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Degenerate poly-Bell polynomials and numbers

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Abstract

Numerous mathematicians have studied 'poly' as one of the generalizations to special polynomials, such as Bernoulli, Euler, Cauchy, and Genocchi polynomials. In relation to this, in this paper, we introduce the degenerate poly-Bell polynomials emanating from the degenerate polyexponential functions which are called the poly-Bell polynomials when $\lambda \to 0$. Specifically, we demonstrate that they are reduced to the degenerate Bell polynomials if $k = 1$. We also provide explicit representations and combinatorial identities for these polynomials, including Dobinski-like formulas, recurrence relationships, etc.

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Keywords: Bell polynomials and numbers; Modified degenerate polyexponential functions; Degenerate poly-Bernoulli polynomials; Degenerate poly-Euler polynomials

1 Introduction

Although various mathematicians studied 'poly' as one of the generalizations of Bernoulli, Euler, Genocchi, and Cauchy polynomials [1, 4, 6–8, 11, 14, 16, 18, 20], the 'poly' for Bell polynomials has not been studied so far. Furthermore, in recent years, a lot of research has been conducted on various degenerate versions of many special polynomials and numbers, accumulating in a renewed interest for mathematicians various special polynomials and numbers [3, 9, 11, 12, 14–18]. For instance, Kim and Kim [8] reappraised the polyexponential functions in relation to polylogarithm functions, expanding upon the research which was first conducted by Hardy [5].

With this in mind, in this paper, we define the degenerate poly-Bell polynomials through their degenerate polyexponential functions, reducing them to the degenerate Bell polynomials if $k = 1$. Hence, we define the poly-Bell polynomials when $\lambda \to 0$, providing explicit expressions and identities involving those polynomials.

In recent years, much research has been done for various degenerate versions of many special polynomials and numbers. Moreover, various special polynomials and numbers regained interest of mathematicians, and quite a few results have been discovered [3, 9, 11, 12, 14–18]. The polyexponential functions were reconsidered by Kim [8] in view of an inverse to the polylogarithm functions which were first studied by Hardy [5]. In this paper, we define the degenerate poly-Bell polynomials by means of the degenerate poly-exponential functions, and they are reduced to the degenerate Bell polynomials if $k = 1$. In particular, when $\lambda \to 0$, we call them the poly-Bell polynomials. We also provide explicit representations and combinatorial identities for these polynomials, including Dobinski-like formulas, recurrence relationships, etc.

The Bell polynomials $B_n(x) = \sum_{k=0}^{n} S_2(n,k)x^n$ are natural extensions of the Bell numbers which are a number of ways to partition a set with n elements into nonempty subsets. It is well known that the generating function of the Bell polynomials is given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x)\frac{t^n}{n!} \quad \text{(see [3, 11, 15, 19])}.$$

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \quad \text{and} \quad e_\lambda(t) = \sum_{n=0}^{\infty} (1)_{n,\lambda}\frac{t^n}{n!} \quad \text{(see [3, 7, 9, 11, 12, 14–18])}, \tag{1}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

The fully degenerate Bell polynomials are given by

$$e_\lambda\big(x\big(e_\lambda(t)-1\big)\big) = \sum_{n=0}^{\infty} \mathrm{bel}_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [3])}. \tag{2}$$

When $\lambda \to 0$, $\mathrm{bel}_{n,\lambda}(x) = \mathrm{bel}_n(x)$.

Carlitz considered the degenerate Bernoulli polynomials which are given by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [2])}. \tag{3}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

The degenerate Genocchi polynomials are given by

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{G}_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [12, 16])}. \tag{4}$$

When $x = 0$, $\mathcal{G}_{n,\lambda} = \mathcal{G}_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

Kim and Kim introduced the modified polyexponential function as

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k} \quad (k \in \mathbb{Z}) \text{ (see [8])}. \tag{5}$$

By (5), we see that $\mathrm{Ei}_1(x) = e^x - 1$.

The degenerate polyexponential function is given by

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}x^n}{(n-1)!n^k} \quad (k \in \mathbb{Z}) \text{ (see [11, 17])}. \tag{6}$$

We note that $\mathrm{Ei}_{1,\lambda}(x) = e_\lambda(x) - 1$.

The degenerate poly-Bernoulli polynomials are defined by

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\quad\text{(see [14, 17])}.\tag{7}$$

When $x=0$, $\beta_{n,\lambda}^{(k)}=\beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

Since $\mathrm{Ei}_{1,\lambda}(\log_\lambda(1+t))=t$, $\beta_{n,\lambda}^{(1)}(x)$ are the degenerate Bernoulli polynomials.

The degenerate poly-Genocchi polynomials are given by

$$\frac{2\,\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\quad\text{(see [16])},\tag{8}$$

and $\mathcal{G}_{0,\lambda}^{(k)}(x)=0$. When $x=0$, $\mathcal{G}_{n,\lambda}^{(k)}=\mathcal{G}_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

When $k=1$, $\mathcal{G}_{n,\lambda}^{(1)}(x)$ are the degenerate Genocchi polynomials.

In [9], the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l\quad(n\geq 0).\tag{9}$$

As an inversion formula of (9), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda}\quad(n\geq 0)\ \text{(see [11, 15])}.\tag{10}$$

From (9) and (10), we note that

$$\frac{1}{k!}\bigl(e_\lambda(t)-1\bigr)^k=\sum_{n=k}^{\infty}S_{2,l}(n,k)\frac{t^n}{n!}\quad\text{(see [11, 15])}\tag{11}$$

and

$$\frac{1}{k!}\bigl(\log_\lambda(1+t)\bigr)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\quad\text{(see [11, 17])},\tag{12}$$

where

$$\log_\lambda(t)=\frac{1}{\lambda}\bigl(t^\lambda-1\bigr)\quad\text{(see [17])}\tag{13}$$

is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda(t))=t$.

2 Degenerate poly-Bell polynomials and numbers

In this section, we define the degenerate poly-Bell polynomials by using of the degenerate polyexponential functions and give explicit expressions and identities involving these polynomials.

We define the degenerate poly-Bell polynomials $\mathrm{bel}_{n,\lambda}^{(k)}(x)$, which arise from the degenerate polyexponential functions to be

$$1+\mathrm{Ei}_{k,\lambda}\big(x\big(e_\lambda(t)-1\big)\big)=\sum_{n=0}^{\infty}\mathrm{bel}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} \tag{14}$$

and $\mathrm{bel}_{0,\lambda}^{(k)}(x)=1$.

When $x=1$, $\mathrm{bel}_{n,\lambda}^{(k)}=\mathrm{bel}_{n,\lambda}^{(k)}(1)$ are called the degenerate poly-Bell numbers.

When $k=1$, from (6), we note that

$$\begin{aligned}1+\mathrm{Ei}_{1,\lambda}\big(x\big(e_\lambda(t)-1\big)\big)&=1+\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}(x(e_\lambda(t)-1))^n}{(n-1)!n}\\&=\sum_{n=0}^{\infty}\frac{(1)_{n,\lambda}(x(e_\lambda(t)-1))^n}{n!}\\&=e_\lambda\big(x\big(e_\lambda(t)-1\big)\big)=\sum_{n=0}^{\infty}\mathrm{bel}_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{15}$$

Combining with (14) and (15), we have

$$\mathrm{bel}_{n,\lambda}^{(1)}(x)=\mathrm{bel}_{n,\lambda}(x).$$

When $\lambda\to 0$, $\mathrm{bel}_{n}^{(k)}(x)$ are called the poly-Bell polynomials.

Theorem 1 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$\mathrm{bel}_{n,\lambda}^{(k)}(x)=\sum_{l=1}^{n}\frac{(1)_{l,\lambda}}{l^{k-1}}S_2(n,l)x^l.$$

Proof From (6) and (11), we observe that

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}\big(x\big(e_\lambda(t)-1\big)\big)&=\sum_{l=1}^{\infty}\frac{(1)_{l,\lambda}x^l}{l^{k-1}}\frac{1}{l!}\big(e_\lambda(t)-1\big)^l\\&=\sum_{l=1}^{\infty}\frac{(1)_{l,\lambda}x^l}{l^{k-1}}\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\frac{(1)_{l,\lambda}}{l^{k-1}}S_{2,\lambda}(n,l)x^l\right)\frac{t^n}{n!}.\end{aligned} \tag{16}$$

Combining with (14) and (16), we have the desired result. □

Theorem 2 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$\sum_{m=1}^{n}\mathrm{bel}_{m,\lambda}^{(k)}(x)S_{1,\lambda}(n,m)=\frac{(1)_{m,\lambda}x^m}{n^{k-1}}. \tag{17}$$

In particular, when $x = 1$,

$$\sum_{m=1}^{n} \mathrm{bel}_{m,\lambda}^{(k)} S_{1,\lambda}(n,m) = \frac{(1)_{m,\lambda}}{n^{k-1}}.$$

Proof By replacing t with $\log_\lambda(1+t)$ in (14), the left-hand side is

$$1 + \mathrm{Ei}_{k,\lambda}(xt) = 1 + \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(xt)^n}{(n-1)!n^k} = 1 + \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}x^n}{n^{k-1}} \frac{t^n}{n!}. \tag{18}$$

On the other hand, from (12), the right-hand side is

$$\begin{aligned} \sum_{m=0}^{\infty} \mathrm{bel}_{m,\lambda}^{(k)}(x) \frac{(\log_\lambda(1+t))^m}{m!} &= 1 + \sum_{m=1}^{\infty} \mathrm{bel}_{m,\lambda}^{(k)}(x) \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \mathrm{bel}_{m,\lambda}^{(k)}(x) S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{19}$$

Combining with coefficients of (18) and (19), we get what we want. □

Theorem 3 (Dobinski-like formulas) *For* $k \in \mathbb{Z}$ *and* $n \geq 1$, *we have*

$$\mathrm{bel}_{n,\lambda}^{(k)}(x) = \sum_{h=1}^{\infty} \sum_{m=0}^{h} \binom{m}{h} \frac{(-1)^{h-m}(1)_{h,\lambda}(m)_{n,\lambda}}{(h-1)!h^k}.$$

Proof From (1) and (6), we observe that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathrm{bel}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \sum_{h=1}^{\infty} \frac{(1)_{h,\lambda}x^h}{(h-1)!h^k} \bigl(e_\lambda(t) - 1\bigr)^h \\ &= \sum_{h=1}^{\infty} \frac{(1)_{h,\lambda}x^h}{(h-1)!h^k} \sum_{m=0}^{h} \binom{h}{m} (-1)^{h-m} e_\lambda^m(t) \\ &= \sum_{h=1}^{\infty} \frac{(1)_{h,\lambda}x^h}{(h-1)!h^k} \sum_{m=0}^{h} \binom{h}{m} (-1)^{h-m} \sum_{n=0}^{\infty} (m)_{n,\lambda} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{h=1}^{\infty} \sum_{m=0}^{h} \binom{h}{m} \frac{(-1)^{h-m}(1)_{h,\lambda}(m)_{n,\lambda}}{(h-1)!h^k} \right) \frac{t^n}{n!} \\ &= \sum_{h=1}^{\infty} \sum_{m=0}^{h} \binom{h}{m} \frac{(-1)^{h-m}(1)_{h,\lambda}}{(h-1)!h^k} \\ &\quad + \sum_{n=1}^{\infty} \left(\sum_{h=1}^{\infty} \sum_{m=0}^{h} \binom{h}{m} \frac{(-1)^{h-m}(1)_{h,\lambda}(m)_{n,\lambda}}{(h-1)!h^k} \right) \frac{t^n}{n!}. \end{aligned} \tag{20}$$

By comparing with coefficients on both sides of (20), we get

$$\sum_{h=1}^{\infty}\sum_{m=0}^{h}\binom{h}{m}\frac{(-1)^{h-m}(1)_{h,\lambda}}{(h-1)!h^k}=0 \quad \text{and}$$

$$\mathrm{bel}_{n,\lambda}^{(k)}(x)=\sum_{h=1}^{\infty}\sum_{m=0}^{h}\binom{m}{h}\frac{(-1)^{h-m}(1)_{h,\lambda}(m)_{n,\lambda}}{(h-1)!h^k} \quad (n\geq 1). \qquad \square$$

Theorem 4 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$\sum_{m=0}^{n-1}\binom{n}{m}(1)_{n-m,\lambda}\,\mathrm{bel}_{m+1,\lambda}^{(k)}(x)=\sum_{m=1}^{n}\binom{n}{m}(1-\lambda)_{n-m,\lambda}\,\mathrm{bel}_{m,\lambda}^{(k-1)}(x).$$

Proof Differentiating with respect to t in (14), the left-hand side of (14) is

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Ei}_{k,\lambda}\big(x\big(e_\lambda(t)-1\big)\big)&=\frac{\partial}{\partial t}\sum_{n=1}^{\infty}\frac{x^n(e_\lambda(t)-1)^n}{(n-1)!n^k}\\&=\frac{e_\lambda^{1-\lambda}(t)}{e_\lambda(t)-1}\sum_{n=1}^{\infty}\frac{x^n(e_\lambda(t)-1)^n}{(n-1)!n^{k-1}}\\&=\frac{e_\lambda^{1-\lambda}(t)}{e_\lambda(t)-1}\mathrm{Ei}_{k-1,\lambda}\big(x\big(e_\lambda(t)-1\big)\big)\\&=\frac{e_\lambda^{1-\lambda}(t)}{e_\lambda(t)-1}\sum_{n=1}^{\infty}\mathrm{bel}_{n,\lambda}^{(k-1)}(x)\frac{t^n}{n!}.\end{aligned}\tag{21}$$

On the other hand, the right-hand side of (14) is

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}\mathrm{bel}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\right)=\sum_{n=1}^{\infty}\mathrm{bel}_{n,\lambda}^{(k)}(x)\frac{t^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\mathrm{bel}_{n+1,\lambda}^{(k)}(x)\frac{t^n}{n!}.\tag{22}$$

Combining with (21) and (22), we get

$$\big(e_\lambda(t)-1\big)\sum_{m=0}^{\infty}\mathrm{bel}_{m+1,\lambda}^{(k)}(x)\frac{t^m}{m!}=e_\lambda^{1-\lambda}(t)\sum_{m=1}^{\infty}\mathrm{bel}_{m,\lambda}^{(k-1)}(x)\frac{t^m}{m!}.\tag{23}$$

From (23), we have

$$\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\sum_{m=0}^{\infty}\mathrm{bel}_{m+1,\lambda}^{(k)}(x)\frac{t^m}{m!}=\sum_{i=0}^{\infty}(1-\lambda)_{i,\lambda}\frac{t^i}{i!}\sum_{m=1}^{\infty}\mathrm{bel}_{m,\lambda}^{(k-1)}(x)\frac{t^m}{m!}.\tag{24}$$

From (24), we get

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}(1)_{n-m,\lambda}\,\mathrm{bel}_{m+1,\lambda}^{(k)}(x)\frac{t^n}{n!}=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\binom{n}{m}(1-\lambda)_{n-m,\lambda}\,\mathrm{bel}_{m,\lambda}^{(k-1)}(x)\frac{t^n}{n!}.\tag{25}$$

By comparing with coefficients on both sides of (25), we have

$$\sum_{m=0}^{n-1}\binom{n}{m}(1)_{n-m,\lambda}\,\mathrm{bel}_{m+1,\lambda}^{(k)}(x)=\sum_{m=1}^{n}\binom{n}{m}(1-\lambda)_{n-m,\lambda}\,\mathrm{bel}_{m,\lambda}^{(k-1)}(x). \tag{26}$$

□

Theorem 5 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$\mathrm{bel}_{n,\lambda}^{(k)}(x)=\sum_{j=1}^{n}\sum_{h=1}^{j}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)S_{2,\lambda}(n,j)S_{2,\lambda}(j,h)x^{j}.$$

Proof From (6) and (12), we observe that

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}\bigl(\log_{\lambda}(1+t)\bigr)&=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_{\lambda}(1+t))^{m}}{(m-1)!m^{k}}\\&=\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\frac{(\log_{\lambda}(1+t))^{m}}{m!}=\sum_{n=1}^{\infty}\Biggl(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}}\Biggr)\frac{t^{n}}{n!}.\end{aligned} \tag{27}$$

By replacing t with $e_{\lambda}(x(e_{\lambda}(t)-1))-1$ in (27), we get

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}\bigl(x\bigl(e_{\lambda}(t)-1\bigr)\bigr)&=\sum_{h=1}^{\infty}\Biggl(\sum_{m=1}^{h}\frac{(1)_{m,\lambda}S_{1,\lambda}(h,m)}{m^{k-1}}\Biggr)\frac{1}{h!}\bigl(e_{\lambda}\bigl(x\bigl(e_{\lambda}(t)-1\bigr)\bigr)-1\bigr)^{h}\\&=\sum_{h=1}^{\infty}\Biggl(\sum_{m=1}^{h}\frac{(1)_{m,\lambda}S_{1,\lambda}(h,m)}{m^{k-1}}\Biggr)\sum_{j=h}^{\infty}S_{2,\lambda}(j,h)x^{j}\sum_{n=j}^{\infty}S_{2,\lambda}(n,j)\frac{t^{n}}{n!}\\&=\sum_{n=1}^{\infty}\Biggl(\sum_{j=1}^{n}\sum_{h=1}^{j}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)S_{2,\lambda}(n,j)S_{2,\lambda}(j,h)x^{j}\Biggr)\frac{t^{n}}{n!}.\end{aligned} \tag{28}$$

Combining with (14) and (28), we get what we want. □

For the next theorem, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!}&=\frac{\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{e_{\lambda}(t)-1}e_{\lambda}^{x}(t)\\&=\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(k)}\frac{t^{m}}{m!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^{l}}{l!}=\sum_{n=0}^{\infty}\Biggl(\sum_{m=0}^{n}\binom{n}{m}(x)_{n-m,\lambda}\beta_{m,\lambda}^{(k)}\Biggr)\frac{t^{n}}{n!}.\end{aligned} \tag{29}$$

By comparing with coefficients on both sides of (29), we get

$$\beta_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\binom{n}{m}(x)_{n-m,\lambda}\beta_{m,\lambda}^{(k)}. \tag{30}$$

Theorem 6 *For $n\geq 1$, we have*

$$\mathrm{bel}_{n,\lambda}^{(k)}(x)=\sum_{d=1}^{n}\sum_{h=1}^{d}\bigl(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\bigr)S_{2,\lambda}(n,d)S_{2,\lambda}(d,h)x^{d},$$

where $\beta_{n,\lambda}^{(k)}$ are the degenerate poly-Bernoulli numbers.

Proof From (1), (7), and (30), we observe that

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}\big(\log_\lambda(1+t)\big) &= \big(e_\lambda(t)-1\big)\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\right)\\
&= \left(\sum_{m=0}^{\infty}\frac{(1)_{m,\lambda}}{m!}t^m-1\right)\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\beta_{m,\lambda}^{(k)}-\beta_{n,\lambda}^{(k)}\right)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}\big(\beta_{n,\lambda}^{(k)}(1)-\beta_{n,\lambda}^{(k)}\big)\frac{t^n}{n!}.
\end{aligned}
\tag{31}
$$

By replacing t with $e_\lambda(x(e_\lambda(t)-1))-1$ in (31), we get

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}\big(x\big(e_\lambda(t)-1\big)\big) &= \sum_{h=1}^{\infty}\big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\big)\frac{(e_\lambda(x(e_\lambda(t)-1))-1)^h}{h!}\\
&= \sum_{h=1}^{\infty}\big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\big)\sum_{d=h}^{\infty}S_{2,\lambda}(d,h)\frac{1}{d!}\big(x\big(e_\lambda(t)-1\big)\big)^d\\
&= \sum_{d=1}^{\infty}\sum_{h=1}^{d}\big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\big)S_{2,\lambda}(d,h)x^d\sum_{n=d}^{\infty}S_{2,\lambda}(n,d)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}\left(\sum_{d=1}^{n}\sum_{h=1}^{d}\big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\big)S_{2,\lambda}(n,d)S_{2,\lambda}(d,h)x^d\right)\frac{t^n}{n!}.
\end{aligned}
\tag{32}
$$

From (14) and (32), we get the desired result. □

Theorem 7 *For $k \in \mathbb{Z}$ and $n \geq 1$, we have*

$$
\mathrm{bel}_{n,\lambda}^{(k)}(x) = \sum_{j=1}^{n}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{n}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{2,\lambda}(n,j)S_{2,\lambda}(j,h)x^j,
$$

where $\beta_{n,\lambda}^{(k)}$ are the degenerate poly-Bernoulli numbers.

Proof From (1) and (7), we observe that

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}\big(\log_\lambda(1+t)\big) &= (e_\lambda-1)\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\right)\\
&= \sum_{m=1}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\binom{n}{m}(1)_{m,\lambda}\beta_{n-m,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{33}
$$

By replacing t with $e_\lambda(x(e_\lambda(t)-1))-1$ in (33), from (11), we get

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}\big(x\big(e_\lambda(t)-1\big)\big) &= \sum_{h=1}^{\infty}\left(\sum_{m=1}^{h}\binom{n}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}\right)\frac{1}{h!}\big(e_\lambda\big(x\big(e_\lambda(t)-1\big)\big)-1\big)^h \\ &= \sum_{h=1}^{\infty}\left(\sum_{m=1}^{h}\binom{n}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}\right)\sum_{j=h}^{\infty}S_{2,\lambda}(j,h)x^j\sum_{n=j}^{\infty}S_{2,\lambda}(n,j)\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{n}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{2,\lambda}(n,j)S_{2,\lambda}(j,h)x^j\right)\frac{t^n}{n!}.\end{aligned}\tag{34}$$

Combining with (14) and (34), we get the desired result. □

Theorem 8 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$\mathrm{bel}_{n,\lambda}^{(k)}(x)=\frac{1}{2}\sum_{j=1}^{n}\sum_{h=1}^{j}\left(\sum_{i=1}^{h}\binom{h}{i}(1)_{i,\lambda}\mathcal{G}_{h-i,\lambda}^{(k)}+2\mathcal{G}_{h,\lambda}^{(k)}\right)S_{2,\lambda}(j,h)S_{2,\lambda}(n,j)x^j,$$

where $\mathcal{G}_{n,\lambda}^{(k)}$ are the degenerate poly-Genocchi numbers.

Proof From (1) and (8), we have

$$\begin{aligned}2\,\mathrm{Ei}_{k,\lambda}\big(\log_\lambda(1+t)\big) &= \big(e_\lambda(t)+1\big)\left(\sum_{m=0}^{\infty}\mathcal{G}_{m,\lambda}^{(k)}\frac{t^m}{m!}\right) \\ &= \left(\sum_{i=1}^{\infty}(1)_{i,\lambda}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}\mathcal{G}_{m,\lambda}^{(k)}\frac{t^m}{m!}\right)+2\sum_{m=0}^{\infty}\mathcal{G}_{m,\lambda}^{(k)}\frac{t^m}{m!} \\ &= \sum_{h=1}^{\infty}\left(\sum_{i=1}^{h}\binom{h}{i}(1)_{i,\lambda}\mathcal{G}_{h-i,\lambda}^{(k)}+2\mathcal{G}_{h,\lambda}^{(k)}\right)\frac{t^h}{h!}.\end{aligned}\tag{35}$$

By replacing t with $e_\lambda(x(e_\lambda(t)-1))-1$ in (35), we get

$$\begin{aligned}&2\,\mathrm{Ei}_{k,\lambda}\big(x\big(e_\lambda(t)-1\big)\big) \\ &\quad= \sum_{h=1}^{\infty}\left(\sum_{i=1}^{h}\binom{h}{i}(1)_{i,\lambda}\mathcal{G}_{h-i,\lambda}^{(k)}+2\mathcal{G}_{h,\lambda}^{(k)}\right)\frac{(e_\lambda(x(e_\lambda(t)-1))-1)^h}{h!} \\ &\quad= \sum_{h=1}^{\infty}\left(\sum_{i=1}^{h}\binom{h}{i}(1)_{i,\lambda}\mathcal{G}_{h-i,\lambda}^{(k)}+2\mathcal{G}_{h,\lambda}^{(k)}\right)\sum_{j=h}^{\infty}S_{2,\lambda}(j,h)x^j\sum_{n=j}^{\infty}S_{2,\lambda}(n,j)\frac{t^n}{n!} \\ &\quad= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{h=1}^{j}\left(\sum_{i=1}^{h}\binom{h}{i}(1)_{i,\lambda}\mathcal{G}_{h-i,\lambda}^{(k)}+2\mathcal{G}_{h,\lambda}^{(k)}\right)S_{2,\lambda}(j,h)S_{2,\lambda}(n,j)x^j\frac{t^n}{n!}.\end{aligned}\tag{36}$$

From (14) and (36), we get what we want. □

3 Further remarks

Remark 1 For $\lambda \in (0,1)$, let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$ if the probability mass function of X is given by

$$P_\lambda(j) = P\{X_\lambda = j\} = e_\lambda^{-1}(\alpha)\frac{\alpha^j(1)_{j,\lambda}}{j!} \quad \text{(see [9, 13])}, \tag{37}$$

where $j = 0,1,2,3,\ldots$. It is easy to show that

$$\sum_{j=0}^{\infty} P_\lambda(j) = e_\lambda^{-1}(\alpha)\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}(1)_{j,\lambda} = e_\lambda^{-1}(\alpha)e_\lambda(\alpha) = 1 \quad \text{(see [9, 13])}. \tag{38}$$

Let $f(x)$ be a real variable function on X_λ.

From (17) of Theorem 2, we observe that

$$\begin{aligned} E[X_\lambda] &= \sum_{j=0}^{\infty} jP_\lambda(j) = e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\frac{1}{(j-1)!}(1)_{j,\lambda}\alpha^j \\ &= e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\frac{1}{(j-1)!}j^{k-1}\sum_{h=0}^{j} S_{1,\lambda}(j,h)\operatorname{bel}_{h,\lambda}^{(k)}(\alpha) \\ &= e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\sum_{h=0}^{j}\frac{j^{k-1}}{(j-1)!}S_{1,\lambda}(j,h)\operatorname{bel}_{h,\lambda}^{(k)}(\alpha). \end{aligned} \tag{39}$$

In addition, for $n \in \mathbb{N}$, we also obtain the moments of X_λ as follows:

$$\begin{aligned} E\left[X_\lambda^n\right] &= \sum_{j=0}^{\infty} j^n P_\lambda(j) = e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\frac{j^{n-1}}{(j-1)!}(1)_{j,\lambda}\alpha^j \\ &= e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\frac{j^{n-1}}{(j-1)!}j^{k-1}\sum_{h=0}^{j} S_{1,\lambda}(j,h)\operatorname{bel}_{h,\lambda}^{(k)}(\alpha) \\ &= e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\sum_{h=0}^{j}\frac{j^{n+k-2}}{(j-1)!}S_{1,\lambda}(j,h)\operatorname{bel}_{h,\lambda}^{(k)}(\alpha). \end{aligned} \tag{40}$$

Thus, we have the following theorem.

Theorem 9 *For $\lambda \in (0,1)$, let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$ if the probability mass function of X. Then the expectation and the moments of X_λ are*

$$E[X_\lambda] = e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\sum_{h=0}^{j}\frac{j^{k-1}}{(j-1)!}S_{1,\lambda}(j,h)\operatorname{bel}_{h,\lambda}^{(k)}(\alpha),$$

$$E\left[X_\lambda^n\right] = e_\lambda^{-1}(\alpha)\sum_{j=0}^{\infty}\sum_{h=0}^{j}\frac{j^{n+k-2}}{(j-1)!}S_{1,\lambda}(j,h)\operatorname{bel}_{h,\lambda}^{(k)}(\alpha),$$

respectively, where $\operatorname{bel}_{h,\lambda}^{(k)}(\alpha)$ are the degenerate poly-Bell polynomials.

Remark 2 Kim and Lee introduced a new type of the degenerate Bell polynomials defined by

$$e_\lambda\big(x\big(e^t-1\big)\big)=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}\quad\text{(see [10])}.\tag{41}$$

When $\lim_{\lambda\to 0} e_\lambda(x(e^\lambda-1))=\exp(x(e^t-1))=\sum_{n=0}^{\infty}\mathrm{bel}_n(x)\frac{t^n}{n!}$.

We can also consider a new type of degenerate poly-Bell polynomials by

$$1+\mathrm{Ei}_{k,\lambda}\big(x\big(e^t-1\big)\big)=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\tag{42}$$

and $\mathrm{Bel}_{0,\lambda}^{(k)}(x)=1$.

When $x=1$, $\mathrm{Bel}_{n,\lambda}^{(k)}=\mathrm{Bel}_{n,\lambda}^{(k)}(1)$ are called the degenerate poly-Bell numbers.

When $k=1$, from (42), we note that

$$\begin{aligned}1+\mathrm{Ei}_{1,\lambda}\big(x\big(e^t-1\big)\big)&=1+\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}(x(e^t-1))^n}{(n-1)!n}\\&=\sum_{n=0}^{\infty}\frac{(1)_{n,\lambda}(x(e^t-1))^n}{(n-1)!n}\\&=e_\lambda\big(x\big(e^t-1\big)\big)=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned}\tag{43}$$

From (41) and (43), we have

$$\mathrm{Bel}_{n,\lambda}^{(1)}(x)=\mathrm{Bel}_{n,\lambda}(x).$$

When $\lambda\to 0$, $\mathrm{Bel}_n^{(k)}(x)$ are called a new type of the poly-Bell polynomials.

We can obtain similar results in the same way.

4 Conclusion

To summarize, in this paper, $\lambda\to 0$ is defined as poly-Bell polynomials by introducing the degenerate poly-Bell polynomials through a degenerate polyexponential function and reducing it to a degenerate Bell polynomial for $k=1$. We derived Dobinski-like formula in Theorem 3, recurrence relation in Theorem 4, and combinatorial identities for these polynomials. As for Theorem 1, the explicit formula demonstrated the relationship with Stirling numbers of the second kind according to k. To conclude, there are various methods for studying special polynomials and numbers, including: generating functions, combinatorial methods, umbral calculus, differential equations, and probability theory. We are now interested in continuing our research into the application of 'poly' versions of certain special polynomials and numbers in the fields of physics, science, and engineering as well as mathematics.

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