to 0$ and therefore $\log \lambda_{q_i}$ would be infinite. See the following example:

Assume we perturb a map f obeying SWKTG in a unique point $q \in \mathbb{Z}$ multiplying by $\lambda_q = 0$. In this case (4) becomes

$$F_\lambda(k) = \begin{cases} f(k), & \text{if } k \neq q, \\ 0, & \text{if } k = q. \end{cases}$$

Thus

$$F_\lambda(t,n) = \sum_{\substack{k\in\mathbb{Z}\\ k\neq q}} f^{\frac{1}{n}}(k)\,\mathrm{sinc}(t-k) + 0 = f(t,n) - f^{\frac{1}{n}}(q)\,\mathrm{sinc}(t-q)$$

and, by (3),

$$\lim_{n\to\infty} F_\lambda(t,n) = 1 - \mathrm{sinc}(t-q).$$

Therefore, pointwise we would have

$$\lim_{n\to\infty} \bigl(F_\lambda(t,n)\bigr)^n = \begin{cases} 0, & \text{if } \mathrm{sinc}(t-q) > 0, \\ \infty, & \text{if } \mathrm{sinc}(t-q) < 0, \\ f(t), & \text{if } \mathrm{sinc}(t-q) = 0. \end{cases}$$

3 Types of perturbations

Let $\boldsymbol{\lambda}$ be a sequence of positive real numbers. We shall distinguish several type of perturbations.

Definition 6 We say that a sequence $\boldsymbol{\lambda}$ is admissible for a map f obeying SWKTG, if for every $t \in \mathbb{R}$ there exists the limit

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right)^n,$$

defining therefore a map $F_{\boldsymbol{\lambda}}$ obeying SWKTG. We shall denote by Λ_f the set of admissible sequences for f

$$\Lambda_f = \left\{\boldsymbol{\lambda} : \mathbb{Z} \longrightarrow \mathbb{R}^+; F_{\boldsymbol{\lambda}} \text{ is SWKTG}\right\}.$$

Remark 7 The set of admissible perturbation is not universal it depends on the function that we want to perturb. In general, $\Lambda_f \neq \Lambda_g$.

Proof Consider for instance the sequence $\boldsymbol{\lambda} = \{e^{k^2}\}_{k\in\mathbb{Z}}$. Then $\boldsymbol{\lambda}$ is admissible for the Gaussian map $g(t) = e^{-t^2}$ because

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}} g^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right)^n = \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}\operatorname{sinc}(t-k)\right)^n = 1,$$

but it is not admissible for the unity map because

$$\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}}\operatorname{sinc}(t-k) = \sum_{k\in\mathbb{Z}} e^{\frac{k^2}{n}}\operatorname{sinc}(t-k)$$

is not convergent. □

Note that for every f obeying SWKTG we can characterize Λ_f by

$$\Lambda_f = \left\{\boldsymbol{\lambda} = \left\{\frac{g(k)}{f(k)}\right\}_{k\in\mathbb{Z}}; g \text{ is SWKTG}\right\}.$$

Therefore, only with the definition of admissible perturbation we arrive at the triviality that all maps obeying SWKTG are perturbations ones from other via some proper admissible perturbation. In other words, the families Λ_f are too big in the sense that we can go from a map f to all other SWKTG maps. We endeavor to link to each map f obeying SWKTG a subfamily from Λ_f in the sense that the linked function has some kind of proximity to f. This has inspired us to define two new kind of perturbations.

In the sequel we shall consider as universal set of perturbing sequences the set

$$\Lambda := \left\{\boldsymbol{\lambda} : \mathbb{Z} \longrightarrow \mathbb{R}^+; \prod_{k\in\mathbb{Z}}\boldsymbol{\lambda}_k^{\operatorname{sinc}(t-k)} \in (0,\infty)\right\},$$

introduced in [3].

Definition 8 We say that a perturbation $\boldsymbol{\lambda} \in \Lambda$ is a perturbation compatible with f obeying SWKTG, if the map $f\sigma_{\boldsymbol{\lambda}}$ is SWKTG. We shall denote by Λ_f^* the family of such sequences, i.e.,

$$\Lambda_f^* = \{\boldsymbol{\lambda} \in \Lambda; f\sigma_{\boldsymbol{\lambda}} \text{ is SWKTG}\}.$$

Example 9 In [4], it was proven that the family of bounded sequences

$$\mathcal{L} = \left\{\boldsymbol{\lambda} : \mathbb{Z} \longrightarrow \mathbb{R}^+; \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty\right\}$$

is a subset of Λ_u^*, $u(t)$ being the unity function.

In our searching of a similarity between a map f obeying SWKTG and its perturbed map $F_{\boldsymbol{\lambda}}$ via a compatible perturbation, an interesting property of the sequence $\boldsymbol{\lambda}$ is the stability, i.e., $\lim_{t\to\infty} \sigma_{\boldsymbol{\lambda}}(t) = 1$.

With this idea in mind, we state the following definition.

Definition 10 A sequence $\boldsymbol{\lambda}$ is said to be a stable perturbation of a map f obeying SWKTG if it is a compatible perturbation with f and is stable. We shall denote by $\widetilde{\Lambda}_f$ the family of such sequences.

Remark 11 Note that in the case of finite perturbations the three notions of being stable, admissible and compatible for a map f obeying SWKTG are the same.

Proposition 12 *If we consider sequences $\boldsymbol{\lambda}$ with infinitely many terms different from unity, then given a map f obeying SWKTG we have*

$$\widetilde{\Lambda}_f \subset \Lambda_f^* \subset \Lambda_f.$$

Proof As a direct consequence of the definition it is trivial to state that $\widetilde{\Lambda}_f \subseteq \Lambda_f^* \subseteq \Lambda_f$. In order to see that the previous inclusions never are equalities, we shall consider the unit function $u(t) = 1$. Let $\boldsymbol{\lambda}$ be such that $\lambda_k = e^{(-1)^k}$. It is easy to note that

$$\sigma_{\boldsymbol{\lambda}}(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)} = e^{\cos \pi t}$$

which is an analytic map which obeys SWKTG, thus $\boldsymbol{\lambda} \in \Lambda_u^*$. However, $\boldsymbol{\lambda} \notin \widetilde{\Lambda}_u$, because

$$\lim_{t\to\infty} \sigma_{\boldsymbol{\lambda}}(t) \neq 1.$$

Therefore, $\widetilde{\Lambda}_u \subset \Lambda_u^*$.

On the other hand, let $\boldsymbol{\lambda}$ be such that $\lambda_k = e^{-k^2}$. In [1] it was proven that

$$\lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} e_k^{\frac{-k^2}{n}} \operatorname{sinc}(t-k)\right)^n = e^{-t^2},$$

then $\boldsymbol{\lambda} \in \Lambda_u$. However, $\boldsymbol{\lambda} \notin \Lambda_u^*$ because $\prod_{k\in\mathbb{Z}} \lambda_k{}^{\operatorname{sinc}(t-k)}$ does not converge. Therefore, $\Lambda_u^* \subset \Lambda_u$ ending the proof. □

Proposition 13 *All constant sequence is admissible and compatible with all map f obeying SWKTG and the unique stable of such sequences is the unity one, i.e.,*

$$\left\{\boldsymbol{\lambda} = \{c\}_{k\in\mathbb{Z}}; c \in \mathbb{R}^+\right\} \in \bigcap_{f\in SWKTG} \Lambda_f^* \subseteq \bigcap_{f\in SWKTG} \Lambda_f.$$

If $c \neq 1$ then $\boldsymbol{\lambda} = \{c\}_{k\in\mathbb{Z}} \notin \widetilde{\Lambda}_f$.

Proof It is clear that, if we perturb a map f obeying SWKTG with a constant and positive sequence $\boldsymbol{\lambda} = \{c\}_{k\in\mathbb{Z}}$, then clearly the perturbed map obeys SWKTG because $F_{\boldsymbol{\lambda}} = cf$ and therefore $\boldsymbol{\lambda}$ is admissible for f.

Moreover, since

$$\sigma_{\boldsymbol{\lambda}}(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)} = c,$$

we have $F_{\boldsymbol{\lambda}} = f\sigma_{\boldsymbol{\lambda}}$ and therefore $\boldsymbol{\lambda}$ is compatible with f. However, the unique stable constant sequence is the unity sequence. □

4 Perturbations in an infinite number of points

In the case of finite perturbations we have found that the perturbations of a map f obeying SWKTG are of the type

$$f(t)\prod_{k=1}^{m} \lambda_{q_k}{}^{\operatorname{sinc}(t-q_k)}.$$

Moreover, for a map f obeying SWKTG perturbed in a finite number of points by a sequence $\boldsymbol{\lambda}$, we have the following facts too:

a) The new sequence $\{\lambda_k f(k)\}_{k\in\mathbb{Z}}$ generates a function $F_{\boldsymbol{\lambda}}$ which obeys SWKTG, i.e., $\boldsymbol{\lambda}$ is admissible for f.

b) The relation between f and $F_{\boldsymbol{\lambda}}$ is of the form $F_{\boldsymbol{\lambda}} = f\sigma_{\boldsymbol{\lambda}}$, being

$$\sigma_{\boldsymbol{\lambda}}(t) = \prod_{k=1}^{m} \lambda_{q_k}{}^{\operatorname{sinc}(t-q_k)},$$

i.e., $\boldsymbol{\lambda}$ is compatible for f.

c) Since $\sigma_{\boldsymbol{\lambda}}$ tends to 1 at the infinity, the quotient between f and its perturbation $F_{\boldsymbol{\lambda}}$ tends to 1 too. Thus, $\boldsymbol{\lambda}$ is stable.

Therefore, in a natural way we shall try to study when is possible to have the three previous properties in the case that the number of points where the perturbation is performed be infinite in such a way that we have a perturbed map obeying SWKTG.

In short, our aim will be to find conditions on $\boldsymbol{\lambda}$ such that if f obeys SWKTG this forces $F_{\boldsymbol{\lambda}}$ to be of the form

$$F_{\boldsymbol{\lambda}}(t) = f(t)\prod_{k\in\mathbb{Z}} \lambda_k{}^{\operatorname{sinc}(t-k)} = f(t)\sigma_{\boldsymbol{\lambda}}(t)$$

and $F_{\boldsymbol{\lambda}}$ obeys SWKTG. In other words, we endeavor to study the sets Λ_f^* and $\widetilde{\Lambda}_f$ of the perturbations compatible and stable respect to f, respectively.

In the rest of this paper we will center in the study of Λ_f^*.

We shall consider that a map f obeying SWKTG is perturbed in all integers $k \in \mathbb{Z}$ multiplying by a positive real number λ_k the value of the map in such points and allowing that λ_k can be equal 1, which means not doing any modification.

Remark 14 Inspired in the proof of Theorem 4 it is clear that to be able to repeat the arguments used in the case of finite perturbations it is needed that, if f obeys SWKTG and $\boldsymbol{\lambda} \in \Lambda$, the following conditions hold:

A) $\displaystyle\sum_{k\in\mathbb{Z}} \bigl(\lambda_k f(k)\bigr)^{\frac{1}{n}} \operatorname{sinc}(t-k) < \infty,$

B) $\displaystyle\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \bigl(\lambda_k f(k)\bigr)^{\frac{1}{n}} \operatorname{sinc}(t-k) = 1,$

C) $\displaystyle\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \bigl(\lambda_k^{\frac{1}{n}} - 1\bigr) f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) = 0,$

D) $\displaystyle\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} n\bigl(\lambda_k^{\frac{1}{n}} - 1\bigr) f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) = \sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k).$

Note that the previous relations satisfy the following relations.

Lemma 15 *Let f be a map obeying SWKTG and $\boldsymbol{\lambda} \in \Lambda$. Then* D) $\Longrightarrow$ C) $\Longleftrightarrow$ B) $\Longrightarrow$ A).

Proof The first implication comes from

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \bigl(\lambda_k^{\frac{1}{n}} - 1\bigr) f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) \\
&\quad= \lim_{n\to\infty} \frac{1}{n} \sum_{k\in\mathbb{Z}} n\bigl(\lambda_k^{\frac{1}{n}} - 1\bigr) f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) \\
&\quad= \left(\lim_{n\to\infty} \frac{1}{n}\right) \left(\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k)\right) = 0
\end{aligned}$$

because $\boldsymbol{\lambda} \in \Lambda$. To obtain the equivalence of C) and B) it is enough to note that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \bigl(\lambda_k f(k)\bigr)^{\frac{1}{n}} \operatorname{sinc}(t-k) \\
&\quad= \sum_{k\in\mathbb{Z}} \bigl(\lambda_k^{\frac{1}{n}} - 1\bigr) f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) + \sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) \\
&\quad= \sum_{k\in\mathbb{Z}} \bigl(\lambda_k^{\frac{1}{n}} - 1\bigr) f^{\frac{1}{n}}(k) \operatorname{sinc}(t-k) + f(t,n).
\end{aligned}$$

Therefore, the series that appear in both conditions have the same character. Thus, taking limits in the previous expression and using $\lim_{n\to\infty} f(t,n) = 1$ we have

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \bigl(\lambda_k f(k)\bigr)^{\frac{1}{n}} \operatorname{sinc}(t-k)$$

$$= \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}\bigl(\lambda_k^{\frac{1}{n}}-1\bigr)f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)+f(t,n)\right)$$
$$= \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}\bigl(\lambda_k^{\frac{1}{n}}-1\bigr)f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right)+1,$$

and the proof is completed because the last implication is trivial. □

Therefore, by the previous lemma, we can claim that to guarantee that the method used in the case of finite perturbations works in the case of infinite ones, it is enough, taking $\boldsymbol{\lambda}\in\Lambda$, to prove condition D).

Here is the statement of our main results in this section.

Theorem 16 *Let* $\boldsymbol{\lambda}=\{\lambda_k\}_{k\in\mathbb{Z}}\in\Lambda$, f *be a map obeying SWKTG and* $F_{\boldsymbol{\lambda}}$ *defined in* (1). *Then* $\boldsymbol{\lambda}\in\Lambda_f^*$ *if and only if* $\boldsymbol{\lambda}$ *obeys condition* D).

Proof Note that $F_{\boldsymbol{\lambda}}$ is a perturbation of f in the sense previously stated because for every $k\in\mathbb{Z}$

$$F_{\boldsymbol{\lambda}}(k)=\lambda_k f(k).$$

First we shall prove that D) is a necessary condition. Assume that $F_{\boldsymbol{\lambda}}=f\sigma_{\boldsymbol{\lambda}}$ obeys SWKTG, i.e.,

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right)^n=f(t)\sigma_{\boldsymbol{\lambda}}(t).$$

Applying logarithms and using, by (3), the equivalence of the logarithm we have

$$\begin{aligned}&\lim_{n\to\infty}n\left(\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)-1\right)\\&\quad=\log\bigl(f(t)\sigma_{\boldsymbol{\lambda}}(t)\bigr)=\log f(t)+\sum_{k\in\mathbb{Z}}\log\lambda_k\operatorname{sinc}(t-k).\end{aligned}\tag{6}$$

Analogously, since f obeys SWKTG

$$\lim_{n\to\infty}n\left(\sum_{k\in\mathbb{Z}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)-1\right)=\log f(t).\tag{7}$$

By some calculation it is clear that the series of condition D) can be written in the form

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}n\bigl(\lambda_k^{\frac{1}{n}}-1\bigr)f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\\&\quad=n\left(\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)-\sum_{k\in\mathbb{Z}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right)\\&\quad=n\left(\sum_{k\in\mathbb{Z}}\lambda_k^{\frac{1}{n}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)-1\right)-n\left(\sum_{k\in\mathbb{Z}}f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)-1\right).\end{aligned}$$

Therefore, taking limits at the previous expression, by Eqs. (6) and (7), the condition D) is obeyed.

Now, we shall see that the condition D) is sufficient. Note that the convergences of the infinite product and the series that appear when SWKTG is applied are guaranteed.

By (3), $f(t,n) \neq 0$ for every $n \geq n_0$. Then we have

$$\begin{aligned} F_{\lambda}(t,n) &= \sum_{k\in\mathbb{Z}} \big(\lambda_k f(k)\big)^{\frac{1}{n}} \operatorname{sinc}(t-k) \\ &= \sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k) + \sum_{k\in\mathbb{Z}} \big(\lambda_k^{\frac{1}{n}} - 1\big) f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k) \\ &= f(t,n) + \sum_{k\in\mathbb{Z}} \big(\lambda_k^{\frac{1}{n}} - 1\big) f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k) \\ &= f(t,n)\left(1 + \sum_{k\in\mathbb{Z}} \frac{\lambda_k^{\frac{1}{n}} - 1}{f(t,n)} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k)\right). \end{aligned}$$

Therefore, if we define

$$H(t,n) := \sum_{k\in\mathbb{Z}} \frac{\lambda_k^{\frac{1}{n}} - 1}{f(t,n)} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k),$$

we write

$$\big(F_{\lambda}(t,n)\big)^n = \big(f(t,n)\big)^n \big(1 + H(t,n)\big)^n. \tag{8}$$

By (3) and C) (by Lemma 15: D) $\Longrightarrow$ C))

$$\lim_{n\to\infty} H(t,n) = 0,$$

using the limit equivalences we have

$$\begin{aligned} \lim_{n\to\infty} \log\big(1 + H(t,n)\big)^n &= \lim_{n\to\infty} n\log\big(1 + H(t,n)\big) = \lim_{n\to\infty} nH(t,n) \\ &= \lim_{n\to\infty} n \sum_{k\in\mathbb{Z}} \frac{\lambda_k^{\frac{1}{n}} - 1}{f(t,n)} f^{\frac{1}{n}}(k)\operatorname{sinc}(t-k), \end{aligned}$$

from which it is deduced, by D), that

$$\lim_{n\to\infty} \log\big(1 + H(t,n)\big)^n = \sum_{k\in\mathbb{Z}} \log\lambda_k \operatorname{sinc}(t-k) = \log\prod_{k\in\mathbb{Z}} {\lambda_k}^{\operatorname{sinc}(t-k)}.$$

Thus,

$$\lim_{n\to\infty} \big(1 + H(t,n)\big)^n = \prod_{k\in\mathbb{Z}} {\lambda_k}^{\operatorname{sinc}(t-k)}. \tag{9}$$

Moreover, since by hypothesis the map f obeys SWKTG we have

$$\lim_{n\to\infty} \big(f(t,n)\big)^n = f(t).$$

Therefore, taking limits when n tends to infinity in Eq. (8), we obtain by the previous equality plus (9)

$$\lim_{n\to\infty}\left(F_{\lambda}(t,n)\right)^{n} = f(t)\prod_{k\in\mathbb{Z}}{\lambda_k}^{\operatorname{sinc}(t-k)}$$

finishing the proof of the theorem. □

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