Alsheekhhussain et al. *Advances in Difference Equations* (2021) 2021:330
https://doi.org/10.1186/s13662-021-03475-w

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Asymptotically periodic behavior of solutions to fractional non-instantaneous impulsive semilinear differential inclusions with sectorial operators

Zainab Alsheekhhussain[1*], JinRong Wang[2] and Ahmed Gamal Ibrahim[3]

*Correspondence:
za.hussain@uoh.edu.sa
[1]Department of Mathematics, Faculty of Science, University of Ha'il, Hail, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this paper, we prove two results concerning the existence of S-asymptotically ω-periodic solutions for non-instantaneous impulsive semilinear differential inclusions of order $1 < \alpha < 2$ and generated by sectorial operators. In the first result, we apply a fixed point theorem for contraction multivalued functions. In the second result, we use a compactness criterion in the space of bounded piecewise continuous functions defined on the unbounded interval $J = [0, \infty)$. We adopt the fractional derivative in the sense of the Caputo derivative. We provide three examples illustrating how the results can be applied.

MSC: Primary 26A33; 34A08; secondary 34A60

Keywords: Differential inclusions of fractional order; Sectorial operators; Asymptotically periodic solutions; Caputo fractional derivative; Non-instantaneous impulses

1 Introduction

Fractional calculus has become a well-established branch of mathematical analysis. It has many applications in engineering and science. Much work has appeared studying various models involving fractional differential boundary value problems and providing solutions to those models using analytical methods or numerical methods. We highlight some recent work involving fractional differential equations. Agarwal et al. [1] investigated existence and uniqueness results on time scales for fractional nonlocal thermistor problems in the conformable sense. Sunarto et al. [2] developed a numerical method using a quarter-sweep and PAOR to solve a one-dimensional time-fractional mathematical physics model. Rezapour et al. [3] showed the existence and uniqueness of solutions for a general multi-term fractional BVP involving the generalized ψ-RL operators. Then they suggested two numerical algorithms, namely, the Dafterdar-Gejji and Jafari method (DGJIM) and the Adomian decomposition method (ADM) in which a series of approximate solutions converge to the exact ones. Agarwal et al. [4] discussed the existence and uniqueness of solutions for a nonlocal problem with integral transmitting condition for mixed parabolic-hyperbolic type equations with Caputo fractional derivative. Agarwal et al. [5] provided a detailed description of impulsive fractional differential equations using Lyapunov functions and overviewed results for the stability in Caputo's sense. Khan et al. [6] focused on the existence and uniqueness of solutions and Hyers–Ulam stability for ABC-fractional DEs with p-Laplacian operator. Khan et al. [7] studied the stability and numerical simulation of a fractional order plant-nectar-pollinator model. Khan et al. [8] proved the existence and Hyers–Ulam stability of solutions to a class of hybrid fractional differential equations with p-Laplacian operator.

The problem of existence of non-constant periodic solutions for fractional order models has became one of the most interesting topics to conduct research on. This is particularly due to the differences between systems of integers order and systems of fractional orders in terms of the existence of non-constant periodic solutions. In much work, such as [9–14], the authors have shown that non-constant periodic solutions of fractional order systems do not exist contrary to the case where the order of the system is an integer. Therefore, the concept of an asymptotically periodic solution for fractional differential equations or inclusions is introduced and discussed in much work. For example, in [12, 15–17], the authors considered semilinear differential equations of order $\alpha \in (0,1)$ generated by a C_0-semigroup, while the papers [18–20] addressed semilinear differential equations of order $\alpha \in (0,1)$ generated by sectorial operators. Moreover, the asymptotically periodic solutions for delayed fractional differential equations with almost sectorial operator of order $\alpha \in (0,1)$ are examined in [21]. Rogovchenko et al. [22] studied the asymptotic properties of solutions for a certain classes of second order nonlinear differential equations. Very recently, Wang et al. [23] discussed the asymptotic behavior of solutions to time-fractional neutral functional differential equations of order $\alpha \in (0,1)$.

For more information regarding this subject, we refer the reader to [24, 25].

It is worth noting that the problems discussed in all cited work above do not contain impulse effects, whether it is instantaneous or non-instantaneous, and the nonlinear term is a single-valued function.

Let $\alpha \in (1,2)$, E be a Banach space, m be a natural number, $\omega > 0$, $J = [0,\infty)$,

$$0 = s_0 < \theta_1 \leq s_1 < \theta_2 < \cdots < \theta_m \leq s_m = \omega < \theta_{m+1} = \omega + \theta_1 \leq s_{m+1} = \omega + s_1 < \cdots,$$

with $\lim_{i\to\infty} \theta_i = \infty$, $s_{m+i} = s_i + \omega$, and $\theta_{m+i} = \theta_i + \omega$; $i \in \mathbb{N} = \{1,2,3,\ldots\}$ and $A : D(A) \subseteq E \to E$ be a sectorial operator of type $\{M,\varphi,\alpha,\mu\}$, where $M > 0$, $\varphi \in (0,\frac{\pi}{2})$ and $\mu \in \mathbb{R}$. Moreover, let $F : J \times E \to 2^E - \{\phi\}$, be a multivalued function, $g_i : [\theta_i, s_i] \times E \longrightarrow E$; $i \in \mathbb{N}$, $x_0 \in D(A)$ and $x_1 \in E$.

Motivated by the above cited work, we prove two results concerning the existence of S-asymptotically ω-periodic mild solutions to the following non-instantaneous impulsive semilinear differential inclusion:

$$\begin{cases} {}^cD^{\alpha}_{s_i,\theta}x(\theta) \in Ax(\theta) + F(\theta,x(\theta)), & \text{a.e. } \theta \in (s_i,\theta_{i+1}], i \in \mathbb{N}\cup\{0\}, \\ x(\theta_i^+) = g_i(\theta_i, x(\theta_i^-)), & i \in \mathbb{N}, \\ x(\theta) = g_i(\theta, x(\theta_i^-)), & \theta \in (\theta_i s_i], i \in \mathbb{N}, \\ x(0) = x_0, & x'(0) = x_1, \end{cases} \tag{1}$$

where ${}^{c}D^{\alpha}_{s_i,\theta}x(\theta)$ is the Caputo derivative of the function x at the point θ and with lower limit at s_i [26]. In the first result, we apply a fixed point theorem for contraction multivalued functions, and, in the second result, we use a compactness criterion in the space of bounded piecewise continuous functions defined on the unbounded interval $J = [0,\infty)$. Our work generalizes much recent work such as [18–20] to the case where there are impulse effects, and the right-hand side is a multivalued function.

To the best of our knowledge, there is no work on S-asymptotic ω-periodic behavior of solutions to fractional non-instantaneous impulsive differential inclusions with order $\alpha \in (1,2)$ and generated by sectorial operators, and this fact is the main goal in the present paper.

To clarify the advantage of this study, we mention that two methods have been provided to demonstrate the existence of S-asymptotic ω-periodic solutions for semilinear fractional differential inclusions in the presence of non-instantaneous impulse effects, and in which the nonlinear part is a multivalued function, and the linear part is a sectorial operator. Moreover, the technique presented in this paper can be used to generalize the work in [12, 15–21, 23–25] to the case where the linear part is a sectorial operator, the nonlinear part is a multivalued function, and there is impulse effects. In addition, Problem (1) can be investigated on time scales using the arguments in [1], and using the arguments in [3, 6, 8], one can examine the asymptotic periodic solutions for Problem (1) when the Caputo derivative is replaced by the ψ-Caputo derivative, ψ-RL derivative, Atangana–Baleanu derivative or p-Laplacian operator. Also, the technique used in this paper can be applied to study the asymptotic periodic solutions for many fractional differential equations or inclusions generated by sectorial operators or almost sectorial operators.

For more information related to fractional differential equations and inclusions with non-instantaneous impulse effects, we refer the reader to [27–31]. See [32–35] for more information about semilinear differential equations and inclusions with sectorial operators.

It is worth noting that Refs. [36–44] contain very important and interesting topics in mathematics as well as their applications such as differential equations, fractional calculus and ABC-fuzzy-Volterra integro-differential equation.

The paper is organized as follows. Section 2 includes definitions and basic information that we need to prove our results. In Sect. 3, we provide two existence results of S-asymptotic ω-periodic solutions for Problem (1). In Sect. 4, we give three examples to illustrate our theoretical results.

2 Preliminaries and notations

Let $J_0 = [0,t_1]$ and $J_i = (\theta_i,\theta_{i+1}]$, $i \in \mathbb{N}$. It is known that the vector spaces

$$PC(J,E) := \left\{x : J \to E, x \text{ is bounded}, x|_{J_i} \in C(J_i,E)\right\}$$

and

$$SAP_{\omega}PC(J,E) := \left\{x : J \to E, x \text{ is bounded}, x|_{J_i} \in C(J_i,E),\ \lim_{\theta\to\infty}\left\|x(\theta+\omega) - x(\theta)\right\| = 0\right\}$$

are Banach spaces endowed with the norm

$$\|x\| := \max_{\theta\in J}\left\|x(\theta)\right\|.$$

Definition 1 ([45]) Let $M > 0$, $\varphi \in (0, \frac{\pi}{2})$, and $\mu \in \mathbb{R}$. A closed linear operator $A : D(A) \subseteq E \to E$ with dense domain is called sectorial of type $\{M, \varphi, \alpha, \mu\}$ if:

(i) $\delta^\alpha \notin S_\varphi + \mu \Rightarrow \|R(\delta^\alpha, A)\|$ exists, where $R(\delta^\alpha, A)$ is the α-resolvent operator of A defined by

$$R(\delta^\alpha, A) := (\delta^\alpha I - A)^{-1}$$

and

$$S_\varphi + \mu = \{\mu + \delta^\alpha : \delta \in \mathbb{C}, |\mathrm{Arg}(-\delta^\alpha)| < \varphi\}.$$

(ii)

$$\|R(\delta^\alpha, A)\| \leq \frac{M}{|\delta^\alpha - \mu|}, \quad \delta^\alpha \notin S_\varphi + \mu.$$

Remark 1 ([45], Remark 2.1) If A is a sectorial operator of type $\{M, \varphi, \alpha, \mu\}$, then it is the infinitesimal generator of a α-resolvent family of operators $\{T_\alpha(\theta) : \theta \geq 0\}$ in E defined by

$$T_\alpha(\theta) = \frac{1}{2\pi i} \int_\gamma e^{\delta\theta} R(\delta^\alpha, A)\, d\delta, \tag{2}$$

where γ is a suitable path and $\delta^\alpha \notin S_\varphi + \mu$ for $\delta \in \gamma$.

Definition 2 ([45], Definition 3.1) Let A be a sectorial operator of type $\{M, \varphi, \alpha, \mu\}$ and $f \in L^1([0, b], E)$. A continuous function $u : [0, b] \to E$ is called a mild solution to the Cauchy problem:

$$\begin{cases} {}^cD^\alpha_{0,\theta} x(\theta) = Ax(\theta) + f(\theta), & \theta \in [0, b], \\ x(0) = x_0, \qquad x'(0) = x, \end{cases}$$

if

$$x(\theta) = S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 + \int_0^\theta T_\alpha(\theta - \tau)f(\theta)\, d\tau, \quad \theta \in [0, b],$$

where

$$S_\alpha(\theta) = \frac{1}{2\pi i} \int_\gamma e^{\delta\theta} \delta^{\alpha-1} R(\delta^\alpha, A)\, d\delta,$$
$$K_\alpha(\theta) = \frac{1}{2\pi i} \int_\gamma e^{\delta\theta} \delta^{\alpha-2} R(\delta^\alpha, A)\, d\delta,$$

and $T_\alpha(\theta)$ is given by (2).

The following lemma provides estimates on $\|S_\alpha(\theta)\|$, $\|K_\alpha(\theta)\|$ and $\|T_\alpha(\theta)\|$.

Lemma 1 ([45], Theorems 3.3, 3.4) *Let A be a sectorial operator of type $\{M,\varphi,\alpha,\mu\}$. Suppose $\mu<0$. Then $L=L(M,\varphi,\alpha)>0$ such that*

$$\begin{aligned} &\|S_\alpha(\theta)\| \le \frac{L}{1+|\mu|\theta^\alpha}, \qquad \|K_\alpha(\theta)\| \le \frac{L(\theta+1)}{1+|\mu|\theta^\alpha}, \quad \textit{and} \\ &\|T_\alpha(\theta)\| \le \frac{L\theta^{\alpha-1}}{1+|\mu|\theta^\alpha}, \quad \forall\theta>0. \end{aligned} \tag{3}$$

Remark 2 ([19], Remark 3) In view of (6), we get:

(i)

$$\|S_\alpha(\theta)\| \le L, \quad \forall\theta>0. \tag{4}$$

(ii)

$$\begin{aligned} \|K_\alpha(\theta)\| &\le L+\frac{L\theta}{1+|\mu|\theta^\alpha} \\ &\le \begin{cases} L+L\theta, & \text{if } 0<\theta<1, \\ L+\frac{L}{|\mu|\theta^{\alpha-1}}, & \text{if } \theta\ge 1 \end{cases} \\ &\le L\left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right), \quad \forall\theta>0. \end{aligned} \tag{5}$$

(iii) As in (ii), we derive

$$\begin{aligned} \|T_\alpha(\theta)\| &\le \frac{L\theta^{\alpha-1}}{1+|\mu|\theta^\alpha} \\ &\le L\max\left\{1,\frac{1}{|\mu|}\right\}, \quad \forall\theta>0. \end{aligned} \tag{6}$$

Based on Definition 2, we can give the definition of an S-asymptotically ω-periodic mild solution for Problem (1).

Definition 3 A function $x\in SAP_\omega PC(J,E)$ is called an S-asymptotically ω-periodic mild solution for Problem (1) if it has the form

$$x(\theta)=\begin{cases} S_\alpha(\theta)x_0+K_\alpha(\theta)x_1 \\ \quad +\int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta\in[0,\theta_1], \\ g_i(\theta,x(\theta_i^-)), \quad \theta\in(\theta_i,s_i], i\in\mathbb{N}, \\ S_\alpha(\theta-s_i)g_i(s_i,x(\theta_i^-))+K_\alpha(\theta-s_i)g_i'(s_i,x(\theta_i^-)) \\ \quad -\int_0^{s_i} T_\alpha(s_i-\tau)f(\tau)\,d\tau \\ \quad +\int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta\in[s_i,\theta_{i+1}], i\in\mathbb{N}, \end{cases}$$

where $f(\tau)\in F(\tau,x(\tau))$, a.e. for $\tau>0$.

3 Existence of S-asymptotically ω-periodic mild solutions for Problem (1)

In order to give the first result, we need the following lemma which is due to [46].

Lemma 2 *Let (X,d) be a metric space and G be a contraction multivalued function from X to the family of non-empty closed subsets of X. Then G has a fixed point.*

For notations about multivalued functions we refer the reader to [47].

Theorem 1 *Suppose the following assumptions are satisfied.*

(HA) *$A : D(A) \to E$ is a sectorial operator of type $\{M,\varphi,\alpha,\mu\}$, where $M > 0$, $\varphi \in (0,\frac{\pi}{2})$, and $\mu \in \mathbb{R}$.*

(HF) *$F : J \times E \to P_{ck}(E)$ is a multivalued function such that*

(i) *For any $z \in E$, the multivalued function $\theta \to F(\cdot,z)$ is strongly measurable.*

(ii) *For any $x \in PC(J,E)$, the set*

$$S^1_{F(\cdot,x(\cdot))} := \big\{\psi : J \to E, \psi \text{ is locally integrable and } \psi(\tau) \in F\big(\tau,x(\tau)\big), \\ \text{a.e. } \tau \in J\big\}$$

is not empty.

(iii) *There is a continuous function $L_1 : J \to (0,\infty)$ such that*

$$h\big(F(\theta,z_1),F(\theta,z_2)\big)\| \le L_1(\theta)\|z_1 - z_2\|, \quad \forall\theta \in J, z_1,z_2 \in E, \tag{7}$$

where h is the Hausdorff distance.

(iv) *There is a continuous function $L_2 : J \to (0,\infty)$ such that*

$$h\big(F(\theta + \omega,z),F(\theta,z)\big)\| \le L_2(\theta)\|1 + x\|, \quad \forall\theta \in J, z \in E. \tag{8}$$

(v) *The function $\sigma(\tau) := \|F(\tau,0)\| = \sup_{z\in F(\tau,0)} \|z\|$ is continuous, bounded on J and satisfies the relation*

$$\lim_{\theta\to\infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(\theta - \tau)^\alpha}\sigma(\tau)\,d\tau = 0. \tag{9}$$

(Hg_i) *For any $i \in \mathbb{N}$, $g_i : [\theta_i,s_i] \times E \longrightarrow E$ such that, for any $x \in E$, the function $\theta \mapsto g_i(\theta,x)$ is differentiable at s_i and*

(i)

$$\lim_{\substack{\theta\to\infty \\ i\to\infty}} \big\|g_{i+m}(\theta + \omega,z) - g_i(\theta,z)\big\| = 0, \quad \forall z \in E. \tag{10}$$

(ii) *There is $N > 0$ such that for any $i \in \mathbb{N}$*

$$\big\|g_i(\theta,z_1) - g_i(\theta,z_2)\big\| \le N\|z_1 - z_2\|, \quad \forall\theta \in J, \forall z_1,z_2 \in E. \tag{11}$$

(iii) *There is $\mathcal{N} > 0$ such that for any $i \in \mathbb{N}$*

$$\big\|g_i'(s_i,z_1) - g_i'(s_i,z_2)\big\| \le \mathcal{N}\|z_1 - z_2\|, \quad \forall z_1,z_2 \in E. \tag{12}$$

(iv) *There is* $\kappa_1 > 0$ *such that*

$$\sup_{i\in\mathbb{N}}\sup_{\theta\in J}\left\|g_i(\theta,z)\right\| \le \kappa_1\left(\|z\|+1\right), \quad \forall z\in E. \tag{13}$$

(v) *There is* $\kappa_2 > 0$ *with*

$$\sup_{i\in\mathbb{N}}\left\|g_i'(s_i,z)\right\| \le \kappa_2\left(\|z\|+1\right), \quad \forall z\in E. \tag{14}$$

Then Problem (1) *has an S-asymptotically* ω*-periodic mild solution provided that the following conditions are verified*:

$$\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}L_1(\tau)\,d\tau = 0, \tag{15}$$

$$\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}L_2(\tau)\,d\tau = 0, \tag{16}$$

and

$$L\left(2\xi + N. + \left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right)\mathcal{N}\right) < 1, \tag{17}$$

where $\xi = \sup_{\theta\in J}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}L_1(\tau)\,d\tau$.

Proof Due to (HF)(ii), one can define a multivalued function Θ on $SAP_\omega PC(J,E)$ in the following manner: an element $y\in\Theta(x)$ if and only if

$$y(\theta) = \begin{cases} S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta\in[0,\theta_1], \\ g_i(\theta,x(\theta_i^-)), \quad \theta\in(\theta_i,s_i], i\in\mathbb{N}, \\ S_\alpha(\theta-s_i)g_i(s_i,x(\theta_i^-)) + K_\alpha(\theta-s_i)g_i'(s_i,x(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i-\tau)f(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta\in[s_i,\theta_{i+1}], i\in\mathbb{N}, \end{cases} \tag{18}$$

where $f\in S^1_{F(\cdot,x(\cdot))}$.

Obviously, $y\in PC(J_i,E)$. We clarify that, if $x\in SAP_\omega PC(J,E)$, then $\Theta(x)$ is a closed subset of $SAP_\omega PC(J,E)$. We do this in the following steps.

Step 1. We demonstrate that, if $y\in\Theta(x)$, we have

$$\lim_{\theta\to\infty}\left\|y(\theta+\omega)-y(\theta)\right\| = 0. \tag{19}$$

Since $x\in SAP_\omega PC(J,E)$,

$$\lim_{\theta\to\infty}\left\|x(\theta+\omega)-x(\theta)\right\| = 0. \tag{20}$$

Now, we consider two cases.

(i) If $\theta \in (\theta_i, s_i]$ for some $i \in \mathbb{N}$, then $\theta + \omega \in [\theta_i + \omega, s_i + \omega,] = [\theta_{i+m}, s_{i+m}]$, and hence, using (21) and (18), it yields

$$\begin{aligned}
&\left\|g_{i+m}\left(\theta + \omega, x\left(\theta_{i+m}^-\right)\right) - g_i\left(\theta, x\left(\theta_i^-\right)\right)\right\| \\
&\quad \le \left\|g_{i+m}\left(\theta + \omega, x\left(\theta_{i+m}^-\right)\right) - g_{i+m}\left(\theta, x\left(\theta_{i+m}^-\right)\right)\right\| \\
&\qquad + \left\|g_{i+m}\left(\theta, x\left(\theta_{i+m}^-\right)\right) - g_i\left(\theta, x\left(\theta_i^-\right)\right)\right\| \\
&\quad \le \left\|g_{i+m}\left(\theta + \omega, x\left(\theta_{i+m}^-\right)\right) - g_{i+m}\left(\theta, x\left(\theta_{i+m}^-\right)\right)\right\| \\
&\qquad + N\left\|x\left(\theta_{i+m}^-\right) - x\left(\theta_i^-\right)\right\|.
\end{aligned} \tag{21}$$

(ii) If $\theta \in [s_i, \theta_{i+1}]$ for some $i \in \mathbb{N}$, then $\theta + \omega \in [s_i + \omega, \theta_{i+1} + \omega] = [s_{i+m}, \theta_{i+m+1}]$. Using (4) and arguing as in (21), it follows that

$$\begin{aligned}
&\left\|S_\alpha\left(\theta + \omega - (s_i + \omega)\right)g_{i+m}\left(s_i + \omega, x\left(\theta_{i+m}^-\right)\right) - S_\alpha(\theta - s_i)g_i\left(s_i, x\left(\theta_i^-\right)\right)\right\| \\
&\quad = \left\|S_\alpha(\theta - s_i)\right\|\left\|g_{i+m}\left(s_i + \omega, x\left(\theta_{i+m}^-\right)\right) - g_i\left(s_i, x\left(\theta_i^-\right)\right)\right\| \\
&\quad \le L\left\|g_{i+m}\left(s_i + \omega, x\left(\theta_{i+m}^-\right)\right) - g_i\left(s_i, x\left(\theta_i^-\right)\right)\right\|.
\end{aligned} \tag{22}$$

Moreover, in view of (3) and (14), we arrive at

$$\begin{aligned}
&\left\|K_\alpha\left(\theta + \omega - (s_i + \omega)\right)g'_{i+m}\left(s_i + \omega, x\left(\theta_{i+m}^-\right)\right) - K_\alpha(\theta - s_i)g'_i\left(s_i, x\left(\theta_i^-\right)\right)\right\| \\
&\quad = \left\|K_\alpha(\theta - s_i)g'_{i+m}\left(s_{i+m}, x\left(\theta_{i+m}^-\right)\right) - K_\alpha(\theta - s_i)g'_i\left(s_i, x\left(\theta_i^-\right)\right)\right\| \\
&\quad \le \left\|K_\alpha(\theta - s_i)\right\|\left\|g'_{i+m}\left(s_{i+m}, x\left(\theta_{i+m}^-\right)\right) - g'_i\left(s_i, x\left(\theta_i^-\right)\right)\right\| \\
&\quad \le 2\kappa_2\left(\|x\| + 1\right)\left\|K_\alpha(\theta - s_i)\right\| \\
&\quad \le 2\kappa_2\left(\|x\| + 1\right)\frac{L(\theta - s_i + 1)}{1 + |\mu|(\theta - s_i)^\alpha}.
\end{aligned} \tag{23}$$

Next,

$$\begin{aligned}
&\left\|\int_0^{\theta+\omega} T_\alpha(\theta + \omega - \tau)f(\tau)\,d\tau - \int_0^{\theta} T_\alpha(\theta - \tau)f(\tau)\,d\tau\right\| \\
&\quad = \left\|\int_{-\omega}^{\theta} T_\alpha(\theta - \tau)f(\tau + \omega)\,d\tau - \int_0^{\theta} T_\alpha(\theta - \tau)f(\tau)\,d\tau\right\| \\
&\quad \le \int_{-\omega}^{0}\left\|T_\alpha(\theta - \tau)\right\|\left\|f(\tau + \omega)\right\|\,d\tau \\
&\qquad + \int_0^{\theta}\left\|T_\alpha(\theta - \tau)\right\|\left\|f(\tau + \omega) - f(\tau)\right\|\,d\tau \\
&\quad = I_1 + I_2.
\end{aligned} \tag{24}$$

Let $\tau \in [-\omega, 0]$ be fixed. Since $F(\tau + \omega, 0)$ is compact, there is $\nu_{\tau+\theta} \in F(\tau + \omega, 0)$ such that

$$\begin{aligned}
\left\|f(\tau + \omega) - \nu_{\tau+\omega}\right\| &= d(f(\tau + \omega), F(\tau + \omega, 0) \\
&\le h\left(F\left(\tau + \omega, x(\tau + \omega)\right), F(\tau + \omega, 0)\right).
\end{aligned} \tag{25}$$

From (7), (HF)(v) and (25), we get

$$\begin{aligned}\|f(\tau+\omega)\| &\leq h\big(F\big(\tau+\omega, x(\tau+\omega)\big), F(\tau+\omega,0)\big) + \|v_{\tau+\omega}\| \\ &\leq L_1(\tau+\omega)\|x(\tau+\omega)\| + \sigma(\tau+\omega) \\ &\leq \|x\|L_1(\tau+\omega) + \sigma(\tau+\omega). \end{aligned} \tag{26}$$

Since L_1 and σ are continuous on J, there are two positive numbers ω_1, ω_2 such that

$$\sup_{t\in[0,\omega]} |L_1(t)| \leq \omega_1, \quad \text{and} \quad \sup_{t\in[0,\omega]} |\sigma(\theta)| \leq \omega_2. \tag{27}$$

Then, from (3), (26) and (27), we have

$$\begin{aligned} I_1 &= \left\| \int_{-\omega}^{0} T_\alpha(\theta-\tau) f(\tau+\omega)\, d\tau \right\| \\ &\leq \big(\omega_1\|x\| + \omega_2\big) L \int_{-\omega}^{0} \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\, d\tau \\ &\leq \frac{(\omega_1\|x\|+\omega_2)L}{|\mu|} \int_{-\omega}^{0} \frac{|\mu|(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\, d\tau \\ &= \frac{(\omega_1\|x\|+\omega_2)L}{\alpha|\mu|} \ln \frac{1+|\mu|(\theta+\omega)^\alpha}{1+|\mu|\theta^\alpha}. \end{aligned} \tag{28}$$

Next, let $\tau \in [0,\theta]$ be fixed. From the fact that $F(\tau+\omega, x(\tau))$ is compact, there are $z_{\tau+\omega}, z_\tau \in F(\tau, x(\tau+\omega))$ such that $d(f(\tau+\omega), z_{\tau+\omega}) = d(f(\tau+\omega), F(\tau, x(\tau+\omega)))$ and $d(f(\tau), z_\tau) = d(f(\tau, F(\tau, x(\tau+\omega))))$. Then, by (12), (13) and (25), we arrive at

$$\begin{aligned} &\|f(\tau+\omega) - f(\tau)\| \\ &\quad\leq \|f(\tau+\omega) - z_{\tau+\omega}\| + \|z_{\tau+\omega} - z_\tau\| + \|z_\tau - f(\tau)\| \\ &\quad\leq d(f(\tau+\omega), F\big(\tau+\omega, x(\tau)\big) + \|z_{\tau+\omega} - z_\tau\| \\ &\qquad + d(f(\tau, F\big(\tau+\omega, x(\tau)\big) \\ &\quad\leq h(F((\tau+\omega), x\big((\tau+\omega)\big), F\big(\tau, x(\tau+\omega)\big) \\ &\qquad + 2\big\|F\big(\tau, x(\tau+\omega)\big)\big\| + h\big(F\big(\tau, x(\tau+\omega)\big), F\big(\tau, x(\tau)\big)\big) \\ &\quad\leq L_2(\tau)\|1 + x(\tau+\omega)\| + 2\|F(\tau,0)\| + 2h\big(F\big(\tau, x(\tau+\omega)\big), F(\tau,0)\big) \\ &\qquad + L_1(\tau)\|x(\tau+\omega) - x(\tau)\| \\ &\quad\leq L_2(\tau)\big(1+\|x\|\big) + 2\sigma(\tau) + 3L_1(\tau)\|x\|. \end{aligned} \tag{29}$$

By (3) and (29), we arrive at

$$\begin{aligned} I_2 &= \int_0^\theta \|T_\alpha(\theta-\tau)\| \|f(\tau+\omega) - f(\tau)\|\, d\tau \\ &\leq 3L\|x\| \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha} L_1(\tau)\, d\tau \end{aligned}$$

$$+ 2L \int_0^{\theta} \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^{\alpha}} \sigma(\tau)\,d\tau$$
$$+ L\bigl(1+\|x\|\bigr) \int_0^{\theta} \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^{\alpha}} L_2(\tau)\,d\tau. \tag{30}$$

By arguing as in (24), (28) and (30), one can show

$$\begin{aligned}
&\left\| \int_0^{s_i+\omega} T_\alpha(s_i+\omega-\tau) f(\tau)\,d\tau - \int_0^{s_i} T_\alpha(s_i-\tau) f(\tau)\,d\tau \right\| \\
&\quad\le \frac{(\omega_1\|x\|+\omega_2)L}{\alpha|\mu|} \ln \frac{1+|\mu|(s_i+\omega)^{\alpha}}{1+|\mu|s_i^{\alpha}} \\
&\qquad + 3L\|x\| \int_0^{s_i} \frac{(s_i-\tau)^{\alpha-1}}{1+|\mu|(s_i-\tau)^{\alpha}} L_1(\tau)\,d\tau \\
&\qquad + 2L \int_0^{s_i} \frac{(s_i-\tau)^{\alpha-1}}{1+|\mu|(s_i-\tau)^{\alpha}} \sigma(\tau)\,d\tau \\
&\qquad + L\bigl(1+\|x\|\bigr) \int_0^{s_i} \frac{(s_i-\tau)^{\alpha-1}}{1+|\mu|(s_i-\tau)^{\alpha}} L_2(\tau)\,d\tau.
\end{aligned} \tag{31}$$

Since $s_i \to \infty$ when $\theta \to \infty$, we can derive (19) from (9), (10), (15), (16), (20), (21)–(23), (28), (30) and (31).

Then, due to (21)–(24), (28), (30) and (31), we arrive at (19).

Step 2. In this step, we show that, if $x \in SAP_\omega PC(J,E)$ and $y \in \Theta(x)$, then y is bounded.

(i) Let $\theta \in [0,\theta_1]$. Then, using (4)–(6), we get

$$\begin{aligned}
\bigl\|y(\theta)\bigr\| &\le L\|x_0\| + L\|x_1\| \left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right) \\
&\quad + L\max\left\{1,\frac{1}{|\mu|}\right\} \int_0^{\theta} \bigl\|f(\tau)\bigr\|\,d\tau.
\end{aligned} \tag{32}$$

On the hand, from (7), (9) and (27), we derive

$$\begin{aligned}
\bigl\|f(\tau)\bigr\| &\le \bigl\|F\bigl(\tau,x(\tau)\bigr)\bigr\| \le \bigl\|F(\tau,0)\bigr\| + h\bigl(F(\tau,0),F\bigl(\tau,x(\tau)\bigr)\bigr) \\
&\le \sigma(\tau) + L_1(\tau)\|x\| \le \omega_1 + \omega_2\|x\|.
\end{aligned} \tag{33}$$

Then, by (32) and (33), we conclude that

$$\begin{aligned}
\sup_{\theta\in[0,\theta_1]} \bigl\|y(\theta)\bigr\| &\le L\|x_0\| + L\|x_1\| \left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right) \\
&\quad + L\max\left\{1,\frac{1}{|\mu|}\right\} \bigl[\omega_1+\omega_2\|x\|\bigr)\bigr]\omega.
\end{aligned} \tag{34}$$

(ii) Let $\theta \in (\theta_i, s_i]; i \in \mathbb{N}$. In view of (18) and (19), it follows that

$$\bigl\|g_i\bigl(\theta,x(\theta_i^-)\bigr)\bigr\| \le \kappa_1\bigl(\|x\|+1\bigr) \tag{35}$$

and

$$\bigl\|g_i'\bigl(s_i,x(\theta_i^-)\bigr)\bigr\| \le \kappa_2\bigl(\|x\|+1\bigr). \tag{36}$$

(iii) Let $\theta \in [s_i, \theta_{i+1}]$; $i \in \mathbb{N}$. In view of (4), (5), (35) and (36), we have

$$\begin{aligned}
&\sup_{\theta\in[s_i,\theta_{i+1}]} \left\| S_\alpha(\theta - s_i) g_i\big(s_i, x\big(\theta_i^-\big)\big) + K_\alpha(\theta - s_i) g_i'\big(s_i, x\big(\theta_i^-\big)\big) \right\| \\
&\quad \le L\kappa_1\big(\|x\| + 1\big) + L \max\left\{1, \frac{1}{|\mu|}\right\} \kappa_2\big(\|x\| + 1\big).
\end{aligned} \tag{37}$$

Furthermore, using (3) and (33), we arrive at

$$\begin{aligned}
&\left\| \int_0^\theta T_\alpha(\theta - \tau) f(\tau)\, d\tau \right\| \\
&\quad \le L \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(\theta - \tau)^\alpha} \big\|f(\tau)\big\|\, d\tau \\
&\quad \le L \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(\theta - \tau)^\alpha} \sigma(\tau)\, d\tau \\
&\qquad + \|x\| L \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(\theta - \tau)^\alpha} L_1(\tau)\, d\tau.
\end{aligned} \tag{38}$$

From (9), (15) and (38), there is $\xi > 0$ with

$$\sup_{\theta\in J} \left\| \int_0^\theta T_\alpha(\theta - \tau) f(\tau)\, d\tau \right\| \le \varsigma \tag{39}$$

and

$$\sup_{i\in\mathbb{N}} \left\| \int_0^{s_i} T_\alpha(s_i - \tau) f(\tau)\, d\tau \right\| \le \varsigma. \tag{40}$$

As a result of (34), (35), (37), (39) and (40), we conclude that y is bounded on J.

Hence, Θ is a multivalued function from $SAP_\omega PC(J, E)$ to the non-empty subsets of $SAP_\omega PC(J, E)$.

Step 3. The values of Θ are closed.

To show this, let $x \in SAP_\omega PC(J, E)$ and $y_n \in \Theta(x)$, $\forall n \ge 1$, with $y_n \to y$ in $SAP_\omega PC(J, E)$. Then we have $f_n \in S^1_{F(\cdot, x(\cdot))}$ such that

$$y_n(\theta) = \begin{cases}
S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\
\quad + \int_0^\theta T_\alpha(\theta - \tau) f_n(\tau)\, d\tau, \quad \theta \in [0, \theta_1], \\
g_i(\theta, x(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\
S_\alpha(\theta - s_i) g_i(s_i, x(\theta_i^-)) + K_\alpha(\theta - s_i) g_i'(s_i, x(\theta_i^-)) \\
\qquad - \int_0^{s_i} T_\alpha(s_i - \tau) f_n(\tau)\, d\tau \\
\qquad + \int_0^\theta T_\alpha(\theta - \tau) f_n(\tau)\, d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}.
\end{cases} \tag{41}$$

Let θ be a fixed point in $[0, \theta_1]$ and put $J_\theta = [0, \theta]$. Then

$$\begin{aligned}
\big\|f_n(\tau)\big\| &\le \big\|F\big(\tau, x(\tau)\big)\big\| \le \big\|F(\tau, 0)\big\| + L_1(\tau)\big\|x(\tau)\big\| \\
&\le \sigma(\tau) + L_1(\tau)\|x\|, \quad \forall \tau \in J_\theta.
\end{aligned} \tag{42}$$

This relation with the fact that σ and L_1 are continuous guarantee that the family $\{f_n : n \geq 1\}$ is bounded in $L^2(J_\theta, E)$, and hence, by Mazur's lemma, there is a sequence $(z_n)_{n\geq 1}$ of convex combinations of f_n with $z_n \to f$ strongly in $L^2(J_\theta, E)$. Hence, we can suppose, without loss of generality, that $z_n(\theta) \to f(\theta)$, a.e. $\theta \in J_\theta$. Moreover, from (6) and (42), we get

$$\begin{aligned} &\bigl\| T_\alpha(\theta-\tau)f_n(\tau)\bigr\| \\ &\quad \leq L\max\left\{1, \frac{1}{|\mu|}\right\}\bigl(\sigma(\tau)+L_1(\tau)\bigr)\|x\|, \quad \forall \tau \in J_\theta. \end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem and the continuity of $T_\alpha(\theta - \tau)$, we obtain

$$\begin{aligned} &\lim_{n\to\infty}\int_0^\theta T_\alpha(\theta-\tau)z_n(\tau)\,d\tau \\ &\quad = \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} y_n(\theta) = S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [0,\theta_1]. \tag{43}$$

Similarly, one can show that, for any $\theta \in [s_i, \theta_{i+1}]$; $i \in \mathbb{N}$, we have

$$\begin{aligned} \lim_{n\to\infty} y_n(\theta) ={}& S_\alpha(\theta-s_i)g_i\bigl(s_i, x\bigl(\theta_i^-\bigr)\bigr) + K_\alpha(\theta-s_i)g_i'\bigl(s_i, x\bigl(\theta_i^-\bigr)\bigr) \\ & - \int_0^{s_i} T_\alpha(s_i-\tau)f(\tau)\,d\tau + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau. \end{aligned} \tag{44}$$

Remark that (HF)(iv) implies that $f(\theta) \in F(\theta, x(\theta))$, a.e. $\theta \in J$. So, from (43) and (44), we arrive at

$$y(\theta) = \begin{cases} S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ S_\alpha(\theta-s_i)g_i(s_i, x(\theta_i^-)) + K_\alpha(\theta-s_i)g_i'(s_i, x(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i-\tau)f(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases}$$

Then $y \in \Theta(x)$.

Step 4. Θ is a contraction.

Let $u_1, u_2 \in SAP_\omega PC(J,E)$ and $y_1 \in \Theta(u_1)$. Then we have $f_1 \in S^1_{F(\cdot,u_1(\cdot))}$ such that

$$y_1(\theta) = \begin{cases} S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f_1(\tau)\,d\tau, \quad \theta \in [0,\theta_1], \\ g_i(\theta, u_1(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta - s_i)g_i(s_i, u_1(\theta_i^-)) + K_\alpha(\theta - s_i)g_i'(s_i, u_1(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i - \tau)f_1(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f_1(\tau)\,d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases} \tag{45}$$

Consider the multivalued function $\Psi : J \to 2^E$ defined by

$$\Psi(\theta) = \{z \in E : \|z - f_1(\theta)\| \le L_1(\theta)\|u_1(\theta) - u_2(\theta)\|, \text{a.e. } \theta \in J\}.$$

We clarify that the values of Ψ are non-empty. Let $\theta \in J$. From (5), we get

$$h\big(F(\theta, u_1(\theta)), F(\theta, u_2(\theta))\big) \le L_1(\theta)\|u_1(\theta) - u_2(\theta)\|.$$

So, there is $z_\theta \in F(\theta, u_2(\theta))$ such that

$$\|f_1(\theta) - z_\theta\| \le h\big(F(\theta, u_1(\theta)), F(\theta, u_2(\theta))\big) \le L_1(\theta)\|u_1(\theta) - u_2(\theta)\|,$$

which leads to $\Psi(\theta) \neq \phi$; $\theta \in J$. Moreover, the set $\Lambda(\theta) = \Psi(\theta) \cap F(\theta, u_2(\theta))$; $\theta \in J$ is not empty. Because the functions f_1, L_1, u_1, u_2 are measurable, Proposition 3.4 in [47] or (Corollary 1.3.1(a) in [48]) guarantees that the multivalued map $\theta \to \Lambda(\theta)$ is measurable. Notice that the set $\Psi(\theta)$; $\theta \in J$ is closed. Consequently, the values of Λ are non-empty and compact, and hence, there exists a measurable selection f_2 for Λ with

$$\|f_1(\theta) - f_2(\theta)\| \le L_1(\theta)\|u_1(\theta) - u_2(\theta)\|, \quad \text{a.e. } \theta \in J. \tag{46}$$

We define $y_2 : J \to E$ as follows:

$$y_2(\theta) = \begin{cases} S_\alpha(\theta)x_0 + K_q(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f_2(\tau)\,d\tau, \quad \theta \in [0,\theta_1], \\ g_i(\theta, u_2(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta - s_i)g_i(s_i, u_2(\theta_i^-)) + K_\alpha(\theta - s_i)g_i'(s_i, u_2(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i - \tau)f_2(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f_2(\tau)\,d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases} \tag{47}$$

Obviously, $y_2 \in \Theta(u_1)$. Now, we estimate the quantity $\|y_1 - y_2\|$. To do this, we consider three cases.

Case 1. $\theta \in [0, \theta_1]$. In view of (3), (15), (47), (46) and (47), we have

$$\begin{aligned} &\|y_1(\theta) - y_2(\theta)\| \\ &\le \| \int_0^\theta \|T_\alpha(\theta-\tau)\| \|f_1(\tau) - f_2(\tau)\|\,d\tau \end{aligned}$$

$$\leq L\|u_1 - u_2\| \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha} L_1(\tau)\,d\tau$$
$$\leq \xi L\|u_1 - u_2\|. \tag{48}$$

Case 2. $\theta \in (\theta_i, s_i]$. Using (11), we get

$$\begin{aligned}
&\|y_1(\theta) - y_2(\theta)\| \\
&\quad = \|g_i(\theta, u_1(\theta_i^-)) - g_i(\theta, u_2(\theta_i^-))\| \\
&\quad \leq N\|u_1(\theta_i^-) - u_2(\theta_i^-)\| \leq N\|u_1 - u_2\|.
\end{aligned} \tag{49}$$

Case 3. $\theta \in [s_i, \theta_{i+1}]$, $i \in \mathbb{N}$. From (4) and (49), we derive

$$\begin{aligned}
&\|S_\alpha(\theta - s_i)g_i(s_i, u_1(\theta_i^-)) - S_\alpha(\theta - s_i)g_i(s_i, u_2(\theta_i^-))\| \\
&\quad \leq L\|g_i(s_i, u_1(\theta_i^-)) - g_i(s_i, u_2(\theta_i^-))\| \leq LN\|u_1 - u_2\|.
\end{aligned} \tag{50}$$

By (4) and (12), we arrive at

$$\begin{aligned}
&\|K_\alpha(\theta - s_i)g_i'(s_i, u_1(\theta_i^-)) - K_\alpha(\theta - s_i)g_i'(s_i, u_2(\theta_i^-))\| \\
&\quad \leq \|K_\alpha(\theta - s_i)\| \|g_i'(s_i, u_1(\theta_i^-)) - g_i'(s_i, u_2(\theta_i^-))\| \\
&\quad \leq L\left(1 + \max\left\{1, \frac{1}{|\mu|}\right\}\right)\mathcal{N}\|u_1 - u_2\|.
\end{aligned} \tag{51}$$

Moreover, as in (48), one can show that

$$\begin{aligned}
&\left\|\int_0^\theta T_\alpha(\theta-\tau)f_1(\tau)\,d\tau - \int_0^\theta T_\alpha(\theta-\tau)f_2(\tau)\,d\tau\right\| \\
&\quad \leq L\xi\|u_1 - u_2\|
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\left\|\int_0^{s_i} T_\alpha(s_i-\tau)f_1(\tau)\,d\tau - \int_0^{s_i} T_\alpha(s_i-\tau)f_2(\tau)\,d\tau\right\| \\
&\quad \leq L\xi\|u_1 - u_2\|.
\end{aligned} \tag{53}$$

Now, by (48)–(53), we conclude that

$$\|y_1 - y_2\| \leq \|u - v\|L\left(2\xi + N. + \left(1 + \max\left\{1, \frac{1}{|\mu|}\right\}\right)\mathcal{N}\right). \tag{54}$$

As a consequence of (17), Eq. (54) becomes

$$\|y_1(\theta) - y_2(\theta)\| < \vartheta\|u - v\|, \tag{55}$$

where $\vartheta = L(2\xi + N. + (1 + \max\{1, \frac{1}{|\mu|}\})\mathcal{N}) < 1$. By interchange o the role of y_1 and y_2 in the above discussion, we conclude that Θ is a contraction. So, applying Lemma 2, shows that Θ has a fixed point which is an S-asymptotically ω-periodic mild solution to Problem (1). □

Remark 3 If there is no impulse effect, then $N = \mathcal{N} = 0$, and hence inequality (17) becomes $2L\xi < 1$.

Now, we present another result concerning the existence of S-asymptotically ω-periodic solutions for Problem (1).

We need the following fixed point theorem for multivalued functions and a compactness criterion in $PC(J,E)$.

Lemma 3 ([49], Corollary 3.3.1) *Let W be a closed convex subset of a Banach space X and $N : E \to P_{ck}(W)$ be a closed multifunction which is ϑ-condensing on every bounded subset of W, where ϑ is a non-singular measure of noncompactness defined on subsets of W, then the set of fixed points for N is non-empty.*

Lemma 4 ([50], Lemma 1.2) *Let $D \subseteq PC(J,E)$. Assume that*

(i) $\mathrm{Lim}_{\theta\to\infty} \|u(\theta)\| = 0$, *uniformly for $u \in D$.*

(ii) *The set $D_{|\overline{J_i}}$ is equicontinuous for every $i \in \mathbb{N}$, where*

$$D_{|\overline{J_i}} = \{y^* \in C(\overline{J}_i, E) : y^*(t) = y(t), t \in J_i = (t_i, t_{i+1}], y^*(t_i) = y(t_i^+), y \in D\}.$$

(iii) *For any $i \in \mathbb{N}$, and any $\theta \in J$, the set $\{y^*(t) : y^* \in D_{|\overline{J_i}}\}$ is relatively compact in E.*

Then D is relatively compact in $PC(J,E)$.

Theorem 2 *Assume that (HA) and the following conditions are verified:*

$(HF)^*$ $F : J \times E \to P_{ck}(E)$ *such that:*

(i) *For every $x \in E$, the multivalued function $t \longrightarrow F(t,x)$ is measurable.*

(ii) *For almost $t \in J$, the multivalued function $x \longrightarrow F(t,x)$ is upper semicontinuous.*

(iii) *For any $x \in PC(J,E)$, the set*

$$S^1_{F(\cdot,x(\cdot))} := \{\psi : J \to E, \psi \text{ is locally integrable and } \psi(\tau) \in F(\tau, x(\tau)), \text{ a.e.} \tau \in J\}$$

is not empty.

(iv) *There exists a continuous function $\varphi : J \to (0,\infty)$ with*

$$\|F(t,z)\| \leq \varphi(t)(1 + \|z\|) \quad \forall (t,z) \in J \times E.$$

$(Hg_i)^*$ *For any $i \in \mathbb{N}$, $g_i : [\theta_i, s_i] \times E \longrightarrow E$ $(i \in \mathbb{N})$ is uniformly continuous on bounded sets and for any $z \in E$, the function $\theta \mapsto g_i(\theta, z)$ is continuously differentiable at s_i such that* (10), (11), (14) *and the following conditions are satisfied:*

(i) *There is a bounded continuous function $h^* : J \to J$ with $\lim_{\theta\to\infty} h^*(\theta) = 0$ and*

$$\|g_i(\theta,z)\| \leq h^*(\theta)(\|z\| + 1), \quad \forall (i,\theta,z) \in \mathbb{N} \times [t_i, s_i] \times E. \tag{56}$$

(ii) *For any* $t \in J$, *the function.* $z \to g_i(t,z)$ *is compact.*

Then Problem (1) *has an S-asymptotically* ω*-periodic mild solution provided that the families* $\{S_\alpha(t) : t > 0\}$, $\{K_\alpha(t) : t > 0\}$ *and* $\{T_\alpha(t) : t > 0\}$ *are compact,*

$$\lim_{\theta\to\infty} \int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\bigl(\varphi(\tau+\omega)+\varphi(t)\bigr) = 0, \tag{57}$$

$$\sup_{\theta\in J} \int_0^\theta \varphi(t)\,d\tau < \frac{1}{4L\max\{1,\frac{1}{|\mu|}\}}, \tag{58}$$

and

$$\eta + L\eta + L\kappa_2 \max\left\{1, \frac{1}{|\mu|}\right\} < \frac{1}{2}, \tag{59}$$

where $\eta = \sup_{t\in J}\|h^*(t)\|$, *and* κ_2 *is as defined in* (14).

Proof Due to $(HF)^*$(iii), we can consider a multioperator Θ on $SAP_\omega PC(J,E)$ defined as in (18).

In the following steps we show that Θ satisfies the assumptions of Lemma 3.

Step 1. In this step, we demonstrate that, if $x \in SAP_\omega PC(J,E)$, $y \in \Theta(x)$, then $\lim_{\theta\to\infty}\|y(\theta+\omega)-y(\theta)\| = 0$.

Notice that condition $(Hg_i)^*$(i) implies (Hg_i)(ii), so Eqs. (21)–(23) are satisfied. Now, let $f \in S^1_{F(\cdot,x(\cdot))}$ and $\theta \in [s_1,\theta_{i+1}]$; $i > 1$. As in (24), we have

$$\begin{aligned}
&\left\| \int_0^{\theta+\omega} T_\alpha(\theta+\omega-\tau)f(\tau)\,d\tau - \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau \right\| \\
&\quad\le \int_{-\omega}^0 T_\alpha(\theta-\tau)\bigl\|f(\tau+\omega)\bigr\|\,d\tau \\
&\qquad + \int_0^\theta T_\alpha(\theta-\tau)\bigl\|f(\tau+\omega)-f(\tau)\,d\tau\bigr\| \\
&\quad= I_1 + I_2.
\end{aligned} \tag{60}$$

From the continuity of φ, there is ρ such that $\sup_{t\in[0,\omega|}\varphi(t) \le \rho$, and hence

$$\sup_{\tau\in[-\omega,0]} \bigl\|f(\tau+\omega)\bigr\| \le \rho\bigl(1+\|x\|\bigr). \tag{61}$$

Then, using (3), $(HF)^*$(iv) and (61), we get

$$\begin{aligned}
I_1 &\le \bigl(1+\|x\|\bigr)\rho L \int_{-\omega}^0 \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\,d\tau \\
&= \frac{(1+\|x\|)\rho L}{\alpha|\mu|} \ln\frac{1+|\mu|(\theta+\omega)^\alpha}{1+|\mu|\theta^\alpha}.
\end{aligned} \tag{62}$$

Also, in view of (3) and $(HF)^*$(iv), we arrive at

$$I_2 \le \int_0^\theta \big\| T_\alpha(\theta-\tau)\big\| \big\| f(\tau+\omega)-f(\tau)\big\| \, d\tau \\ \le 2L\big(1+\|x\|\big)\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\big(\varphi(\tau+\omega)+\varphi(t)\big)\, d\tau. \tag{63}$$

Then, by (60), (62) and (63), it follows that

$$\left\| \int_0^{\theta+\omega} T_\alpha(\theta+\omega-\tau)f(\tau)\, d\tau - \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\, d\tau \right\| \\ \le \frac{(1+\|x\|)\rho L}{\alpha|\mu|}\ln\frac{1+|\mu|(\theta+\omega)^\alpha}{1+|\mu|\theta^\alpha} \\ + 2L\big(1+\|x\|\big)\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\big(\varphi(\tau+\omega)+\varphi(t)\big)\, d\tau. \tag{64}$$

Similarly,

$$\left\| \int_0^{s_i+\omega} T_\alpha(s_i+\omega-\tau)f(\tau)\, d\tau - \int_0^{s_i} T_\alpha(s_i-\tau)f(\tau)\, d\tau \right\| \\ \le \frac{(1+\|x\|)\rho L}{\alpha|\mu|}\ln\frac{1+|\mu|(s_i+\omega)^\alpha}{1+|\mu|s_i^\alpha} \\ + 2L\big(1+\|x\|\big)\int_0^{s_i} \frac{(s_i-\tau)^{\alpha-1}}{1+|\mu|(s_i-\tau)^\alpha}\big(\varphi(\tau+\omega)+\varphi(t)\big)\, d\tau. \tag{65}$$

Then, from (21)–(23), (57), (64) and (65), we derive $\lim_{\theta\to\infty}\|y(\theta+\omega)-y(\theta)\| = 0$.

Step 2. Put $D_\lambda = \{u \in SAP_\omega PC(J,E) : \|u\| \le \lambda\}$, where

$$\lambda = \frac{L\|x_0\| + L(\|x_1\|+\kappa_2)(1+\max\{1,\frac{1}{|\mu|}\}) + \eta + L\eta + \frac{1}{2}}{1-[\eta+L\eta+L\kappa_2\max\{1,\frac{1}{|\mu|}\}+\frac{1}{2}]}. \tag{66}$$

Due to Eq. (59), λ is well defined. In this step, we show that, if $x \in D_\lambda$ and $y \in \Theta(x)$, then $\|y\| \le \lambda$.

(i) Let $\theta \in [0,\theta_1]$. Then from (4)–(6) and $(HF)^*$(iv), we get

$$\big\|y(\theta)\big\| \le L\|x_0\| + L\|x_1\|\left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right) \\ + \big(1+\|x\|\big)L\max\left\{1,\frac{1}{|\mu|}\right\}\int_0^{\theta_1}\varphi(t)\, d\tau \\ \le L\|x_0\| + L\|x_1\|\left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right) + \frac{1+\|x\|}{4} \\ \le L\|x_0\| + L\|x_1\|\left(1+\max\left\{1,\frac{1}{|\mu|}\right\}\right) + \frac{1+\lambda}{4}. \tag{67}$$

(ii) Let $\theta \in (\theta_i, s_i]$, for some $i \in \mathbb{N}$. In view of (56), we have

$$\big\|g_i\big(\theta, x(\theta_i^-)\big)\big\| \le \eta(\lambda+1), \tag{68}$$

where $\eta = \sup_{t\in J} \|h^*(t)\|$. Moreover, from (15), it follows that

$$\sup_{i\in\mathbb{N}} \big\| g_i'\big(s_i, x\big(\theta_i^-\big)\big)\big\| \le \kappa_2(\lambda + 1). \tag{69}$$

Next, let $\theta \in [s_i, \theta_{i+1}]$. In view of (4), (5) and (69), we arrive at

$$\begin{aligned}
&\big\| S_\alpha(\theta - s_i) g_i\big(s_i, x\big(\theta_i^-\big)\big) + K_\alpha(\theta - s_i) g_i'\big(s_i, x\big(\theta_i^-\big)\big)\big\| \\
&\quad \le \left(L\eta + L\kappa_2 \max\left\{1, \frac{1}{|\mu|}\right\}\right)(\lambda + 1).
\end{aligned} \tag{70}$$

Furthermore, by (6) and $(HF)^*$(iv), and (58), we obtain

$$\begin{aligned}
&\left\| \int_0^\theta T_\alpha(\theta - \tau) f(\tau)\, d\tau \right\| \\
&\quad \le L \max\left\{1, \frac{1}{|\mu|}\right\}\big(1 + \|x\|\big) \int_0^\theta \varphi(t)\, d\tau \\
&\quad \le \big(1 + \|x\|\big) L \max\left\{1, \frac{1}{|\mu|}\right\} \sup_{\theta\in J} \int_0^\theta \varphi(t)\, d\tau < \frac{\lambda + 1}{4}.
\end{aligned} \tag{71}$$

Similarly, we obtain

$$\sup_{i\in\mathbb{N}} \left\| \int_0^{s_i} T_\alpha(s_i - \tau) f(\tau)\, d\tau \right\| \le \frac{\lambda + 1}{4}. \tag{72}$$

As a result of (67), (68), (70)- (72), we arrive at

$$\begin{aligned}
\sup_{\theta\in J} \big\| y(\theta)\big\| &\le L\|x_0\| + L\|x_1\| \left(1 + \max\left\{1, \frac{1}{|\mu|}\right\}\right) + \frac{1 + \lambda}{4} \\
&\quad + \eta(\lambda + 1) + \left(L\eta + L\kappa_2 \max\left\{1, \frac{1}{|\mu|}\right\}\right)(\lambda + 1) \\
&\quad + \frac{\lambda + 1}{4} \\
&\le L\|x_0\| + L\|x_1\| \left(1 + \max\left\{1, \frac{1}{|\mu|}\right\}\right) + \eta \\
&\quad + L\eta + L\kappa_2 \max\left\{1, \frac{1}{|\mu|}\right\} + \frac{1}{2} \\
&\quad + \lambda\left[\eta + L\eta + L\kappa_2 \max\left\{1, \frac{1}{|\mu|}\right\} + \frac{1}{2}\right] \le \lambda.
\end{aligned}$$

Therefore, our aim in this step is achieved.

Now, as a result of Steps 1 and 2, Θ is a multivalued function from $D_\lambda \subseteq SAP_\omega PC(J, E)$ to the non-empty subsets of D_λ.

Step 3. Θ is closed (its graph is closed) on D_λ.

Let $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ be two sequences in D_λ with $x_n \to x$, $y_n \to y$ and $y_n \in \Theta(x_n)$, $\forall n \geq 1$. Then we have $f_n \in S^1_{F(\cdot,x_n(\cdot))}$ such that

$$y_n(\theta) = \begin{cases} S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f_n(\tau)\,d\tau, \quad \theta \in [0,\theta_1], \\ g_i(\theta, x_n(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ S_\alpha(\theta - s_i)g_i(s_i, x_n(\theta_i^-)) + K_\alpha(\theta - s_i)g_i'(s_i, x_n(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i - \tau)f_n(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f_n(\tau)\,d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases}$$

Let θ be a fixed point in $[0,\theta_1]$ and $J_\theta = [0,\theta]$. In view of $(HF)^*$(iv), we have

$$\|f_n(t)\| \leq \varphi(t)(1+\lambda), \quad \text{a.e. } t \in [0,\theta]. \tag{73}$$

Using similar arguments to Step 3 in the proof of Theorem 1, one can show, by (73), that $f_n \rightharpoonup f$ weakly in $L^2(J_\theta, E)$ and there is a sequence of convex combinations (z_n) of (f_n) such that $z_n \to f$, a.e. $t \in J_\theta$, and

$$\lim_{n\to\infty} \int_0^\theta T_\alpha(\theta-\tau)z_n(\tau)\,d\tau = \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau; \quad \theta \in [0,\theta_1]. \tag{74}$$

Now, due to the continuity of $g_i(\theta,\cdot)$, $g_i'(s_i,\cdot)$, $S_\alpha(\theta - s_i)$ and $K_\alpha(\theta - s_i)$, we arrive at

$$\lim_{n\to\infty} g_i\big(\theta, x_n\big(\theta_i^-\big)\big) = g_i\big(\theta, x\big(\theta_i^-\big)\big) \tag{75}$$

and

$$\begin{aligned} &\lim_{n\to\infty} S_\alpha(\theta - s_i)g_i\big(s_i, x_n\big(\theta_i^-\big)\big) + K_\alpha(\theta - s_i)g_i'\big(s_i, x_n\big(\theta_i^-\big)\big) \\ &\quad = S_\alpha(\theta - s_i)g_i\big(s_i, x\big(\theta_i^-\big)\big) + K_\alpha(\theta - s_i)g_i'\big(s_i, x\big(\theta_i^-\big)\big). \end{aligned} \tag{76}$$

Noting that (z_n) is a subsequence of (f_n), and hence by (74)–(76), there is a subsequence of (y_n) that converge to

$$y^*(t) = \begin{cases} S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ S_\alpha(\theta - s_i)g_i(s_i, x(\theta_i^-)) + K_\alpha(\theta - s_i)g_i'(s_i, x(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i - \tau)f(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases}$$

Because $y_n \to y$, we arrive at $y = y^*$. Moreover, $(HF)^*$(ii) ensures that $f(\tau) \in F(\tau, x(\tau))$, a.e. $\tau \in J$. So, $y \in \Theta(x)$.

Step 4. $\lim_{\theta\to\infty}\|y(\theta)\| = 0$ uniformly on D_λ.

Let $x \in D_\lambda$ and $y \in \Theta(x)$. According to the definition of Θ, there is $f \in S^1_{F(\cdot,x(\cdot))}$ such that y is given by (18). We consider two cases:

Case 1. $\theta \in (\theta_i, s_i]$, $i \in \mathbb{N}$. By (56), we get

$$\|y(\theta)\| = \|g_i(\theta, x(\theta_i^-))\| \leq (\lambda+1)\|h^*(\theta)\|. \tag{77}$$

Case 2. $\theta \in [s_i, \theta_{i+1}]$, $i \in \mathbb{N}$. In view of (3), (15), (56) and $(HF)^*$(iv), we find

$$\begin{aligned}
\|y(\theta)\| &= \|S_\alpha(\theta - s_i) g_i(s_i, x(\theta_i^-))\| \\
&\quad + \|K_\alpha(\theta - s_i) g_i'(s_i, x(\theta_i^-))\| \\
&\quad + \left\| \int_0^{s_i} T_\alpha(s_i - \tau) f(\tau)\, d\tau \right\| \\
&\quad + \left\| \int_0^{\theta} T_\alpha(\theta - \tau) f(\tau)\, d\tau \right\| \\
&\leq (\lambda+1)\eta \frac{L}{1+|\mu|(\theta - s_i)^\alpha} + (\lambda+1)\kappa_2 \frac{L(\theta - s_i + 1)}{1+|\mu|(\theta - s_i)^\alpha} \\
&\quad + L(1+\lambda) \int_0^{s_i} \frac{L(s_i - \tau)^{\alpha-1}}{1+|\mu|(s_i - \tau)^\alpha} \varphi(\tau)\, d\tau \\
&\quad + L(1+\lambda) \int_0^{\theta_i} \frac{L(\theta_i - \tau)^{\alpha-1}}{1+|\mu|(\theta_i - \tau)^\alpha} \varphi(\tau)\, d\tau.
\end{aligned} \tag{78}$$

It follows from (77) (78) and (57) that $\lim_{\theta\to\infty} \|y(\theta)\| = 0$ uniformly on D_λ.

Step 5. Let $D = \Theta(D_\lambda)$. In this step, we claim that the set $D_{|\overline{J_i}}$ is equicontinuous for every $i \in \mathbb{N}$, where

$$D_{|\overline{J_i}} = \{y^* \in C(\overline{J_i}, E) : y^*(\theta) = y(\theta), t \in J_i =]\theta_i, \theta_{i+1}], y^*(\theta_i) = y(\theta_i^+), y \in D\}.$$

Let $y^* \in D_{|\overline{J_i}}$. Then we have $x \in D$ and $f \in S^1_{F(\cdot,x(\cdot))}$ such that

$$y(\theta) = \begin{cases} S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ S_\alpha(\theta - s_i) g_i(s_i, x(\theta_i^-)) + K_\alpha(\theta - s_i) g_i'(s_i, x(\theta_i^-)) \\ \quad - \int_0^{s_i} T_\alpha(s_i - \tau) f(\tau)\,d\tau \\ \quad + \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}, \end{cases}$$

and $y^*(\theta_i) = y(\theta_i^+)$.

Case 1. Let $\theta_1, \theta_2 \in \overline{J_0} = [0, \theta_1]$ with $\theta_1 < \theta_2$. We have

$$\begin{aligned}
\|y^*(\theta_2) - y^*(\theta_1)\| &= \|y(\theta_2) - y(\theta_1)\| \\
&\leq \|S_\alpha(\theta_2) - S_\alpha(\theta_1)\| \|x_0\| + \|K_\alpha(\theta_2) - K_\alpha(\theta_1)\| \|x_1\| \\
&\quad + \left\| \int_0^{\theta_2} T_\alpha(\theta_2 - \tau) f(s)\, ds - \int_0^{\theta_1} T_\alpha(\theta_1 - \tau) f(\tau)\, ds \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \left\|S_\alpha(\theta_2)-S_\alpha(\theta_1)\right\|\|x_0\| + \left\|K_\alpha(\theta_2)-K_\alpha(\theta_1)\right\|\|x_1\| \\
&\quad + \int_{\theta_1}^{\theta_2} \left\|T_\alpha(\theta_2-\tau)\right\|\left\|f(\tau)\right\| d\tau \\
&\quad + \int_0^{\theta_1} \left\|T_\alpha(\theta_2-\tau)-T_\alpha(\theta_1-\tau)\right\|\left\|f(\tau)\right\| d\tau \\
&= Q_1 + Q_2 + Q_3. \qquad (79)
\end{aligned}$$

From the compactness of the families $\{S_\alpha(\theta):\theta>0\}$ and $\{K_\alpha(\theta):\theta>0\}$, we get

$$\lim_{\theta_2\to\theta_1} Q_1 = \lim_{\theta_2\to\theta_1} \left\|S_\alpha(\theta_2)-S_\alpha(\theta_1)\right\|\|x_0\| + \left\|K_\alpha(\theta_2)-K_\alpha(\theta_1)\right\|\|x_1\| = 0, \qquad (80)$$

and from the continuity of φ, we arrive at

$$\begin{aligned}
\lim_{\theta_2\to\theta_1} Q_2 &= \lim_{\theta_2\to\theta_1} \int_{\theta_1}^{\theta_2} \left\|T_\alpha(\theta_2-\tau)\right\|\left\|f(\tau)\right\| d\tau \\
&\leq (1+\lambda)L\max\left\{1,\frac{1}{|\mu|}\right\} \lim_{\theta_2\to\theta_1}\int_{\theta_1}^{\theta_2}\varphi(\tau)\,d\tau = 0. \qquad (81)
\end{aligned}$$

Moreover, since the family $\{T_\alpha(\theta):\theta>0\}$ is compact, we get

$$\begin{aligned}
\lim_{\theta_2\to\theta_1} Q_3 &= \lim_{\theta_2\to\theta_1} \int_0^{\theta_1} \left\|T_\alpha(\theta_2-\tau)-T_\alpha(\theta_1-\tau)\right\|\left\|f(\tau)\right\| d\tau \\
&\leq (1+\lambda)\lim_{\theta_2\to\theta_1}\int_0^{\theta_1}\left\|T_\alpha(\theta_2-\tau)-T_\alpha(\theta_1-\tau)\right\|\varphi(\tau)\,d\tau \\
&= 0. \qquad (82)
\end{aligned}$$

Equations (79)–(82) lead to $\lim_{\theta_2\to\theta_1}\|y^*(\theta_2)-y^*(\theta_1)\| = 0$.

Case 2. Let $\theta_1,\theta_2\in(\theta_i,s_i]$ $(i\in\mathbb{N})$ with $\theta_1<\theta_2$. From the fact that g_i is uniformly continuous on bounded sets,

$$\begin{aligned}
\lim_{\theta_2\to\theta_1}\left\|y^*(\theta_2)-y^*(\theta_1)\right\| &= \lim_{\theta_2\to\theta_1}\left\|y(\theta_2)-y(\theta_1)\right\| \\
&\leq \lim_{\theta_2\to\theta_1}\left\|g_i\big(\theta_2,x(\theta_i^-)\big)-g_i\big(\theta,x(\theta_i^-)\big)\right\| \\
&\leq \lim_{\theta_2\to\theta_1}\sup_{\|z\|\leq\lambda}\left\|g_i(\theta_2,z)-g_i(\theta_1,z)\right\| = 0. \qquad (83)
\end{aligned}$$

Case 3. Let $\theta_1,\theta_2\in(s_i,\theta_{i+1}]$ $(i\in\mathbb{N})$ with $\theta_1<\theta_2$.

$$\begin{aligned}
\lim_{\theta_2\to\theta_1}\left\|y^*(\theta_2)-y^*(\theta_1)\right\| &= \lim_{\theta_2\to\theta_1}\left\|y(\theta_2)-y(\theta_1)\right\| \\
&\leq \lim_{\theta_2\to\theta_1}\left\|S_\alpha(\theta_2-s_i)-S_\alpha(\theta_1-s_i)\right\|\left\|g_i\big(s_i,x(\theta_i^-)\big)\right\| \\
&\quad + \lim_{\theta_2\to\theta_1}\|K_\alpha(\theta_2-s_i)-K_\alpha(\theta_1-s_i)\|\left\|g_i'\big(s_i,x(\theta_i^-)\big)\right\| \\
&\quad + \lim_{\theta_2\to\theta_1}\left\|\int_0^{\theta_2}T_\alpha(\theta_2-\tau)f(s)\,ds-\int_0^{\theta_1}T_\alpha(\theta_1-\tau)f(s)\,ds\right\|. \qquad (84)
\end{aligned}$$

Again, from the compactness of the families $\{S_\alpha(\theta):\theta>0\}$ and $\{K_\alpha(\theta):\theta>0\}$, we have

$$\begin{aligned}
&\lim_{\theta_2\to\theta_1}\big\|S_\alpha(\theta_2-s_i)-S_\alpha(\theta_1-s_i)\big\|\,\big\|g_i\big(s_i,x\big(\theta_i^-\big)\big)\big\| \\
&\quad=\lim_{\theta_2\to\theta_1}\big\|K_\alpha(\theta_2-s_i)-K_\alpha(\theta_1-s_i)\big\|\,\big\|g_i'\big(s_i,x\big(\theta_i^-\big)\big)\big\| \\
&\quad=0.
\end{aligned}\tag{85}$$

Furthermore, by repeating the arguments employed in Case 1, one can show that

$$\lim_{\theta_2\to\theta_1}\left\|\int_0^{\theta_2}T_\alpha(\theta_2-\tau)f(s)\,ds-\int_0^{\theta_1}T_\alpha(\theta_1-\tau)f(s)\,ds\right\|=0.\tag{86}$$

Equations (84) and (86) ensure that $\lim_{\theta_2\to\theta_1}\|y^*(\theta_2)-y^*(\theta_1)\|$. Therefore, $T_{|\overline{J_i}}$ is equicontinuous for any $i\in\mathbb{N}$.

Step 6. Our goal in this step is showing that, for any $i\in\mathbb{N}$ and any $\theta\in J_i$, the set $Z_\theta^i:=\{y(\theta):y\in D=\Theta(D_\lambda)\}$ is relatively compact in E.

Case 1. Let $\theta\in J_0=[0,\theta_1]$. If $\theta=0$, then $Z_\theta^1=\{x_0\}$ is compact. Let $\theta\in(0,\theta_1]$. be a fixed point. We have

$$\begin{aligned}
Z_\theta^1&=\big\{y(\theta):y\in\Theta(D_\lambda)\big\} \\
&=\left\{S_\alpha(\theta)x_0+K_\alpha(\theta)x_1+\int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau:f\in S^1_{F(\cdot,x(\cdot))},x\in D_\lambda\right\}.
\end{aligned}$$

Now, for any $\delta\in(0,\theta)$, let

$$Z_{\theta,\delta}^1:=\left\{S_\alpha(\theta)x_0+K_\alpha(\theta)x_1+\int_0^{\theta-\delta}T_\alpha(\theta-\tau)f(\tau)\,d\tau:f\in S^1_{F(\cdot,x(\cdot))},x\in D_\lambda\right\}.$$

Notice that $\|f(\tau)\|\leq(1+\lambda)\varphi(\tau)$, for any $\tau\in[0,\theta-\delta]$, any $n\geq1$. Because φ is continuous on $[0,\theta]$, the set $\{f_n(\tau):n\geq1\}$ is bounded, and hence, by the compactness of $T_\alpha(\theta-\tau)$; $\tau\in[0,\theta-\delta]$, the set $Z_{\theta,\delta}^1$ is relatively compact in E. Moreover, for any $x\in D_\lambda$ and any $f\in S^1_{F(\cdot,x(\cdot))}$, we get from (6), $(HF)^*$(iv) and the continuity of φ,

$$\begin{aligned}
&\lim_{\delta\to0}\left\|\int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau-\int_0^{\theta-\delta}T_\alpha(\theta-\tau)f(\tau)\,d\tau\right\| \\
&\quad\leq L\max\left\{1,\frac{1}{|\mu|}\right\}(1+\lambda)\lim_{\delta\to0}\int_{\theta-\delta}^\theta\varphi(\tau)\,d\tau \\
&\quad=0.
\end{aligned}$$

Then there exist relatively compact sets that can be arbitrarily approximated to the set Z_θ^i. Then it is relatively compact in E.

Case 2. Let $\theta\in[s_i,\theta_i]$; $i\in\mathbb{N}$ be a fixed point. Since the set $\{x(\theta_i^-):x\in D_\lambda\}$ is bounded, then by $(Hg_i)^*$(ii), the set $\{y(\theta):y\in\Theta(x),x\in D_\lambda\}=\{g_i(\theta,x(\theta_i^-)):x\in D_\lambda\}$ is relatively compact in E.

Case 3. Let $\theta\in[\theta_i,s_{i+1}]$; $i\in\mathbb{N}$ be a fixed point. As in Case 2, the set $\{g_i(\theta,x(\theta_i^-)):x\in D_\lambda\}$ is relatively compact in E, and hence, by the compactness of $S_\alpha(\theta-s_i)$, we conclude that

$\{S_\alpha(\theta - s_i)g_i(s_i, x(\theta_i^-)) : x \in D_\lambda\}$ is relatively compact in E. Also, due to the compactness of $K_\alpha(t)$; $t > 0$, the set $\{K_\alpha(\theta - s_i)g_i'(s_i, x(\theta_i^-)) : x \in D_\lambda\}$ is relatively compact in E. Furthermore, using the same arguments as in Case 1, one can show that the sets $\{\int_0^\theta T_\alpha(\theta - \tau)f(\tau)\,d\tau : f \in S^1_{F(\cdot,x(\cdot))}, x \in D_\lambda\}$ and $\{\int_0^{s_i} T_\alpha(s_i - \tau)f(\tau)\,d\tau : f \in S^1_{F(\cdot,x(\cdot))}, x \in D_\lambda\}$ are relatively compact. As a result of this discussion, we conclude that, for any $i \in \mathbb{N}$ and any $t \in J_i$, the set $Z^i_\theta = \{y(\theta) : y \in \Theta(D_\lambda)\}$ is relatively compact in E.

Now, according to Lemma 4, Steps 4, 5 and 6 imply the set $T = \Theta(D_\lambda)$ is relatively compact in $SAP_\omega PC(J, E)$.

Now, as a result of Steps 1–6, we conclude that Θ is closed and completely continuous from D_λ to the family on non-empty convex compact of D_λ. Applying Lemma 3, Θ has a fixed point which is an S-asymptotically ω-periodic mild solution to Problem (1). □

Remark 4 For any $t \geq 0$, we have

$$\frac{t^{\alpha-1}}{1+|\mu|t^\alpha} \leq \begin{cases} \frac{2}{|\mu|(1+t)}; & \text{if } 0 < |\mu| \leq 1, \\ \frac{2}{1+t}; & \text{if } |\mu| > 1. \end{cases} \tag{87}$$

To clarify this, we consider the following cases:

Case 1. Let $0 < |\mu| \leq 1$. Then

$$\frac{t^{\alpha-1}}{1+|\mu|t^\alpha} \leq \begin{cases} t^{\alpha-1} \leq 1 \leq \frac{2}{|\mu|(1+t)}; & \text{if } 0 \leq t \leq 1, \\ \frac{t^{\alpha-1}}{|\mu|t^\alpha} = \frac{1}{|\mu|t} \leq \frac{2}{|\mu|(1+t)}; & \text{if } t > 1. \end{cases}$$

So,

$$\frac{t^{\alpha-1}}{1+|\mu|t^\alpha} \leq \frac{2}{|\mu|(1+t)}, \quad \forall t \geq 0.$$

Case 2. Let $|\mu| > 1$. In this case, we have

$$\frac{t^{\alpha-1}}{1+|\mu|t^\alpha} \leq \begin{cases} t^{\alpha-1} \leq 1 \leq \frac{2}{1+t}; & \text{if } 0 \leq t \leq 1, \\ \frac{t^{\alpha-1}}{|\mu|t^\alpha} = \frac{1}{|\mu|t} \leq \frac{1}{t} \leq \frac{2}{1+t}; & \text{if } t > 1, \end{cases}$$

which yields

$$\frac{t^{\alpha-1}}{1+|\mu|t^\alpha} \leq \frac{2}{1+t}, \quad \forall t \geq 0.$$

Then (87) holds.

4 Examples

Example 1 Let $\alpha = \frac{3}{2}$, $m = 4$, $\omega = 2\pi$, $J = [0, \infty)$, $s_i = i\frac{\pi}{2}$; $i \in \{0\} \cup \mathbb{N}$, and $\theta_i = (2i-1)\frac{\pi}{4}$; $i \in \mathbb{N}$. Observe that, for $i \in \mathbb{N}$, $s_{i+m} = s_{i+4} = (i+4)\frac{\pi}{2} = s_i + 2\pi$, and $\theta_{i+4} = (2(i+4)-1)\frac{\pi}{4} = \theta_i + 2\pi$. Let $\Omega = \{s = (s_1, s_2) : s_1^2 + s_2^2 \leq 1\}$, and $E = L^2(\Omega)$. Define an operator $A : D(A) \subseteq E \to E$ by

$$A(u) := \Delta u - u, \tag{88}$$

with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is known that (see [19]) A is a sectorial operator of type $\{M, \varphi, \alpha, \mu\}$ with $\mu = -1$ and $L = 3$. Let Z be a non-empty, compact and convex subset of E, $\upsilon = \sup_{z\in Z} \|z\|$. Consider the multivalued function $F : J \times E \to 2^E$ defined by

$$F(\theta, u) := \{v_{\theta,u,z} : z \in Z\}, \tag{89}$$

where for any $z \in Z$ any $(\theta, u) \in J \times E$, $v_{\theta,u,z} : \Omega \to \mathbb{R}$; $v_{\theta,u,z}(s) = \frac{1+\rho \sin u(s)}{(\theta+1)\upsilon} \|z\|$; $\rho > 0$. Since Z is a non-empty, convex and compact subset, the values of F also are. Moreover,

$$\|F(\theta, 0)\| = \sup_{z\in Z} \frac{1}{(\theta+1)\upsilon} \|z\| \leq \frac{1}{\theta+1} = \sigma(\theta), \quad \forall \theta \in J, \tag{90}$$

$$\begin{aligned}
h\big(F(\theta,u), F(\theta,v)\big) &\leq \left(\int_\Omega \left|v_{\theta,u,z}(s) - v_{\theta,v,z}(s)\right|^2 ds\right)^{\frac{1}{2}}, \quad \forall z \in Z \\
&= \frac{\rho}{\theta+1} \int_\Omega \left|\sin u(s) - \sin v(s)\right|^2 dx)^{\frac{1}{2}} \\
&\leq \frac{\rho}{\theta+1} \int_\Omega \left|u(s) - v(s)\right|^2 dx)^{\frac{1}{2}} \\
&= \frac{\rho}{\theta+1} \|u - v\|,
\end{aligned} \tag{91}$$

and

$$\begin{aligned}
h\big(F(\theta+2\pi,u), F(\theta,u)\big) &\leq \left(\int_\Omega \left|v_{\theta+2\pi,u,z}(s) - v_{\theta,u,z}(s)\right|^2 ds\right)^{\frac{1}{2}}, \quad \forall z \in Z \\
&= \int_\Omega \left|\frac{\rho \sin u(s) + 1}{\theta + 1 + 2\pi} - \frac{\rho \sin u(s) + 1}{\theta+1}\right|^2 dx)^{\frac{1}{2}} \\
&\leq \frac{1}{(\theta+1+2\pi)(\theta+1)} \left(\int_\Omega 2\pi \left|\rho \sin u(s) + 1\right|^2 ds\right)^{\frac{1}{2}} \\
&\leq \frac{2\pi(\rho+1)}{(\theta+1+2\pi)(\theta+1)} \big(\|u\| + 1\big) = L_2(\theta)\big(\|u\| + 1\big).
\end{aligned} \tag{92}$$

Due to (91) and (92), Eqs. (7) and (8) are verified with $L_1(\theta) = \frac{\rho}{\theta+1}$ and $L_2(\theta) = \frac{2\pi(\rho+1)}{(\theta+1+2\pi)(\theta+1)}$; $\theta \in [0, \infty)$.

Next, in view of (87), and since $|\mu| = 1$, it follows that

$$\frac{(\theta-\tau)^{\alpha-1}}{1 + |\mu|(\theta-\tau)^\alpha} \leq \frac{2}{1+\theta-\tau}, \quad \forall \theta \geq 0, \tau \in [0, \theta].$$

Therefore,

$$\begin{aligned}
&\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha} L_1(\tau)\, d\tau \\
&\leq 2\rho \int_0^\theta \frac{1}{1+\theta-\tau} \frac{1}{\tau+1}\, d\tau \\
&= 2\rho \int_0^\theta \frac{1}{\theta+2}\left[\frac{1}{1+\theta-\tau} + \frac{1}{\tau+1}\right] d\tau \\
&= 4\rho \frac{\ln|1+\theta|}{(\theta+2)},
\end{aligned}$$

which yields $\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}L_1(\tau)\,d\tau = 0$. Similarly, $\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}\sigma(\tau)\,d\tau = 0$. Hence, (9) and (15) are verified. Moreover,

$$\xi = \sup_{\theta\in J}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}L_1(\tau)\,d\tau \le 4\rho. \tag{93}$$

Now, again from (87), one has

$$\begin{aligned}
&\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^\alpha}L_2(\tau)\,d\tau\\
&\quad\le(\rho+1)\lim_{\theta\to\infty}\int_0^\theta \frac{1}{1+\theta-\tau}\left(\frac{2\pi}{(\tau+1+2\pi)(\tau+1)}\right)d\tau\\
&\quad\le(\rho+1)\lim_{\theta\to\infty}\int_0^\theta \left[\frac{1}{1+\theta-\tau}\frac{1}{\tau+1}-\frac{1}{1+\theta-\tau}\frac{1}{\tau+1+2\pi}\right]d\tau\\
&\quad=(\rho+1)\lim_{\theta\to\infty}\left[\frac{1}{\theta+2}\int_0^\theta\left[\frac{1}{1+\theta-\tau}+\frac{1}{\tau+1}\right]d\tau\right.\\
&\qquad\left.-\frac{1}{\theta+2+2\pi}\int_0^\theta\left[\frac{1}{1+\theta-\tau}+\frac{1}{\tau+1+2\pi}\right]d\tau\right]\\
&\quad=(\rho+1)\lim_{\theta\to\infty}\frac{2\ln|1+\theta|}{\theta+2}\\
&\qquad-(\rho+1)\lim_{\theta\to\infty}\frac{1}{\theta+2+2\pi}\bigl[\ln|1+\theta|+\ln|1+2\pi+\theta|-\ln(2\pi+1)\bigr]\\
&\quad=0,
\end{aligned} \tag{94}$$

which means that (16) holds.

Furthermore, for any $i\in\mathbb{N}$, define $g_i:[t_i,s_i]\times E\to E$ by

$$g_i(\theta,u)(s) := \frac{\varrho u(s)\sin i\theta}{i};\quad (\theta,u)\in[\theta_i,s_i]\times E, s\in\Omega, \tag{95}$$

where ϱ is a positive real number. Obviously for any $u\in E$, $\frac{d}{d\theta}(g_i(\theta,u))(s)=\varrho(\cos i\theta)u(s)$; $t\in J$. Notice that, for any $u\in E$,

$$\begin{aligned}
&\lim_{\substack{\theta\to\infty\\ i\to\infty}}\bigl\|g_{i+m}(\theta+2\pi,u)-g_i(\theta,u)\bigr\|\\
&\quad=\lim_{\substack{\theta\to\infty\\ i\to\infty}}\left(\int_\Omega\left|\frac{\varrho u(s)\sin(i+m)(\theta+2\pi)}{i+m}-\frac{\varrho u(s)\sin(i\theta)}{i}\right|^2 ds\right)^{\frac12}\\
&\quad\le\lim_{i\to\infty}\frac{2\varrho}{i}\|u\|=0.
\end{aligned}$$

Then (10) is satisfied. Moreover, for any $u_1,u_2\in E$,

$$\begin{aligned}
&\bigl\|g_i(\theta,u_1)-g_i(\theta,u_2)\bigr\|\\
&\quad=\frac{\varrho}{i}\left(\int_\Omega\bigl|u_1(s)\sin i\theta-u_2(s)\sin i\theta\bigr|^2 ds\right)^{\frac12}\\
&\quad\le\varrho\|u_1-u_2\|
\end{aligned} \tag{96}$$

and

$$\left\|g_i'(s_i,u_1)-g_i'(s_i,u_2)\right\| = \left(\int_\Omega \varrho^2\left|u_1(s)(\cos i\theta)-u_2(s)(\cos i\theta)\right|^2 ds\right)^{\frac{1}{2}} \\ \leq \varrho\|u_1-u_2)\|. \tag{97}$$

From (96) and (97), it follows that (11) and (12) hold with $N=\mathcal{N}=\varrho$. In addition,

$$\left\|g_i(\theta,u)\right\| = \left(\int_\Omega \left|\frac{\varrho u(s)\sin i\theta}{i}\right|^2 ds\right)^{\frac{1}{2}} \leq \frac{\varrho}{i}\|u\| \leq \varrho\left(\|u\|+1\right) \tag{98}$$

and

$$\left\|g_i'(s_i,z)\right\| = \left(\int_\Omega \left|\varrho u(s)\cos is_i\right|^2 ds\right)^{\frac{1}{2}} \leq \varrho\|u\| \leq \varrho\left(\|u\|+1\right). \tag{99}$$

By (98) and (99), we arrive at (13) and (14) are verified where $\kappa_1=\kappa_1=\varrho$.

Now, all the assumptions of Theorem 1 are satisfied, so the problem

$$\begin{cases} {}^cD^\alpha_{s_i,\theta}x(\theta,s)\in Ax(\theta,s)+F(\theta,x(\theta)), & \text{a.e. } \theta\in(s_i,\theta_{i+1}], i\in\mathbb{N}\cup\{0\}, s\in\Omega, \\ x(\theta_i^+,s)=g_i(\theta_i,x(\theta_i^-,s)), & i\in\mathbb{N}, s\in\Omega, \\ x(\theta,s)=g_i(\theta,x(\theta_i^-,s)), & \theta\in(\theta_i s_i], i\in\mathbb{N}, s\in\Omega, \\ x(0,s)=x_0(s), \quad x'(0)=x_1(s); & s\in\Omega, \\ x(\theta,s)=0, & \theta\in J, s\in\partial\Omega, \end{cases}$$

has an S-asymptotically 2π-periodic mild solution, $x:J\to L^2(\Omega)$, provided that

$$8\rho+3\varrho<\frac{1}{3}, \tag{100}$$

where A, F, g_i are given by (88), (90) and (94). By choosing ρ and ϱ sufficiently small, we can arrive at (100).

Example 2 Let A, α, E, m, $\omega=2\pi$, J, s_i, θ_i; $i\in\mathbb{N}$ be as in Example 1, Z be a non-empty convex compact subset of E, the families $\{S_\alpha(t):t>0\}$, $\{K_\alpha(t):t>0\}$ and $\{T_\alpha(t):t>0\}$ are compact [19]. Define a multivalued function $F:J\times E\to P_{ck}(E)$ by

$$F(\theta,u)=\frac{\rho(\|u\|+1)}{\upsilon(1+\theta)^{\frac{3}{2}}}Z, \tag{101}$$

where $\rho>0$ and $\upsilon=\operatorname{Sup}\{\|z\|:z\in Z\}$. Clearly for every $x\in E$, $\theta\to F(\theta,x)$ is strongly measurable, $F(\theta,\cdot)$ is upper semicontinuous and, for any $u\in PC(J,E)$, the function $f(\theta)=\frac{\rho(\|u\|+1)}{\upsilon(1+\theta)^{\frac{3}{2}}}z_0$; $z_0\in Z$ is locally integrable and $f(\theta)\in F(\theta,u(\theta))$; $\theta\in J$. Moreover, in view of (101), for any $u\in E$ and any $\theta\in J$,

$$\left\|F(\theta,u)\right\| \leq \frac{\rho(\|u\|+1)}{(1+\theta)^{\frac{3}{2}}}=\varphi(\theta)\left(1+\|u\|\right), \tag{102}$$

where

$$\varphi(\theta)=\frac{\rho}{(1+\theta)^{\frac{3}{2}}};\quad \theta\in J. \tag{103}$$

We show that (57) and (58) are verified. In view of (87), and by arguing as in (94), one has

$$\begin{aligned}
&\lim_{\theta\to\infty}\int_0^{\theta}\frac{(\theta-\tau)^{\alpha-1}}{1+|\mu|(\theta-\tau)^{\alpha}}\big(\varphi(\tau+2\pi)+\varphi(\tau)\big)\,d\tau\\
&\quad=\rho\lim_{\theta\to\infty}\int_0^{\theta}\left[\frac{1}{1+\theta-\tau}\left(\frac{1}{(\tau+1+2\pi)^{\frac{3}{2}}}+\frac{1}{(\tau+1)^{\frac{3}{2}}}\right)\right]d\tau\\
&\quad\le\rho\lim_{\theta\to\infty}\int_0^{\theta}\left[\frac{1}{1+\theta-\tau}\left(\frac{1}{\tau+1+2\pi}+\frac{1}{\tau+1}\right)\right]d\tau\\
&\quad\le 2\rho\lim_{\theta\to\infty}\int_0^{\theta}\frac{1}{1+\theta-\tau}\frac{1}{\tau+1}\,d\tau=0.
\end{aligned}$$

Then (57) holds. Furthermore,

$$\sup_{\theta\in J}\int_0^{\theta}\varphi(\tau)\,d\tau=\sup_{\theta\in J}\frac{2\rho}{\sqrt{1+\theta}}=2\rho. \tag{104}$$

Next, let $K:D(K)=E\to E$ be a linear bounded compact operator and for any $i\in\mathbb{N}$, define $g_i:[\theta_i,s_i]\times E\to E$ by

$$g_i(\theta,u)(s)=\frac{(Ku)(s)}{i(1+\theta)},\quad \forall s\in\Omega. \tag{105}$$

Notice that

$$g_i'(s_i,u)(s)=\frac{-(Ku)(s)}{i(1+s_i)^2},\quad \forall s\in\Omega. \tag{106}$$

In view of (105) and (106), we get

$$\begin{aligned}
&\lim_{\substack{\theta\to\infty\\ i\to\infty}}\big\|g_{i+m}(\theta+2\pi,u)-g_i(\theta,u)\big\|\\
&\quad=\|K\|\lim_{\substack{\theta\to\infty\\ i\to\infty}}\Big(\int_{\Omega}\left|\frac{u(s)}{(i+2\pi)(1+\theta+2\pi)}-\frac{u(s)}{i(1+\theta)}\right|^2ds\Big)^{\frac{1}{2}}\\
&\quad=\|K\|\|u\|\lim_{\substack{\theta\to\infty\\ i\to\infty}}\left|\frac{1}{(i+2\pi)(1+\theta+2\pi)}-\frac{1}{i(1+\theta)}\right|=0,\\
&\big\|g_i(\theta,u_1)-g_i(\theta,u_2)\big\|\\
&\quad=\left(\int_{\Omega}\left|\frac{(Ku_1)(s)}{i(1+\theta)}-\frac{(Ku_2)(s)}{i(1+\theta)}\right|^2ds\right)^{\frac{1}{2}}\\
&\quad\le\frac{\|K|}{i(1+\theta)}\|u_1-u_2\|\le\|K\|\|u_1-u_2\|,
\end{aligned}$$

$$\begin{aligned}\|g_i(\theta,u)\| &= \left(\int_\Omega \left|\frac{(Ku)(s)}{i(1+\theta)}\right|^2 ds\right)^{\frac{1}{2}}\\ &\le \frac{\|K\|\|u\|}{1+\theta} = h^*(\theta)\|u\|,\end{aligned}$$

and

$$\begin{aligned}\|g_i'(s_i,u)\| &= \left(\int_\Omega \left|\frac{(Ku)(s)}{i(1+s_i)^2}\right|^2 ds\right)^{\frac{1}{2}}\\ &\le \frac{\|K\|\|u\|}{i(1+s_i)^2} \le \|K\|\big(\|u\|+1\big).\end{aligned}$$

Then (10), (11), (14) and (56) are verified with $N = \mathcal{N} = \|K\|$ and $h^*(\theta) = \frac{\|K\|}{1+\theta}$; $\theta \in J$ and $\eta = \kappa_2 = \|K\|$. Notice that, by the compactness of K, the function $g_i(\theta,\cdot)$; $i \in \mathbb{N}$ is compact, and hence all assumptions of Theorem 2 are satisfied. So, by applying Theorem 2, Problem (1), where A as be in example (1) and F, g_i are given by (102) and (105), has an S-asymptotically 2π-periodic mild solution provided that $2\rho < \frac{1}{4L} = \frac{1}{12}$ and $\|k\| < \frac{1}{14}$.

Example 3 Let A, α, E, m, $\omega = 2\pi$, J, Ω, s_i, θ_i;i $\in \mathbb{N}$ be as in Example 1, Z a non-empty convex compact subset of E and x_0, x_1 two fixed elements of E. For any $(\theta,s) \in (J,\Omega)$ and any $x : J \to E$, we denote $x(\theta)(s)$ by $x(\theta,s)$. Consider the impulsive semilinear differential inclusion:

$$\begin{cases} {}^cD^\alpha_{s_i,\theta}x(\theta,s) \in \Delta x(\theta,s) - x(\theta,s) + \frac{\cos x(\theta,s)}{40(1+\theta)}Z, \\ \quad \text{a.e. } \theta \in (s_i,\theta_{i+1}], i \in \mathbb{N}\cup\{0\}, s \in \Omega, \\ x(\theta_i^+,s) = \frac{x(\theta_i^-,s)\sin i\theta_i}{40i}, \quad i \in \mathbb{N}, s \in \Omega, \\ x(\theta,s) = \frac{x(\theta_i^-,s)\sin i\theta}{40i}, \quad \theta \in (\theta_i s_i], i \in \mathbb{N}, s \in \Omega, \\ x(0,s) = x_0(s), \qquad x'(0) = x_1(s); \quad s \in \Omega, \\ x(\theta,s) = 0, \quad \theta \in J, s \in \partial\Omega. \end{cases} \tag{107}$$

Let $F : J \times E \to 2^E$ be defined by

$$F(\theta,u) = \frac{\cos u(s)}{40(1+\theta)}Z. \tag{108}$$

Obviously, F verifies (i) and (ii) of (HF). Moreover,

$$\|F(\theta,0)\| \le \frac{1}{40(1+\theta)} = \sigma(\theta), \quad \forall\theta \in J, \tag{109}$$

$$h\big(F(\theta,u),F(\theta,v)\big) \le \frac{1}{40(1+\theta)}\|u-v\|, \tag{110}$$

and

$$\begin{aligned} h\big(F(\theta+2\pi,u),F(\theta,u)\big) &\le \frac{1}{40(\theta+1+2\pi)(\theta+1)}\|u\| \\ &\le \frac{1}{40(\theta+1+2\pi)(\theta+1)}\big(\|u\|+1\big). \end{aligned} \tag{111}$$

It shows that (7) and (8) are satisfied with $L_1(\theta) = \frac{1}{40(1+\theta)}$ and $L_2(\theta) = \frac{1}{40(\theta+1+2\pi)(\theta+1)}$; $\theta \in [0, \infty)$. Define

$$g_i(\theta, u)(s) := \frac{u(s)\sin i\theta}{40i}; \quad (\theta, u) \in [\theta_i, s_i] \times E, s \in \Omega. \tag{112}$$

Using the same arguments as in Example 1, one can demonstrate that (9) and (10)–(16) are obtained with $N = \mathcal{N} = \kappa_1 = \kappa_1 = \frac{1}{40}$ and $\xi \le \frac{1}{10}$. Notice that $L = 3$. So, $L(2\xi + N. + (1 + \max\{1, \frac{1}{|\mu|}\})\mathcal{N}) < 1$ By applying Theorem 1, Problem (107) has an S-asymptotically 2π-periodic mild solution.

Conclusion Two existence results of S-asymptotically ω-periodic of mild solutions to non-instantaneous impulsive semilinear differential inclusions of order $1 < \alpha < 2$ and generated by sectorial operators are given This work generalizes much recent work such as [18–20] to the case when there are impulse effects and the right-hand side is a multivalued function. Moreover, our technique can be used to develop the work in [12, 15–17, 21, 23–25] to the case when the linear part is a sectorial operator, the nonlinear part is a multivalued function and we have impulse effects. There are many directions for future work, for example:

1— With the help of technique in [1], we study the existence of solutions for Problem (1) on a time scales.

2— Investigation an existence theorem for a nonlinear singular-delay-fractional differential equation considered in [42, 43], when it contains a sectorial operator as a linear term and the nonlinear term becomes a multivalued function instead of single-valued function.

3— With the help of technique in [3], discuss the numerical solutions for Problem (1) on a closed bounded interval.

4— Study the S-asymptotically periodic solutions to Problem (1) when the sectorial operator is replaced by almost sectorial.

5— Study the S-asymptotically periodic solutions to Problem (1) when it involves p-Laplacian operator φ_p as well as when the Caputo derivative is replaced by the ψ-Caputo or ψ-Riemann–Liouville derivative. For contributions on BVP involving the ψ-Riemann–Liouville derivative, see [3] and for references on BVP containing the p-Laplacian operator φ_p, see [6, 8].

Acknowledgements
This research has been funded by the Scientific Research Deanship at University of Ha'il—Kingdom of Saudi Arabia through project number BA-2012.

Funding
Not applicable.

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors contributed equally and read and approved the final version of the manuscript.

Authors' information
Not applicable.

Author details
[1]Department of Mathematics, Faculty of Science, University of Ha'il, Hail, Saudi Arabia. [2]Department of Mathematics, Guizhou University, Guiyang, China. [3]Department of Mathematics and Statistics, College of Science, King Faisal University, Al-Ahsa, Saudi Arabia.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 19 February 2021 Accepted: 22 June 2021 Published online: 13 July 2021

References

1. Agarwal, P., Ammi, M.S., Asad, J.: Existence and uniqueness results on time scales for fractional nonlocal thermistor problem in the conformable sense. Adv. Differ. Equ. **2021**, 162 (2021)
2. Sunarto, A., Agarwal, P., Sulaiman, J., Chew, J.V., Aruchunan, E.: Iterative method for solving one-dimensional fractional mathematical physics model via quarter-sweep and PAOR. Adv. Differ. Equ. **2021**, 147 (2021)
3. Rezapour, S., Etemad, S., Tellab, B., Agarwal, P., Garcia Guirao, J.L.: Numerical solutions caused by DGJIM and ADM methods for multi-term fractional BVP involving the generalized ψ-RL-operators. Symmetry **13**(4), 532 (2021)
4. Agarwal, P., Berdyshev, A., Karimov, E.: Solvability of a non-local problem with integral transmitting condition for mixed type equation with Caputo fractional derivative. Results Math. **71**(3), 1235–1257 (2017)
5. Agarwal, R., Hristova, S., O'Regan, D.: A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations. Fract. Calc. Appl. Anal. **19**(2), 290–318 (2016)
6. Khan, H., Jarad, F., Abdeljawad, T., Khan, A.: A singular ABC-fractional differential equation with p-Laplacian operator. Chaos Solitons Fractals **129**, 56–61 (2019)
7. Khan, A., Gomez-Aguilar, J.F., Abdeljawad, T., Khan, H.: Stability and numerical simulation of a fractional order plant-nectar-pollinator model. Alex. Eng. J. **59**(1), 49–59 (2020)
8. Khan, H., Tunc, C., Chen, W., Khan, A.: Existence theorems and Hyers–Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator. J. Appl. Anal. Comput. **8**(4), 1211–1226 (2018)
9. Diblik, J., Fečkan, M., Pospish, M.: Non-existence of periodic solutions and S-asymptotically periodic solutions in fractional difference equations. Appl. Math. Comput. **257**, 230–240 (2015)
10. Kang, Y.M., Xie, Y., Lu, J.C., Jiang, J.: On the non-existence of non-constant exact periodic solutions in a class of the Caputo fractional-order dynamical system. Nonlinear Dyn. **82**, 1259–1267 (2015)
11. Kaslik, E., Srivastava, H.M., Trujillo, J.J.: Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions. Nonlinear Anal., Real World Appl. **13**, 1489–1497 (2012)
12. Ren, L., Wang, J.R., Fečkan, M.: Asymptotically periodic behavior solutions for Caputo type fractional evolution equations. Fract. Calc. Appl. Anal. **27**(5), 1294–1312 (2018)
13. Tavazoel, M.S., Haeri, M.: A proof for nonexistence of periodic solutions in time invariant fractional order system. Automatica **45**, 1886–1890 (2009)
14. Wang, J.-R., Fečkan, M., Zhou, Y.: Non-existence of periodic solutions and asymptotically periodic solutions of fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **18**, 246–256 (2013)
15. Li, F., Liang, J., Wang, H.: S-Asymptotically θ-periodic solution for fractional differential equations of order $q \in (0,1)$ with finite delay. Adv. Differ. Equ. **2017**, 83 (2017)
16. Maghsoodi, S., Neamaty, A.: Existence and uniqueness of S-asymptotically w-periodic solution for fractional semilinear problem. J. Appl. Comput. Math. **8**(2), 1–5 (2019)
17. Mu, J., Zhou, Y., Peng, L.: Periodic solutions and S-asymptotically periodic solutions to fractional evolution equations. Discrete Dyn. Nat. Soc. **2017**, Article ID 1364532 (2017)
18. Cuevas, C., César de Souza, J.: S-Asymptotically T-periodic solutions of semilinear fractional integro-differential equations. Appl. Math. Lett. **22**, 865–870 (2009)
19. Ren, L., Wang, J.R., O'Regan, D.: Asymptotically periodic behavior of solutions of fractional evolution equations of order $1 < \alpha < 2$. Math. Slovaca **69**, 599–610 (2019)
20. Wu, Z.-H.: Asymptotic periodicity for a class of fractional integro-differential equations. J. Nonlinear Sci. Appl. **9**, 506–517 (2016)
21. Wang, H., Li, F.: S-Asymptotically θ-periodic solutions for delay fractional differential equations with almost sectorial operator. Adv. Differ. Equ. **2016**, 315 (2016)
22. Rogovchenko, S.P., Rogovchenko, Y.V.: Asymptotic behavior of solutions of second order nonlinear differential equations. Port. Math. **57**(1), 17–33 (2000)
23. Wang, D., Xiao, A., Sun, S.: Asymptotic behavior of solutions to time fractional neutral functional differential equations. J. Comput. Appl. Math. **382**, 113086 (2021)
24. Abou-El-Ela, A.M., Sadek, A.-R., Mahmoud, A.M., Farghaly, E.S.: S-Asymptotic stability of solutions for a certain non-autonomous second-order stochastic delay differential equation. Turk. J. Math. **41**, 576–584 (2017)
25. Cuevas, C., Pierri, M., Sepulveda, A.: Weighted S-asymptotically T-periodic solutions of a class of fractional differential equations. Adv. Differ. Equ. **2011**, 584874 (2011)

26. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
27. Hernandez, E., O'Regan, D.: On a new class of abstract impulsive differential equation. Proc. Am. Math. Soc. **141**, 1641–1649 (2013)
28. Wang, J.-R., Li, M., O'Regan, D.: Robustness for nonlinear evolution equation with non-instantaneous effects. Bull. Sci. Math. **159**, 102827 (2020)
29. Wang, J.-R., Ibrahim, A.G., O'Regan, D.: Global attracting solutions to Hilfer fractional non-instantaneous impulsive semilinear differential inclusions of Sobolev type and with nonlocal conditions. Nonlinear Anal., Model. Control **24**(5), 775–803 (2019)
30. Wang, J.-R., Ibrahim, A.G., O'Regan, D.: Hilfer type fractional differential switched inclusions with non-instantaneous impulsive and nonlocal conditions. Nonlinear Anal., Model. Control **23**(6), 921–941 (2018)
31. Wang, J.-R., Ibrahim, A.G., O'Regan, D., Elmandouha, A.A.: Nonlocal fractional semilinear differential inclusions with noninstantaneous impulses and of order $1 < \alpha < 2$. J. Nonlinear Sci. Numer. Simul. (2020). https://doi.org/10.1515/ijnsns-2019-0179
32. Chalishajar, D.N., Malar, K., Ilavaras, R.: Existence and controllability results of impulsive fractional neutral integro-differential equation with sectorial operator and infinite delay. AIP Conf. Proc. **2159**, 030006 (2019)
33. Wang, J.-R., Ibrahim, A.G., Fečkan, M.: Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces. Appl. Math. Comput. **257**, 103–112 (2015)
34. Yan, Z.: Approximate controllability of fractional impulsive partial neutral stochastic differential inclusions with state-dependent delay and fractional sectorial operators. Numer. Funct. Anal. Optim. **7**(12), 1590–1639 (2016)
35. Yan, Z., Han, L.: Optimality of fractional impulsive partial stochastic differential systems with analytic sectorial operators and controls. Optimization **68**(4), 853–894 (2019)
36. Agarwal, P., Agarwal, R.P., Ruzhansky, M.: Special Functions and Analysis of Differential Equations. Chapman & Hall/CRC, Boca Raton (2020)
37. Agarwal, P., Dragomir, S.S., Jleli, M., Samet, B.: Advances in Mathematical Inequalities and Applications. Birkhäuser, Basel (2018)
38. Agarwal, P., Ramadan, M., Osheba, H.S., Chu, Y.M.: Study of hybrid orthonormal functions method for solving second kind fuzzy Fredholm integral equations. Adv. Differ. Equ. **2020**, 533 (2020)
39. Ruzhansky, M., Cho, Y.J., Agarwal, P., Area, I.: Advances in Real and Complex Analysis with Applications. Birkhäuser, Basel (2017)
40. Ali, M.A., Abbas, M., Budak, H., Agarwal, P., Murtaza, G., Chu, Y.M.: New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Adv. Differ. Equ. **2021**, 64 (2021)
41. Wu, S., Li, C., Agarwal, P.: Relaxed modulus-based matrix splitting methods for the linear complementarity problem. Symmetry **13**(3), 503 (2021)
42. Khan, H., Khan, A., Abdeljawad, T., Alkhazzan, A.: Existence results in Banach space for a nonlinear impulsive system. Adv. Differ. Equ. **2019**, 18 (2019)
43. Khan, H., Tunc, C., Khan, A.: Green function's properties and existence theorems for nonlinear singular-delay-fractional differential equations. Discrete Contin. Dyn. Syst., Ser. S **13**(9), 2475–2487 (2020)
44. Khan, H., Gomez-Aguillar, J.F., Abdeljawad, T., Khan, A.: Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation. Fractals **28**(8), 2040048 (2020)
45. Shu, X.B., Wang, Q.: The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$. Comput. Math. Appl. **64**, 2100–2110 (2012)
46. Covitz, H., Nadler, S.B.: Multivalued contraction mapping in generalized metric spaces. Isr. J. Math. **8**, 5–11 (1970)
47. Casting, C., Vladier, M.: Convex Analysis and Measurable Multifunctions. Lect. Notes in Math., vol. 580. Springer, Berlin (1977)
48. Hiai, F., Umegaki, H.: Integrals conditional expectation and martingales of multivalued functions. J. Multivar. Anal. **7**, 149–182 (1977)
49. Kamenskii, M., Obukhowskii, V., Zecca, P.: Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces. de Gruyter, Berlin (2001)
50. Hernandez, E.: Global solutions for abstract impulsive neutral differential equations. Math. Comput. Model. **53**, 196–204 (2011)