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On a fractional problem of Lane–Emden type: Ulam type stabilities and numerical behaviors

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Abstract

In this work, we study some types of Ulam stability for a nonlinear fractional differential equation of Lane–Emden type with anti periodic conditions. Then, by using a numerical approach for the Caputo derivative, we investigate behaviors of the considered problem.

MSC: 30C45; 39B72; 39B82

Keywords: Lane–Emden equation; Caputo derivative; Random differential equation; Ulam stability; Fourth Runge–Kutta method

1 Introduction

The theory of singular fractional boundary value problems has become an area of research investigation in the last three decades (see [1, 3, 6, 7, 16, 21]). One of the equations describing this type of problems is the very important Lane–Emden equation, which was published by Lane in 1870 [18] and detailed by Emden [8]. Lane–Emden differential equations are singular initial value problems of the second order, they describe a variety of phenomena in mathematical physics and astrophysics such as aspects of the stellar structure. For more information and some applications, one can consult Refs. [2, 13, 23].

The classical Lane–Emden equation has the following form [5, 8]:

$$x''(t) + \frac{a}{t}x'(t) + f\big(t, x(t)\big) = g(t), \quad t \in [0,1],$$

under the conditions

$$x(0) = A, \qquad x'(0) = B,$$

where A and B are constants and f and g are continuous real functions.

The above problem has attracted many researchers attention. In fact, in [20], the authors have used the method of collocation to study the following Lane–Emden problem:

$$\begin{cases} D^{\alpha}y(t) + \frac{k}{t^{\alpha-\beta}}D^{\beta}y(t) + f(t,y(t)) = g(t), \quad t \in [0,1], \\ k \geq 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \end{cases}$$

Ibrahim [15] has been concerned with the stability of Ulam Hyers for the following fractional Lane–Emden problem:

$$\begin{cases} D^{\beta}(D^{\alpha} + \frac{a}{t})u(t) + f(t,u(t)) = g(t), \\ u(0) = \mu, \quad u(1) = \nu, \\ 0 < \alpha, \beta \leq 1, \quad 0 \leq t \leq 1, \quad a \geq 0, \end{cases}$$

under the conditions: D^{γ} is the Caputo derivative, f is a continuous function and $g \in C([0,1])$.

Very recently, Y. Gouari et al. [10] have investigated the following nonlocal fractional problem of Lane–Emden type:

$$\begin{cases} D^{\beta}(D^{\alpha} + \frac{k}{t^{\lambda}})y(t) + \Delta_1 f(t,y(t),D^{\delta}y(t)) + \Delta_2 g(t,y(t),I^{\rho}y(t)) + h(t,y(t)) = l(t), \\ y(0) = 0, \quad y(1) = b\int_0^{\eta} y(s)\,ds, \quad 0 < \eta < 1, \quad I^{q}y(u) = y(1), \quad 0 < u < 1, \\ k > 0, \quad 0 < \lambda \leq 1, \quad 1 \leq \beta \leq 2, \quad 0 \leq \alpha, \delta \leq 1, \quad t \in]0,1[, \end{cases}$$

Motivated by the above cited papers, in [25] we have proved the existence and uniqueness of solutions by application of the Banach contraction principle for the following anti periodic fractional differential problem:

$$\begin{cases} D^{\alpha}D^{\beta}y(t) + \frac{k}{t^{\lambda}}D^{\alpha}y(t) + a_1F(t,y(t),D^{\gamma}y(t),J^{p}y(t)) \\ \quad + a_2G(t,y(t),D^{\gamma}y(t)) + a_3H(t,y(t)) = L(t). \\ y(0) + y(1) = 0, \quad y'(0) + y'(1) = 0, \quad D^{\gamma}(0) + D^{\gamma}(1) = 0, \\ k > 0, \quad 1 \leq \beta \leq 2, \quad 0 \leq \gamma \leq \alpha \leq 1, \quad 0 < \lambda < 1, \quad p > 0, \quad t \in [0,1], \end{cases} \tag{1}$$

where $I := [0,1]$, the derivatives of the problem are in the sense of Caputo, J^{p} denotes the Riemann–Liouville integral of order p and $F : I \times \mathbb{R}^3 \to \mathbb{R}$, $G : I \times \mathbb{R}^2 \to \mathbb{R}$, $H : I \times \mathbb{R}^2 \to \mathbb{R}$ and $L : I \to \mathbb{R}$ are four given functions, and $\mathbb{R}$ being the set of real numbers.

In this work, we continue studying the above problem by investigating certain types of Ulam stability for the problem (1). Then, using a numerical approach of the derivative Caputo, we analyze certain behavior of the problem by means of the fourth-order Runge–Kutta integrator method.

2 Preliminaries

We present some necessary lemmas and theorems which will be used in this paper.

As it is proved in our last work [25], the integral solution of (1) is given by the following auxiliary result.

Lemma 1 *Let $L_1 \in C([0,1])$, $t \in I$, $0 \le \gamma \le \alpha \le 1$, $1 < \beta < 2$. Then the integral solution of the problem*

$$\begin{cases} D^{\alpha}(D^{\beta})y(t) + (\frac{k}{t^{\lambda}})D^{\alpha}y(t) = L_1(t), \\ y(0) + y(1) = 0, \qquad y'(0) + y'(1) = 0, \qquad D^{\gamma}(0) + D^{\gamma}(1) = 0, \end{cases} \tag{2}$$

is given by the following expression:

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \left[L_1(u) - \frac{k}{u^{\lambda}} D^{\alpha} y(u)\right] du\, ds \\ &\quad + \left[K_1 t^{\beta} + K_2 t + K_3\right]\left[\int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \left[L_1(u) - \frac{k}{u^{\lambda}} D^{\alpha} y(u)\right] du\, ds\right] \\ &\quad + \left[K_4 t^{\beta} + K_5 t - K_6\right]\left[\int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \left[L_1(u) - \frac{k}{u^{\lambda}} D^{\alpha} y(u)\right] du\, ds\right] \\ &\quad - [K_7]\left[\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \left[L_1(u) - \frac{k}{u^{\lambda}} D^{\alpha} y(u)\right] du\, ds\right], \end{aligned} \tag{3}$$

where

$$\begin{aligned} L_1(u) &= L(u) - a_1 F\big(u, y(u), D^{\gamma} y(u), J^{p} y(u)\big) - a_2 G\big(u, y(u), D^{\gamma} y(u)\big) \\ &\quad - a_3 H\big(u, y(u)\big) - \frac{k}{u^{\lambda}} D^{\alpha} y(u) \end{aligned}$$

and

$$\begin{aligned} K_1 &= \frac{\Gamma(\beta-\gamma+1)}{\beta[\Gamma(\beta-\gamma+1) - 2\Gamma(\beta)\Gamma(2-\gamma)]}, \\ K_2 &= \frac{\Gamma(\beta)\Gamma(2-\gamma)}{2\Gamma(\beta)\Gamma(2-\gamma) - \Gamma(\beta-\gamma+1)}, \\ K_3 &= \frac{\Gamma(\beta+1)\Gamma(2-\gamma) - \Gamma(\beta-\gamma+1)}{2\beta\Gamma(\beta-\gamma+1) - 4\Gamma(\beta+1)\Gamma(2-\gamma)}, \\ K_4 &= \frac{2\Gamma(2-\gamma)\Gamma(\beta-\gamma+1)}{\beta[\Gamma(\beta-\gamma+1) - 2\Gamma(\beta)\Gamma(2-\gamma)]}, \\ K_5 &= \frac{\Gamma(2-\gamma)\Gamma(\beta-\gamma+1)}{2\Gamma(\beta)\Gamma(2-\gamma) - \Gamma(\beta-\gamma+1)}, \\ K_6 &= \frac{2\Gamma(2-\gamma)\Gamma(\beta-\gamma+1) - \beta\Gamma(2-\gamma)\Gamma(\beta-\gamma+1)}{2\beta\Gamma(\beta-\gamma+1) - 4\Gamma(\beta+1)\Gamma(2-\gamma)}, \\ K_7 &= \frac{\Gamma(\beta-\gamma+1) - 2\Gamma(\beta)\Gamma(2-\gamma)}{2\Gamma(\beta-\gamma+1) - 4\Gamma(\beta)\Gamma(2-\gamma)}. \end{aligned}$$

Before presenting our main results, we shall introduce also the Banach space

$$X := \big\{y \in C(I,\mathbb{R}), D^{\alpha} y \in C(I,\mathbb{R}), D^{\gamma} y \in C(I,\mathbb{R})\big\},$$

and the norm

$$\|y\|_X = \max\big\{\|y\|_\infty, \|D^{\alpha} y\|_\infty, \|D^{\gamma} y\|_\infty\big\},$$

where

$$\|x\|_\infty = \sup_{t\in I}\big|x(t)\big|, \qquad \big\|D^\alpha x\big\|_\infty = \sup_{t\in I}\big|D^\alpha x(t)\big|, \qquad \big\|D^\gamma x\big\|_\infty = \sup_{t\in I}\big|D^\gamma x(t)\big|.$$

Also, we consider the following hypotheses:

(H1): There exist nonnegative constants W_i, $i = 1,\dots,6$, such that for each $t\in I$ and for all $x_1,x_2,x_3,y_1,y_2,y_3\in\mathbb{R}$ we have

$$\begin{aligned}
&\big|F(t,x_1,x_2,x_3) - F(t,y_1,y_2,y_3)\big| \le W_1|x_1-y_1| + W_2|x_2-y_2| + W_3|x_3-y_3|,\\
&\big|G(t,x_1,x_2) - G(t,y_1,y_2)\big| \le W_4|x_1-y_1| + W_5|x_2-y_2|,\\
&\big|H(t,x_1) - H(t,y_1)\big| \le W_6|x_1-y_1|.
\end{aligned}$$

The following quantities are also needed in this paper:

$$\begin{aligned}
N_1 ={}& (a_1W_{1,2} + a_2W_{4,5} + a_3W_6)\bigg[\frac{1+|K_7|}{\Gamma(\alpha+\beta+1)}\\
&+ \frac{|K_1|+|K_2|+|K_3|}{\Gamma(\alpha+\beta)} + \frac{|K_4|+|K_5|+|K_6|}{\Gamma(\alpha+\beta-\gamma+1)}\bigg]\\
&+ a_1W_3\bigg[\frac{1+|K_7|}{\Gamma(\alpha+\beta+p+1)} + \frac{|K_1|+|K_2|+|K_3|}{\Gamma(\alpha+\beta+p)} + \frac{|K_4|+|K_5|+|K_6|}{\Gamma(\alpha+\beta-\gamma+p+1)}\bigg]\\
&+ |k|\Gamma(1-\lambda)\bigg[\frac{1+|K_7|}{\Gamma(\alpha+\beta-\lambda+1)} + \frac{|K_1|+|K_2|+|K_3|}{\Gamma(\alpha+\beta-\lambda)} + \frac{|K_4|+|K_5|+|K_6|}{\Gamma(\alpha+\beta-\gamma-\lambda+1)}\bigg],\\
N_2 ={}& (a_1W_{1,2} + a_2W_{4,5} + a_3W_6)\bigg[\frac{1+|K_7|}{\Gamma(\beta+1)} + \frac{|K_1|\Gamma(\beta+1)\Gamma(2-\alpha) + |K_2|\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(\beta-\alpha+1)\Gamma(2-\alpha)}\\
&+ \frac{|K_4|\Gamma(\beta+1)\Gamma(2-\alpha) + |K_5|\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\gamma+1)\Gamma(\beta-\alpha+1)\Gamma(2-\alpha)}\bigg]\\
&+ a_1W_3\bigg[\frac{1+|K_7|}{\Gamma(\beta+p+1)} + \frac{|K_1|\Gamma(\beta+1)\Gamma(2-\alpha) + |K_2|\Gamma(\beta-\alpha+1)}{\Gamma(\beta+p)\Gamma(\beta-\alpha+1)\Gamma(2-\alpha)}\\
&+ \frac{|K_4|\Gamma(\beta+1)\Gamma(2-\alpha) + |K_5|\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\gamma+p+1)\Gamma(\beta-\alpha+1)\Gamma(2-\alpha)}\bigg]\\
&+ |k|\Gamma(1-\lambda)\bigg[\frac{1+|K_7|}{\Gamma(\beta-\lambda+1)} + \frac{|K_1|\Gamma(\beta+1)\Gamma(2-\alpha) + |K_2|\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\lambda)\Gamma(\beta-\alpha+1)\Gamma(2-\alpha)}\\
&+ \frac{|K_4|\Gamma(\beta+1)\Gamma(2-\alpha) + |K_5|\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\gamma-\lambda+1)\Gamma(\beta-\alpha+1)\Gamma(2-\alpha)}\bigg],\\
N_3 ={}& (a_1W_{1,2} + a_2W_{4,5} + a_3W_6)\bigg[\frac{1+|K_7|}{\Gamma(\alpha+\beta-\gamma+1)}\\
&+ \frac{|K_1|\Gamma(\beta+1)\Gamma(2-\gamma) + |K_2|\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\beta-\gamma+1)\Gamma(2-\gamma)}\\
&+ \frac{|K_4|\Gamma(\beta+1)\Gamma(2-\gamma) + |K_5|\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-2\gamma+1)\Gamma(\beta-\gamma+1)\Gamma(2-\gamma)}\bigg]\\
&+ a_1W_3\bigg[\frac{1+|K_7|}{\Gamma(\alpha+\beta-\gamma+p+1)} + \frac{|K_1|\Gamma(\beta+1)\Gamma(2-\gamma) + |K_2|\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+p)\Gamma(\beta-\gamma+1)\Gamma(2-\gamma)}
\end{aligned}$$

$$+\frac{|K_4|\Gamma(\beta+1)\Gamma(2-\gamma)+|K_5|\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-2\gamma+p+1)\Gamma(\beta-\gamma+1)\Gamma(2-\gamma)}\Bigg]$$
$$+|k|\Gamma(1-\lambda)\Bigg[\frac{1+|K_7|}{\Gamma(\alpha+\beta-\gamma-\lambda+1)}+\frac{|K_1|\Gamma(\beta+1)\Gamma(2-\gamma)+|K_2|\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma-\lambda)\Gamma(\beta-\gamma+1)\Gamma(2-\gamma)}$$
$$+\frac{|K_4|\Gamma(\beta+1)\Gamma(2-\gamma)+|K_5|\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-2\gamma-\lambda+1)\Gamma(\beta-\gamma+1)\Gamma(2-\gamma)}\Bigg],$$

where $W_{1,2} := \max(W_1, W_2)$ and $W_{4,5} := \max(W_4, W_5)$.

We recall the following result [25], which allows us to study the stability phenomena of the considered problem.

Theorem 2 ([25]) *Assume that* (H1) *holds and suppose that* $0 < N < 1$, *where* $N = \max(N_1, N_2, N_3)$. *Then the problem* (1) *has a unique solution on* I.

3 Ulam type stabilities

The notion of the stability problem of functional equations originated from a problem of Stanislaw Ulam [26], posed in 1940: When can we assert that approximate solution of a functional equation can be approximated by a solution of the corresponding equation. In 1941, Hyers [14] solved it. This approach can guarantee that there exists a close exact solution useful in many applications. For more details on the recent advances on the Hyers–Ulam stability (see for example [9, 11, 24, 27]).

In order to study some types of Ulam stability for the problem (1), we consider the following fractional differential equation:

Let $1 \le \beta \le 2$, $0 \le \gamma \le \alpha \le 1$ and ϵ a positive real numbers and the function $T \in C(I, \mathbb{R}^+)$. We consider the following fractional differential equation:

$$D^\alpha D^\beta y(t) + \frac{k}{t^\lambda} D^\alpha y(t) + a_1 F\big(t, y(t), D^\gamma y(t), J^p y(t)\big) + a_2 G\big(t, y(t), D^\gamma y(t)\big) + a_3 H\big(t, y(t)\big) = L(t), \quad t \in I, \tag{4}$$

and the following fractional differential inequality:

$$\bigg|D^\alpha D^\beta x(t) + \frac{k}{t^\lambda} D^\alpha x(t) + a_1 F\big(t, x(t), D^\gamma x(t), J^p x(t)\big) + a_2 G\big(t, x(t), D^\gamma x(t)\big) + a_3 H\big(t, x(t)\big) - L(t)\bigg| \le \epsilon, \quad t \in I, \tag{5}$$

$$\bigg|D^\alpha D^\beta x(t) + \frac{k}{t^\lambda} D^\alpha x(t) + a_1 F\big(t, x(t), D^\gamma x(t), J^p x(t)\big) + a_2 G\big(t, x(t), D^\gamma x(t)\big) + a_3 H\big(t, x(t)\big) - L(t)\bigg| \le \epsilon T(t), \quad t \in I, \tag{6}$$

$$\bigg|D^\alpha D^\beta x(t) + \frac{k}{t^\lambda} D^\alpha x(t) + a_1 F\big(t, x(t), D^\gamma x(t), J^p x(t)\big) + a_2 G\big(t, x(t), D^\gamma x(t)\big) + a_3 H\big(t, x(t)\big) - L(t)\bigg| \le T(t), \quad t \in I. \tag{7}$$

Definition 3 The problem (1) is Ulam–Hyers stable, if there exists a real number $S > 0$, such that, for each $\epsilon > 0$, $t \in I$, and for each $x \in X$ solution of (5), there exists a solution

$y \in X$ of (4) (with the same conditions as in (1)), such that

$$\|x - y\|_X \le S\epsilon, \quad t \in I.$$

Definition 4 The problem (1) is generalized Ulam–Hyers stable, if there exists an increasing function $Z \in C(\mathbb{R}^+, \mathbb{R}^+)$, $Z(0) = 0$, such that, for all $\epsilon > 0$, and for each solution $x \in X$ of (5), there exists a solution $y \in X$ of (4) (with the same conditions as in (1)), such that

$$\|x - y\|_X \le Z(\epsilon), \quad t \in I.$$

Definition 5 The problem (1) is Ulam–Hyers–Rassias stable, if there exists a function $T \in C(\mathbb{I}, \mathbb{R}^+)$ and $\sigma > 0$, such that for each $\epsilon > 0$ and for all solutions $x \in X$ of (6) there exists a solution $y \in X$ of (4) (with the same conditions as in (1)), such that

$$\left|x(t) - y(t)\right| \le \sigma\epsilon T(t), \quad t \in I.$$

Definition 6 The problem (1) is generalized Ulam–Hyers–Rassias stable, if there exists a function $T \in C(\mathbb{I}, \mathbb{R}^+)$ and $\sigma > 0$, such that for all solutions $x \in X$ of (7) there exists a solution $y \in X$ of (4) (with the same conditions as in (1)), such that

$$\left|x(t) - y(t)\right| \le \sigma T(t), \quad t \in I.$$

Now, we are ready to prove the following result.

Theorem 7 *Assume that* (H1) *is fulfilled and* $N = \max(N_1, N_2, N_3) < 1$. *Then the problem* (1) *is Ulam–Hyers stable in* X.

Proof Let us note

$$\begin{aligned}
O = \bigg| &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \bigg[L(s) - a_1 F\big(s, x(s), D^\gamma x(s), J^p x(s)\big) \\
&- a_2 G\big(s, x(s), D^\gamma x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^\lambda} D^\alpha x(s) \bigg] \,du\,ds \\
&+ \big[K_1 t^\beta + K_2 t + K_3\big] \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times \bigg[L(s) - a_1 F\big(s, x(s), D^\gamma x(s), J^p x(s)\big) \\
&- a_2 G\big(s, x(s), D^\gamma x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^\lambda} D^\alpha x(s) \bigg] \,du\,ds \\
&+ \big[K_4 t^\beta + K_5 t - K_6\big] \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times \bigg[L(s) - a_1 F\big(s, x(s), D^\gamma x(s), J^p x(s)\big) \\
&- a_2 G\big(s, x(s), D^\gamma x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^\lambda} D^\alpha x(s) \bigg] \,du\,ds
\end{aligned}$$

$$
\begin{aligned}
&- [K_7] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, x(s), D^\gamma x(s), J^p x(s)\bigr) \\
&- a_2 G\bigl(s, x(s), D^\gamma x(s)\bigr) - a_3 H\bigl(s, x(s)\bigr) - \frac{k}{s^\lambda} D^\alpha x(s)\biggr]\, du\, ds \\
&- \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, y(s), D^\gamma y(s), J^p y(s)\bigr) \\
&- a_2 G\bigl(s, y(s), D^\gamma y(s)\bigr) - a_3 H\bigl(s, y(s)\bigr) - \frac{k}{s^\lambda} D^\alpha y(s)\biggr]\, du\, ds \\
&- \bigl[K_1 t^\beta + K_2 t + K_3\bigr] \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, y(s), D^\gamma y(s), J^p y(s)\bigr) \\
&- a_2 G\bigl(s, y(s), D^\gamma y(s)\bigr) - a_3 H\bigl(s, y(s)\bigr) - \frac{k}{s^\lambda} D^\alpha y(s)\biggr]\, du\, ds \\
&- \bigl[K_4 t^\beta + K_5 t - K_6\bigr] \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times \biggl[L(s) - a_1 F\bigl(s, y(s), D^\gamma y(s), J^p y(s)\bigr) \\
&- a_2 G\bigl(s, y(s), D^\gamma y(s)\bigr) - a_3 H\bigl(s, y(s)\bigr) - \frac{k}{s^\lambda} D^\alpha y(s)\biggr]\, du\, ds \\
&+ [K_7] \times \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, y(s), D^\gamma y(s), J^p y(s)\bigr) \\
&- a_2 G\bigl(s, y(s), D^\gamma y(s)\bigr) - a_3 H\bigl(s, y(s)\bigr) - \frac{k}{s^\lambda} D^\alpha y(s)\biggr]\, du\, ds \biggr|,
\end{aligned}
$$

$$
\begin{aligned}
M_1 = {}& \biggl| x(t) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, x(s), D^\gamma x(s), J^p x(s)\bigr) \\
&- a_2 G\bigl(u, x(s), D^\gamma x(s)\bigr) - a_3 H\bigl(s, x(s)\bigr) - \frac{k}{s^\lambda} D^\alpha x(s)\biggr]\, du\, ds \\
&- \bigl[K_1 t^\beta + K_2 t + K_3\bigr] \\
&\times \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, x(s), D^\gamma x(s), J^p x(s)\bigr) \\
&- a_2 G\bigl(u, x(s), D^\gamma x(s)\bigr) - a_3 H\bigl(s, x(s)\bigr) - \frac{k}{s^\lambda} D^\alpha x(s)\biggr]\, du\, ds \\
&- \bigl[K_4 t^\beta + K_5 t - K_6\bigr] \\
&\times \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, x(s), D^\gamma x(s), J^p x(s)\bigr) \\
&- a_2 G\bigl(u, x(s), D^\gamma x(s)\bigr) - a_3 H\bigl(s, x(s)\bigr) - \frac{k}{s^\lambda} D^\alpha x(s)\biggr]\, du\, ds \\
&+ [K_7] \\
&\times \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \biggl[L(s) - a_1 F\bigl(s, x(s), D^\gamma x(s), J^p x(s)\bigr) \\
&- a_2 G\bigl(u, x(s), D^\gamma x(s)\bigr) - a_3 H\bigl(s, x(s)\bigr) - \frac{k}{s^\lambda} D^\alpha x(s)\biggr]\, du\, ds \biggr|,
\end{aligned}
$$

$$\begin{aligned}
M_2 = \Bigg| & D^{\alpha}x(t) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(u, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \\
& - \left[\frac{K_1 \Gamma(\beta+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} + \frac{K_2 t^{1-\alpha}}{\Gamma(2-\alpha)} \right] \\
& \times \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(s, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \\
& - \left[\frac{K_4 \Gamma(\beta+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} + \frac{K_5 t^{1-\alpha}}{\Gamma(2-\alpha)} \right] \\
& \times \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(s, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \\
& + [K_7] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(s, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \Bigg|, \\
M_3 = \Bigg| & D^{\gamma}x(t) - \int_0^t \frac{(t-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta-\gamma)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) - a_2 \\
& \times G\big(u, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \\
& - \left[\frac{K_1 \Gamma(\beta+1) t^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} + \frac{K_2 t^{1-\gamma}}{\Gamma(2-\gamma)} \right] \\
& \times \int_0^1 \frac{(1-s)^{\alpha+\beta-\gamma-2}}{\Gamma(\alpha+\beta-\gamma-1)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(s, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \\
& - \left[\frac{K_4 \Gamma(\beta+1) t^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} + \frac{K_5 t^{1-\gamma}}{\Gamma(2-\gamma)} \right] \\
& \times \int_0^1 \frac{(1-s)^{\alpha+\beta-2\gamma-1}}{\Gamma(\alpha+\beta-2\gamma)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(s, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \\
& + [K_7] \int_0^t \frac{(t-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta-\gamma)} \bigg[L(s) - a_1 F\big(s, x(s), D^{\gamma}x(s), J^p x(s)\big) \\
& - a_2 G\big(s, x(s), D^{\gamma}x(s)\big) - a_3 H\big(s, x(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha}x(s) \bigg] ds \Bigg|.
\end{aligned}$$

Let now $x \in X$ be a solution of (5). Then, by integrating (5), we obtain

$$M_1 \leq \frac{\epsilon t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Thanks to Theorem 2, the unique solution of (1) is given by

$$
\begin{aligned}
y(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \Big[L(s) - a_1 F\big(s, y(s), D^{\gamma} y(s), J^{p} y(s)\big) \\
& - a_2 G\big(s, y(s), D^{\gamma} y(s)\big) - a_3 H\big(s, y(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha} y(s) \Big] \, du \, ds \\
& + \big[K_1 t^{\beta} + K_2 t + K_3\big] \\
& \times \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \Big[L(s) - a_1 F\big(s, y(s), D^{\gamma} y(s), J^{p} y(s)\big) \\
& - a_2 G\big(s, y(s), D^{\gamma} y(s)\big) - a_3 H\big(s, y(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha} y(s) \Big] \, du \, ds \\
& + \big[K_4 t^{\beta} + K_5 t - K_6\big] \\
& \times \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \Big[L(s) - a_1 F\big(s, y(s), D^{\gamma} y(s), J^{p} y(s)\big) \\
& - a_2 G\big(s, y(s), D^{\gamma} y(s)\big) - a_3 H\big(s, y(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha} y(s) \Big] \, du \, ds \\
& - [K_7] \times \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \Big[L(s) - a_1 F\big(s, y(s), D^{\gamma} y(s), J^{p} y(s)\big) \\
& - a_2 G\big(s, y(s), D^{\gamma} y(s)\big) - a_3 H\big(s, y(s)\big) - \frac{k}{s^{\lambda}} D^{\alpha} y(s) \Big] \, du \, ds.
\end{aligned}
$$

Then, from all $t \in I$, we get

$$\big|x(t) - y(t)\big| \leq \frac{\epsilon t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + O.$$

This implies that

$$\|x - y\|_{\infty} \leq \frac{\epsilon}{\Gamma(\alpha+\beta+1)} + N_1 \|x - y\|_X. \tag{8}$$

By integrating and differentiating (5), we get

$$M_2 \leq \frac{\epsilon t^{\beta}}{\Gamma(\beta+1)}.$$

Similarly, we show that

$$\big\|D^{\alpha} x - D^{\alpha} y\big\|_{\infty} \leq \frac{\epsilon}{\Gamma(\beta+1)} + N_2 \|x - y\|_X. \tag{9}$$

On the other hand, we have

$$M_3 \leq \frac{\epsilon t^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}.$$

Also, we have

$$\big\|D^{\gamma} x - D^{\gamma} y\big\|_{\infty} \leq \frac{\epsilon}{\Gamma(\alpha+\beta-\gamma+1)} + N_3 \|x - y\|_X. \tag{10}$$

Using the inequalities (8), (9) and (10), we get

$$\|x-y\|_X \leq \max\left(\frac{\epsilon}{\Gamma(\alpha+\beta+1)}, \frac{\epsilon}{\Gamma(\beta+1)}, \frac{\epsilon}{\Gamma(\alpha+\beta-\gamma+1)}\right) + N\|x-y\|_X.$$

Thus,

$$\|x-y\|_X \leq S\epsilon,$$

such that

$$S = \frac{\max(\frac{1}{\Gamma(\alpha+\beta+1)}, \frac{1}{\Gamma(\beta+1)}, \frac{1}{\Gamma(\alpha+\beta-\gamma+1)})}{1-N} > 0.$$

Consequently, the problem (1) shows the Ulam–Hyers stability. □

Taking $Z(\epsilon) = S\epsilon$, we can state that the problem (1) is generalized Ulam–Hyers stable.

In the following, we introduce the following hypothesis to study Rassias stability.

(H2): $T \in C(\mathbb{I}, \mathbb{R}^+)$ is continuous, nondecreasing function, and there exists $\lambda_{T,\alpha} > 0$ such that $J^\alpha T(t) \leq \lambda_{T,\alpha} T(t)$ for each $t \in I$.

We present the following result.

Theorem 8 *Assume that* (H1)–(H2) *are satisfied and* $N := \max(N_1, N_2, N_3) < 1$. *Then the problem* (1) *is Ulam–Hyers–Rassias stable in* X.

Proof Let $x \in X$ be a solution of (6). Then, by integrating (6), we obtain

$$M_1 \leq \epsilon J^\beta J^\alpha T(t).$$

Let y be the unique solution of the problem (1). Then, for each $t \in I$, we have

$$\left|x(t)-y(t)\right| \leq \epsilon J^\beta J^\alpha T(t) + O.$$

In view of (H2), we have

$$\left|x(t)-y(t)\right| \leq \epsilon J^\beta J^\alpha T(t) + N_1\|x-y\|_X \leq \epsilon\lambda_{T,\beta+\alpha} T(t) + N_1\|x-y\|_X,$$

$$\text{which implies that } \left|x(t)-y(t)\right| \leq \epsilon\lambda_{T,\beta+\alpha} T(t) + N_1\|x-y\|_X. \tag{11}$$

On the other hand, by integrating and differentiating (6), we get

$$M_2 \leq \epsilon J^\beta T(t).$$

Also, we can show that

$$\left|D^\alpha x - D^\alpha y\right| \leq \epsilon\lambda_{T,\beta} T(t) + N_2\|x-y\|_X. \tag{12}$$

We have also

$$M_3 \leq \epsilon J^{\alpha+\beta-\gamma} T(t).$$

By the same arguments as before, we observe that

$$\left|D^{\gamma}x(t)-D^{\gamma}y(t)\right|\leq\epsilon\lambda_{T,\alpha+\beta-\gamma}T(t)+N_3\|x-y\|_X. \tag{13}$$

Using the inequalities (11), (12) and (13) yields

$$\begin{cases}|x(t)-y(t)|\leq\epsilon\max(\lambda_{T,\alpha+\beta},\lambda_{T,\beta},\lambda_{T,\alpha+\beta-\gamma})T(t)+N_1\|x-y\|_X, & t\in I,\\ |D^{\alpha}x(t)-D^{\alpha}y(t)|\leq\epsilon\max(\lambda_{T,\alpha+\beta},\lambda_{T,\beta},\lambda_{T,\alpha+\beta-\gamma})T(t)+N_2\|x-y\|_X, & t\in I,\\ |D^{\gamma}x(t)-D^{\gamma}y(t)|\leq\epsilon\max(\lambda_{T,\alpha+\beta},\lambda_{T,\beta},\lambda_{T,\alpha+\beta-\gamma})T(t)+N_3\|x-y\|_X, & t\in I.\end{cases}$$

Hence, it follows that there exists a real number

$$\sigma=\frac{\max(\lambda_{T,\alpha+\beta},\lambda_{T,\beta},\lambda_{T,\alpha+\beta-\gamma})}{1-N},$$

such that

$$\|x-y\|_X\leq\sigma\epsilon T(t),\quad t\in I.$$

Consequently, the problem (1) shows the Ulam–Hyers–Rassias stability. □

4 Numerical simulations

In this section, we recall a numerical approach for the Caputo derivative. Then, for some fixed parameters, we investigate behavior of the above fractional Lane–Emden problem. To do this, we shall first obtain a reduced fractional differential system that is equivalent to our studied problem. Using a fourth-order Runge–Kutta integrator, the numerical simulations recover the convective behavior of the integer model in astrophysics [4]. In order to ensure the effect of the fractional order in Lane–Emden dynamics, we consider judicious values for α and β.

- Hydrodynamic simulations of giant stars, where the stellar profiles can be modeled in [12, 22, 28] as

  $$\frac{1}{t}\frac{d}{dt}\left(t^2\frac{dy}{dt}+t^2\frac{g_c(at)}{4a\pi Gp_0}\right)+y^n=0,$$

  where y is the polytropic temperature with index n, $t\equiv\frac{r}{a}$, and p_0 the central gas density. For $r\leq\frac{h}{2}$ and $x\equiv\frac{r}{h}$, the smoothed gravitational force of the core is defined by

  $$g_c(r):=Gm_c\frac{x(\frac{32}{3}+x^2(\frac{-192}{5}+32x))}{h^2}.$$

- Self-similar profiles of nonlinear wave equations in flat space-time were modeled in [4, 17] as

  $$\left(1-t^2\right)\frac{d^2y}{dt^2}+\left(\frac{A}{t}+Bt\right)\frac{dy}{dt}-Cy+Dy^E=0.$$

4.1 Numerical approach for Caputo derivative

In this subsection, we presented an important numerical approach for the Riemann–Liouville fractional integral and the Caputo derivative; we recall the theorems of [6, 19].

Theorem 9 *Assume that* $y \in \mathcal{C}^1([0,1],\mathbb{R})$. *The fractional integration approach is given by*

$$J^{\alpha}y(t_i) \simeq \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{i} y(t_j)\sigma_j(\alpha), \quad i = 0,\dots,n+1,$$

where

$$\sigma_j(\alpha) = \begin{cases} (n+2-j)^{(\alpha+1)} + (n-j)^{(\alpha+1)} - 2(n-j+1)^{(\alpha+1)}, & j = 1,\dots,i-1, \\ (n)^{(\alpha+1)} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \textit{and } 1, j = i. \end{cases}$$

Theorem 10 *Assume that* $y \in \mathcal{C}^1([0,1],\mathbb{R})$ *and* $0 < \alpha \leq 1$. *Then we have*

$$D^{\alpha}y(t_i) \simeq \frac{h^{1-\alpha}}{\Gamma(1-\alpha+2)}\sum_{j=0}^{i} y^{(j)}(t_j)\sigma_j(1-\alpha), \quad i = 0,\dots,n,$$

where

$$y^{(j)} = \begin{cases} \frac{y_1-y_0}{h}, \quad j = 0, \qquad \frac{y_{j+1}-y_{j-1}}{2h}, \quad j = 1,\dots,i-1, \qquad \frac{y_i-y_{i-1}}{h}, \quad j = i. \end{cases}$$

4.2 Simulation for Lane–Emden behaviors

We note that the problem (1) can be reduced to the following system:

$$\begin{aligned} &D^{\beta}y(t) = z(t), \\ &D^{\alpha}z(t) = -\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr) \\ &\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t). \end{aligned}$$

In order to achieve the mentioned phenomena, we take $1 < \alpha + \beta \leq 2$, and $\lambda = 1$. Taking into account our problem parameters, three cases can be observed:

Case 1: $\alpha = \beta = 1$, we get

$$\begin{aligned} &Dy(t) = z(t), \\ &Dz(t) = -\frac{k}{t^{\lambda}}Dy(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr) \\ &\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t). \end{aligned}$$

Case 2: $0 < \alpha \leq 1$, $\beta = 1$, we obtain

$$\begin{aligned} &Dy(t) = z(t), \\ &D^{\alpha}z(t) = -\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr) \\ &\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t). \end{aligned}$$

As a consequence,

$$\begin{aligned}
&Dy(t) = z(t),\\
&Dz(t) = D^{1-\alpha}\biggl(-\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr)\\
&\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t)\biggr).
\end{aligned}$$

Case 3: $0 < \alpha \le 1$, $1 \le \beta \le 2$, we have

$$\begin{aligned}
&D^{\beta}y(t) = z(t),\\
&D^{\alpha}z(t) = -\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr)\\
&\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t),
\end{aligned}$$

that is,

$$\begin{aligned}
&J^{2-\beta}D\bigl[Dy(t)\bigr] = z(t),\\
&D^{\alpha}z(t) = -\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr)\\
&\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Dy(t) = z(t),\\
&J^{2-\beta}Dz = w(t),\\
&D^{\alpha}w(t) = -\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr)\\
&\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&Dy(t) = z(t),\\
&Dz(t) = D^{2-\beta}w(t),\\
&Dw(t) = D^{1-\alpha}\biggl(-\frac{k}{t^{\lambda}}D^{\alpha}y(t) - a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr)\\
&\qquad - a_2G\bigl(t,y(t),D^{\gamma}y(t)\bigr) - a_3H\bigl(t,y(t)\bigr) + L(t)\biggr).
\end{aligned}$$

I: As a first simulation, we consider the hydrodynamic simulations of giant stars, where $k = 2$, $p = \gamma = 0.01$, and f, H, G, H, L are given by

$$a_1F\bigl(t,y(t),D^{\gamma}y(t),J^{p}y(t)\bigr) = \frac{16a^4m_c}{\pi p_0h^6}t^4 + \frac{289}{51t}\bigl(J^{p}y(t)\bigr)^{n},$$

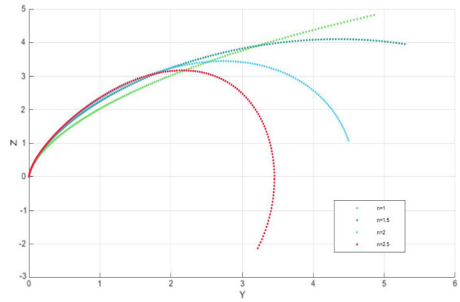


Figure 1 Numerical simulation of Case 1 for different values of the polytropic index n and $\alpha = \beta = 1$

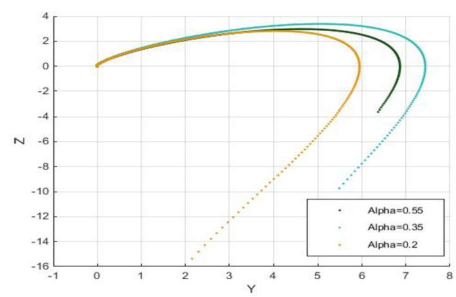


Figure 2 Numerical simulations of Case 2 for $\alpha = \{0.55, 0.35, 0.2\}$ and $\beta = 1$

$$a_2 G\big(t, y(t), D^{\gamma} y(t)\big) = -\frac{48a^3 m_c}{\pi p_0 h^5} t^3 - \frac{663}{255t}\big(D^{\gamma} y(t)\big)^n,$$

$$a_3 H\big(t, y(t)\big) = \frac{32a^4 m_c}{\pi p_0 h^6} t^4 - \frac{527}{255t}\big(y(t)\big)^n,$$

$$L(t) = \frac{8am_c}{\pi p_0 h^3} t.$$

For the first case, with initial conditions $(0,0)$, $h = 0.001$, and $n = \{1, 1.5, 2, 2.5\}$, the numerical simulations are carried out only by the fourth-order Runge–Kutta method, for specific parameters, we have Fig. 1.

Remark 11 Through ongoing evaluation, we observe that the change in value of n has no impact on the attitude of the remaining cases.

For the following simulation we take $n = 1.5$ as it is more adequate. Now, to ensure that all three cases are convenient, we should be looking for a suitable fractional order.

For the second case, with initial conditions $(0,0)$, $h = 0.001$, and $\alpha = \{0.55, 0.35, 0.2\}$, numerical simulations are realized by a combination of the Caputo approach and the fourth-order Runge–Kutta method, we acquire Fig. 2. By comparing the above result with the one of the first case, we conclude that both cases are adequate for $\alpha = 0.35$ (see Fig. 3).

For the third case, with initial conditions $(0,0,0.5)$, $h = 0.001$, $\beta = \{1.45, 1.3, 1.05\}$, for any β value, we take $\alpha = \{0.55, 0.35, 0.2\}$. Numerical simulations are carried out by a combination of the Caputo approach and the fourth-order Runge–Kutta method, we see, according to Figs. 4–6, that $\beta = 1.3$ is the valid value. It is obvious from Fig. 7 that $\alpha = 0.35$ is the appropriate value.

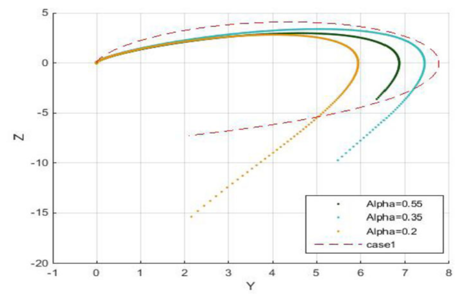


Figure 3 Comparative simulation (Case 1–Case 2)

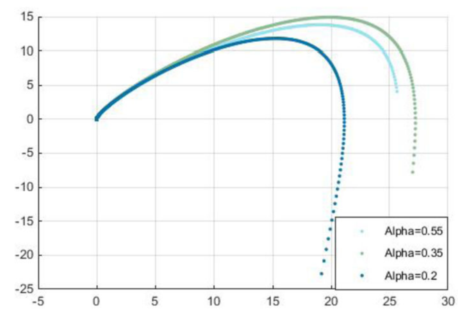


Figure 4 Numerical simulations of Case 3 for $\beta = 1.45$ and $\alpha = \{0.55, 0.35, 0.2\}$

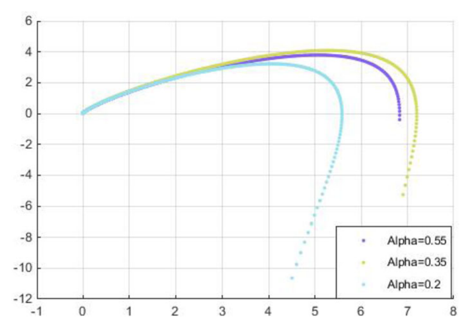


Figure 5 Numerical simulations of Case 3 for $\beta = 1.3$ and $\alpha = \{0.55, 0.35, 0.2\}$

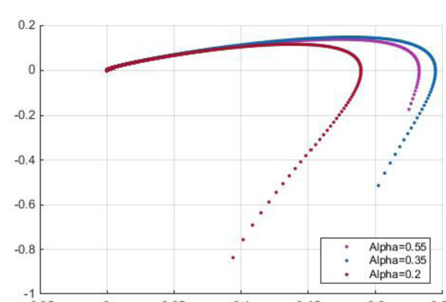


Figure 6 Numerical simulations of Case 3 for $\beta = 1.05$ and $\alpha = \{0.55, 0.35, 0.2\}$

II: As a second simulation, we consider self-similar profiles of nonlinear wave equation in flat space-time, where $k = A$, $p = \gamma = 0.01$, and f, H, G, H, L are given by

$$a_1 F\big(t, y(t), D^{\gamma} y(t), J^{p} y(t)\big) = \frac{C}{1 - t^2} J^{p} y(t),$$

Figure 7 Comparative simulation (Case 1–Case 3)

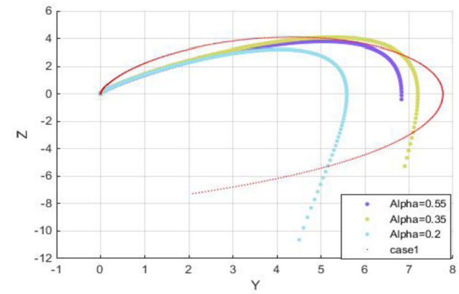


Figure 8 Numerical simulation of Case 1

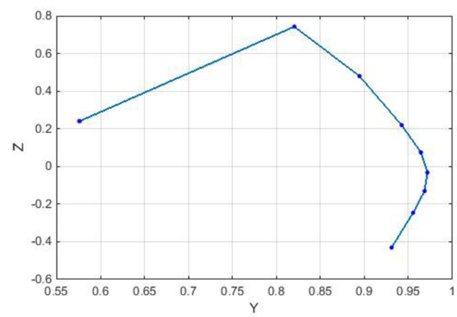
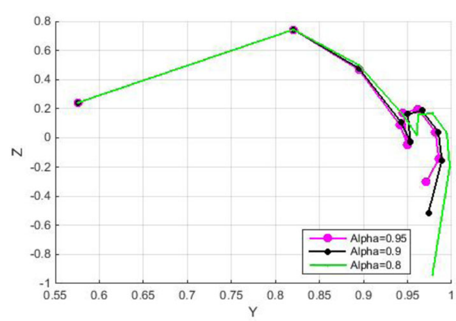


Figure 9 Numerical simulation of Case 2



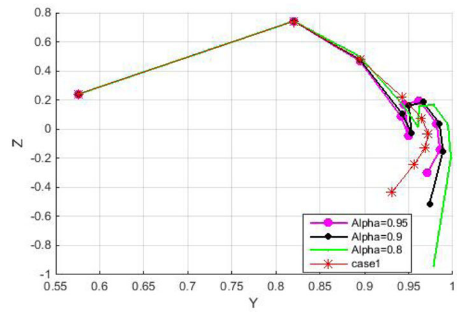
$$
\begin{aligned}
&a_2 G\big(t, y(t), D^\gamma y(t)\big) = -\frac{Bt}{1-t^2} D^\gamma y(t), \\
&a_3 H\big(t, y(t)\big) = -\frac{D}{1-t^2}\big(y(t)\big)^E, \\
&L(t) = 0,
\end{aligned}
$$

with initial conditions $(0.576037116, 0.24090)$, and $A = 2$, $B = \frac{-25}{12}$, $C = \frac{1}{4}$, $D = 1$, $E = 2$, $h = 0.01$. The integration for the first case is carried out by the fourth-order Runge–Kutta method, now, we are trying to determine an appropriate fractional order (see Fig. 8).

For the second case, we take the same data as above, and $\alpha = \{0.95, 0.9, 0.8\}$. Numerical simulations are realized by a combination of the Caputo approach and the fourth-order Runge–Kutta method (see Fig. 9).

Comparing our outcome to that in the first case, we summarize that the two cases are consistent in terms of $\alpha = 0.95$ (see Fig. 10).

Figure 10 Comparative simulation (Case 1–Case 2)



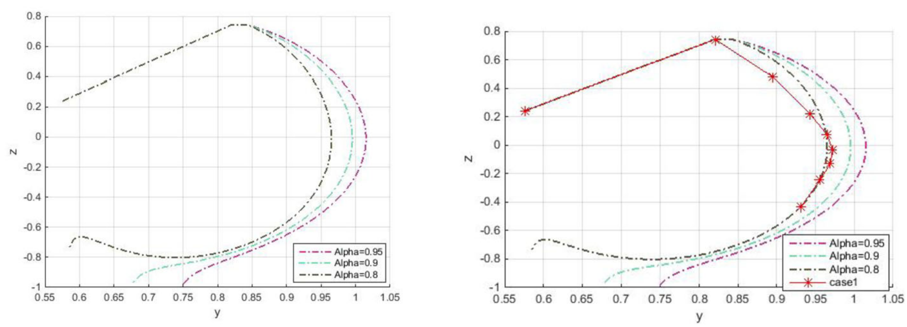


Figure 11 Numerical simulations of Case 3 for $\beta = 1.2$ and different values of α, on the left side. Comparative simulation (Case 1–Case 3), on the right side

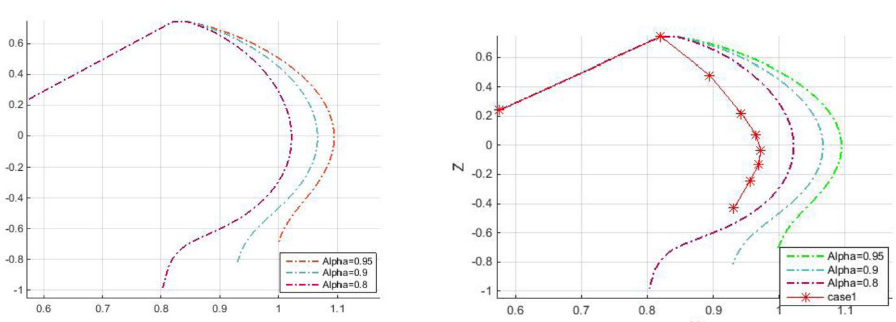


Figure 12 Numerical simulations of Case 3 for $\beta = 1.15$ and different values of α, on the left side. Comparative simulation (Case 1–Case 3), on the right side

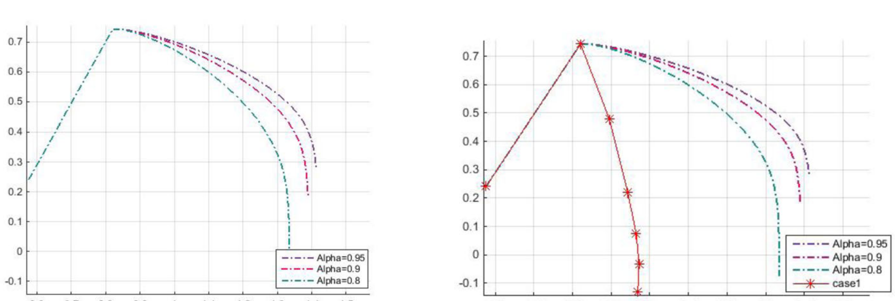


Figure 13 Numerical simulations of Case 3 for $\beta = 1.05$ and different values of α, on the left side. Comparative simulation (Case 1–Case 3), on the right side

For the third case, with initial conditions $(0.576037116, 0.24090, 0)$, and $h = 0.01$, $\beta = \{1.2, 1.15, 1.05\}$, for each β, we take $\alpha = \{0.95, 0.9, 0.8\}$. Numerical simulations are carried out by a combination of the Caputo approach and the fourth-order Runge–Kutta method.

It appeared from Figs. 11–13 that $\beta = 1.2$ and $\alpha = 0.8$ are the most acceptable values too.

5 Conclusions

In this manuscript, we study some types of Ulam stability for a nonlinear fractional differential equation of Lane–Emden type with antiperiodic conditions. Then, by using a numerical approach for the Caputo derivative, we investigate the behaviors of the considered system.

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