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New oscillation theorems for a class of even-order neutral delay differential equations

Mona Anis^{1*} and Osama Moaaz^{1,2}

*Correspondence:

mona_anis1985@yahoo.com

¹Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
Full list of author information is available at the end of the article

Abstract

In this work, we study the oscillatory behavior of even-order neutral delay differential equations $v^n(l) + b(l)u(\eta(l)) = 0$, where $l \geq l_0$, $n \geq 4$ is an even integer and $v = u + a(u \circ \mu)$. By deducing a new iterative relationship between the solution and the corresponding function, new oscillation criteria are established that improve those reported in (T. Li, Yu.V. Rogovchenko in *Appl. Math. Lett.* 61:35–41, 2016).

MSC: 34C10; 34K11

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1 Introduction

In this paper, we consider the even-order neutral differential equations

$$v^n(t) + b(t)u(\eta(t)) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, $n \geq 4$ is an even natural number and $v := u + a \cdot (u \circ \mu)$. Moreover, we suppose $a, b, \eta, \mu \in C([t_0, \infty), \mathbb{R})$, $0 < a(t) \leq a_0$, $b(t) \geq 0$, $\mu(t) \leq t$, $b(t)$ is not identically zero for large t , and $\lim_{t \rightarrow \infty} \mu(t) = \lim_{t \rightarrow \infty} \eta(t) = \infty$.

By a solution of Eq. (1.1), we mean a function $u \in C([t_*, \infty), \mathbb{R})$, $t_* \geq t_0$, which has the property $v \in C^n([t_0, \infty), \mathbb{R})$, and $u(t)$ satisfies Eq. (1.1) on $[t_*, \infty)$. We only focus on solutions of Eq. (1.1), which exist on $[t_0, \infty)$ and satisfy

$$\sup\{|u(t)| : t_u \leq t\} > 0 \quad \text{for every } t \geq t_u.$$

As is customary, a solution of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative on $[t_0, \infty)$ and otherwise, it is termed nonoscillatory.

The importance of studying neutral delay differential equations comes from their emergence when modeling many phenomena in different applied sciences, see [2, 3]. The qualitative theory of various classes of neutral differential equations has become an important area of research due to the fact that such equations arise in a variety of real world prob-

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lems such as in the study of non-Newtonian fluid theory and porous medium problems; see [4].

Very recently, a great development was found in the study of the oscillatory properties of solutions of second order neutral-delay differential equations; see for examples [5–17]. It would be interesting to extend this development to higher-order differential equations.

In 2016, Li and Rogovchenko [1] studied the oscillatory behavior of solutions of neutral delay equation (1.1). They used an approach similar to that used in [18], and established the relationship between u and v to have the form

$$u(t) \geq \frac{1}{a(\mu^{-1}(t))} \left(v(\mu^{-1}(t)) - \frac{v(\mu^{-1}(\mu^{-1}(t)))}{a(\mu^{-1}(\mu^{-1}(t)))} \right). \tag{1.2}$$

By using the comparison with the first-order delay equations, they obtained improved criteria over the previous ones in the literature.

In this paper, by improving the relationship (1.2), we establish a new criterion that improves the results in [1]. An example is given to illustrate the importance of our results.

In order to discuss our main results, we need the following auxiliary lemmas.

Lemma 1.1 ([19]) *Assume that $\psi \in C^n([t_0, \infty), (0, \infty))$ and $\psi^{(n)}(t)\psi^{(n-1)}(t) \leq 0$ for $t \geq t_1$. If $\lim_{t \rightarrow \infty} \psi(t) \neq 0$, then there exists a $t_\lambda \in [t_1, \infty)$ such that*

$$\psi(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |\psi^{(n-1)}(t)|, \tag{1.3}$$

for all $t \in [t_\lambda, \infty)$ and $\lambda \in (0, 1)$.

Lemma 1.2 ([20]) *Assume that the $\psi \in C^{(k+1)}([t_0, \infty))$ with $\psi^{(i)}(t) > 0$ for $i = 0, 1, 2, \dots, k$ and $\psi^{(k+1)}(t) \leq 0$ for all $t \geq t_1$. Then there exists a $t_\lambda \in [t_1, \infty)$ such that*

$$\frac{h(t)}{h'(t)} \geq \frac{\lambda t}{k},$$

for all $t \in [t_\lambda, \infty)$ and $\lambda \in (0, 1)$.

Lemma 1.3 ([21]) *If $\psi \in C^n([t_0, \infty), (0, \infty))$, $\psi^{(n)}(t)$ is eventually of one sign for large t , then there exist a $t_u \geq t_0$ and an integer $t \in [0, n]$ with $(-1)^{n+t}\psi^{(n)}(t) \geq 0$, such that $t > 0$ yields*

$$\psi^{(k)}(t) > 0, \quad t \geq t_u, k = 0, 1, \dots, t - 1,$$

and

$$t \leq n - 1 \quad \text{yields} \quad (-1)^{t+k}\psi^{(k)}(t) > 0, \quad t \geq t_u, k = t, t + 1, \dots, n - 1.$$

2 Main results

Through this section, we will be using the next notation: $\mu^{[-1]} := \mu^{-1}$, $\mu^{[-h+1]} := \mu^{-1} \circ \mu^{-[h]}$ for $h = 1, 2, \dots$,

$$\tilde{a}(t) := \sum_{k=1}^{n/2} \frac{1}{\sum_{m=1}^{2k-1} a(\mu^{-[m]}(t))} \left(1 - \frac{1}{a(\mu^{-[2k]}(t))} \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{(n-1)/\lambda_0} \right),$$

$$\widehat{a}(t) := \sum_{k=1}^{n/2} \frac{1}{\sum_{m=1}^{2k-1} a(\mu^{-[m]}(t))} \left(1 - \frac{1}{a(\mu^{-[2k]}(t))} \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{1/\lambda_2} \right),$$

and

$$B(t) = \min\{b(\eta^{-1}(t)), b(\eta^{-1}(\mu(t)))\},$$

where n is an even positive integer and $\lambda_1, \lambda_2 \in (0, 1)$.

Lemma 2.1 *Assume that u is an eventually positive solution. Then, we have two cases for the derivatives of v as*

- (1) $v(t) > 0, \quad v'(t) > 0, \quad v''(t) > 0, \quad v^{(n-1)}(t) > 0;$
- (2) $v(t) > 0, \quad (-1)^{k+1} v^{(k)}(t) > 0 \quad \text{for all } k \in \{1, 2, \dots, n-3\}.$

Proof From the definition of v , we get that $v(t) > 0$ for large t . From Eq. (1.1), $v^{(n)}(t) \leq 0$. Based on the facts that n is even and $v''(t) \leq 0$, cases (1) and (2) are deduced directly from Lemma 1.3. □

Theorem 2.1 *Assume that $\mu'(t) > 0$ and there exists an even integer m such that*

$$\frac{1}{a(\mu^{-[2k]}(t))} \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{(n-1)/\lambda_0} \leq 1, \tag{2.1}$$

for all $k = 1, 2, \dots, n/2$. Suppose that there exist functions $\chi \in C^1([t_0, \infty), \mathbb{R})$ and $\varkappa \in C^1([t_0, \infty), \mathbb{R})$ satisfying

$$\varkappa(t) \leq \eta(t), \quad \varkappa(t) < \mu(t), \quad \lim_{t \rightarrow \infty} \varkappa(t) = \infty \tag{2.2}$$

and

$$\chi(t) \leq \eta(t), \quad \chi(t) < \mu(t), \quad \chi'(t) \geq 0, \quad \lim_{t \rightarrow \infty} \chi(t) = \infty. \tag{2.3}$$

If there exist $\lambda_i \in (0, 1), i = 0, 1, 2$, such that the first-order delay equations

$$G'(t) + \frac{\lambda_1}{(n-1)!} b(t) \widetilde{a}(\eta(t)) (\mu^{-1}(\varkappa(t)))^{n-1} G(\mu^{-1}(\varkappa(t))) = 0 \tag{2.4}$$

and

$$\phi'(t) + \frac{\lambda_2}{(n-3)!} \mu^{-1}(\chi(t)) \phi(\mu^{-1}(\chi(t))) \int_t^\infty (s-3)^{n-3} b(s) \widehat{a}(\eta(s)) \, ds = 0 \tag{2.5}$$

are oscillatory, every solution of Eq. (1.1) is oscillatory.

Proof Assume that Eq. (1.1) has an eventually positive solution u . It follows from (1.1) that $v^n(t) = -b(t)u(\eta(t)) \leq 0$. Thus, using Lemma 2.1, we see that there are two cases for the derivatives of v for large t , either (1) or (2).

Assume that (1) holds. Since v is an increasing positive function, we obtain $\lim_{t \rightarrow \infty} v(t) \neq 0$. Therefore, by virtue of Lemma 1.1, we get

$$v(t) \geq \frac{\lambda}{(n-1)!} t^{(n-1)} v^{(n-1)}(t), \tag{2.6}$$

for every $\lambda \in (0, 1)$ and for all large t . It follows from the definition of $v(t)$ that

$$u(t) = \frac{1}{a(\mu^{-1}(t))} (v(\mu^{-1}(t)) - u(\mu^{-1}(t)))$$

and

$$u(t) = \frac{v(\mu^{-1}(t))}{a(\mu^{-1}(t))} - \frac{1}{a(\mu^{-1}(t))} \left(\frac{v(\mu^{-1}(\mu^{-1}(t)))}{a(\mu^{-1}(\mu^{-1}(t)))} - \frac{u(\mu^{-1}(\mu^{-1}(t)))}{a(\mu^{-1}(\mu^{-1}(t)))} \right).$$

If we repeat the previous procedure, then there exists an even positive integer n such that

$$\begin{aligned} u(t) &= \sum_{k=1}^n \frac{(-1)^{k+1}}{\sum_{m=1}^k a(\mu^{-[m]}(t))} v(\mu^{-[k]}(t)) + \frac{1}{\sum_{m=1}^n a(\mu^{-[m]}(t))} u(\mu^{-[n]}(t)) \\ &\geq \sum_{k=1}^n \frac{(-1)^{k+1}}{\sum_{m=1}^k a(\mu^{-[m]}(t))} v(\mu^{-[k]}(t)) \\ &\geq \sum_{k=1}^{n/2} \frac{1}{\sum_{m=1}^{2k-1} a(\mu^{-[m]}(t))} \left(v(\mu^{-[2k-1]}(t)) - \frac{1}{a(\mu^{-[2k]}(t))} v(\mu^{-[2k]}(t)) \right). \end{aligned} \tag{2.7}$$

Now, using Lemma 1.2, we obtain

$$\frac{v(t)}{v'(t)} \geq \frac{\lambda_0 t}{n-1},$$

for all $\lambda_0 \in (0, 1)$ and $t \geq t_1$, and so

$$\begin{aligned} \left(\frac{v(t)}{t^{(n-1)/\lambda_0}} \right)' &= \frac{v'(t)}{t^{(n-1)/\lambda_0}} - \frac{n-1}{\lambda_0} \frac{v(t)}{t^{(n-1)/\lambda_0+1}} \\ &= \frac{tv'(t) - \frac{n-1}{\lambda_0} v(t)}{t^{(n-1)/\lambda_0+1}} \leq 0. \end{aligned} \tag{2.8}$$

Taking into account that $\mu(t) \leq t$, we get $\mu^{-[2k-1]}(t) \leq \mu^{-[2k]}(t)$. Thus, from (2.8), we find

$$v(\mu^{-[2k]}(t)) \leq v(\mu^{-[2k-1]}(t)) \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{(n-1)/\lambda_0},$$

which with (2.7) gives

$$\begin{aligned} u(t) &\geq \sum_{k=1}^{n/2} \frac{1}{\sum_{m=1}^{2k-1} a(\mu^{-[m]}(t))} \\ &\quad \times \left(1 - \frac{1}{a(\mu^{-[2k]}(t))} \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{(n-1)/\lambda_0} \right) v(\mu^{-[2k-1]}(t)). \end{aligned} \tag{2.9}$$

Since $v'(t) > 0$ and $\mu^{-[2k-1]}(t) > \mu^{-1}(t)$, we have that $v(\mu^{-[2k-1]}(t)) > v(\mu^{-1}(t))$ for all $k = 1, 2, \dots, n/2$. Therefore, (2.9) becomes

$$\begin{aligned} u(t) &\geq v(\mu^{-1}(t)) \sum_{k=1}^{n/2} \frac{1}{\sum_{m=1}^{2k-1} a(\mu^{-[m]}(t))} \left(1 - \frac{1}{a(\mu^{-[2k]}(t))} \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{(n-1)/\lambda_0} \right) \\ &= \tilde{a}(t)v(\mu^{-1}(t)), \end{aligned}$$

which, with the facts that $\varkappa(t) \leq \eta(t)$ and $\mu'(t) > 0$, gives

$$u(\eta(t)) \geq \tilde{a}(\eta(t))v(\mu^{-1}(\eta(t))) \geq \tilde{a}(\eta(t))v(\mu^{-1}(\varkappa(t))).$$

Then, Eq. (1.1) will become

$$v^n(t) + b(t)\tilde{a}(\eta(t))v(\mu^{-1}(\varkappa(t))) \leq 0. \tag{2.10}$$

Now, using Lemma 1.1, we arrive at

$$v(t) \geq \frac{\lambda_1}{(n-1)!} t^{n-1} v^{(n-1)}(t), \tag{2.11}$$

for all $\lambda_1 \in (0, 1)$. It follows from Eqs. (2.10) and (2.11) that

$$v^n(t) + \frac{\lambda_1}{(n-1)!} b(t)\tilde{a}(\eta(t))(\mu^{-1}(\varkappa(t)))^{n-1} v^{(n-1)}(\mu^{-1}(\varkappa(t))) \leq 0.$$

Clearly, $G(t) := v^{n-1}(t)$ is a positive solution of the first-order delay differential inequality

$$G'(t) + \frac{\lambda_1}{(n-1)!} b(t)\tilde{a}(\eta(t))(\mu^{-1}(\varkappa(t)))^{n-1} G(\mu^{-1}(\varkappa(t))) \leq 0. \tag{2.12}$$

It follows from [22] that Eq. (2.6) also has a positive solution for all $\lambda_0, \lambda_1 \in (0, 1)$, but this contradicts our assumption.

Assume that (2) holds. It follows from Lemma 1.2 that

$$v(t) \geq \lambda_2 t v'(t), \tag{2.13}$$

for all $\lambda_3 \in (0, 1)$ and $t \geq t_1 \geq t_0$, and so

$$\left(\frac{v(t)}{t^{1/\lambda_2}} \right)' = \frac{v'(t)}{t^{1/\lambda_2}} - \frac{1}{\lambda_2} \frac{v(t)}{t^{(1+\lambda_2)/\lambda_2}} = \frac{1}{\lambda_2} \frac{\lambda_2 t v'(t) - v(t)}{t^{(1+\lambda_2)/\lambda_2}} \leq 0.$$

Thus, from the fact that $\mu^{-[2k]}(t) \leq \mu^{-[2k-1]}(t)$, we conclude that

$$v(\mu^{-[2k]}(t)) \leq \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{1/\lambda_2} v(\mu^{-[2k-1]}(t)). \tag{2.14}$$

Combining (2.7) and (2.14), we obtain

$$\begin{aligned}
 u(t) &\geq \sum_{k=1}^{n/2} \frac{1}{\sum_{m=1}^{2k-1} a(\mu^{-[m]}(t))} \\
 &\quad \times \left(1 - \frac{1}{a(\mu^{-[2k]}(t))} \left(\frac{\mu^{-[2k]}(t)}{\mu^{-[2k-1]}(t)} \right)^{1/\lambda_2} \right) v(\mu^{-[2k-1]}(t)).
 \end{aligned}
 \tag{2.15}$$

Since $\mu^{-[2k-1]}(t) \geq \mu^{-1}(t)$ for all $k = 1, 2, \dots, n/2$, (2.15) becomes

$$u(t) \geq \widehat{a}(t)v(\mu^{-1}(t)).
 \tag{2.16}$$

Therefore, (1.1) will be

$$v''(t) + b(t)\widehat{a}(\eta(t))v(\mu^{-1}(\eta(t))) \leq 0,$$

which, with $\chi(t) \leq \eta(t)$ and $v'(t) > 0$, give

$$v''(t) + b(t)\widehat{a}(\eta(t))v(\mu^{-1}(\chi(t))) \leq 0.
 \tag{2.17}$$

Integrating (2.17) from t to ∞ consecutively $n - 2$ times and using the properties of derivatives in case (2), we get

$$v''(t) + \frac{1}{(n-3)!}v(\mu^{-1}(\chi(t))) \int_t^\infty (s-3)^{n-3}b(s)\widehat{a}(\eta(s)) \, ds \leq 0.
 \tag{2.18}$$

By setting $\phi(t) = v'(t)$ and using (2.13), we conclude that $\phi(t)$ is a positive solution of the first-order delay differential inequality,

$$\phi'(t) + \frac{\lambda_2}{(n-3)!}\mu^{-1}(\chi(t))\phi(\mu^{-1}(\chi(t))) \int_t^\infty (s-3)^{n-3}b(s)\widehat{a}(\eta(s)) \, ds \leq 0.
 \tag{2.19}$$

It follows from [22] that the Eq. (2.5) also has a positive solution, which contradicts our assumption. Therefore, the proof of this theorem is complete. \square

Corollary 2.1 *Assume that there exist an even integer m and functions $\varkappa \in C^1([t_0, \infty), \mathbb{R})$, $\chi \in C^1([t_0, \infty), \mathbb{R})$ such that (2.1)–(2.3) hold. If*

$$\liminf_{t \rightarrow \infty} \int_{\mu^{-1}(\varkappa(t))}^t b(s)\widetilde{a}(\eta(s))(\mu^{-1}(\varkappa(s)))^{n-1} \, ds > \frac{(n-1)!}{\lambda_1 e}
 \tag{2.20}$$

and

$$\liminf_{t \rightarrow \infty} \int_{\mu^{-1}(\chi(t))}^t \mu^{-1}(\chi(\varrho)) \left(\int_\varrho^\infty (s-3)^{n-3}b(s)\widehat{a}(\eta(s)) \, ds \right) d\varrho > \frac{(n-3)!}{\lambda_2 e},
 \tag{2.21}$$

for some $\lambda_i \in (0, 1)$, $i = 0, 1, 2$, every solution of Eq. (1.1) is oscillatory.

Proof Applying a well-known criterion [23, Theorem 2] for first-order delay differential equations (2.4) and (2.5) to be oscillatory, we obtain immediately the criteria (2.20) and (2.21), respectively. \square

Remark 2.2 Combining Theorem 2.1 and the results reported in [24–26] for the oscillation of Eqs. (2.4) and (2.5), one can derive various oscillation criteria for Eq. (1.1).

By using a Riccati transformation, we obtain the following criterion.

Theorem 2.3 *Assume that*

$$(\eta^{-1}(t))' \geq \eta_0 > 0, \quad \eta^{-1}(\mu(t)) \geq t, \quad \mu'(t) \geq \mu_0 > 0, \tag{2.22}$$

and there exist an even integer m and a function $\chi \in C^1([t_0, \infty), \mathbb{R})$ such that (2.1), (2.3) and (2.21) hold. If there exist $\lambda_0 \in (0, 1)$ and a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left(\eta_0 \rho(s) B(s) - \frac{(n-2)!}{4\lambda_0} \left(1 + \frac{a_0}{\mu_0} \right) \frac{(\rho'(s))^2}{s^{(n-2)} \rho(s)} \right) ds = \infty, \tag{2.23}$$

then every solution of Eq. (1.1) is oscillatory.

Proof Assume that Eq. (1.1) has an eventually positive solution u . It follows from (1.1) that $v^n(t) = -b(t)u(\eta(t)) \leq 0$. Thus, using Lemma 2.1, we have that there are two cases for the derivatives of v for large t , either (1) or (2).

Assume that (1) holds. From Eq. (1.1), we obtain

$$\frac{1}{(\eta^{-1}(t))'} (v^{(n-1)}(\eta^{-1}(t)))' + b(\eta^{-1}(t))u(t) = 0 \tag{2.24}$$

and

$$\frac{1}{(\eta^{-1}(\mu(t)))'} (v^{(n-1)}(\eta^{-1}(\mu(t))))' + b(\eta^{-1}(\mu(t)))u(\mu(t)) = 0. \tag{2.25}$$

Combining (2.24) and (2.25), and using (2.22), we find

$$\begin{aligned} & \frac{1}{\eta_0} (v^{(n-1)}(\eta^{-1}(t)))' + \frac{a_0}{\eta_0 \mu_0} (v^{(n-1)}(\eta^{-1}(\mu(t))))' \\ & + b(\eta^{-1}(t))u(t) + a_0 b(\eta^{-1}(\mu(t)))u(\mu(t)) \leq 0, \end{aligned}$$

and so

$$\frac{1}{\eta_0} (v^{(n-1)}(\eta^{-1}(t)))' + \frac{a_0}{\eta_0 \mu_0} (v^{(n-1)}(\eta^{-1}(\mu(t))))' + B(t)(u(t) + a_0 u(\mu(t))) \leq 0.$$

Then,

$$\frac{1}{\eta_0} \left(v^{(n-1)}(\eta^{-1}(t)) + \frac{a_0}{\mu_0} v^{(n-1)}(\eta^{-1}(\mu(t))) \right)' + B(t)v(t) \leq 0. \tag{2.26}$$

Now, we define the Riccati transformation as

$$\varpi(t) := \rho(t) \frac{v^{(n-1)}(\eta^{-1}(t))}{v(t)}.$$

Thus, $\varpi(t) > 0$ for $t \geq t_1 \geq t_0$, and

$$\varpi'(t) = \frac{\rho'(t)}{\rho(t)} \varpi(t) - \rho(t) \left(\frac{(v^{(n-1)}(\eta^{-1}(t)))v'(t)}{v^2(t)} - \frac{(v^{(n-1)}(\eta^{-1}(t)))'}{v(t)} \right). \tag{2.27}$$

Using Lemma 1.1 and the fact that $v^{(n)} \leq 0$, we arrive at

$$\begin{aligned} v'(t) &\geq \frac{\lambda_0}{(n-2)!} t^{n-2} v^{(n-1)}(t) \geq \frac{\lambda_0}{(n-2)!} t^{n-2} v^{(n-1)}(\eta^{-1}(\mu(t))) \\ &\geq \frac{\lambda_0}{(n-2)!} t^{n-2} v^{(n-1)}(\eta^{-1}(t)). \end{aligned} \tag{2.28}$$

Hence, (2.27) yields

$$\varpi'(t) \leq \frac{\rho'(t)}{\rho(t)} \varpi(t) - \frac{\lambda_0}{(n-2)!} \frac{t^{(n-2)}}{\rho(t)} \varpi^2(t) + \rho(t) \frac{(v^{(n-1)}(\eta^{-1}(t)))'}{v(t)}. \tag{2.29}$$

Next, we define function

$$\omega(t) := \rho(t) \frac{v^{(n-1)}(\eta^{-1}(\mu(t)))}{v(t)}.$$

Then $\omega(t) > 0$ for $t \geq t_1 \geq t_0$ and

$$\begin{aligned} \omega'(t) &= \frac{\rho'(t)}{\rho(t)} \omega(t) - \rho(t) \left(\frac{(v^{(n-1)}(\eta^{-1}(\mu(t))))v'(t)}{v^2(t)} - \frac{(v^{(n-1)}(\eta^{-1}(\mu(t))))'}{v(t)} \right) \\ &\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\lambda_0}{(n-2)!} \frac{t^{(n-2)}}{\rho(t)} \omega^2(t) + \rho(t) \frac{(v^{(n-1)}(\eta^{-1}(\mu(t))))'}{v(t)}. \end{aligned} \tag{2.30}$$

Combining (2.29) and (2.30), we get

$$\begin{aligned} \varpi'(t) + \frac{a_0}{\mu_0} \omega'(t) &\leq \frac{\rho'(t)}{\rho(t)} \varpi(t) - \frac{\lambda_0}{(n-2)!} \frac{t^{(n-2)}}{\rho(t)} \varpi^2(t) + \frac{a_0}{\mu_0} \left[\frac{\rho'(t)}{\rho(t)} \omega(t) \right. \\ &\quad \left. - \frac{\lambda_0}{(n-2)!} \frac{t^{(n-2)}}{\rho(t)} \omega^2(t) \right] - \eta_0 \rho(t) B(t). \end{aligned}$$

Using the fact that

$$Hy - Ky^2 \leq \frac{1}{4} \frac{H^2}{K}, \quad K > 0,$$

we obtain

$$\varpi'(t) + \frac{a_0}{\mu_0} \omega'(t) \leq \frac{(n-2)!}{4\lambda_0} \left(1 + \frac{a_0}{\mu_0} \right) \frac{(\rho'(t))^2}{t^{(n-2)}\rho(t)} - \eta_0 \rho(t) B(t).$$

Integrating the above inequality from t_1 to t , we have

$$\int_{t_1}^t \left(\eta_0 \rho(s) B(s) - \frac{(n-2)!}{4\lambda_0} \left(1 + \frac{a_0}{\mu_0} \right) \frac{(\rho'(s))^2}{s^{(n-2)}\rho(s)} \right) ds \leq \varpi(t_1) + \frac{a_0}{\mu_0} \omega(t_1),$$

which contradicts (2.23).

Assume that case (2) holds. If we are back to the proof of Corollary 2.1, then we get a contradiction with (2.21). Hence, the proof is complete. \square

Next, we give an example to illustrate our main results.

Example 2.1 Consider a fourth-order neutral delay differential equation

$$(u(t) + a_0u(\beta t))^{(4)} + \frac{b_0}{t^4}u(\delta t) = 0, \quad t \geq 1, \tag{2.31}$$

where $a_0, b_0 > 0$ and $0 < \delta \leq \beta < 1$. It is easy to see that $B(t) = (b_0\delta^4)/t^4$,

$$\tilde{a}(t) = \left(1 - \frac{1}{\beta^{3/\lambda_0} a_0}\right) \sum_{k=1}^{n/2} \frac{1}{a_0^{2k-1}} := A_0$$

and

$$\widehat{a}(t) := \left(1 - \frac{1}{\beta^{1/\lambda_2} a_0}\right) \sum_{k=1}^{n/2} \frac{1}{a_0^{2k-1}} := A_1.$$

By choosing $\varkappa(t) = \chi(t) = \delta t$, we see that (2.2) and (2.3) hold, and conditions (2.20) and (2.21) reduce to

$$b_0 \ln \frac{\beta}{\delta} > \frac{6\beta^3}{\delta^3 A_0 e}$$

and

$$b_0 \ln \frac{\beta}{\delta} > \frac{12\beta}{\delta A_1 e},$$

respectively. Thus, from Corollary 2.1, we see that every solution of Eq. (2.31) is oscillatory if

$$b_0 > \max \left\{ \frac{6\beta^3}{\delta^3 A_0 e \ln \beta/\delta}, \frac{12\beta}{\delta A_1 e \ln \beta/\delta} \right\}. \tag{2.32}$$

Moreover, the condition (2.23) reduces to

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left(\delta^4 \eta_0 b_0 - \frac{9}{2\lambda_0} \left(1 + \frac{a_0}{\mu_0}\right) \right) \frac{1}{s} ds = \infty,$$

when

$$b_0 > \frac{9}{2} \left(1 + \frac{a_0}{\mu_0}\right) \frac{1}{\delta^4 \eta_0}.$$

Thus, from Theorem 2.3, we see that every solution of Eq. (2.31) is oscillatory if

$$b_0 > \max \left\{ \frac{9}{2} \left(1 + \frac{a_0}{\mu_0}\right) \frac{1}{\delta^4 \eta_0}, \frac{6\beta^3}{\delta^3 A_0 e \ln \beta/\delta} \right\}.$$

Remark 2.4 Although the results of Li and Rogovchenko in [1] improved their previous results, they used Lemma 1.2 with $\lambda = 1$ (and this is inaccurate); see Remark 12 in [20]. Theorem 2.1, with $n = 2$, is a correction of Theorem 2.1 in [1]. Moreover, our results improve the results in [1], since the iterative nature of the two functions $\tilde{a}(t)$ and $\hat{a}(t)$ enables us to test for oscillations, even when the previously known results fail to apply. Let us consider a special case of (2.31), namely,

$$(u(t) + 10u(0.9t))^{(4)} + \frac{110}{t^4}u(0.5t) = 0. \quad (2.33)$$

We note that the condition (2.32) fail to apply on (2.33) when $n = 2, 4$ (consequently, the results in [1] also fail). But, at $n = 6$, the condition (2.32) is satisfied. Therefore, our results improve the previous results in the literature.

Remark 2.5 It would be of interest to further investigate Eq. (1.1) with different neutral coefficients; see [27] and [28] for more details. It would also be interesting to extend this development to higher-order nonlinear neutral differential equations.

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Authors' contributions

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Author details

¹Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt. ²Section of Mathematics, International Telematic University Uninettuno, CorsoVittorio Emanuele II, 39, 00186 Roma, Italy.

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