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Some new formulas of complete and incomplete degenerate Bell polynomials

Dae San Kim[1], Taekyun Kim[2], Si-Hyeon Lee[2] and Jin-Woo Park[3*]

*Correspondence:
a0417001@daegu.ac.kr
[3]Department of Mathematics Education, Daegu University, Daegu, Republic of Korea
Full list of author information is available at the end of the article

Abstract

The aim of this paper is to study the complete and incomplete degenerate Bell polynomials, which are degenerate versions of the complete and incomplete Bell polynomials, and to derive some properties and identities for those polynomials. In particular, we introduce some new polynomials associated with the incomplete degenerate Bell polynomials. In fact, they are the coefficients of the reciprocal of the power series given by 1 plus the one appearing as the exponent of the generating function of the complete degenerate Bell polynomials.

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1 Introduction

In recent years, we have seen that studying degenerate versions of many special polynomials and numbers, which was initiated in [2] by Carlitz, yielded very fruitful and interesting results. Especially, it is amusing to note that these studies are not just restricted to polynomials but also include transcendental functions like gamma functions.

The aim of this paper is to further study the complete and incomplete degenerate Bell polynomials (see (10), (12)) which are degenerate versions of the complete and incomplete Bell polynomials. In more detail, we deduce recurrence relations for them. As a corollary, we get a recurrence relation for the degenerate Stirling numbers of the second kind. We consider the problem of finding the reciprocal power series of the invertible formal power series which is equal to 1 plus the power series appearing as the exponent of the generating function of the complete degenerate Bell polynomials in (10). This leads us to introduce the new polynomials $T_{n,\lambda}(a_1, a_2, \ldots, a_n)$ (see (21)) that are associated with the incomplete degenerate Bell polynomials. As a corollary, this gives us an expression for the reciprocal of the degenerate exponential function $e_\lambda(at)$. In addition, we obtain some identities regarding the degenerate Stirling numbers. For the rest of this section, we recall the facts that are needed throughout this paper.

For any $\lambda \in \mathbb{R}$, it is known that the degenerate exponential function is defined as

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n \quad \text{(see [2, 6, 8, 12, 14])}, \tag{1}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

In particular, for $x = 1$, we write $e_\lambda(t) = e_\lambda^1(t)$.

In [10], the degenerate Bell polynomials are given by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2}$$

When $x = 1$, $\mathrm{Bel}_{n,\lambda} = \mathrm{Bel}_{n,\lambda}(1)$ are called the degenerate Bell numbers. When $\lambda = 1$, the falling factorial sequence is given by $(x)_0 = 1$, $(x)_n = (x)_{n,1} = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

It is well known that the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l \quad (n \geq 0) \text{ (see [1, 5, 15])}. \tag{3}$$

As the inversion formula of (3), the Stirling numbers of the second kind are defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad (n \geq 0) \text{ (see [3, 9, 12, 15])}. \tag{4}$$

Recently, the degenerate Stirling numbers of the first kind were given by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n \geq 0) \text{ (see [9, 15])}. \tag{5}$$

As the inversion formula of (5), the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0) \text{ (see [6, 7])}. \tag{6}$$

Here we recall that the degenerate Stirling numbers of the first kind and those of the second kind satisfy the orthogonality relations. Namely, they are related by the following:

$$\sum_{l=0}^{n} S_2(n,l) S_1(l,k) = \delta_{n,k}, \qquad \sum_{l=0}^{n} S_1(n,l) S_2(l,k) = \delta_{n,k}, \tag{7}$$

where $\delta_{n,k}$ is Kronecker's delta.

As is well known, the complete Bell polynomials are defined by

$$\exp\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right) = \sum_{n=0}^{\infty} B_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!}, \tag{8}$$

where $B_n(1,1,\ldots,1) = \mathrm{Bel}_n$, $(n \geq 0)$, are the ordinary Bell numbers given by

$$e^{e^t-1} = \sum_{n=0}^{\infty} \mathrm{Bel}_n \frac{t^n}{n!} \quad \text{(see [3, 4])}.$$

For $k \geq 0$, the incomplete Bell polynomials are given by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})\frac{t^n}{n!}, \tag{9}$$

where

$$\begin{aligned} &B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) \\ &\quad= \sum_{\substack{l_1+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}} \frac{n!}{l_1!l_2!\cdots l_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{l_1}\left(\frac{x_2}{2!}\right)^{l_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{l_{n-k+1}}, \\ &B_{n,k}(1,1,\ldots,1) = S_2(n,k) \quad (n \geq k \geq 0) \text{ (see [3, 4])}. \end{aligned}$$

In [10], the complete degenerate Bell polynomials are constructed by

$$\exp\left(\sum_{i=1}^{\infty} (1)_{i,\lambda} x_i \frac{t^i}{i!}\right) = \sum_{n=0}^{\infty} B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}, \tag{10}$$

where

$$B_n^{(\lambda)}(x_1,x_2,\ldots,x_n) = \sum_{l_1+2l_2+\cdots+nl_n=n} \frac{n!}{l_1!l_2!\cdots l_n!}\left(\frac{(1)_{1,\lambda}x_1}{1!}\right)^{l_1}\left(\frac{(1)_{2,\lambda}x_2}{2!}\right)^{l_2}\cdots\left(\frac{(1)_{n,\lambda}x_n}{n!}\right)^{l_n}.$$

Note that

$$B_n^{(\lambda)}(1,1,\ldots,1) = \mathrm{Bel}_{n,\lambda} \quad (n \geq 0) \text{ (see [7, 9–11])}. \tag{11}$$

In the light of (9), the incomplete degenerate Bell polynomials are given by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty} (1)_{i,\lambda} x_i \frac{t^i}{i!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1})\frac{t^n}{n!}, \tag{12}$$

where

$$\begin{aligned} &B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1}) \\ &\quad= \sum_{\substack{l_1+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}} \frac{n!}{l_1!l_2!\cdots (l_{n-k+1})!} \\ &\qquad\times \left(\frac{(1)_{1,\lambda}x_1}{1!}\right)^{l_1}\left(\frac{(1)_{2,\lambda}x_2}{2!}\right)^{l_2}\cdots\left(\frac{(1)_{n-k+1,\lambda}x_{n-k+1}}{(n-k+1)!}\right)^{l_{n-k+1}}, \end{aligned}$$

and

$$B_{0,0}^{(\lambda)}(x_1,x_2,\ldots,x_{n+1})=1,\qquad B_{n,0}^{(\lambda)}(x_1,x_2,\ldots,x_{n+1})=0\quad(n\in\mathbb{N})\text{ (see [11, 13]).}$$

From (12), we note that

$$B_{n,k}^{(\lambda)}(1,1,\ldots,1)=S_{2,\lambda}(n,k)\quad(n,k\geq 0)\text{ (see [11, 13]).}\tag{13}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}&=\exp\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)^k\\&=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}B_{n,k}^{(\lambda)}(x_1,\ldots,x_{n-k+1})\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1})\right)\frac{t^n}{n!}.\end{aligned}\tag{14}$$

By comparing the coefficients on both sides of (14), we get

$$B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)=\sum_{k=0}^{n}B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1})\quad(n\geq 0)\text{ (see [9, 11, 13]).}\tag{15}$$

2 Complete and incomplete degenerate Bell polynomials

From (10), we note that

$$\begin{aligned}\exp\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)&=\sum_{n=0}^{\infty}B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}\\&=1+\sum_{n=1}^{\infty}B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}.\end{aligned}\tag{16}$$

Taking the derivative with respect to t on both sides of (16), we have

$$\begin{aligned}\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^{i-1}}{(i-1)!}\exp\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)&=\sum_{n=1}^{\infty}B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)\frac{t^{n-1}}{(n-1)!}\\&=\sum_{n=0}^{\infty}B_{n+1}^{(\lambda)}(x_1,x_2,\ldots,x_{n+1})\frac{t^n}{n!}.\end{aligned}\tag{17}$$

On the other hand,

$$\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^{i-1}}{(i-1)!}\exp\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)\tag{18}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} (1)_{i+1,\lambda} x_{i+1} \frac{t^i}{i!} \sum_{l=0}^{\infty} B_l^{(\lambda)}(x_1, x_2, \dots, x_l) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} (1)_{i+1,\lambda} x_{i+1} B_{n-i}^{(\lambda)}(x_1, x_2, \dots, x_{n-i}) \right) \frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by (17) and (18), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$
B_{n+1}^{(\lambda)}(x_1, x_2, \dots, x_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} (1)_{i+1,\lambda} x_{i+1} B_{n-i}^{(\lambda)}(x_1, x_2, \dots, x_{n-i}).
$$

For $k, n \in \mathbb{Z}$ with $n \geq k \geq 1$, taking the derivative with respect to t on both sides of (12), we get

$$
\begin{aligned}
&\sum_{n=k}^{\infty} B_{n,k}^{(\lambda)}(x_1, \dots, x_{n-k+1}) \frac{t^{n-1}}{(n-1)!} \\
&= \frac{1}{(k-1)!} \left(\sum_{i=0}^{\infty} (1)_{i,\lambda} x_i \frac{t^i}{i!} \right)^{k-1} \sum_{i=1}^{\infty} (1)_{i,\lambda} x_i \frac{t^{i-1}}{(i-1)!} \\
&= \sum_{l=k-1}^{\infty} B_{l,k-1}^{(\lambda)}(x_1, \dots, x_{l-k+2}) \frac{t^l}{l!} \sum_{i=1}^{\infty} (1)_{i,\lambda} x_i \frac{t^{i-1}}{(i-1)!} \\
&= \sum_{n=k}^{\infty} \sum_{i=1}^{n-k+1} \binom{n-1}{i-1} (1)_{i,\lambda} x_i B_{n-i,k-1}^{(\lambda)}(x_1, \dots, x_{n-i-k+2}) \frac{t^{n-1}}{(n-1)!} \\
&= \sum_{n=k}^{\infty} \sum_{i=0}^{n-k} \binom{n-1}{i} (1)_{i+1,\lambda} x_{i+1} B_{n-1-i,k-1}^{(\lambda)}(x_1, \dots, x_{n-i-k+1}) \frac{t^{n-1}}{(n-1)!}.
\end{aligned} \tag{19}
$$

Therefore, by comparing the coefficients on both sides of (19), we obtain the following theorem.

Theorem 2 *For $k, n \in \mathbb{Z}$ with $n \geq k \geq 1$, we have*

$$
B_{n,k}^{(\lambda)}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{i=0}^{n-k} \binom{n-1}{i} (1)_{i+1,\lambda} x_{i+1} B_{n-1-i,k-1}^{(\lambda)}(x_1, x_2, \dots, x_{n-i-k+1}).
$$

Recalling (13), we obtain the following corollary.

Corollary 3 *For $n, k \in \mathbb{Z}$ with $n \geq k \geq 1$, we have*

$$
S_{2,\lambda}(n,k) = \sum_{i=0}^{n-k} \binom{n-1}{i} (1)_{i+1,\lambda} S_{2,\lambda}(n-1-i, k-1).
$$

We need the following lemma for the next result, which is stated without proof in [3, p. 136].

Lemma 4 *For $n,k \in \mathbb{Z}$ with $n \geq k \geq 1$, we have*

$$B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1}) = \frac{1}{k}\sum_{l=k-1}^{n-1}\binom{n}{l}(1)_{n-l,\lambda}x_{n-l}B_{l,k-1}^{(\lambda)}(x_1,\ldots,x_{l-k+2})$$
$$= \frac{1}{k}\sum_{l=0}^{n-k}\binom{n}{l+1}(1)_{l+1,\lambda}x_{l+1}B_{n-l-1,k-1}^{(\lambda)}(x_1,\ldots,x_{n-l-k+1}).$$

Proof From (12), we see that

$$\sum_{n=k}^{\infty}B_{n,k}^{(\lambda)}(x_1,\ldots,x_{n-k+1})\frac{t^n}{n!} = \frac{1}{k}\frac{1}{(k-1)!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)^{k-1}\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}$$
$$= \frac{1}{k}\sum_{l=k-1}^{\infty}B_{l,k-1}^{(\lambda)}(x_1,\ldots,x_{l-k+2})\frac{t^l}{l!}\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!},$$

from which the first identity follows.

The second identity is deduced from the first by replacing l by $n-1-l$. □

For a given formal power series $\sum_{l=0}^{\infty}(1)_{l,\lambda}a_l\frac{t^l}{l!}$, with $a_0 = 1$, we want to determine the reciprocal power series $\sum_{m=0}^{\infty}(1)_{m,\lambda}b_m\frac{t^m}{m!}$ satisfying $\sum_{l=0}^{\infty}(1)_{l,\lambda}a_l\frac{t^l}{l!}\sum_{m=0}^{\infty}(1)_{m,\lambda}b_m\frac{t^m}{m!} = 1$.

Theorem 5 *Assume $\sum_{l=0}^{\infty}(1)_{l,\lambda}a_l\frac{t^l}{l!}\sum_{m=0}^{\infty}(1)_{m,\lambda}b_m\frac{t^m}{m!} = 1$, with $a_0 = 1$. Then we have*

$$(1)_{0,\lambda}b_0 = 1,$$
$$(1)_{n,\lambda}b_n = \sum_{k=1}^{n}B_{n,k}^{(\lambda)}(a_1,\ldots,a_{n-k+1})(-1)^k k! \quad (n \geq 1). \tag{20}$$

In other words, we have

$$\frac{1}{1+a_1(1)_{1,\lambda}\frac{t}{1!}+a_2(1)_{2,\lambda}\frac{t^2}{2!}+a_3(1)_{3,\lambda}\frac{t^3}{3!}+\cdots} = \sum_{n=0}^{\infty}T_{n,\lambda}(a_1,a_2,\ldots,a_n)\frac{t^n}{n!},$$

where the new polynomials $T_{n,\lambda}(a_1,a_2,\ldots,a_n)$ $(n \geq 0)$, associated with the incomplete degenerate Bell polynomials, are defined by

$$T_{n,\lambda}(a_1,a_2,\ldots,a_n) = \sum_{k=1}^{n}B_{n,k}^{(\lambda)}(a_1,a_2,\ldots,a_{n-k+1})(-1)^k k! \quad (n \geq 1), \tag{21}$$
$$T_{0,\lambda}(a_1,a_2,\ldots,a_n) = 1.$$

Proof We observe first that $1 = \sum_{n=0}^{\infty}(\sum_{l=0}^{n}\binom{n}{l}(1)_{l,\lambda}a_l(1)_{n-l,\lambda}b_{n-l})\frac{t^n}{n!}$. Thus we have

$$(1)_{0,\lambda}b_0 = 1, \qquad \sum_{l=0}^{n}\binom{n}{l}(1)_{l,\lambda}a_l(1)_{n-l,\lambda}b_{n-l} = 0 \quad (n \geq 1). \tag{22}$$

We show the identity in (20) by induction on $n \geq 1$. From (22), we note that

$$(1)_{n,\lambda} b_n = -\sum_{l=1}^{n} \binom{n}{l} (1)_{l,\lambda} a_l (1)_{n-l,\lambda} b_{n-l}. \tag{23}$$

If $n = 1$, we note from (23) that $(1)_{1,\lambda} b_1 = -(1)_{1,\lambda} a_1 = \sum_{k=1}^{1} B_{1,k}^{(\lambda)}(a_1)(-1)^k k!$.

Assume that $n > 1$ and that (20) holds for all positive integers smaller than n. From (23) and Lemma 4, we have

$$\begin{aligned}
(1)_{n,\lambda} b_n &= -\sum_{l=1}^{n} \binom{n}{l} (1)_{l,\lambda} a_l (1)_{n-l,\lambda} b_{n-l} \\
&= -\sum_{l=1}^{n-1} \binom{n}{l} (1)_{l,\lambda} a_l \sum_{k=1}^{n-l} B_{n-l,k}^{(\lambda)}(a_1, \ldots, a_{n-l-k+1})(-1)^k k! - (1)_{n,\lambda} a_n \\
&= -\sum_{k=1}^{n-1} (-1)^k k! \sum_{l=1}^{n-k} \binom{n}{l} (1)_{l,\lambda} a_l B_{n-l,k}^{(\lambda)}(a_1, \ldots, a_{n-l-k+1}) - (1)_{n,\lambda} a_n \\
&= -\sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{l=0}^{n-k} \binom{n}{l+1} (1)_{l+1,\lambda} a_{l+1} B_{n-l-1,k-1}^{(\lambda)}(a_1, \ldots, a_{n-l-k+1}) \\
&= \sum_{k=1}^{n} (-1)^k k! \frac{1}{k} \sum_{l=0}^{n-k} \binom{n}{l+1} (1)_{l+1,\lambda} a_{l+1} B_{n-l-1,k-1}^{(\lambda)}(a_1, \ldots, a_{n-l-k+1}) \\
&= \sum_{k=1}^{n} (-1)^k k! B_{n,k}^{(\lambda)}(a_1, \ldots, a_{n-k+1}).
\end{aligned}$$

□

Letting $a_i = a^i$ for all integers $i \geq 0$, we obtain the following corollary.

Corollary 6 *The following identity holds true*:

$$\begin{aligned}
\frac{1}{e_\lambda(at)} &= \sum_{k=0}^{\infty} T_{k,\lambda}\left(a^1, a^2, \ldots, a^k\right) \frac{t^k}{k!} \\
&= 1 + \sum_{k=1}^{\infty} \left(\sum_{l=1}^{k} B_{k,l}^{(\lambda)}\left(a^1, a^2, \ldots, a^{k-l+1}\right)(-1)^l l! \right) \frac{t^k}{k!}.
\end{aligned}$$

From (6), we can easily derive the following equation:

$$\frac{1}{k!}\left(e_\lambda(t) - 1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!} \quad (k \geq 0). \tag{24}$$

Now, we observe that

$$\begin{aligned}
e_{-\lambda}(t) - 1 = e_\lambda^{-1}(-t) - 1 &= \frac{1}{e_\lambda(-t)}\left(1 - e_\lambda(-t)\right) \\
&= \frac{1 - e_\lambda(-t)}{1 - (1 - e_\lambda(-t))} = \sum_{k=1}^{\infty} \left(1 - e_\lambda(-t)\right)^k
\end{aligned} \tag{25}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}(-1)^k\big(e_\lambda(-t)-1\big)^k = \sum_{k=1}^{\infty}(-1)^k k!\frac{1}{k!}\big(e_\lambda(-t)-1\big)^k \\
&= \sum_{k=1}^{\infty}(-1)^k k!\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)(-1)^n\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^k k!(-1)^n S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.
\end{aligned}
$$

On the other hand

$$
e_{-\lambda}(t)-1 = \sum_{n=0}^{\infty}(1)_{n,-\lambda}\frac{t^n}{n!} - 1 = \sum_{n=1}^{\infty}(1)_{n,-\lambda}\frac{t^n}{n!}. \tag{26}
$$

From (25) and (26), we note that

$$
\sum_{k=1}^{n}(-1)^{n-k}k!S_{2,\lambda}(n,k) = (1)_{n,-\lambda} \quad (n \geq 1). \tag{27}
$$

The rising λ-factorial sequence is defined by

$$
\langle x\rangle_{0,\lambda} = 1, \qquad \langle x\rangle_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots\big(x+(n-1)\lambda\big) \quad (n \geq 1), \text{ (see [7])}. \tag{28}
$$

Therefore, by (27) and (28), we obtain the following theorem, the second of which follows from the orthogonality relations in (7) for the degenerate Stirling numbers.

Theorem 7 *For $n \in \mathbb{N}$, we have*

$$
\sum_{k=1}^{n}(-1)^{n-k}k!S_{2,\lambda}(n,k) = \langle 1\rangle_{n,\lambda}, \qquad \sum_{k=1}^{n}(-1)^{n-k}\langle 1\rangle_{k,\lambda}S_{1,\lambda}(n,k) = n!.
$$

Note that

$$
\begin{aligned}
T_{n,\lambda}(1,1,\dots,1) &= \sum_{k=1}^{n}B_{n,k}^{(\lambda)}(1,1,\dots,1)(-1)^k k! \\
&= \sum_{k=1}^{n}S_{2,\lambda}(n,k)(-1)^k k! \\
&= (-1)^n\sum_{k=1}^{n}S_{2,\lambda}(n,k)(-1)^{n-k}k! \\
&= (-1)^n\langle 1\rangle_{n,\lambda} \quad (n \geq 1).
\end{aligned} \tag{29}
$$

Therefore, by (29), we obtain the following corollary.

Corollary 8 *For $n \in \mathbb{N}$, we have*

$$
T_{n,\lambda}(1,1,\dots,1) = (-1)^n\langle 1\rangle_{n,\lambda}.
$$

3 Conclusion

In recent years, we have seen that degenerate versions of many special polynomials and numbers were investigated by means of various different tools like generating functions, combinatorial methods, umbral calculus techniques, probability theory, p-adic analysis, special functions, analytic number theory and differential equations. Studying them has been rewarding; yielding many interesting results not only in combinatorics and number theory but also in probability, differential equations and symmetric identities. Moreover, they have potential applications in engineering and the sciences.

In this paper, we studied the complete and incomplete degenerate Bell polynomials which are degenerate versions of the complete and incomplete Bell polynomials and obtained some identities and properties as to such polynomials. Above all, we considered the problem of finding the reciprocal power series of the invertible formal power series which is equal to 1 plus the power series appearing as the exponent of the generating function of the complete degenerate Bell polynomials. This led us to introduce the new polynomials $T_{n,\lambda}(a_1, a_2, \ldots, a_n)$ that are associated with the incomplete degenerate Bell polynomials. As a corollary, this gave us an expression for the reciprocal of the degenerate exponential function $e_\lambda(at)$.

It is one of our future projects to continue to explore various degenerate versions of some special polynomials and numbers and to find many applications in mathematics, science and engineering.

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Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; TK and DSK wrote the paper; DSK completed the revision of the article; JWP and SHL checked the errors of the article. All authors have read and agreed to the published version of the manuscript.

Author details
[1]Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea. [3]Department of Mathematics Education, Daegu University, Daegu, Republic of Korea.

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