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Applications of Orlicz–Pettis theorem in vector valued multiplier spaces of generalized weighted mean fractional difference operators

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Abstract

In this study, we deal with some new vector valued multiplier spaces $S_{G_h}(\sum_k z_k)$ and $S_{wG_h}(\sum_k z_k)$ related with $\sum_k z_k$ in a normed space Y. Further, we obtain the completeness of these spaces via weakly unconditionally Cauchy series in Y and Y^*. Moreover, we show that if $\sum_k z_k$ is unconditionally Cauchy in Y, then the multiplier spaces of G_h-almost convergence and weakly G_h–almost convergence are identical. Finally, some applications of the Orlicz–Pettis theorem with the newly formed sequence spaces and unconditionally Cauchy series $\sum_k z_k$ in Y are given.

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Keywords: Almost convergence; Generalized weighted mean operator $G(u,v)$; Weakly unconditionally Cauchy series; Unconditionally Cauchy series; Orlicz–Pettis theorem

1 Introduction and preliminaries

Consider Ω as the space of real (or complex) valued sequences. Consider Y to be a sequence space with linear topology. Then Y is said to be a K-space provided that each of the maps $p_i : Y \to \mathbb{R}$ defined by $p_i(z) = z_i$ is continuous $\forall i \in \mathbb{N}$. A K-space Y, where Y is a complete linear space, is called FK space. A normed FK space is called BK space. An FK space Y is said to have the property AK if for every sequence $y = (y_n)_{n\geq 1} \in Y$

$$y = \lim_{n\to\infty} \sum_{k=1}^{n} y_k e^k,$$

where $e^k = (0, 0, 0, \ldots, 1, 0, \ldots)$ such that 1 is in the kth-position $\forall k \in \mathbb{N}$. The spaces of bounded, convergent, and null sequences, which are denoted by ℓ_∞, c, and c_0, respectively, are BK spaces which are endowed with the sup norm $\|y\|_\infty = \sup_{k\in\mathbb{N}} |y_k|$. By ℓ_1, we denote the space of absolutely summable sequences, bs and cs are the spaces consisting of all bounded and convergent series. Let Y and Z be two sequence spaces and $\mathcal{A} = (a_{nk})_{n,k\in\mathbb{N}}$

be an infinite matrix. Then, for $z = (z_k) \in Y$, we have $\mathcal{A} : Y \to Z$ which is defined as

$$(\mathcal{A}z)_n = \sum_k a_{nk} z_k. \tag{1.1}$$

If $\sum_k a_{nk} z_k$ converges for each $n \in \mathbb{N}$, then we call $\mathcal{A}z$ the $\mathcal{A}$-transform of z. Thus, $\mathcal{A} \in (Y, Z)$ iff the series in (1.1) converges $\forall n \in \mathbb{N}$ and $\mathcal{A}z \in Z$. A sequence $z = (z_k)$ is called $\mathcal{A}$–summable to $p \in \mathbb{C}$ (the set of complex numbers) if $(\mathcal{A}z)$ converges to p. For a detailed study about recent results in summability theory, one can refer to [8, 24, 33]. The Euler gamma functions are represented by $\Gamma(\gamma)$ where $\gamma \in (0, \infty)$ is defined as an improper integral such as $\Gamma(\gamma) = \int_0^\infty e^{-t} t^{\gamma-1}\, dt$. Let $(\gamma)_k$ be the generalized factorial function which is defined in terms of Euler gamma function as

$$(\gamma)_k = \begin{cases} 1, & k = 0, \\ \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \gamma(\gamma+1)(\gamma+2)(\gamma+3)\cdots(\gamma+k-1), & k \in \mathbb{N}, \end{cases}$$

where $\mathbb{N}$ is denoted by a set of all positive integers. Kizmaz [20] gave the idea of difference sequences spaces which was generalized by Et and Colak [15]. Recently, many specialists like Ahmad and Mursaleen [2], Tripathy [32], Altay and Basar [4] studied difference sequences spaces. For a detailed study about the difference sequence spaces, one can refer to [27, 28]. Furthermore, Baliarsingh ([6, 7]) defined the generalized fractional difference operator Δ^γ, which is given as

$$\left(\Delta^\gamma z\right)_k = \sum_{i=0}^\infty \frac{(-1)^i \Gamma(\gamma+1)}{i!\,\Gamma(\gamma-i+1)} z_{k+i} \quad (k \in \mathbb{N}_0),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $z \in \Omega$. In [25] the difference operator Δ^γ, $\Delta^{(\gamma)}$, $\Delta^{-\gamma}$, $\Delta^{(-\gamma)}$ is defined from Ω to Ω as follows:

$$\left(\Delta^\gamma z\right)_k = \sum_{i=0}^\infty \frac{(-\gamma)_i}{i!} z_{k+i}, \tag{1.2}$$

$$\left(\Delta^{(\gamma)} z\right)_k = \sum_{i=0}^\infty \frac{(-\gamma)_i}{i!} z_{k-i}, \tag{1.3}$$

$$\left(\Delta^{-\gamma} z\right)_k = \sum_{i=0}^\infty \frac{(\gamma)_i}{i!} z_{k+i}, \tag{1.4}$$

$$\left(\Delta^{(-\gamma)} z\right)_k = \sum_{i=0}^\infty \frac{(\gamma)_i}{i!} z_{k-i}. \tag{1.5}$$

It is being assumed throughout that the above defined summations are convergent for $z \in \Omega$. For a detailed study of fractional difference operator, one may refer to [6]. Recently, Mohiuddine et al. [23] studied linear isomorphic spaces of fractional-order difference operators. A lot of research has been made in this field, one can refer to [1, 17, 34].

Let Y be a Banach space. Then $\sum_k z_k \in Y$ is called unconditionally convergent (uc) or unconditionally Cauchy (uC) if $\sum_k z_{\pi(k)}$ is convergent (or Cauchy, resp.) for every $\pi \in \mathbb{N}$, where π is the permutation. Further, $\sum_k z_k \in Y$ is called weakly unconditionally Cauchy

(wuC) if the sequence $(\sum_{k=1}^{n} z_{\pi(k)})$ is weakly Cauchy sequence or, alternatively, $\sum_k z_k$ is wuC iff $\sum_k |z^*(z_k)| < \infty$ $\forall z^* \in Y^*$, the space of all linear and bounded (continuous) functionals defined on Y. For a detailed study, one can refer to [10]. Using the completeness property of a subspace of ℓ_∞ obtained by almost convergence, a depiction of wuC and uc series along with a new form of the Orlicz–Pettis theorem was presented by Aizpuru et al. [3]. Recently, a vector valued multiplier space through Cesàro convergence was introduced by Altay and Kama [5]. Esi [11] investigated some classes of generalized paranormed sequence spaces associated with multiplier sequences. Tripathy and Mahanta [31] also studied vector valued sequences associated with multiplier sequences. Furthermore, Karakus and Basar introduced the multiplier spaces $S_\Lambda(\mathbb{T})$, $S_{w\Lambda}(\mathbb{T})$ and studied some new multiplier spaces by using generalization of almost summability in [18, 19]. To know more about multiplier spaces, one may refer to [13, 14, 16, 29]. Lorentz proved that a sequence $z = (z_k) \in \ell_\infty$ is said to be almost convergent to $L \in \mathbb{C}$ and is denoted by $f - \lim z_k = L$ iff

$$\lim_{m\to\infty} \sum_{k=0}^{m} \frac{z_{n+k}}{m+1} = L$$

uniformly in n. For a detailed study of almost convergence of the sequence spaces, one can refer to [12, 22, 35]. A sequence $z = (z_k) \in \ell_\infty$ is called $F_{\mathcal{A}}$-summable if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} z_{k+m} = L$$

uniformly in $m \in \mathbb{N}$.

Altay and Basar [4] first studied generalized weighted mean operator $G(p,q)$ which was further enlarged to a difference operator $G(p,q,\Delta)$ by Polat et al. [26]. Later, Demiriz and Cakan [9] introduced generalized weighted mean of order m as $G(p,q,\Delta^m)$. Consider a set of all sequences $\boldsymbol{U}$ and $p = (p_n)$ such that $p_n \neq 0$ $\forall n \in \mathbb{N}$ and $\frac{1}{p} = (\frac{1}{p_n})$, $\forall p \in \boldsymbol{U}$. As defined by Nayak et al. [25], the generalized weighted fractional difference mean or factorable fractional difference matrix $G(p,q,\Delta^{(\gamma)}) = (g_{nk}^{\Delta^{(\gamma)}})$ is defined as follows:

$$g_{nk}^{\Delta^{(\gamma)}} = \begin{cases} \sum_{i=k}^{n} p_n \frac{(-\gamma)_{i-k}}{(i-k)!} q_i, & \text{when } 1 \leq k \leq n; \\ 0, & \text{when } k > n, \end{cases}$$

where $i,k,n \in \mathbb{N}$ such that p_n depends on n and q_k on k.

Let us consider $h = (h_k)$ to be a strictly increasing sequence of positive real numbers such that

$$0 < h_1 < h_2 < h_3 < \cdots \quad \text{and} \quad \lim_{k\to\infty} h_k = \infty. \tag{1.6}$$

It is being assumed throughout that any term with a negative subscript is zero. The matrix $G(p,q,\Delta^{(\gamma)},h) = (g_{hnk}^{\Delta^{(\gamma)}})$ is given by

$$g_{hnk}^{\Delta^{(\gamma)}} = \begin{cases} \frac{1}{h_n} \sum_{i=k}^{n} p_n \frac{(-\gamma)_{i-k}}{(i-k)!} q_i, & \text{when } 1 \leq k \leq n; \\ 0, & \text{when } k > n. \end{cases}$$

A sequence $z = (z_k) \in \Omega$ is called G_h-convergent to $a \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k=1}^{n} p_n q_k \Delta^{(\gamma)} z_k = a, \quad \forall n \in \mathbb{N}$$

or

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k=1}^{n} p_n \left(\sum_{i=k}^{n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \right) z_k = a, \quad \forall n \in \mathbb{N}.$$

Before going to our main results, we present some lemmas. For details, one may refer to [30].

Lemma 1.1

(i) *Let Y be a normed space. Then $\sum_k z_k$ is said to be wuC series iff*

$$\begin{aligned} H &= \sup_{n\in\mathbb{N}} \left\{ \left\| \sum_{k=1}^{n} t_k z_k \right\| : |t_k| \le 1 \right\} \\ &= \sup_{n\in\mathbb{N}} \left\{ \left\| \sum_{k=1}^{n} \epsilon_k z_k \right\| : |\epsilon_k| \in \{-1, 1\} \right\} \\ &= \sup_{n\in\mathbb{N}} \left\{ \sum_{k=1}^{n} \left| z^*(z_k) \right| : \forall z^* \in B_{Y^*} \right\}, \end{aligned}$$

where $H \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers and B_{Y^*} represents the closed unit ball of Y^*.

(ii) *Suppose that Y is a normed space. Then a formal series $\sum_k z_k$ in Y is called uC (or wuC) iff, for any $(a_n) \in \ell_\infty$, $\sum_k a_k z_k$ converges, i.e., $\sum_k z_k$ is an ℓ_∞–(respectively a c_0–) multiplier convergent series.*

2 Main results

Definition 2.1 Consider Y to be a normed space and $h = (h_n)$ to be the sequence fulfilling property (1.6). Then $z = (z_k)$ is called G_h-almost convergent (or wG_h-almost convergent) to $z_0 \in Y$ if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k=m}^{m+n} p_{m+n} \left(\sum_{i=k}^{m+n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \right) z_k = z_0$$

uniformly in $m \in \mathbb{N}$ or

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k=m}^{m+n} p_{m+n} \left(\sum_{i=k}^{m+n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \right) z^*(z_k) = z^*(z_0)$$

uniformly in $m \in \mathbb{N}$, $\forall z^* \in Y^*$, where $z_0 \in Y$ is the G_h-limit (or weak G_h-limit) of $z = (z_k)$ and is denoted by $G_h - \lim_{n\to\infty} z_n = z_0$ or ($wG_h - \lim_{n\to\infty} z_n = z_0$).

Let $\Omega(Y)$ be the Y-valued sequence space. Then the spaces of all G_h-almost convergent and wG_h-almost convergent sequences in Y are denoted by $G_h(Y)$ and $wG_h(Y)$, respectively, which are defined as

$$G_h(Y) = \Bigg\{(z_k) \in \Omega(Y) : \lim_{n\to\infty} \frac{1}{h_n} \sum_{k=m}^{m+n} p_{m+n} \Bigg(\sum_{i=k}^{m+n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \Bigg) z_k, \\ \text{uniformly exists in } m \in \mathbb{N} \Bigg\}$$

and

$$wG_h(Y) = \Bigg\{z^*(z_k) \in \Omega(Y) : \lim_{n\to\infty} \frac{1}{h_n} \sum_{k=m}^{m+n} p_{m+n} \Bigg(\sum_{i=k}^{m+n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \Bigg) z^*(z_k), \\ \text{uniformly exists in } m \in \mathbb{N} \Bigg\}.$$

We may consider this definition as a generalization of almost convergence given by Lorentz [21].

Proposition 2.2 *Suppose that Y is a normed space. If $z = (z_k)$ is G_h-almost convergent in Y, then $z \in \ell_\infty(Y)$.*

Proof Since $z = (z_k)$ is an G_h-almost convergent sequence in Y, then $\exists z_0 \in Y$, $\forall \varepsilon > 0$ and $n'_0 \in \mathbb{N}$ such that

$$\Bigg\| \frac{1}{h_n} \sum_{k=m}^{m+n} p_{m+n} \Bigg(\sum_{i=k}^{m+n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \Bigg) z_k - z_0 \Bigg\| < \varepsilon,$$

$\forall m \in \mathbb{N}$ and $n \geq n_0$, which implies that

$$\Bigg\| \frac{1}{h_n} \sum_{k=m}^{m+n} p_{m+n} \Bigg(\sum_{i=k}^{m+n} \frac{(-\gamma)_{i-k}}{(i-k)!} q_i \Bigg) z_k \Bigg\| \leq \|z_0\| + \varepsilon,$$

$\exists Z > 0$ such that

$$\begin{aligned} \frac{p_m}{h_{n'_0}} q_m \big\| \Delta^{(\gamma)} z_m \big\| &= \Bigg\| \frac{h_{n'_0+1}}{h_{n'_0}} p_{m+n'_0+1} \sum_{k=m}^{m+n'_0+1} \frac{q_k}{h_{n'_0+1}} \Delta^{(\gamma)} z_k - p_{m+n'_0+1} \sum_{k=m+1}^{m+n'_0+1} \frac{q_k}{h_{n'_0}} \Delta^{(\gamma)} z_k \Bigg\| \\ &\leq \Bigg\| \frac{h_{n'_0+1}}{h_{n'_0}} p_{m+n'_0+1} \sum_{k=m}^{m+n'_0+1} \frac{q_k}{h_{n'_0+1}} \Delta^{(\gamma)} z_k \Bigg\| + \Bigg\| p_{m+n'_0+1} \sum_{k=m+1}^{m+n'_0+1} \frac{q_k}{h_{n'_0}} \Delta^{(\gamma)} z_k \Bigg\| \\ &\leq \Bigg(\frac{h_{n'_0+1}}{h_{n'_0}} + 1 \Bigg) \big(\|z_0\| + \varepsilon \big), \end{aligned}$$

which yields that

$$\big\| \Delta^{(\gamma)} z_m \big\| \leq \Bigg(\frac{h_{n'_0+1} + h_{n'_0}}{p_m q_m} \Bigg) \big(\|z_0\| + \varepsilon \big) = Z.$$

There exists an analog of Proposition 2.2 in weak topologies as, by the Banach–Mackey theorem, a weak bounded subset of Y is also bounded. □

Proposition 2.3 *Let Y be the normed space. If $z = (z_k)$ is a wG_h-almost convergent sequence, then $(z_k) \in \ell_\infty(Y)$.*

Definition 2.4 Suppose that Y is a normed space and $h = (h_n)$ is the sequence fulfilling property (1.6). Then $\sum_k z_k \in Y$ is called G_h-almost convergent to $z_0 \in Y$ if

$$\lim_{n\to\infty}\left\| \frac{1}{h_n}\sum_{k=m}^{m+n} p_{m+n}q_k\Delta^\gamma s_k - z_0\right\| = 0$$

uniformly in $m \in \mathbb{N}$, where $\Delta^\gamma s_k = \sum_{j=1}^{k} \Delta^\gamma z_j \ \forall k \in \mathbb{N}$. So, we use the notation $G_h - \sum_k z_k = z_0$ for G_h-almost convergence. By some easy calculation, we have $G_h - \sum_k z_k = z_0$ iff

$$\lim_{n\to\infty}\left[\frac{1}{h_n}\sum_{k=1}^{m} p_m q_k \Delta^{(\gamma)} z_k + \frac{1}{h_n}\sum_{k=1}^{n} p_{m+n}q_{m+k}\Delta^{(\gamma)} z_{m+k}\right] = z_0,$$

i.e.,

$$\lim_{n\to\infty}\left[\frac{1}{h_n}\sum_{k=1}^{m} p_m\left(\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\right)z_k + \frac{1}{h_n}\sum_{k=1}^{n} p_{m+n}\left(\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\right)z_{m+k}\right] = z_0$$

in the norm topology, uniformly in $m \in \mathbb{N}$ $\forall m, n, k \in \mathbb{N}$. We can write $wG_h - \sum_k z_k = z_0$ if the series is weakly G_h-almost convergent to z_0 in the weak topology. To obtain the definition given in [3], we will take $h_n = n + 1$, $p_{n+m} = 1$, $\gamma = 0$ such that $q_k = \Delta q_{m+n} z_k$, where $q_n = n$, $\forall n \in \mathbb{N}$.

3 Multiplier spaces of G_h-almost convergence

This particular section deals with multiplier spaces of G_h–almost convergence and gives a theorem related to completeness through *wuC* series.

Definition 3.1 Suppose that Y is the normed space such that $\sum_k z_k$ belongs to Y. Then the Y-valued multiplier space of G_h-almost convergence of $\sum_k z_k$ is defined as

$$S_{G_h}\left(\sum_k z_k\right) = \left\{y = (y_k) \in \ell_\infty : \sum_k z_k y_k is G_h\text{-almost convergent}\right\}$$

equipped with $\boldsymbol{S}$ (summing operator), and the sup norm is also defined by

$$\boldsymbol{S} : S_{G_h}\left(\sum_k z_k\right) \to Y, \qquad y = (y_k) \mapsto \boldsymbol{S}(y) = G_h - \sum_k z_k y_k. \tag{3.1}$$

Theorem 3.2 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then the following are identical:*

(i) *$\sum_k z_k$ is wuC.*

(ii) $S_{G_h}(\sum_k z_k)$ *is complete.*
(iii) $c_0 \subseteq S_{G_h}(\sum_k z_k)$.

Proof (i) ⇒ (ii) Since $\sum_k z_k$ is *wuC* series in Y, then from Lemma 1.1 the following supremum is greater than zero, i.e., $Q > 0$ such that

$$Q = \sup_{n\in\mathbb{N}}\left\{\left\|\sum_{k=1}^{n} t_k z_k\right\| : |t_k| \leq 1\right\}.$$

Now, let $t^n \in S_{G_h}(\sum_k z_k)$, where $t^n = (t_k^n)$ such that $\lim_{n\to\infty}\|t^n - t^0\| = 0$ with $t^0 \in \ell_\infty$. We wish to prove that $t^0 \in S_{G_h}(\sum_k z_k)$. Let $y_n = G_h - \sum_k t_k^n z_k$, then $y_n \in Y$ since $(t_k^n) \in S_{G_h}(\sum_k z_k)$. Now $\forall\varepsilon > 0$, $\exists n_0' \in \mathbb{N}$ and $\nu_1, \nu_2 > n_0'$ such that $\|t^{\nu_1} - t^{\nu_2}\| < \frac{\varepsilon}{3Q}$. Therefore, for $\nu_1, \nu_2 > n_0'$, $\exists n \in \mathbb{N}$ which satisfies the inequalities

$$\left\|y_{\nu_1} - \left[\sum_{k=1}^{m}\frac{p_{m+n}}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i t_k^{\nu_1} z_k + \sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i t_{m+k}^{\nu_1} z_{m+k}\right]\right\| < \frac{\varepsilon}{3}, \tag{3.2}$$

$$\left\|y_{\nu_2} - \left[\sum_{k=1}^{m}\frac{p_{m+n}}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i t_k^{\nu_2} z_k + \sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i t_{m+k}^{\nu_2} z_{m+k}\right]\right\| < \frac{\varepsilon}{3}, \tag{3.3}$$

and

$$\begin{aligned}&\left\|\sum_{k=1}^{m}\frac{p_{m+n}}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\left(t_k^{\nu_1} - t_k^{\nu_2}\right)z_k + \sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\left(t_{m+k}^{\nu_1} - t_{m+k}^{\nu_2}\right)z_{m+k}]\right\| \\ &\quad< \frac{\varepsilon}{3},\end{aligned} \tag{3.4}$$

uniformly in $m \in \mathbb{N}$. Thus, $\exists n_0' \in \mathbb{N}$ such that

$$\|y_{\nu_1} - y_{\nu_2}\| \leq (3.2) + (3.3) + (3.4) < \varepsilon$$

$\forall \nu_1, \nu_2 \geq n_0'$. To a further extent, $\exists y_0 \in Y$ such that $y_n \to y_0$ as $n \to \infty$, as Y is complete.

Now, we also have to show that $G_h - \sum_k t_k^0 z_k = y_0$. For this, let $\forall\varepsilon > 0$, we have $\|t^j - t^0\| < \frac{\varepsilon}{3Q}$, and for fixed j

$$\|y_j - y_0\| < \frac{\varepsilon}{3}. \tag{3.5}$$

Hence, $\exists n_0' \in \mathbb{N}$ such that

$$\left\|y_j - \left[\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i t_k^j z_k + \sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i t_{m+k}^j z_{m+k}\right]\right\| < \frac{\varepsilon}{3} \tag{3.6}$$

$\forall n \geq n_0'$, uniformly in $m \in \mathbb{N}$, since

$$y_j = G_h - \sum_k t_k^j z_k \quad \forall j \in \mathbb{N}.$$

From Lemma 1.1, we get

$$\left[\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\frac{(t_k^j-t_k^0)}{\|t^j-t^0\|}z_k+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\frac{(t_{m+k}^j-t_{m+k}^0)}{\|t^j-t^0\|}z_{m+k}\right] \le Q. \tag{3.7}$$

Since $\sum_k z_k$ is a wuC series, so $\forall\varepsilon>0\ \exists n_0'\in\mathbb{N}$ such that

$$\begin{aligned}&\left\|y_0-\left[\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_it_k^0z_k+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_it_{m+k}^0z_{m+k}\right]\right\|\\ &\quad\le(3.5)+(3.6)\\ &\qquad+\left\|\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\big(t_k^j-t_k^0\big)z_k+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i\big(t_{m+k}^j-t_{m+k}^0\big)z_{m+k}\right\|\\ &\quad\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\big\|t^j-t^0\big\|.Q\\ &\quad\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3Q}.Q=\varepsilon,\end{aligned}$$

$\forall n\ge n_0'$ uniformly in $m\in\mathbb{N}$. Therefore, $t^0=(t_k^0)_k\in S_{G_h}(\sum_k z_k)$.

(ii) $\Rightarrow$ (iii) If $S_{G_h}(\sum_k z_k)$ is a complete space with $t=(t_k)$ being an arbitrary sequence in the space c_0, then we need to show that $t=(t_k)\in S_{G_h}(\sum_k z_k)$. Now, since $S_{G_h}(\sum_k z_k)$ is a complete space, then it contains the space of eventually zero sequences c_0. That is, $\phi\subset S_{G_h}(\sum_k z_k)$. Since c_0 is an AK space, we have $t^{[m]}=\sum_{k=1}^m t_ke^k\in S_{G_h}(\sum_k z_k)$. Therefore, $\lim_{m\to\infty}\|t^{[m]}-t\|_\infty=0$. Thus $t=(t_k)\in S_{G_h}(\sum_k z_k)$.

(iii) $\Rightarrow$ (i) Let us consider that a series $\sum_k z_k$ is not wuC, then $\exists z^*\in B_{z^*}$ such that $\sum_{k=1}^\infty|z^*(z_k)|=+\infty$. Since $\sum_{k=1}^\infty|z^*(z_k)|=+\infty$, then there exists m_1 such that $\sum_{k=1}^{m_1}|z^*(z_k)|>n.n$ for $n>1$. Let us define

$$(t_k)=\begin{cases}\frac{1}{n}, & \text{when } z^*(z_k)\ge 0;\\ -\frac{1}{n}, & \text{when } z^*(z_k)<0,\end{cases}$$

for $k=\{1,2,3,\dots\}$, which implies that $\sum_{k=1}^{m_1}t_kz^*(z_k)>n$ and $t_kz^*(z_k)\ge 0$. Let $m_2>m_1$ such that $\sum_{k=m_1+1}^{m_2}t_kz^*(z_k)>n^2.n^2$. Now, we define

$$(t_k)=\begin{cases}\frac{1}{n^2}, & \text{when } z^*(z_k)\ge 0;\\ -\frac{1}{n^2}, & \text{when } z^*(z_k)<0,\end{cases}$$

for $k=\{m_1+1,\dots m_2\}$, which shows that $\sum_{k=m_1+1}^{m_2}t_kz^*(z_k)>n^2$ and $t_kz^*(z_k)\ge 0$. Thus, for arbitrary null sequences $t=(t_k)\in S_{G_h}(\sum_k z_k)$, we have $\sum_k t_kz^*(z_k)\to+\infty$, which is a contradiction since the sequences of partial sums $\{\sum_{k=1}^{\eta}t_kz^*(z_k)\}_{n\in\mathbb{N}}$ should be bounded by the hypothesis. Therefore, our claim is wrong, and hence the series $\sum_k z_k$ must be wuC series.

(ii) $\Rightarrow$ (i) Suppose that $S_{G_h}(\sum_k z_k)$ is a Banach space and $t=(t_k)\in c_0(Y)$, which means $c_0(Y)\subseteq S_{G_h}(\sum_k z_k)$ (already proved), which implies that $\sum_k t_kz_k$ is almost convergent for all $t = (t_k) \in c_0(Y)$. From the monotonicity of $c_0(Y)$, $\sum_k t_k z_k$ is subseries almost convergent, and thus from the Orlicz–Pettis theorem, we get $\sum_k t_k z_k$ is wuC. □

Corollary 3.3 *Let Y be the Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then $\sum_k z_k$ is c_0-multiplier convergent iff $c_0 \subseteq S_{G_h}(\sum_k z_k)$.*

Aizpuru et al. [3] studied $S_{AC}(\sum_k z_k)$ which was given as

$$S_{AC}\left(\sum_k z_k\right) = \left\{t = t_k \in \ell_\infty : AC \sum_k t_k z_k \text{ exists}\right\}.$$

We have $\sum_k z_k$ is almost convergent to $z_0 \in Y$. If $AC \sum_k z_k = z_0$, then $S_{AC}(\sum_k z_k) \subseteq S_{G_h}(\sum_k z_k)$.

Corollary 3.4 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then the following are identical:*

(i) *$\sum_k z_k$ is (wuC).*
(ii) *$c_0(Y) \subseteq S_{G_h}(\sum_k z_k)$.*
(iii) *$S_{G_h}(\sum_k z_k)$ is a Banach space.*
(iv) *$c_0(Y) \subseteq AC \sum_k t_k z_k$.*
(v) *$S_{AC}(\sum_k z_k)$ is a Banach space.*

Theorem 3.5 *Suppose that Y is a normed space. Then Y is complete iff $S_{G_h}(\sum_k z_k)$ is closed in ℓ_∞ for each wuC series $\sum_k z_k$.*

Proof If we consider Y to be complete, then Theorem 3.2 shows that $S_{G_h}(\sum_k z_k)$ is complete for each wuC series $\sum_k z_k$. Conversely, suppose that Y is not complete, then we obtain a series $\sum_k z_k$ with $\|z_k\| < \frac{1}{k2^k}$ and $\sum_k z_k = z^{**} \in Y^{**} \setminus Y$. Thus, we have $G_h - \sum_k z_k = z^{**}$. Let us define the series $\sum_k x_k$, which is wuC, as it is defined that $x_k = kz_k$ for $k \in \mathbb{N}$. Consider a sequence $t = (t_k) \in c_0$ given by $t_k = \frac{1}{k}$ $\forall k \in \mathbb{N}$, then we have $G_h - \sum_k t_k z_k \in Y^{**} \setminus Y$. Therefore, $t \notin S_{G_h}(\sum_k z_k)$, which implies that there exists $\sum_k z_k$ such that $S_{G_h}(\sum_k z_k)$ is not complete. □

Theorem 3.6 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y, then $\sum_k z_k$ is wuC iff $\boldsymbol{S}$ defined in (3.1) is continuous.*

Proof Suppose that $\boldsymbol{S}$ is continuous and I is a set such that

$$I = \left\{\left\|\sum_{k=1}^{n} y_k z_k\right\| : \|y_k\| \leq 1, \forall n \in \mathbb{N}\right\}. \tag{3.8}$$

Thus, we have $Q = \sup_{n\in\mathbb{N}} I \leq \|\boldsymbol{S}\|$ such that $\sum_k z_k$ in Y is wuC as $\phi \subset S_{G_h}(\sum_k z_k)$. Conversely, let $\sum_k z_k$ be wuC series, then $Q = \sup_{n\in\mathbb{N}} I$, since the set I in (3.8) is bounded. If $y = (y_k) \in S_{G_h}(\sum_k z_k)$, then $\|\boldsymbol{S}(y)\| = \|G_h - \sum_k y_k z_k\| \leq Q\|y\|$. We can say that $\boldsymbol{S}$ is continuous. □

As defined in [3], the linear mapping $\boldsymbol{T}$ related with $\sum_k z_k$ in Y is given as

$$\boldsymbol{T}: S_{AC}\left(\sum_k z_k\right) \to Y, \qquad t = (t_k) \to A(t) = AC\sum_k a_k z_k. \tag{3.9}$$

Corollary 3.7 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then the following are identical:*

(i) $\sum_k z_k$ *is* (wuC).

(ii) $\boldsymbol{T}: S_{AC}(\sum_k z_k) \to Y$ *is continuous.*

(iii) $\boldsymbol{S}$ *described in* (3.1) *is continuous.*

4 Multiplier spaces of weak G_h-almost convergence

This particular section deals with multiplier spaces of weak G_h-almost convergence and build on the prior results to weak topologies.

Definition 4.1 Let us consider $\sum_k z_k$ to be the formal series in the normed space Y. Then the Y-valued multiplier space of wG_h-almost convergence of $\sum_k z_k$ is defined as

$$S_{wG_h}\left(\sum_k z_k\right) = \left\{y = (y_k) \in \ell_\infty : \sum_k z_k y_k \text{ is } wG_h\text{-almost convergent}\right\},$$

equipped with $\boldsymbol{S}$ (summing operator), and the sup norm is also defined by

$$\boldsymbol{S}: S_{wG_h}\left(\sum_k z_k\right) \to Y, \qquad y \to \boldsymbol{S}(y) = wG_h - \sum_k z_k y_k. \tag{4.1}$$

Theorem 4.2 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then the following are identical:*

(i) $\sum_k z_k$ *is* (wuC).

(ii) $S_{wG_h}(\sum_k z_k)$ *is a Banach space.*

(iii) $c_0 \subseteq S_{wG_h}(\sum_k z_k)$.

Proof Consider $\sum_k z_k$ is wuC series in Y. Then $\exists Q$ such that $Q = \sup_{n\in\mathbb{N}} I$ as defined in (3.8). If (t_k^n) is a Cauchy sequence in $S_{wG_h}(\sum_k z_k)$, then we have $t^0 = (t_k^0) \in \ell_\infty(Y)$ such that $t^n \to t^0$, as $n \to \infty$. Since $\ell_\infty(Y)$ is a Banach space, we wish to prove that $t^0 \in S_{wG_h}(\sum_k z_k)$. Let $y_n = wG_h - \sum_k t_k^n z_k$, then $y_n \in Y$ since $(t_k^n) \in S_{G_h}(\sum_k z_k)$ for each $n \in \mathbb{N}$. Now, $\forall \varepsilon > 0$ $\exists n_0' \in \mathbb{N}$ such that $\|t^{\nu_1} - t^{\nu_2}\| < \frac{\varepsilon}{3Q}$ $\forall \nu_1, \nu_2 > n_0'$. Thus, for $\nu_1, \nu_2 > n_0'$ $\exists n \in \mathbb{N}$ such that the following inequalities are satisfied for all $y^* \in Y^*$:

$$\begin{aligned} &\Bigg\| y^*(y_{\nu_1}) - \Bigg[\sum_{k=1}^m \frac{p_{m+n}}{h_n}\sum_{i=k}^m \frac{(-\gamma)_{i-k}}{(i-k)!} q_i y^*\big(t_k^{\nu_1} z_k\big) \\ &\quad + \sum_{k=1}^n \frac{p_{m+n}}{h_n}\sum_{i=k}^n \frac{(-\gamma)_{i-k}}{(i-k)!} q_i y^*\big(t_{m+k}^{\nu_1} z_{m+k}\big)\Bigg]\Bigg\| < \frac{\varepsilon}{3}, \end{aligned} \tag{4.2}$$

$$\Bigg\| y^*(y_{\nu_2}) - \Bigg[\sum_{k=1}^m \frac{p_{m+n}}{h_n}\sum_{i=k}^m \frac{(-\gamma)_{i-k}}{(i-k)!} q_i y^*\big(t_k^{\nu_2} z_k\big)$$

$$+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left(t_{m+k}^{\nu_2}z_{m+k}\right)\Bigg]\Bigg\| < \frac{\varepsilon}{3}, \tag{4.3}$$

and

$$\Bigg\|\sum_{k=1}^{m}\frac{p_{m+n}}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left[\left(t_k^{\nu_1}-t_k^{\nu_2}\right)z_k\right]$$

$$+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left[\left(t_{m+k}^{\nu_1}-t_{m+k}^{\nu_2}\right)z_{m+k}\right]\Bigg\| < \frac{\varepsilon}{3} \tag{4.4}$$

uniformly in $m\in\mathbb{N}$. Thus, $\forall\varepsilon>0$

$$\|y_{\nu_1}-y_{\nu_2}\| = \left|y^*(y_{\nu_1})-y^*(y_{\nu_2})\right| \le (4.2)+(4.3)+(4.4) < \varepsilon$$

$\forall\nu_1,\nu_2\ge n_0'$ and $y^*\in Y^*$. To a further extent, $\exists y_0^*\in Y^*$ such that $y_n\to y_0$ as $n\to\infty$, as Y is complete.

Now, we also have to show that $wG_h-\sum_k t_k^0 z_k = y_0$. For this, let $\forall\varepsilon>0$, we have $\|t^j-t^0\| < \frac{\varepsilon}{3Q}$, and for fixed j and $y^*\in Y^*$, we have

$$\left\|y^*(y_j-y_0)\right\| < \frac{\varepsilon}{3}. \tag{4.5}$$

Hence, $\exists n_0'\in\mathbb{N}$ such that

$$\Bigg\|y^*(y_j)-\Bigg[\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left(t_k^j z_k\right)+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left(t_{m+k}^j z_{m+k}\right)\Bigg]\Bigg\|$$

$$< \frac{\varepsilon}{3} \tag{4.6}$$

$\forall n\ge n_0'$, uniformly in $m\in\mathbb{N}$, since

$$y_j = wG_h-\sum_k t_k^j z_k \quad \forall j\in\mathbb{N}.$$

Now, from Lemma 1.1, we get

$$\Bigg[\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\frac{(t_k^j-t_k^0)}{\|t^j-t^0\|}z_k$$

$$+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\frac{(t_{m+k}^j-t_{m+k}^0)}{\|t^j-t^0\|}z_{m+k}\Bigg] \le Q. \tag{4.7}$$

Since $\sum_k z_k$ is wuC, so $\forall\varepsilon>0\ \exists n_0'\in\mathbb{N}$ such that

$$\Bigg\|y^*(y_0)-\Bigg[\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left(t_k^0 z_k\right)+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left(t_{m+k}^0 z_{m+k}\right)\Bigg]\Bigg\|$$

$$\le (4.5)+(4.6)+\Bigg\|\sum_{k=1}^{m}\frac{p_m}{h_n}\sum_{i=k}^{m}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\left[\left(t_k^j-t_k^0\right)z_k\right]$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\frac{p_{m+n}}{h_n}\sum_{i=k}^{n}\frac{(-\gamma)_{i-k}}{(i-k)!}q_i y^*\bigl[\bigl(t^j_{m+k}-t^0_{m+k}\bigr)z_{m+k}\bigr]\Bigg\| \\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\bigl\|t^j-t^0\bigr\|.Q \\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3Q}.Q=\varepsilon
\end{aligned}
$$

$\forall n \ge n_0'$, uniformly in $m \in \mathbb{N}$. Thus,

$$t^0=\bigl(t^0_k\bigr)_k \in S_{wG_h}\Biggl(\sum_k z_k\Biggr).$$

(ii) $\Rightarrow$ (iii) If $S_{wG_h}(\sum_k z_k)$ is complete with $t=(t_k)$ being a sequence in c_0, then we need to prove that $t=(t_k)\in S_{wG_h}(\sum_k z_k)$. Now, since $S_{wG_h}(\sum_k z_k)$ is a complete space, then it contains the space of eventually zero sequences c_0. That is, $\phi \subset S_{wG_h}(\sum_k z_k)$. Since c_0 is an AK space, we have $t^{[m]}=\sum_{k=1}^m t_k e^k \in S_{wG_h}(\sum_k z_k)$. Therefore, $\lim_{m\to\infty}\|t^{[m]}-t\|_\infty=0$. Thus $t=(t_k)\in S_{wG_h}(\sum_k z_k)$.

(iii) $\Rightarrow$ (ii) We can prove this with the same example as given in Theorem 3.2.

(ii) $\Rightarrow$ (i) Suppose that $S_{wG_h}(\sum_k z_k)$ is a Banach space and $t=(t_k)\in c_0(Y)$, which means $c_0(Y)\subseteq S_{wG_h}(\sum_k z_k)$ (already proved), which implies that $\sum_k t_k z_k$ is almost convergent for all $t=(t_k)\in c_0(Y)$. Therefore, from the monotonicity of $c_0(Y)$, $\sum_k t_k z_k$ is subseries almost convergent, and thus we get $\sum_k t_k z_k$ is wuC from the Orlicz–Pettis theorem. □

Corollary 4.3 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then $\sum_k z_k$ is c_0-multiplier convergent iff $c_0 \subseteq S_{wG_h}(\sum_k z_k)$.*

$S_{wG_h}(\sum_k z_k)$ of almost summability related with $\sum_k z_k$ was studied by Aizpuru et al. [3] which is given as

$$S_{wAC}\Biggl(\sum_k z_k\Biggr)=\Biggl\{t=(t_k)\in \ell_\infty : wAC\sum_k t_k z_k \text{ exists}\Biggr\}.$$

Corollary 4.4 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then the following are identical:*

(i) $\sum_k z_k$ *is* (wuC).
(ii) $c_0(Y)\subseteq S_{wG_h}(\sum_k z_k)$.
(iii) $S_{wG_h}(\sum_k z_k)$ *is a Banach space.*
(iv) *For all* $t=(t_k)\in c_0$ *there exists* $wAC\sum_k t_k z_k$.
(v) $S_{wAC}(\sum_k z_k)$ *is a Banach space.*

Theorem 4.5 *Suppose that Y is a normed space. Then Y is complete iff $S_{wG_h}(\sum_k z_k)$ is closed in ℓ_∞ for each wuC series $\sum_k z_k$.*

Proof The proof is similar to Theorem 3.5. So, we omit the details. □

Theorem 4.6 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y, then $\sum_k z_k$ is wuC iff $\mathbf{S}$ defined in (4.1) is continuous.*

Proof The proof is similar to Theorem 3.5. So, we omit the details. □

Corollary 4.7 *Suppose that Y is a Banach space such that the formal series $\sum_k z_k$ belongs to Y. Then the following are identical*:

(i) $\sum_k z_k$ *is* (wuC).

(ii) $\boldsymbol{T} : S_{wAC}(\sum_k z_k) \to Y$ *is continuous.*

(iii) $\boldsymbol{S}$ *described in* (4.1) *is continuous.*

Remark 4.8 Suppose that χ is a linear space and μ_1 and μ_2 are linear topologies on χ such that μ_2 has a neighborhood base at 0 consisting of μ_1 closed sets [in a sense of Wilanski]. If $z = (z_i) \subset \chi$ is a Cauchy sequence converging to z in (χ, μ_1), then it will converge to z in (χ, μ_2).

Proposition 4.9 *Let $\sum_k z_k$ be uC in Y. Then $S_{wG_h}(\sum_k z_k) = S_{G_h}(\sum_k z_k)$.*

Proof Suppose that $y = (y_k) \in S_{wG_h}(\sum_k z_k)$. This implies that the partial sum of $\sum_k y_k z_k$ obtains a Cauchy sequence that is again weakly G_h-convergent. Since the weak topology is connected with the norm topology, it will converge to the same point as in the norm topology. □

5 Orlicz–Pettis theorem for weak G_h-almost convergence

This particular section deals with a new version of the Orlicz–Pettis theorem for a Banach space Y. As noted earlier, the classical form of the Orlicz–Pettis theorem for the normed space claims that a series is subseries convergent in weak topology for the space is subseries convergent to the norm topology for the same space. In addition to that, if Y is complete, then $\sum_k z_k$ is ℓ_∞-multiplier convergent. The Orlicz–Pettis theorem proportionately states that if Y is a Banach space and if $\forall M \subset \mathbb{N}$ there exists a weakly sum $\sum_{k\in M} z_k$, then $\sum_k z_k$ is *uc*.

Theorem 5.1 *Suppose that Y is a Banach space and sum $\sum_{k\in M} z_k$ is wG_h-almost convergent for every $M \subset \mathbb{N}$, then $\sum_k z_k$ is uc.*

Proof From the previous results, we know that $\sum_k z_k$ is wuC. Let $M \subset \mathbb{N}$, then $wG_h - \sum_{k\in M} z_k = z_0$ $\forall z_0 \in Y$. From the classical Orlicz–Pettis theorem and the equalities given below

$$\sum_{k\in M} z^*(z_k) = G_h - \sum_{k\in M} z^*(z_k) = z^*(z_0) \quad \forall z^* \in Y^*,$$

we get $\sum_k z_k$ is *uc* series. □

Corollary 5.2 *Suppose that Y is a Banach space and $\sum_k z_k$ belongs to Y. Then the given assertions are equivalent*:

(i) $\sum_k z_k$ *is uc.*

(ii) $\ell_\infty \subseteq S_{G_h}(\sum_k z_k)$.

(iii) $\ell_\infty \subseteq S_{wG_h}(\sum_k z_k)$.

Here, we remark that if $\sum_k z_k$ is wuC series in Y, then $\sum_k y_k z_k$ is wuC series for all $y_k \in \ell_\infty$. Thus,

$$S_{G_h}\left(\sum_k z_k\right) \subset S_w\left(\sum_k z_k\right),$$

where $S_w(\sum_k z_k) = \{y = (y_k) \in \ell_\infty : w\sum_k y_k z_k \text{ exists}\}$.

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