Even-order differential equation with continuous delay: nonexistence criteria of Kneser solutions

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Abstract

In this paper, we study even-order DEs where we deduce new conditions for nonexistence Kneser solutions for this type of DEs. Based on the nonexistence criteria of Kneser solutions, we establish the criteria for oscillation that take into account the effect of the delay argument, where to our knowledge all the previous results neglected the effect of the delay argument, so our results improve the previous results. The effectiveness of our new criteria is illustrated by examples.

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1 Introduction

There is no doubt that the theory of oscillation of DEs is a fertile study area and has attracted the attention of many researchers recently. This is due to the existence of many important applications of this theory in various fields of applied science, see [18, 19]. In the last decade, it is easy to notice the new research movement that aims to improve and develop the criteria for oscillations of DEs of different orders, see [3–5] and [9–17].

In detail, we consider the even-order delay DE of the form

$$\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)'(\varsigma) + A\left[q \cdot (y \circ g)^{\gamma}; a, b\right](\varsigma) = 0, \quad \varsigma \geq \varsigma_0, \tag{1.1}$$

where $n \geq 4$ is an even natural number, γ is quotient of odd positive integers, and $A[f; a, b](\varsigma) := \int_a^b f(\varsigma, \varrho) \, d\varrho$. Our study is under the following conditions:

($\Omega 1$) $r \in C^1(I_0, (0, \infty))$, $r'(\varsigma) \geq 0$, $\int_{\varsigma_0}^{\infty} r^{-1/\gamma}(\xi) \, d\xi < \infty$, and $I_\vartheta := [\varsigma_\vartheta, \infty)$;

($\Omega 2$) $q \in C(I_0 \times [a, b], [0, \infty))$ and q is not zero on any half line $[T, \infty) \times [a, b]$ for all $T \geq \varsigma_0$;

($\Omega 3$) $g \in C(I_0 \times [a, b], \mathbb{R})$, g has nonnegative partial derivative w.r.t s and $g(\varsigma, s) \leq \varsigma$, $\lim_{\varsigma \to \infty} g(\varsigma, s) = \infty$ for all $s \in [a, b]$.

A solution of (1.1) means a function $y \in C^{(n-1)}(I_y, \mathbb{R})$, $\varsigma_y \geq \varsigma_0$, which satisfies the property $r \cdot (y^{(n-1)})^{\gamma} \in C^1(I_y, \mathbb{R})$; moreover, it satisfies (1.1) on I_y. We consider only the proper solutions y of (1.1), that is, y is not identically zero eventually.

Definition 1.1 A solution y of (1.1) is called a *Kneser solution* if there exists $\varsigma_* \in I_0$ such that $y(\varsigma)y'(\varsigma) < 0$ for all $\varsigma \geq \varsigma_*$. (The set of all eventually positive Kneser solutions of (1.1) is denoted by **K**.)

Definition 1.2 A solution y of (1.1) is said to be *nonoscillatory* if it is positive or negative, ultimately; otherwise, it is said to be *oscillatory*. The equation itself is termed oscillatory if all its solutions oscillate.

Next, let us briefly review a number of closely related results which motivated the present study.

Li and Rogovchenko [9] were concerned with the asymptotic behavior of a class of higher-order sublinear Emden–Fowler delay DEs

$$\left(r(\varsigma)y^{(n-1)}(\varsigma)\right)' + q(\varsigma)y^{\beta}\left(\tau(\varsigma)\right) = 0,$$

where $0 < \beta < 1$ is a ratio of odd natural numbers and $\tau(\varsigma) < \varsigma$. They established two tests for the asymptotic behavior of solutions to the above equations. Moreover, they improved the theorems reported by Li and Rogovchenko [8] and Zhang et al. [20, 22].

Moaaz and Muhib [17] and Zhang et al. [21] presented criteria for oscillation of solutions of the DE

$$\left(r(\varsigma)\left(y'''(\varsigma)\right)^{\gamma}\right)' + f\left(\varsigma, y(\sigma(\varsigma))\right) = 0,$$

where $f(\varsigma, y) \geq h(\varsigma)y^{\beta}$, γ, β are quotients of odd natural numbers and $\sigma(\varsigma) < \varsigma$. Results in [17] are an improvement on some of the results obtained in Zhang et al. [2].

Recently, Moaaz et al. [14] studied the oscillation and the asymptotic behavior of solutions of the DE

$$\left(r(\varsigma)\left(y'''(\varsigma)\right)^{\gamma}\right)' + q(\varsigma)y^{\beta}\left(\sigma(\varsigma)\right) = 0$$

with the middle term

$$\left(r(\varsigma)\left(y'''(\varsigma)\right)^{\gamma}\right)' + p(\varsigma)\left(y'''(\varsigma)\right)^{\gamma} + q(\varsigma)y^{\beta}\left(\sigma(\varsigma)\right) = 0$$

under the condition

$$\int_{\varsigma_0}^{\infty} \frac{1}{r^{1/\gamma}(s)}\, ds = \infty$$

and the condition

$$\int_{\varsigma_0}^{\infty} \left[\frac{1}{r(s)}\exp\left(-\int_{\varsigma_0}^{s}\frac{p(\xi)}{r(\xi)}\,d\xi\right)\right]^{1/\gamma} ds = \infty,$$

where $r'(\varsigma) + p(\varsigma) \geq 0$.

In the paper, we are working on finding new criteria for oscillation of solutions of a class of even-order DEs in a noncanonical case. The paper is organized as follows. In Sect. 2,

we present new conditions for the nonexistence of Kneser solutions of nonlinear even-order DEs with continuous delay arguments. In Sect. 3, we are taking advantage of the new nonexistence criteria of Kneser solutions to create better criteria that ensure all solutions of (1.1) are oscillatory. In Sect. 4, we illustrate the effectiveness of our new criteria with examples.

Now, we provide the lemmas that will be needed during the results.

Lemma 1.1 ([1, Lemma 2.2.3]) *Assume that $\varpi \in C^n(I_0, \mathbb{R}^+)$ and $\varpi^{(n)}$ are of fixed sign and not identically zero on a subray of I_0. Furthermore, suppose that there exists $\varsigma_1 \in I_0$ such that $\varpi^{(n-1)}\varpi^{(n)} \leq 0$ for $\varsigma \in I_1$. If $\lim_{\varsigma \to \infty} \varpi(\varsigma) \neq 0$, then there exists $\varsigma_\lambda \in I_1$ such that*

$$\varpi \geq \frac{\lambda}{(n-1)!}\varsigma^{n-1}\left|\varpi^{(n-1)}\right|$$

for every $\lambda \in (0,1)$ and $\varsigma \in I_\lambda$.

Lemma 1.2 *Let $\varpi(\xi) = D\xi - M(\xi - N)^{(\gamma+1)/\gamma}$, where $M > 0$, D and N are constants. Then the maximum value of ϖ on R at $\xi^* = N + (\gamma D/((\gamma + 1)M))^\gamma$ is*

$$\max_{\xi \in R} \varpi(\xi) = \varpi\left(\xi^*\right) = DN + \frac{\gamma^\gamma}{(\gamma+1)^{(\gamma+1)}}\frac{D^{\gamma+1}}{M^\gamma}. \tag{1.2}$$

2 Nonexistence of Kneser solutions

Firstly, we define the notations $\delta_0(\varsigma) := \int_\varsigma^\infty r^{-1/\gamma}(\xi)\,d\xi$ and $\delta_m(\varsigma) := \int_\varsigma^\infty \delta_{m-1}(\xi)\,d\xi$ for $m = 1, 2, \ldots, n - 2$. The following lemma is an adaptation of Lemma 1.1 in [6] based on n even.

Lemma 2.1 *If y is an eventually positive solution of (1.1), then $(r \cdot y^{(n-1)})' \leq 0$, and one of the following cases holds for ς large enough:*
  (1) $y'(\varsigma) > 0$, $y^{(n-1)}(\varsigma) > 0$ and $y^{(n)}(\varsigma) < 0$;
  (2) $y'(\varsigma) > 0$, $y^{(n-2)}(\varsigma) > 0$ and $y^{(n-1)}(\varsigma) < 0$;
  (3) $(-1)^k y^{(k)}(\varsigma) > 0$ for $k = 1, 2, \ldots, n - 1$.

Remark 2.1 Based on the definition of the class **K**, we note that $y \in $ **K** if and only if y satisfies case (3).

Lemma 2.2 *Assume that $y \in $ **K**. Then y converges to zero if*

$$\int_{\varsigma_0}^\infty \left(\frac{1}{r(\xi)}\int_{\varsigma_0}^\xi A[q;a,b](v)\,dv\right)^{1/\gamma} d\xi = \infty. \tag{2.1}$$

Proof Based on the belonging of y to **K**, we note that y is a positive decreasing function, and so $\lim_{\varsigma \to \infty} y(\varsigma) = \epsilon \geq 0$. Assuming the opposite of that, it is required that $\epsilon > 0$. Then there exists $\varsigma_1 \in I_0$ such that $y(\varsigma) > \epsilon$ for all $\varsigma \geq \varsigma_1$. Thus, from ($\Omega$3), there exists $\varsigma_2 \geq \varsigma_1$ such that $(y \circ g)(\varsigma) > \epsilon$ for $\varsigma \geq \varsigma_2$. From (1.1), we arrive at

$$\left(r \cdot \left(y^{(n-1)}\right)^\gamma\right)'(\varsigma) \leq -\epsilon^\gamma A[q;a,b](\varsigma) \quad \text{for } \varsigma \geq \varsigma_2.$$

Integrating the above inequality from ς_2 to ς, we get

$$\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) \leq \left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma_2) - \epsilon^{\gamma} \int_{\varsigma_2}^{\varsigma} A[q; a, b](v) \, dv,$$

that is,

$$y^{(n-1)}(\varsigma) \leq -\epsilon \left(\frac{1}{r(\varsigma)} \int_{\varsigma_2}^{\varsigma} A[q; a, b](v) \, dv\right)^{1/\gamma}.$$

Integrating the last inequality from ς_2 to ς, we obtain

$$y^{(n-2)}(\varsigma) \leq y^{(n-2)}(\varsigma_2) - \epsilon \int_{\varsigma_2}^{\varsigma} \left(\frac{1}{r(\xi)} \int_{\varsigma_2}^{\xi} A[q; a, b](v) \, dv\right)^{1/\gamma} d\xi.$$

Taking $\lim_{\varsigma \to \infty}$ and assumption (2.1) into account, we get that $y^{(n-2)}(\varsigma) \to -\infty$ as $\varsigma \to \infty$, which is a contradiction. Thus, $\epsilon = 0$. This completes the proof. $\square$

Lemma 2.3 *Assume that* (2.1) *holds. If* $y \in \mathbf{K}$, *then*

$$-\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) \geq \int_{\varsigma_0}^{\varsigma} A[q \cdot (y \circ g)^{\gamma}; a, b](\xi) \, d\xi \tag{2.2}$$

and

$$(-1)^{(k)} y^{(k)}(\varsigma) \geq -\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)^{1/\gamma}(\varsigma) \delta_{n-2-k}(\varsigma) \tag{2.3}$$

for $k = 0, 1, \ldots, n - 2$.

Proof Assume that $y \in \mathbf{K}$ on $[\varsigma_1, \infty)$. Integrating (1.1) from ς_1 to ς and using that fact that $y'(\varsigma) < 0$, we obtain

$$\begin{aligned}
-\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) &\geq -\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma_1) + \int_{\varsigma_1}^{\varsigma} A[q \cdot (y \circ g)^{\gamma}; a, b](\xi) \, d\xi \\
&\geq -\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma_1) + \int_{\varsigma_0}^{\varsigma} A[q \cdot (y \circ g)^{\gamma}; a, b](\xi) \, d\xi \\
&\quad - \int_{\varsigma_0}^{\varsigma_1} A[q \cdot (y \circ g)^{\gamma}; a, b](\xi) \, d\xi \tag{2.4}
\end{aligned}$$

for all $\varsigma \in I_1$. It follows from Lemma 2.2 that y converges to zero. Then there is $\varsigma_2 \in I_1$ such that, for $\varsigma \geq \varsigma_2$,

$$\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma_1) + \int_{\varsigma_0}^{\varsigma_1} A[q \cdot (y \circ g)^{\gamma}; a, b](\xi) \, d\xi < 0,$$

which with (2.4) gives

$$-\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) \geq \int_{\varsigma_0}^{\varsigma} A[q \cdot (y \circ g)^{\gamma}; a, b](\xi) \, d\xi.$$

Next, by using the fact that $(r^{1/\gamma} \cdot y^{(n-1)})' \leq 0$, we see that

$$
\begin{aligned}
y^{(n-2)}\big(g(\varsigma,s)\big) \geq y^{(n-2)}(\varsigma) &\geq \int_{\varsigma}^{\infty} \frac{1}{r^{1/\gamma}(\varrho)}\big(-r^{1/\gamma}(\varrho)y^{(n-1)}(\varrho)\big)\,\mathrm{d}\varrho \\
&\geq -\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)^{1/\gamma}(\varsigma)\delta_0(\varsigma).
\end{aligned}
\tag{2.5}
$$

Integrating (2.5) from ς to ∞ and taking the monotonicity of $y^{(n-3)}(\varsigma)$ into account, we find

$$
-y^{(n-3)}(\varsigma) \geq -\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)^{1/\gamma}(\varsigma)\delta_1(\varsigma).
$$

Integrating again from ς to ∞, we obtain

$$
y^{(n-4)}(\varsigma) \geq -\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)^{1/\gamma}(\varsigma)\delta_2(\varsigma).
$$

Going forward along the same method, we get

$$
(-1)^{(k)}y^{(k)}(\varsigma) \geq -\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)^{1/\gamma}(\varsigma)\delta_{n-2-k}(\varsigma)
$$

for $k = 0, 1, \ldots, n-2$. This completes the proof. $\qquad\square$

Theorem 2.2 *Assume that* (2.1) *holds. If*

$$
\eta := \limsup_{\varsigma \to \infty} \delta_{n-2}(\varsigma)\left(\int_{\varsigma_0}^{\varsigma} A[q;a,b](\xi)\,\mathrm{d}\xi\right)^{1/\gamma} > 1,
\tag{2.6}
$$

then $\mathbf{K} = \varnothing$.

Proof Suppose to the contrary that $y \in \mathbf{K}$ on $[\varsigma_1, \infty)$. From Lemma 2.3, we obtain (2.2) and (2.3) hold. Since g is delay w.s.t ς, we get $y \circ g \geq y$ for $\varsigma \geq \varsigma_2$ and $s \in [a, b]$. Thus, (2.2) becomes

$$
-\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)(\varsigma) \geq y^{\gamma}(\varsigma)\int_{\varsigma_0}^{\varsigma} A[q;a,b](\xi)\,\mathrm{d}\xi,
$$

which with ((2.3), $k = 0$) gives

$$
-\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)(\varsigma) \geq -\big(r \cdot \big(y^{(n-1)}\big)^{\gamma}\big)(\varsigma)\delta_{n-2}^{\gamma}(\varsigma)\int_{\varsigma_0}^{\varsigma} A[q;a,b](\xi)\,\mathrm{d}\xi
$$

or equivalently,

$$
1 \geq \delta_{n-2}^{\gamma}(\varsigma)\int_{\varsigma_0}^{\varsigma} A[q;a,b](\xi)\,\mathrm{d}\xi.
$$

Taking the limsup on both sides of the inequality, we arrive at contradiction with (2.6). This completes the proof. $\qquad\square$

For the next results, we introduce the following additional condition:

(Ω) There is a constant $h > 1$ t such that $\frac{\delta_{n-2}(g(\varsigma,s))}{\delta_{n-2}(\varsigma)} \geq h$ for $\varsigma \geq \varsigma_0$ and $s \in [a,b]$.

Lemma 2.4 *Assume that $y \in \mathbf{K}$, (2.1) hold and η is defined as in (2.6). Then there exists $\varsigma_\varepsilon \geq \varsigma_1$ such that*

$$\frac{d}{d\varsigma}\left(\frac{y(\varsigma)}{\delta_{n-2}^{\eta-\varepsilon}(\varsigma)}\right) \leq 0$$

for any $\varepsilon > 0$ and $\varsigma \geq \varsigma_\varepsilon$. Moreover, if ($\Omega$) holds, then

$$y\big(g(\varsigma,s)\big) \geq h^{\eta-\varepsilon}y(\varsigma) \text{for } \varsigma \geq \varsigma_\varepsilon \text{ and } s \in [a,b]. \tag{2.7}$$

Proof Assume that $y \in \mathbf{K}$ on I_1. From Lemma 2.3, we obtain (2.2) and (2.3) hold. It follows from (2.2) and the fact that $g(\varsigma,s) \leq \varsigma$ that

$$-\big(r \cdot \big(y^{(n-1)}\big)^\gamma\big)(\varsigma) \geq y^\gamma(\varsigma)\int_{\varsigma_0}^\varsigma A[q;a,b](\xi)\,d\xi. \tag{2.8}$$

From the definition of η in Theorem 2.2, there exists $\varsigma_2 \geq \varsigma_1$ such that

$$\delta_{n-2}\left(\int_{\varsigma_0}^\varsigma A[q;a,b](\xi)\,d\xi\right)^{1/\gamma} > \eta_* := \eta - \varepsilon$$

for all $\varepsilon > 0$ and $\varsigma \geq \varsigma_2$. Hence, from ((2.3), $k = 1$), we have

$$\frac{d}{d\varsigma}\left(\frac{y(\varsigma)}{\delta_{n-2}^{\eta_*}(\varsigma)}\right) \leq \frac{\delta_{n-2}^{\eta_*}(\varsigma)r^{1/\gamma}(\varsigma)y^{(n-1)}(\varsigma)\delta_{n-3}(\varsigma)}{\delta_{n-2}^{2\eta_*}(\varsigma)} + \frac{y(\varsigma)\eta_*\delta_{n-2}^{\eta_*-1}(\varsigma)\delta_{n-3}(\varsigma)}{\delta_{n-2}^{2\eta_*}(\varsigma)},$$

which with (2.8) gives

$$\begin{aligned}\frac{d}{d\varsigma}\left(\frac{y(\varsigma)}{\delta_{n-2}^{\eta_*}(\varsigma)}\right) &\leq \frac{-y(\varsigma)\delta_{n-2}^{\eta_*}(\varsigma)\delta_{n-3}(\varsigma)}{\delta_{n-2}^{2\eta_*}(\varsigma)}\left(\int_{\varsigma_0}^\varsigma A[q;a,b](\xi)\,d\xi\right)^{1/\gamma} \\ &\quad + \frac{y(\varsigma)\eta_*\delta_{n-2}^{\eta_*-1}(\varsigma)\delta_{n-3}(\varsigma)}{\delta_{n-2}^{2\eta_*}(\varsigma)} \\ &\leq \frac{y(\varsigma)\delta_{n-3}(\varsigma)}{\delta_{n-2}^{\eta_*+1}(\varsigma)}\left(\eta_* - \delta_{n-2}(\varsigma)\left(\int_{\varsigma_0}^\varsigma A[q;a,b](\xi)\,d\xi\right)^{1/\gamma}\right) \leq 0.\end{aligned}$$

Using this fact, one can easily see that

$$y\big(g(\varsigma,s)\big) \geq y(\varsigma)\left(\frac{\delta_{n-2}(g(\varsigma,s))}{\delta_{n-2}(\varsigma)}\right)^{\eta_*} \geq h^{\eta_*}y(\varsigma).$$

This completes the proof. □

Theorem 2.3 *Assume that (Ω), (2.1) hold and η is defined as in (2.6). If*

$$h^\eta\eta > 1, \tag{2.9}$$

then $\mathbf{K} = \varnothing$.

Proof Suppose to the contrary that $y \in \mathbf{K}$ on I_1. From Lemma 2.3 and 2.4, we obtain that (2.2), (2.3), and (2.7) hold. Combining (2.2) and (2.7), we obtain

$$-\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) \geq y^{\gamma}(\varsigma) h^{\gamma \eta *} \int_{\varsigma_0}^{\varsigma} A[q; a, b](\xi) \, d\xi \tag{2.10}$$

for all $\varepsilon > 0$ and $\varsigma \geq \varsigma_1$. Using ((2.3), $k = 0$), we have

$$-\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) \geq -\left(r \cdot \left(y^{(n-1)}\right)^{\gamma}\right)(\varsigma) \delta_{n-2}^{\gamma}(\varsigma) h^{\gamma \eta *} \int_{\varsigma_0}^{\varsigma} A[q; a, b](\xi) \, d\xi.$$

Taking the limsup on both sides of the latter inequality, we obtain $h^{\eta}\eta \leq 1$. Then we obtain a contradiction with (2.9). This completes the proof. □

Theorem 2.4 *Assume that* (Ω), *(2.1) hold and* η *is defined as in* (2.6). *If*

$$\limsup_{\varsigma \to \infty} \int_{\varsigma_0}^{\varsigma} \left(h^{\gamma \eta} \delta_{n-2}^{\gamma}(\varrho) A[q; a, b](\varrho) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \frac{\delta_{n-3}(\varrho)}{\delta_{n-2}(\varrho)} \right) d\varrho = \infty, \tag{2.11}$$

then $\mathbf{K} = \varnothing$.

Proof Suppose to the contrary that $y \in \mathbf{K}$ on I_1. From Lemmas 2.3 and 2.4, we obtain that (2.2), (2.3), and (2.7) hold. Define the function

$$\omega(\varsigma) := \frac{(r \cdot (y^{(n-1)})^{\gamma})(\varsigma)}{y^{\gamma}(\varsigma)}.$$

Differentiating $\omega(\varsigma)$, we get

$$\omega'(\varsigma) = \frac{(r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma})'}{y^{\gamma}(\varsigma)} - \frac{\gamma r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma} y'(\varsigma)}{y^{\gamma+1}(\varsigma)}.$$

Using (1.1), ((2.3), $k = 1$), and (2.7), we arrive at

$$\omega'(\varsigma) \leq -h^{\gamma \eta *} A[q; a, b](\varsigma) - \gamma \delta_{n-3}(\varsigma) \omega^{(\gamma+1)/\gamma}(\varsigma). \tag{2.12}$$

Multiplying (2.12) by δ_{n-2}^{γ} and integrating the resulting inequality from ς_1 to ς, we obtain

$$\delta_{n-2}^{\gamma}(\varsigma)\omega(\varsigma) - \delta_{n-2}^{\gamma}(\varsigma_1)\omega(\varsigma_1) \leq -\int_{\varsigma_1}^{\varsigma} h^{\gamma \eta *} \delta_{n-2}^{\gamma}(\varrho) A[q; a, b](\varrho) \, d\varrho$$

$$-\int_{\varsigma_1}^{\varsigma} \gamma \delta_{n-2}^{\gamma-1}(\varrho) \delta_{n-3}(\varrho) \omega(\varrho) \, d\varrho$$

$$-\int_{\varsigma_1}^{\varsigma} \gamma \delta_{n-3}(\varrho) \delta_{n-2}^{\gamma}(\varrho) \omega^{(\gamma+1)/\gamma}(\varrho) \, d\varrho.$$

Using the inequality

$$-B\upsilon + A\upsilon^{(\gamma+1)/\gamma} \geq -\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}, \quad A, B > 0,$$

with $A = \delta_{n-3}(\varrho)\delta_{n-2}^\gamma(\varrho)$, $B = \delta_{n-2}^{\gamma-1}(\varrho)\delta_{n-3}(\varrho)$, and $\upsilon = -\omega(\varrho)$, we conclude that

$$
\int_{\varsigma_1}^{\varsigma} \left(h^{\gamma\eta_*}\delta_{n-2}^\gamma(\varrho)A[q;a,b](\varrho) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\frac{\delta_{n-3}(\varrho)}{\delta_{n-2}(\varrho)} \right) d\varrho
$$
$$
\leq \delta_{n-2}^\gamma(\varsigma_1)\omega(\varsigma_1) - \delta_{n-2}^\gamma(\varsigma)\omega(\varsigma). \tag{2.13}
$$

From ((2.3), $k = 0$), one can easily see that $-1 \leq \omega(\varsigma)\delta_{n-2}^\gamma(\varsigma) < 0$, which with (2.13) gives

$$
\int_{\varsigma_1}^{\varsigma} \left(h^{\gamma\eta_*}\delta_{n-2}^\gamma(\varrho)A[q;a,b](\varrho) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\frac{\delta_{n-3}(\varrho)}{\delta_{n-2}(\varrho)} \right) d\varrho < 1 + \delta_{n-2}^\gamma(\varsigma_1)\omega(\varsigma_1).
$$

Taking the limsup on both sides of the latter inequality, we obtain a contradiction with (2.11). This completes the proof. $\qquad\square$

Theorem 2.5 *Assume that* (Ω) *and* (2.1) *hold. If there exists a function* $\rho \in C^1(I_0, (0,\infty))$ *such that*

$$
\limsup_{\varsigma\to\infty} \frac{\delta_{n-2}^\gamma(\varsigma)}{\rho(\varsigma)} \int_{\varsigma_1}^{\varsigma} \left(\rho(\zeta)h^{\gamma\eta}A[q;a,b](\zeta) - \frac{\rho^{-\gamma}(\zeta)(\rho'(\zeta))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\delta_{n-3}^\gamma(\zeta)} \right) d\zeta > 1, \tag{2.14}
$$

then $\mathbf{K} = \varnothing$.

Proof Suppose to the contrary that $y \in \mathbf{K}$ on I_1. Using Lemmas 2.3 and 2.4, we obtain that (2.2), (2.3), and (2.7) hold. From ((2.3), $k = 0$), we obtain

$$
\frac{(r \cdot (y^{(n-1)})^\gamma)(\varsigma)}{y^\gamma(\varsigma)} \geq -\frac{1}{\delta_{n-2}^\gamma(\varsigma)}. \tag{2.15}
$$

Thus, if we define a generalized Riccati substitution as

$$
w(\varsigma) := \rho\left(\frac{r \cdot (y^{(n-1)})^\gamma}{y^\gamma} + \frac{1}{\delta_{n-2}^\gamma} \right), \tag{2.16}
$$

then $w(\varsigma) > 0$ for all $\varsigma \geq \varsigma_1$. Differentiating ω, we have

$$
w'(\varsigma) = \frac{\rho'(\varsigma)}{\rho(\varsigma)}w(\varsigma) + \rho(\varsigma)\frac{(r \cdot (y^{(n-1)})^\gamma)'(\varsigma)}{y^\gamma(\varsigma)}
$$
$$
- \gamma\rho(\varsigma)\frac{(r \cdot (y^{(n-1)})^\gamma)(\varsigma)}{y^{\gamma+1}(\varsigma)}y'(\varsigma)
$$
$$
- \frac{\gamma\rho(\varsigma)\delta_{n-2}'(\varsigma)}{\delta_{n-2}^{\gamma+1}(\varsigma)}. \tag{2.17}
$$

From (1.1), we see that

$$
\left(r(\varsigma)\left(y^{(n-1)}(\varsigma)\right)^\gamma \right)' = -A\left[q \cdot (y \circ g)^\gamma; a, b \right](\varsigma) \leq 0. \tag{2.18}
$$

Using ((2.3), $k = 1$) and (2.18), (2.17) becomes

$$
\begin{aligned}
w'(\varsigma) \leq{} & \frac{\rho'(\varsigma)}{\rho(\varsigma)} w(\varsigma) - \rho(\varsigma) \frac{A[q \cdot (y \circ g)^{\gamma}; a, b](\varsigma)}{y^{\gamma}(\varsigma)} \\
& - \gamma \rho(\varsigma) r(\varsigma) \left(\frac{y^{(n-1)}(\varsigma)}{y(\varsigma)} \right)^{\gamma+1} r^{1/\gamma}(\varsigma) \delta_{n-3}(\varsigma) + \frac{\gamma \rho(\varsigma) \delta_{n-3}(\varsigma)}{\delta_{n-2}^{\gamma+1}(\varsigma)}.
\end{aligned}
\tag{2.19}
$$

Thus, from (2.7), (2.19) yields

$$
\begin{aligned}
w'(\varsigma) \leq{} & \frac{\rho'(\varsigma)}{\rho(\varsigma)} w(\varsigma) - \rho(\varsigma) h^{\gamma(\eta-\varepsilon)} A[q; a, b](\varsigma) + \frac{\gamma \rho(\varsigma) \delta_{n-3}(\varsigma)}{\delta_{n-2}^{\gamma+1}(\varsigma)} \\
& - \gamma \rho(\varsigma) r(\varsigma) \left(\frac{y^{(n-1)}(\varsigma)}{y(\varsigma)} \right)^{\gamma+1} r^{1/\gamma}(\varsigma) \delta_{n-3}(\varsigma).
\end{aligned}
$$

Therefore, from the definition of w, we get

$$
\begin{aligned}
w'(\varsigma) \leq{} & -\rho(\varsigma) h^{\gamma(\eta-\varepsilon)} A[q; a, b](\varsigma) + \frac{\gamma \rho(\varsigma) \delta_{n-3}(\varsigma)}{\delta_{n-2}^{\gamma+1}(\varsigma)} \\
& + \frac{\rho'(\varsigma)}{\rho(\varsigma)} w(\varsigma) - \gamma \frac{\delta_{n-3}(\varsigma)}{\rho^{1/\gamma}(\varsigma)} \left(w(\varsigma) - \frac{\rho(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)} \right)^{1+1/\gamma}.
\end{aligned}
\tag{2.20}
$$

Using inequality (1.2) with

$$
D := \frac{\rho'(\varsigma)}{\rho(\varsigma)}, \qquad M := \gamma \frac{\delta_{n-3}(\varsigma)}{\rho^{1/\gamma}(\varsigma)}, \qquad N := \frac{\rho(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)}
$$

and $\xi := w$, we obtain

$$
\frac{\rho'(\varsigma)}{\rho(\varsigma)} w(\varsigma) \leq \gamma \frac{\delta_{n-3}(\varsigma)}{\rho^{1/\gamma}(\varsigma)} \left(w(\varsigma) - \frac{\rho(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)} \right)^{1+1/\gamma} + \frac{\rho'(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)} + \frac{\rho^{-\gamma}(\varsigma)(\rho'(\varsigma))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\varsigma)},
$$

which, with (2.20), gives

$$
\begin{aligned}
w'(\varsigma) \leq{} & -\rho(\varsigma) h^{\gamma(\eta-\varepsilon)} A[q; a, b](\varsigma) \\
& + \frac{\rho^{-\gamma}(\varsigma)(\rho'(\varsigma))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\varsigma)} + \frac{\rho'(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)} + \frac{\gamma \rho(\varsigma) \delta_{n-3}(\varsigma)}{\delta_{n-2}^{\gamma+1}(\varsigma)}
\end{aligned}
$$

or

$$
w'(\varsigma) \leq -\rho(\varsigma) h^{\gamma(\eta-\varepsilon)} A[q; a, b](\varsigma) + \frac{\rho^{-\gamma}(\varsigma)(\rho'(\varsigma))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\varsigma)} + \frac{d}{d\varsigma} \left(\frac{\rho(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)} \right).
$$

Integrating this inequality from ς_1 to ς, we arrive at

$$
\begin{aligned}
w(\varsigma) - w(\varsigma_1) \leq{} & -\int_{\varsigma_1}^{\varsigma} \left(\rho(\zeta) h^{\gamma(\eta-\varepsilon)} A[q; a, b](\zeta) - \frac{\rho^{-\gamma}(\zeta)(\rho'(\zeta))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\zeta)} \right) d\zeta \\
& + \frac{\rho(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)} - \frac{\rho(\varsigma_1)}{\delta_{n-2}^{\gamma}(\varsigma_1)}.
\end{aligned}
$$

From (2.16), we are led to

$$\int_{\varsigma_1}^{\varsigma} \left(\rho(\zeta) h^{\gamma(\eta-\varepsilon)} A[q;a,b](\zeta) - \frac{\rho^{-\gamma}(\zeta)(\rho'(\zeta))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\zeta)} \right) d\zeta$$

$$\leq -\rho(\varsigma) \frac{r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma}}{y^{\gamma}(\varsigma)} + \rho(\varsigma_1) \frac{r(\varsigma_1)(y^{(n-1)}(\varsigma_1))^{\gamma}}{y^{\gamma}(\varsigma_1)}$$

$$\leq -\rho(\varsigma) \frac{r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma}}{y^{\gamma}(\varsigma)}.$$

In view of (2.15), we get

$$\int_{\varsigma_1}^{\varsigma} \left(\rho(\zeta) h^{\gamma(\eta-\varepsilon)} A[q;a,b](\zeta) - \frac{\rho^{-\gamma}(\zeta)(\rho'(\zeta))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\zeta)} \right) d\zeta \leq \frac{\rho(\varsigma)}{\delta_{n-2}^{\gamma}(\varsigma)}$$

or

$$\frac{\delta_{n-2}^{\gamma}(\varsigma)}{\rho(\varsigma)} \int_{\varsigma_1}^{\varsigma} \left(\rho(\zeta) h^{\gamma(\eta-\varepsilon)} A[q;a,b](\zeta) - \frac{\rho^{-\gamma}(\zeta)(\rho'(\zeta))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_{n-3}^{\gamma}(\zeta)} \right) d\zeta \leq 1.$$

Taking the limsup, we obtain a contradiction. This completes the proof. □

Corollary 2.1 *Assume that* (Ω) *and* (2.1) *hold. If one of the following conditions holds*:

$$\limsup_{\varsigma \to \infty} \delta_{n-2}^{\gamma}(\varsigma) \int_{\varsigma_1}^{\varsigma} h^{\gamma\eta} A[q;a,b](\zeta) d\zeta > 1 \tag{2.21}$$

or

$$\limsup_{\varsigma \to \infty} \delta_{n-2}^{\gamma-1}(\varsigma) \int_{\varsigma_1}^{\varsigma} \left(h^{\gamma\eta} \delta_{n-2}(\zeta) A[q;a,b](\zeta) - \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{\delta_{n-3}(\zeta)}{\delta_{n-2}^{\gamma}(\zeta)} \right) d\zeta > 1, \tag{2.22}$$

or

$$\limsup_{\varsigma \to \infty} \int_{\varsigma_1}^{\varsigma} \left(h^{\gamma\eta} \delta_{n-2}^{\gamma}(\zeta) A[q;a,b](\zeta) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}} \frac{\delta_{n-3}(\varsigma)}{\delta_{n-2}(\varsigma)} \right) d\zeta > 1, \tag{2.23}$$

then $\mathbf{K} = \varnothing$.

Proof By choosing $\rho(\varsigma) = 1$, $\rho(\varsigma) = \delta_2(\varsigma)$, and $\rho(\varsigma) = \delta_2^{\gamma}(\varsigma)$, condition (2.14) in Theorem 2.5 becomes as (2.21), (2.22), and (2.23), respectively. □

3 Oscillation criteria

In this section, we are taking advantage of new nonexistence criteria of Kneser solutions to create better criteria that ensure all solutions of (1.1) are oscillatory.

Theorem 3.1 *Assume that* (Ω) *and* (2.1) *hold and there exists a function* $\rho \in C^1(I_0, (0, \infty))$ *such that* (2.14) *holds. If the DE*

$$\upsilon'(\varsigma) + \left(\frac{\lambda_0 g^{n-1}(\varsigma, a)}{(n-1)!} \right)^{\gamma} \frac{A[q;a,b](\varsigma)}{r(g(\varsigma, a))} \upsilon(g(\varsigma, a)) = 0 \tag{3.1}$$

is oscillatory for some $\lambda_0 \in (0, 1)$ *and there exists a function* $\theta \in C^1(I_0, (0, \infty))$ *such that*

$$\limsup_{\varsigma \to \infty} \int_{\varsigma_0}^{\varsigma} \left(\Phi(\varrho) - \frac{r(\varrho)\theta(\varrho)}{(\gamma+1)^{\gamma+1}} \left(\frac{\theta'(\varrho)}{\theta(\varrho)} + \frac{1+\gamma}{r^{1/\gamma}(\varrho)\delta_0(\varrho)} \right)^{\gamma+1} \right) \mathrm{d}\varrho = \infty \tag{3.2}$$

holds for some $\lambda_1 \in (0, 1)$, *where*

$$\Phi := \theta(\varsigma) \left(\frac{\lambda_1 g^{n-2}(\varsigma, a)}{(n-2)!} \right)^{\gamma} A[q; a, b](\varsigma) + (1 - \gamma) \frac{\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)},$$

then (1.1) *is oscillatory.*

Proof Suppose that there exists a nonoscillatory solution y of (1.1) in I_0. Without loss of generality, we suppose that y is eventually positive. From Lemma 2.1, we have three cases $(1) - (3)$. Since $y > 0$ and $y' > 0$ in cases (1) and (2), we have that $\lim_{\varsigma \to \infty} y(\varsigma) \neq 0$.

Now, let case (1) hold. Using Lemma 1.1, we get

$$y\big(g(\varsigma, s)\big) \geq \frac{\lambda_0 g^{n-1}(\varsigma, s)}{(n-1)! r^{1/\gamma}(g(\varsigma, s))} r^{1/\gamma}\big(g(\varsigma, s)\big) y^{(n-1)}\big(g(\varsigma, s)\big) \tag{3.3}$$

for all $\lambda_0 \in (0, 1)$ and sufficiently large ς. So, from (3.3), we get that $\upsilon(\varsigma) = r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma} > 0$ is a solution of the delay differential inequality

$$\upsilon'(\varsigma) + \left(\frac{\lambda_0 g^{n-1}(\varsigma, a)}{(n-1)!} \right)^{\gamma} \frac{A[q; a, b](\varsigma)}{r(g(\varsigma, a))} \upsilon\big(g(\varsigma, a)\big) \leq 0.$$

From [19, Corollary 1], there exists also a positive solution of (3.1), a contradiction.

Assume that case (2) holds. Note that $r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma}$ is nonincreasing, and so

$$y^{(n-2)}(\nu) - y^{(n-2)}(\varsigma) = \int_{\varsigma}^{\nu} \frac{1}{r^{1/\gamma}(\varrho)} \big(r(\varrho)\big(y^{(n-1)}(\varrho)\big)^{\gamma} \big)^{1/\gamma} \mathrm{d}\varrho$$

$$\leq r^{1/\gamma}(\varsigma) y^{(n-1)}(\varsigma) \int_{\varsigma}^{\nu} \frac{1}{r^{1/\gamma}(\varrho)} \mathrm{d}\varrho.$$

Letting $\nu \to \infty$, we get

$$y^{(n-2)}(\varsigma) \geq -r^{1/\gamma}(\varsigma) y^{(n-1)}(\varsigma) \delta_0(\varsigma). \tag{3.4}$$

Next, we define the function $\Theta(\varsigma)$ by

$$\Theta(\varsigma) := \theta(\varsigma) \left(\frac{r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma}}{(y^{(n-2)}(\varsigma))^{\gamma}} + \frac{1}{\delta_0^{\gamma}(\varsigma)} \right). \tag{3.5}$$

From (3.4), $\Theta(\varsigma) > 0$ for $\varsigma \geq \varsigma_1$. Therefore, we have

$$\Theta'(\varsigma) = \frac{\theta'(\varsigma)}{\theta(\varsigma)} \Theta(\varsigma) + \theta(\varsigma) \left(\frac{(r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma})'}{(y^{(n-2)}(\varsigma))^{\gamma}} - \frac{\gamma r(\varsigma)(y^{(n-1)}(\varsigma))^{\gamma+1}}{(y^{(n-2)}(\varsigma))^{\gamma+1}} - \frac{\gamma \delta_0'(\varsigma)}{\delta_0^{\gamma+1}(\varsigma)} \right),$$

it follows from (1.1) and (3.5) that

$$
\Theta'(\varsigma) \le \frac{\theta'(\varsigma)}{\theta(\varsigma)}\Theta(\varsigma) - \theta(\varsigma)\frac{y^\gamma(g(\varsigma,a))}{(y^{(n-2)}(\varsigma))^\gamma}A[q;a,b](\varsigma)
$$
$$
- \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)}\left(\frac{\Theta(\varsigma)}{\theta(\varsigma)} - \frac{1}{\delta_0^\gamma(\varsigma)}\right)^{(\gamma+1)/\gamma} + \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)}. \tag{3.6}
$$

Using Lemma 1.1, we get

$$
y(g(\varsigma,a)) \ge \lambda_1 \frac{g^{n-2}(\varsigma,a)}{n-2!}y^{(n-2)}(g(\varsigma,a)).
$$

Thus, (3.6) becomes

$$
\Theta'(\varsigma) \le \frac{\theta'(\varsigma)}{\theta(\varsigma)}\Theta(\varsigma) - \theta(\varsigma)\left(\frac{\lambda_1 g^{n-2}(\varsigma,a)}{(n-2)!}\right)^\gamma A[q;a,b](\varsigma) + \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)}
$$
$$
- \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)}\left(\frac{\Theta(\varsigma)}{\theta(\varsigma)} - \frac{1}{\delta_0^\gamma(\varsigma)}\right)^{(\gamma+1)/\gamma}.
$$

Using the inequality

$$
A^{(\gamma+1)/\gamma} - (A-B)^{(\gamma+1)/\gamma} \le \frac{B^{1/\gamma}}{\gamma}[(1+\gamma)A - B], \quad AB \ge 0,
$$

with $A = \Theta(\varsigma)/\theta(\varsigma)$, $B = 1/\delta_0^\gamma(\varsigma)$, we obtain

$$
\Theta'(\varsigma) \le \frac{\theta'(\varsigma)}{\theta(\varsigma)}\Theta(\varsigma) - \theta(\varsigma)\left(\frac{\lambda_1 g^{n-2}(\varsigma,a)}{(n-2)!}\right)^\gamma A[q;a,b](\varsigma) + \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)}
$$
$$
- \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)}\left(\left(\frac{\Theta(\varsigma)}{\theta(\varsigma)}\right)^{(\gamma+1)/\gamma} - \frac{1}{\gamma\delta_0(\varsigma)}\left[(1+\gamma)\frac{\Theta(\varsigma)}{\theta(\varsigma)} - \frac{1}{\delta_0^\gamma(\varsigma)}\right]\right).
$$

Therefore,

$$
\Theta'(\varsigma) \le \left(\frac{\theta'(\varsigma)}{\theta(\varsigma)} + \frac{1+\gamma}{r^{1/\gamma}(\varsigma)\delta_0(\varsigma)}\right)\Theta(\varsigma) - \theta(\varsigma)\left(\frac{\lambda_1 g^{n-2}(\varsigma,a)}{(n-2)!}\right)^\gamma A[q;a,b](\varsigma)
$$
$$
- \frac{\gamma}{r^{1/\gamma}(\varsigma)\theta^{1/\gamma}(\varsigma)}\Theta^{(\gamma+1)/\gamma}(\varsigma) - \frac{\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)} + \frac{\gamma\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)}.
$$

By using the inequality

$$
\nu E - VE^{(\gamma+1)/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\nu^{\gamma+1}}{V^\gamma}, \quad V > 0,
$$

with $\nu = \theta'(\varsigma)/\theta(\varsigma) + (1+\gamma)/(r^{1/\gamma}(\varsigma)\delta_0(\varsigma))$, $V = \gamma/(r^{1/\gamma}(\varsigma)\theta^{1/\gamma}(\varsigma))$, and $E = \Theta(\varsigma)$, we find

$$
\Theta'(\varsigma) \le -\theta(\varsigma)\left(\frac{\lambda_1 g^{n-2}(\varsigma,a)}{(n-2)!}\right)^\gamma A[q;a,b](\varsigma) + (\gamma-1)\frac{\theta(\varsigma)}{r^{1/\gamma}(\varsigma)\delta_0^{\gamma+1}(\varsigma)}
$$
$$
+ \frac{r(\varsigma)\theta(\varsigma)}{(\gamma+1)^{\gamma+1}}\left(\frac{\theta'(\varsigma)}{\theta(\varsigma)} + \frac{1+\gamma}{r^{1/\gamma}(\varsigma)\delta_0(\varsigma)}\right)^{\gamma+1}.
$$

Integrating this inequality from ς_1 to ς, we find

$$\int_{\varsigma_1}^{\varsigma} \left(\Phi(\varrho) - \frac{r(\varrho)\theta(\varrho)}{(\gamma+1)^{\gamma+1}} \left(\frac{\theta'(\varrho)}{\theta(\varrho)} + \frac{1+\gamma}{r^{1/\gamma}(\varrho)\delta_0(\varrho)} \right)^{\gamma+1} \right) d\varrho \leq \Theta(\varsigma_1),$$

which contradicts (3.2).

Next, using Theorem 2.5, it follows from (Ω) and (2.14) that $y \notin \mathbf{K}$, and so y does not satisfy case (3).

This completes the proof. $\qquad\square$

Corollary 3.1 *Assume that* (Ω) *and* (2.1) *hold and there exist functions* $\rho, \theta \in C^1(I_0, (0, \infty))$ *such that* (2.14) *and* (3.2) *hold. If*

$$\liminf_{\varsigma \to \infty} \int_{g(\varsigma,a)}^{\varsigma} \frac{(\lambda_0 g^{n-1}(\xi,a))^{\gamma}}{((n-1)!)^{\gamma} r(g(\xi,a))} A[q;a,b](\xi) \, d\xi > \frac{1}{e}, \tag{3.7}$$

then (1.1) *is oscillatory.*

Proof Applying a well-known criterion [7, Theorem 2] for first-order equation (3.1) to be oscillatory, we obtain immediately criterion (3.7). $\qquad\square$

Theorem 3.2 *Assume that* $n = 4$, (Ω) *and* (2.1) *hold. If there exist functions* $\varphi, \phi, \rho \in C^1(I_0, (0, \infty))$ *such that*

$$\int_{\varsigma_0}^{\infty} \left(\varphi(s) \left(\frac{g(s,a)}{s} \right)^{3\gamma} A[q;a,b](s) - \frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{r(s)(\varphi'(s))^{\gamma+1}}{(\mu s^2 \varphi(s))^{\gamma}} \right) ds = \infty, \tag{3.8}$$

$$\int_{\varsigma_0}^{\infty} \left(\phi(\xi) \int_{\xi}^{\infty} \left(\frac{1}{r(\nu)} \int_{\nu}^{\infty} \frac{g^{\gamma}(s,a)}{s^{\gamma}} A[q;a,b](s) \, ds \right)^{1/\gamma} d\nu - \frac{(\phi'(\xi))^2}{4\phi(\xi)} \right) d\xi$$

$$= \infty, \tag{3.9}$$

$$\lim_{\varsigma \to \infty} \sup \frac{\delta_0^{\gamma}(\varsigma)}{\rho(\varsigma)} \int_{\varsigma_0}^{\varsigma} \left(\rho(\varrho) \left(\frac{\lambda}{2!} g^2(\varrho,a) \right)^{\gamma} A[q;a,b](\varrho) - \frac{r(\varrho)(\rho'(\varrho))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \rho^{\gamma}(\varrho)} \right) d\varrho$$

$$> 1, \tag{3.10}$$

and

$$\limsup_{\varsigma \to \infty} \frac{\delta_2^{\gamma}(\varsigma)}{\rho(\varsigma)} \int_{\varsigma_1}^{\varsigma} \left(\rho(\zeta) h^{\gamma\eta} A[q;a,b](\zeta) - \frac{\rho^{-\gamma}(\zeta)(\rho'(\zeta))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \delta_1^{\gamma}(\zeta)} \right) d\zeta > 1, \tag{3.11}$$

for some $\lambda, \mu \in (0,1)$, *then* (1.1) *is oscillatory.*

Proof Suppose that there exists a nonoscillatory solution y of (1.1) in I_0. Without loss of generality, we suppose that y is eventually positive. Using [1, Lemma 2.2.1], there exist four possible cases:

C1: $\quad y'(\varsigma) > 0, \qquad y''(\varsigma) > 0 \quad \text{and} \quad y'''(\varsigma) > 0;$

C2: $\quad y'(\varsigma) > 0, \qquad y''(\varsigma) < 0 \quad \text{and} \quad y'''(\varsigma) > 0;$

$\text{C3}: \quad y'(\varsigma) > 0, \qquad y''(\varsigma) > 0 \quad \text{and} \quad y'''(\varsigma) < 0;$

$\text{C4}: \quad y'(\varsigma) < 0, \qquad y''(\varsigma) > 0 \quad \text{and} \quad y'''(\varsigma) < 0.$

The proof of the case where C1 or C2 holds is the same as that of [16, Theorem 2.1].

Assume that C3 holds. Proceeding as in the proof of Theorem 3.1, we obtain that (3.6) holds. Thus, we get

$$\Theta'(\varsigma) \le \frac{\rho'(\varsigma)}{\rho(\varsigma)}\Theta(\varsigma) - \frac{\rho(\varsigma)y^\gamma(g(\varsigma,a))}{(y''(\varsigma))^\gamma}A[q;a,b](\varsigma)$$
$$- \frac{\gamma}{r^{1/\gamma}(\varsigma)\rho^{1/\gamma}(\varsigma)}\left(\Theta(\varsigma) - \frac{\rho(\varsigma)}{\delta_0^\gamma(\varsigma)}\right)^{(\gamma+1)/\gamma} - \frac{\gamma\rho(\varsigma)\delta_0'(\varsigma)}{\delta_0^{\gamma+1}(\varsigma)}.$$

Using Lemma 1.2 with $D = \rho'(\varsigma)/\rho(\varsigma), M = \gamma/(r^{1/\gamma}(\varsigma)\rho^{1/\gamma}(\varsigma)), N = \rho(\varsigma)/\delta_0^\gamma(\varsigma),$ and $\xi = \Theta,$ we obtain

$$\Theta'(\varsigma) \le -\frac{\rho(\varsigma)y^\gamma(g(\varsigma,a))}{(y''(\varsigma))^\gamma}A[q;a,b](\varsigma)$$
$$- \frac{\gamma\rho(\varsigma)\delta_0'(\varsigma)}{\delta_0^{\gamma+1}(\varsigma)} + \frac{\rho'(\varsigma)}{\delta_0^\gamma(\varsigma)} + \frac{r(\varsigma)(\rho'(\varsigma))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(\varsigma)}$$

or

$$\Theta'(\varsigma) \le -\frac{\rho(\varsigma)y^\gamma(g(\varsigma,a))}{(y''(\varsigma))^\gamma}A[q;a,b](\varsigma) + \left(\frac{\rho(\varsigma)}{\delta_0^\gamma(\varsigma)}\right)' + \frac{r(\varsigma)(\rho'(\varsigma))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(\varsigma)}.$$

From Lemma 1.1, we have

$$\Theta'(\varsigma) \le -\rho(\varsigma)\left(\frac{\lambda}{2!}g^2(\varsigma,a)\right)^\gamma A[q;a,b](\varsigma) + \left(\frac{\rho(\varsigma)}{\delta_0^\gamma(\varsigma)}\right)' + \frac{r(\varsigma)(\rho'(\varsigma))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(\varsigma)}.$$

Integrating the above inequality from ς_1 to ς, we find

$$\Theta(\varsigma) - \Theta(\varsigma_1) \le -\int_{\varsigma_1}^\varsigma \left(\rho(\varrho)\left(\frac{\lambda}{2!}g^2(\varrho,a)\right)^\gamma A[q;a,b](s) - \frac{r(\varrho)(\rho'(\varrho))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\rho^\gamma(\varrho)}\right)d\varrho$$
$$+ \frac{\rho(\varsigma)}{\delta_0^\gamma(\varsigma)} - \frac{\rho(\varsigma_1)}{\delta_0^\gamma(\varsigma_1)}.$$

From the definition of Θ, we see that

$$\int_{\varsigma_1}^\varsigma \left(\rho(\varrho)\left(\frac{\lambda}{2!}g^2(\varrho,a)\right)^\gamma A[q;a,b](s) - \frac{r(\varrho)(\rho'(\varrho))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\rho^\gamma(\varrho)}\right)d\varrho$$
$$\le -\frac{\rho(\varsigma)r(\varsigma)(y'''(\varsigma))^\gamma}{(y''(\varsigma))^\gamma} + \frac{\rho(\varsigma_1)r(\varsigma_1)(y'''(\varsigma_1))^\gamma}{(y''(\varsigma_1))^\gamma}.$$

This provides

$$\int_{\varsigma_1}^\varsigma \left(\rho(\varrho)\left(\frac{\lambda}{2!}g^2(\varrho,a)\right)^\gamma A[q;a,b](s) - \frac{r(\varrho)(\rho'(\varrho))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\rho^\gamma(\varrho)}\right)d\varrho \le \frac{\rho(\varsigma)}{\delta_0^\gamma(\varsigma)}.$$

Hence,

$$\frac{\delta_0^\gamma(\varsigma)}{\rho(\varsigma)} \int_{\varsigma_1}^\varsigma \left(\rho(\varrho) \left(\frac{\lambda}{2!} g^2(\varrho,a) \right)^\gamma A[q;a,b](s) - \frac{r(\varrho)(\rho'(\varrho))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)} \rho^\gamma(\varrho)} \right) d\varrho \leq 1,$$

which contradicts (3.10).

Next, using Theorem 2.5 with $n = 4$, it follows from (Ω) and (3.11) that $y \notin \mathbf{K}$, and so y does not satisfy case C4.

This completes the proof. □

4 Examples

Example 4.1 Consider the fourth-order DE

$$\left(e^{\gamma\varsigma} \left(y'''(\varsigma) \right)^\gamma \right)' + q_0 e^{\gamma\varsigma} A[y \circ g; \lambda, 1](\varsigma) = 0, \tag{4.1}$$

where $\varsigma \geq 1$, $\lambda \in (0, 1 - 1/e)$, $g(\varsigma, s) = s\varsigma$, and $q_0 > 0$. Then we get $\delta_m(\varsigma) = e^{-\varsigma}$ for $m = 0, 1, 2$. Moreover, it is easy to verify that conditions (2.1), (3.2), and (3.7) are satisfied.

By using the fact that $e^\upsilon > e\upsilon$ for $\upsilon > 0$, we get

$$\frac{\delta_2(g(\varsigma,s))}{\delta_2(\varsigma)} \geq \frac{\delta_2(g(\varsigma,a))}{\delta_2(\varsigma)} = e^{(1-\lambda)\varsigma} > e(1-\lambda)\varsigma \geq e(1-\lambda) := h > 1.$$

From Theorems 2.2 and 2.3, equation (4.1) has no Kneser solutions if

$$\eta = \left(\frac{q_0}{\gamma} \right)^{1/\gamma} (1-\lambda)^{1/\gamma} > 1$$

or

$$h^\eta \eta = \left(e(1-\lambda) \right)^{(1-\lambda)^{1/\gamma} (\frac{q_0}{\gamma})^{1/\gamma}} \left(\frac{q_0}{\gamma} \right)^{1/\gamma} (1-\lambda)^{1/\gamma} > 1$$

holds.

Next, condition (2.14) takes the form

$$\left(e(1-\lambda) \right)^{\gamma(1-\lambda)^{1/\gamma} (\frac{q_0}{\gamma})^{1/\gamma}} q_0 (1-\lambda) > \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1}. \tag{4.2}$$

By Corollary 3.1, equation (4.1) is oscillatory provided that (4.2) holds.

Example 4.2 Consider the fourth-order DE

$$\left(\varsigma^5 y'''(\varsigma) \right)' + q_0 \varsigma A[y \circ g; \lambda_2, 1](\varsigma) = 0, \tag{4.3}$$

where $\varsigma \geq 1$, $\lambda_2 \in (0, 1)$, $g(\varsigma, s) = s\varsigma$, and $q_0 > 0$. Then we have that $\delta_0(\varsigma) = 1/4\varsigma^4$, $\delta_1(\varsigma) = 1/12\varsigma^3$, and $\delta_2(\varsigma) = 1/24\varsigma^2$. Moreover, it is easy to verify that conditions (3.8) and (3.9) are satisfied. Using Theorem 3.2, equation (4.3) is oscillatory if

$$(3.10) \to \frac{\lambda_1}{8} \lambda_2^2 q_0 (1-\lambda_2) > 1$$

and

$$(3.11) \rightarrow \frac{1}{24}\left(\frac{1}{\lambda_2}\right)^{\frac{q_0}{48}(1-\lambda_2)} q_0(1-\lambda_2) > \frac{1}{2}$$

hold.

Remark 4.1 Consider the fourth-order DE (4.3). Condition (3.7) is not satisfied, so Theorem 3.1 cannot be applied. Thus, Theorem 3.2 provides an applicable criterion when Theorem 3.1 fails to apply.

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