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New dynamic Hilbert-type inequalities in two independent variables involving Fenchel–Legendre transform

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Abstract

In this paper, we establish some dynamic Hilbert-type inequalities in two independent variables on time scales by using the Fenchel–Legendre transform. We also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as particular cases. Our results give more general forms of several previously established inequalities.

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1 Introduction

Hilger [1] presented a new calculus, named time scales, to get a unification of discrete and continuous calculus. The monographs by Bohner and Peterson [2, 3] summarize and organize much of time scales calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We assume that $\mathbb{T}$ has the topology inherited from the standard topology on the real numbers $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for any $t \in \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In the previous two definitions, we set $\inf \emptyset = \sup \mathbb{T}$ (i.e., if t is the maximum of $\mathbb{T}$, then $\sigma(t) = t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if t is the minimum of $\mathbb{T}$, then $\rho(t) = t$), where $\emptyset$ is the empty set.

A point $t \in \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$. The points that are right-and left-scattered are called isolated. The points that are right- and left-dense are called dense.

We define the forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$. Then the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined by $f^{\sigma}(t) = f(\sigma(t))$ for $t \in \mathbb{T}$, that is, $f^{\sigma} = f \circ \sigma$. Similarly, the function $f^{\rho} : \mathbb{T} \to \mathbb{R}$ is defined by $f^{\rho}(t) = f(\rho(t))$ for $t \in \mathbb{T}$, that is, $f^{\rho} = f \circ \rho$.

We introduce the set $\mathbb{T}^{\kappa}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

The interval $[a,b]$ in $\mathbb{T}$ is defined by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-closed intervals are defined similarly.

Let $f : \mathbb{T} \to \mathbb{R}$ be a real-valued function on a time scale $\mathbb{T}$. Then, for $t \in \mathbb{T}^{\kappa}$, we define $f^{\Delta}(t)$ to be the number (if it exists) such that for any $\varepsilon > 0$, there is a neighborhood U of t such that, for all $s \in U$, we have

$$\bigl|\bigl[f\bigl(\sigma(t)\bigr) - f(s)\bigr] - f^{\Delta}(t)\bigl[\sigma(t) - s\bigr]\bigr| \leq \varepsilon\bigl|\sigma(t) - s\bigr|.$$

In this case, we say that f is delta-differentiable on $\mathbb{T}^{\kappa}$, provided that $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Now we recall some basic concepts related to partial derivatives on time scales. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be any time scales. Let σ_1 and Δ_1 (σ_2 and Δ_2) denote the forward jump operator and delta differentiation operator on $\mathbb{T}_1$ (resp., $\mathbb{T}_2$). Assume that $u < v$ are points in $\mathbb{T}_1$, $e < f$ are points in $\mathbb{T}_2$, $[u,v)$ is a semiclosed bounded interval in $\mathbb{T}_1$, and $[e,f)$ is a semiclosed bounded interval in $\mathbb{T}_2$. Let us consider a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$,

$$\mathrm{R} = [u,v)_{\mathbb{T}_1} \times [e,f)_{\mathbb{T}_2} = \bigl\{(t_1,t_2) : t_1 \in [u,v)_{\mathbb{T}_1}, t_2 \in [e,f)_{\mathbb{T}_2}\bigr\}.$$

Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \longrightarrow \mathbb{R}$. We say that f has a Δ_1 partial derivative at $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with respect to t_1 if for each $\epsilon > 0$, there exists a neighborhood U_{t_1} of t_1 such that

$$\bigl|\bigl[f\bigl(\sigma_1(t_1),t_2\bigr) - f(s,t_2)\bigr] - f^{\Delta_1}(t_1,t_2)\bigl[\sigma_1(t_1) - s\bigr]\bigr| \leq \epsilon\bigl|\sigma_1(t_1) - s\bigr|$$

for all $s \in U_{t_1}$. We say that f has a Δ_2 partial derivative at $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with respect to t_2 if for each $\epsilon > 0$, there exists a neighborhood U_{t_2} of t_2 such that

$$\bigl|\bigl[f\bigl(t_1,\bigl(\sigma_2(t_2)\bigr) - f(t_1,l)\bigr] - f^{\Delta_2}(t_1,t_2)\bigl[\sigma_2(t_2) - l\bigr]\bigr| \leq \epsilon\bigl|\sigma_2(t_2) - l\bigr|$$

for all $l \in U_{t_2}$.

A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \longrightarrow \mathbb{R}$ is said to be rd-continuous in t_2 if for every $\beta_1 \in \mathbb{T}_1$, the function $f(\beta_1,t_2)$ is rd-continuous on $\mathbb{T}_2$, and it is rd-continuous in t_1 if for every $\beta_2 \in \mathbb{T}_2$, the function $f(t_1,\beta_2)$ is rd-continuous on $\mathbb{T}_1$.

Let CC_{rd} denote the set of functions $f(t_1,t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the following properties:

1. f is rd-continuous in t_1.
2. f is rd-continuous in t_2.

3. If $(x_1,x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_1 right-dense or maximal and x_2 right-dense or maximal, then f is continuous at (x_1,x_2).
4. If x_1 and x_2 are both left-dense, then the limit of $f(t_1,t_2)$ exists as (t_1,t_2) approaches (x_1,x_2) along any path in the region

$$\mathrm{R}_{LL}(x_1,x_2) = \{(t_1,t_2) : t_1 \in [u,x_1) \cap \mathbb{T}_1, t_2 \in [e,x_2) \cap \mathbb{T}_2\}.$$

Next, we write Hölder's inequality and Jensen's inequality in two independent variables on time scales.

Lemma 1.1 (Dynamic Hölder's inequality [4]) *Let $u, v \in \mathbb{T}$ with $u < v$. If $f, g \in CC_{\mathrm{rd}}([u,v]_{\mathbb{T}} \times [u,v]_{\mathbb{T}}, \mathbb{R})$ are integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\int_u^v \int_u^v |f(r,t)g(r,t)| \Delta r \Delta t \leq \left[\int_u^v \int_u^v |f(r,t)|^p \Delta r \Delta t\right]^{\frac{1}{p}} \times \left[\int_u^v \int_u^v |g(r,t)|^q \Delta r \Delta t\right]^{\frac{1}{q}}. \tag{1.1}$$

Lemma 1.2 (Dynamic Jensen's inequality [5]) *Let r, t in the rectangle R, and let $-\infty \leq m, n \leq \infty$. If $f \in CC_{\mathrm{rd}}(\mathbb{R}, (m,n))$ and $\Phi : (m,n) \longrightarrow \mathbb{R}$ is convex, then*

$$\Phi\left(\frac{\int_u^v \int_\omega^s f(r,t) \Delta_1 r \Delta_2 t}{\int_u^v \int_\omega^s \Delta_1 r \Delta_2 t}\right) \leq \frac{\int_u^v \int_\omega^s \Phi(f(r,t)) \Delta_1 r \Delta_2 t}{\int_u^v \int_\omega^s \Delta_1 r \Delta_2 t}. \tag{1.2}$$

Lemma 1.3 (Fubini's thoerem [6]) *Let $(\Omega_1, \Sigma_1, \mu_\Delta)$ and $(\Omega_2, \Sigma_2, \nu_\Delta)$ be two finite-dimensional time scales measure spaces. Further, let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a delta-integrable function. Define the functions*

$$\phi(y) = \int_{\Omega_1} f(x,y)\, d\mu_\Delta(x), \quad y \in \Omega_2,$$

and

$$\psi(x) = \int_{\Omega_2} f(x,y)\, d\nu_\Delta(y), \quad x \in \Omega_1.$$

Then ϕ is delta-integrable on Ω_2, ψ is delta-integrable on Ω_1, and

$$\int_X d\mu_\Delta(x) \int_Y f(x,y)\, d\nu_\Delta(y) = \int_Y d\nu_\Delta(y) \int_X f(x,y)\, d\mu_\Delta(x).$$

Lemma 1.4 ([7]) *Let $y + x \geq 1$, where y and $x \in \mathbb{R}$. For $\alpha \geq \beta \geq \frac{1}{2}$ and $\gamma > 0$, we have*

$$(x+y)^{\frac{1}{\gamma}} \leq \left(|x|^{\frac{1}{2\beta}} + |y|^{\frac{1}{2\beta}}\right)^{\frac{2\alpha}{\gamma}}. \tag{1.3}$$

Definition 1.5 A function $\Phi : [0,\infty) \longrightarrow \mathbb{R}$ is called a submultiplicative function if

$$\Phi(xy) \leq \Phi(x)\Phi(y) \quad \text{for all } x, y \geq 0. \tag{1.4}$$

Now we present the Fenchel–Legendre transform, which will need in the proof of our results. We refer to example to [8–10] for more detail.

Definition 1.6 A function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is called coercive if

$$f(x) \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty.$$

Definition 1.7 Let $h : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $h \not\equiv +\infty$, that is, $\mathrm{Dom}(h) = \{x \in \mathbb{R}^n | h(x) < \infty\} \neq \emptyset$. Then the Fenchel–Legendre transform is defined as

$$h^* : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}, \qquad y \longrightarrow h^*(y) = \sup\bigl\{\langle y, x\rangle - h(x), x \in \mathrm{Dom}(h)\bigr\}, \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. The mapping $h \longrightarrow h^*$ is called the conjugate operation.

The domain of h^* is the set of slopes of all affine functions minorizing the function h over $\mathbb{R}^n$. An equivalent formula for (1.5) is obtained in the next corollary:

Corollary 1.8 *Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be strictly convex differentiable 1-coercive function. Then*

$$h^*(y) = \bigl\langle y, (\nabla h)^{-1}(y)\bigr\rangle - h\bigl((\nabla h)^{-1}(y)\bigr) \tag{1.6}$$

for all $y \in \mathrm{Dom}(h^*)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

Lemma 1.9 (Fenchel–Young inequality [10]) *Let h and h^* be a function and its Fenchel–Legendre transform, respectively. Then*

$$\langle x, y\rangle \leq h(x) + h^*(y) \tag{1.7}$$

for all $y \in \mathrm{Dom}(h^*)$ and $x \in \mathrm{Dom}(h)$.

It is well known that Hilbert's inequalities play a very important role in the mathematical analysis. Several dynamic inequalities of Hilbert type and others are established by different mathematicians; see [4, 9–48].

Hamiaz et al. [20] studied the following discrete inequalities: Suppose $q, p \geq 1$, $\alpha \geq \beta \geq \frac{1}{2}$, and $(b_m)_m \geq 0$ and $(a_n)_n \geq 0$ are sequences of real numbers. Denote $A_n = \sum_{s=1}^{n} a_s$ and $B_m = \sum_{t=1}^{m} b_t$. Then

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r} \frac{A_n^{2p} B_m^{2q}}{h(n) + h^*(m)} &\leq C_1(p,q)\left(\sum_{n=1}^{k}(k-n+1)\bigl(a_n A_n^{p-1}\bigr)^2\right) \\ &\quad \times \left(\sum_{m=1}^{r}(r-m+1)\bigl(b_m B_m^{q-1}\bigr)^2\right)\end{aligned}$$

and

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^pB_m^q}{(|h(n)|^{\frac{1}{2\beta}}+|h^*(m)|^{\frac{1}{2\beta}})^{\alpha}}\le\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^pB_m^q}{\sqrt{h(n)+h^*(m)}}$$
$$\le C_2(p,q,k,r)\left(\sum_{n=1}^{k}(k-n+1)\left(a_nA_n^{p-1}\right)^2\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{m=1}^{r}(r-m+1)\left(b_mB_m^{q-1}\right)^2\right)^{\frac{1}{2}}$$

unless (a_n) or (b_m) is null, where

$$C_1(p,q)=(pq)^2\quad\text{and}\quad C_2(p,q,r,k)=pq\sqrt{kr}.$$

Very recently, El-Deeb et al. [7] studied the time scales version of the above inequalities. They proved that if $A(s):=\int_{t_0}^{s}a(\tau)\Delta\tau$ and $B(t):=\int_{t_0}^{t}b(\tau)\Delta\tau$, where $a(\tau)\ge0$ and $b(\tau)\ge0$ are right-dense continuous, then

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{A^{qK}(\sigma(s))B^{qL}(\sigma(t))}{(|h(\sigma(s)-t_0)|^{\frac{1}{2\beta}}+|h^*(\sigma(t)-t_0)|^{\frac{1}{2\beta}})^{\frac{2q\alpha}{p}}}\Delta s\Delta t$$
$$\le C_1^{**}(L,K,q)\left(\int_{t_0}^{x}\left(\sigma(x)-\sigma(s)\right)\left(a(s)A^{K-1}\left(\sigma(s)\right)\right)^q\Delta s\right)$$
$$\times\left(\int_{t_0}^{y}\left(\sigma(y)-\sigma(t)\right)\left(b(t)B^{L-1}\left(\sigma(t)\right)\right)^q\Delta t\right)$$

and

$$\int_{t_0}^{x}\int_{t_0}^{y}\frac{A^{K}(\sigma(s))B^{L}(\sigma(t))}{(|h(\sigma(s)-t_0)|^{\frac{1}{2\beta}}+|h^*(\sigma(t)-t_0)|^{\frac{1}{2\beta}})^{\frac{2\alpha}{p}}}\Delta s\Delta t$$
$$\le C_2^{**}(L,K,p)\left(\int_{t_0}^{x}\left(\sigma(x)-\sigma(s)\right)\left(A^{K-1}\left(\sigma(s)\right)a(s)\right)^q\Delta s\right)^{\frac{1}{q}}$$
$$\times\left(\int_{t_0}^{y}\left(\sigma(y)-\sigma(t)\right)\left(B^{L-1}\left(\sigma(t)\right)b(t)\right)^q\Delta t\right)^{\frac{1}{q}}$$

for $p,q>1$, $K,L\ge1$, and $s,t,t_0,x,y\in\mathbb{T}$, and

$$C_1^{**}(L,K,q)=(KL)^q\quad\text{and}\quad C_2^{**}(L,K,p)=KL(x-t_0)^{\frac{1}{p}}(y-t_0)^{\frac{1}{p}}.$$

In this paper, motivated by the inequalities mentioned, using the Fenchel–Legendre transform and other vital tools, we prove some new Hilbert-type inequalities in two independent variables on time scales. These inequalities extend and give more general new forms of several previously established inequalities. For example, we generalize the results given in [7, 20] and others.

2 Main results

In the next theorems, we assume that $\alpha \geq \beta \geq \frac{1}{2}$ and $p_1, q_1 > 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$. For $t_0 \in \mathbb{T}_1, \mathbb{T}_2$, we denote the subintervals of $\mathbb{T}_1$, $\mathbb{T}_2$ by $I_x = [t_0, x)_{\mathbb{T}_1}$, $I_z = [t_0, z)_{\mathbb{T}_1}$, $I_y = [t_0, y)_{\mathbb{T}_2}$, and $I_\omega = [t_0, \omega)_{\mathbb{T}_2}$, where $x, z \in \Omega_1 = [t_0, \infty) \cap \mathbb{T}_1$, $y, \omega \in \Omega_2 = [t_0, \infty) \cap \mathbb{T}_2$, and $0 \leq t_0 < r_1 < k_1 < t_1 < s_1$.

Theorem 2.1 *Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be any time scales with $t_0, s_1, k_1, x, z \in \mathbb{T}_1$ and $t_0, t_1, r_1, y, \omega \in \mathbb{T}_2$. Let $f(s_1, t_1) \in CC_{\mathrm{rd}}(I_x \times I_y, \mathbb{R}^+)$ and $g(k_1, r_1) \in CC_{\mathrm{rd}}(I_z \times I_\omega, \mathbb{R}^+)$, and define*

$$F(s_1, t_1) := \int_{t_0}^{s_1} \int_{t_0}^{t_1} f(\xi, \eta) \Delta\xi \Delta\eta \quad \text{and} \quad G(k_1, r_1) := \int_{t_0}^{k_1} \int_{t_0}^{r_1} g(\xi, \eta) \Delta\xi \Delta\eta.$$

Then, for $(s_1, t_1) \in I_x \times I_y$ and $(k_1, r_1) \in I_z \times I_\omega$, we have

$$\begin{aligned} &\int_{t_0}^{x} \int_{t_0}^{y} \biggl(\int_{t_0}^{z} \int_{t_0}^{\omega} \frac{F(s_1, t_1) G(k_1, r_1)}{(|h[(s_1 - t_0)(t_1 - t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1 - t_0)(r_1 - t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1 - 1)}{p_1}}} \Delta k_1 \Delta r_1 \biggr) \Delta s_1 \Delta t_1 \\ &\quad \leq C_1(p_1) \biggl(\int_{t_0}^{x} \int_{t_0}^{y} \bigl(\sigma(x) - s_1\bigr)\bigl(\sigma(y) - t_1\bigr) \bigl[f(s_1, t_1)\bigr]^{p_1} \Delta s_1 \Delta t_1 \biggr)^{\frac{1}{p_1}} \\ &\qquad \times \biggl(\int_{t_0}^{z} \int_{t_0}^{\omega} \bigl(\sigma(z) - k_1\bigr)\bigl(\sigma(\omega) - r_1\bigr) \bigl[g(k_1, r_1)\bigr]^{p_1} \Delta k_1 \Delta r_1 \biggr)^{\frac{1}{p_1}}, \end{aligned} \tag{2.1}$$

where

$$C_1(p_1) = \bigl[(x - t_0)(y - t_0)(z - t_0)(\omega - t_0)\bigr]^{\frac{p_1 - 1}{p_1}}.$$

Proof By the assumptions, applying Hölder's inequality with indices $\frac{p_1}{p_1 - 1}$ and p_1, we have

$$F(s_1, t_1) \leq \bigl[(s_1 - t_0)(t_1 - t_0)\bigr]^{\frac{p_1 - 1}{p_1}} \biggl(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \bigl[f(\xi, \eta)\bigr]^{p_1} \Delta\xi \Delta\eta \biggr)^{\frac{1}{p_1}} \tag{2.2}$$

and

$$G(k_1, r_1) \leq \bigl[(k_1 - t_0)(r_1 - t_0)\bigr]^{\frac{p_1 - 1}{p_1}} \biggl(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \bigl[g(\xi, \eta)\bigr]^{p_1} \Delta\xi \Delta\eta \biggr)^{\frac{1}{p_1}}. \tag{2.3}$$

By multiplying (2.2) and (2.3) we get

$$\begin{aligned} F(s_1, t_1) G(k_1, r_1) &\leq \bigl(\bigl[(s_1 - t_0)(t_1 - t_0)\bigr]\bigl[(k_1 - t_0)(r_1 - t_0)\bigr]\bigr)^{\frac{p_1 - 1}{p_1}} \\ &\quad \times \biggl(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \bigl[f(\xi, \eta)\bigr]^{p_1} \Delta\xi \Delta\eta \biggr)^{\frac{1}{p_1}} \\ &\quad \times \biggl(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \bigl[g(\xi, \eta)\bigr]^{p_1} \Delta\xi \Delta\eta \biggr)^{\frac{1}{p_1}}. \end{aligned} \tag{2.4}$$

Using Lemma 1.9 (for $x, y \geq 0$) in (2.4), we get

$$\begin{aligned} F(s_1,t_1)G(k_1,r_1) \leq{}& \bigl(h\bigl[(s_1-t_0)(t_1-t_0)\bigr]+h^*\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr)^{\frac{p_1-1}{p_1}} \\ &\times \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl[f(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}} \\ &\times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl[g(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}. \end{aligned} \tag{2.5}$$

Using Lemma 1.4 in (2.5) gives

$$\begin{aligned} F(s_1,t_1)G(k_1,r_1) \leq{}& \bigl(\bigl|h\bigl[(s_1-t_0)(t_1-t_0)\bigr]\bigr|^{\frac{1}{2\beta}}+\bigl|h^*\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr|^{\frac{1}{2\beta}}\bigr)^{\frac{2\alpha(p_1-1)}{p_1}} \\ &\times \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl[f(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}} \\ &\times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl[g(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}. \end{aligned} \tag{2.6}$$

Dividing both sides of (2.6) by $(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}$, we get

$$\begin{aligned} &\frac{F(s_1,t_1)G(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}} \\ &\quad\leq \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl[f(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl[g(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}. \end{aligned} \tag{2.7}$$

Integrating both sides of (2.7) firstly with respect to r_1 and k_1 and then with respect to s_1 and t_1, and applying Hölder's inequality with indices $\frac{p_1}{p_1-1}$ and p_1, we obtain

$$\begin{aligned} &\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{F(s_1,t_1)G(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\,\Delta r_1\right)\Delta s_1\,\Delta t_1 \\ &\quad\leq \bigl[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\bigr]^{\frac{p_1-1}{p_1}} \\ &\qquad\times\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl[f(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta s_1\,\Delta t_1\right)^{\frac{1}{p_1}} \\ &\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl[g(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta k_1\,\Delta r_1\right)^{\frac{1}{p_1}} \\ &\quad= C_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl[f(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta s_1\,\Delta t_1\right)^{\frac{1}{p_1}} \\ &\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl[g(\xi,\eta)\bigr]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta k_1\,\Delta r_1\right)^{\frac{1}{p_1}}. \end{aligned} \tag{2.8}$$

Applying Fubini's theorem to the right-hand side of (2.8), we have

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{F(s_1,t_1)G(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\quad\leq C_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}(x-s_1)(y-t_1)\big[f(s_1,t_1)\big]^{p}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}(z-k_1)(\omega-r_1)\big[g(k_1,r_1)\big]^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}.\end{aligned}$$

Using the relations $\sigma(x)\geq x$, $\sigma(y)\geq y$, $\sigma(\omega)\geq\omega$, and $\sigma(z)=z$, we obtain

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{F(s_1,t_1)G(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\quad\leq C_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\big[f(s_1,r_1)\big]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\big(\sigma(z)-k_1\big)\big(\sigma(\omega)-r_1\big)\big[g(k_1,t_1)\big]^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}.\end{aligned}$$

This completes the proof. □

In the particular case of Theorem 2.1 where $\mathbb{T}_1=\mathbb{T}_1=\mathbb{R}$, we have $\sigma(y)=y$, $\sigma(x)=x$, $\sigma(\omega)=\omega$, and $\sigma(z)=z$, and we get the following result.

Corollary 2.2 *Let $f(s_1,t_1)$ and $g(k_1,r_1)$ be real-valued continuous functions, and define*

$$F(s_1,t_1):=\int_{0}^{s_1}\int_{0}^{t_1}f(\xi,\eta)\,d\xi\,d\eta\quad\text{and}\quad G(k_1,r_1):=\int_{0}^{k_1}\int_{0}^{r_1}g(\xi,\eta)\,d\xi\,d\eta.$$

Then for $(s_1,t_1)\in I_x\times I_y$ and $(k_1,r_1)\in I_z\times I_\omega$, we have that

$$\begin{aligned}&\int_{0}^{x}\int_{0}^{y}\left(\int_{0}^{z}\int_{0}^{\omega}\frac{F(s_1,t_1)G(k_1,r_1)}{(|h[(s_1)(t_1)]|^{\frac{1}{2\beta}}+|h^*[(k_1)(r_1)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\,dk_1\,dr_1\right)ds_1\,dt_1\\&\quad\leq C_1^*(p_1)\left(\int_{0}^{x}\int_{0}^{y}(x-s_1)(y-t_1)\big[f(s_1,t_1)\big]^{p_1}\,ds_1\,dt_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{0}^{z}\int_{0}^{\omega}(z-k_1)(\omega-r_1)\big[g(k_1,r_1)\big]^{p_1}\,dk_1\,dr_1\right)^{\frac{1}{p_1}},\end{aligned}$$

where

$$C_1^*(p_1)=\big[(x)(y)(z)(\omega)\big]^{\frac{p_1-1}{p_1}}.$$

In the particular case of Theorem 2.1 where $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, we have $\sigma(x)=x+1$, $\sigma(y)=y+1$, $\sigma(w)=\omega+1$, and $\sigma(z)=z+1$, and we get the following result.

Corollary 2.3 *Let $\{a_{m_1,n_1}\}_{0\leq m_1,n_1\leq N}$ and $\{b_{k_1,r_1}\}_{0\leq k_1,r_1\leq N}$ be nonnegative sequences of real numbers, and define*

$$A_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} a_{\xi,\eta}, \quad \textit{and} \quad B_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} b_{\xi,\eta}.$$

Then

$$\begin{aligned}
&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1}\frac{A_{s_1,t_1}B_{k_1,r_1}}{(|h(s_1t_1)|^{\frac{1}{2\beta}}+|h^*(k_1r_1)|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\right)\\
&\quad\leq C_2(p_1)\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\big((m_1+1)-s_1\big)\big((n_1+1)-t_1\big)(a_{s_1,t_1})^{p_1}\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1}\big((z_1+1)-k_1\big)\big((\omega_1+1)-r_1\big)(b_{k_1,r_1})^{p_1}\right)^{\frac{1}{p_1}},
\end{aligned}$$

where

$$C_2(p_1) = (m_1n_1z_1\omega_1)^{\frac{p_1-1}{p_1}}.$$

Corollary 2.4 *Under the assumptions of Theorem 2.1, we have*

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{F(s_1,t_1)G(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\leq C_1(p_1)\left\{h\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\big[f(s_1,t_1)\big]^{p_1}\Delta s_1\Delta t_1\right)\right.\\
&\qquad\left.+h^*\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\big(\sigma(z)-k_1\big)\big(\sigma(\omega)-r_1\big)\big[g(k_1,r_1)\big]^{p_1}\Delta k_1\Delta r_1\right)\right\}^{\frac{1}{p_1}}.
\end{aligned}$$

Proof Using (1.7) in (2.1), we get the desired result. □

Theorem 2.5 *Under the assumptions of Theorem 2.1, let $p(\xi,\eta)$ and $q(\xi,\eta)$ be two positive functions. Let $\Psi\geq 0$ and $\Phi\geq 0$ be submultiplicative convex functions on $[0,\infty)$. Define*

$$P(s_1,t_1) := \int_{t_0}^{s_1}\int_{t_0}^{t_1} p(\xi,\eta)\Delta\xi\Delta\eta \quad \textit{and} \quad Q(k_1,r_1) := \int_{t_0}^{k_1}\int_{t_0}^{r_1} q(\xi,\eta)\Delta\xi\Delta\eta.$$

Then, for $(s_1,t_1)\in I_x\times I_y$ and $(k_1,r_1)\in I_z\times I_\omega$, we have

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\leq D_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\left(p(s_1,t_1)\Phi\left[\frac{f(s_1,t_1)}{p(s_1,t_1)}\right]\right)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\big(\sigma(z)-k_1\big)\big(\sigma(\omega)-r_1\big)\left(q(k_1,r_1)\Psi\left[\frac{g(k_1,r_1)}{q(k_1,r_1)}\right]\right)^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}, \qquad (2.9)
\end{aligned}$$

where

$$D_1(p_1) = \left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}}\Delta s_1\Delta t_1\right)^{\frac{p_1-1}{p_1}} \times \left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\right)^{\frac{p_1}{p_1-1}}\Delta k_1\Delta r_1\right)^{\frac{p_1-1}{p_1}}.$$

Proof Since Φ is a convex submultiplicative function, by applying Jensen's inequality we get that

$$\begin{aligned}\Phi\bigl(F(s_1,t_1)\bigr) &= \Phi\left(\frac{P(s_1,t_1)\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)\frac{f(\xi,\eta)}{p(\xi,\eta)}\Delta\xi\Delta\eta}{\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)\Delta\xi\Delta\eta}\right)\\ &\le \Phi\bigl(P(s_1,t_1)\bigr)\Phi\left(\frac{\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)\frac{f(\xi,\eta)}{p(\xi,\eta)}\Delta\xi\Delta\eta}{\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)\Delta\xi\Delta\eta}\right)\\ &\le \frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)\Phi\left(\frac{f(\xi,\eta)}{p(\xi,\eta)}\right)\Delta\xi\Delta\eta.\end{aligned} \tag{2.10}$$

From Hölder's inequality with indices $\frac{p_1}{p_1-1}$ and p_1 we have

$$\begin{aligned}\Phi\bigl(F(s_1,t_1)\bigr) &\le \frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\bigl[(s_1-t_0)(t_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\\ &\quad\times\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}.\end{aligned} \tag{2.11}$$

Analogously,

$$\begin{aligned}\Psi\bigl(G(k_1,r_1)\bigr) &\le \frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\bigl[(k_1-t_0)(r_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\\ &\quad\times\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}.\end{aligned} \tag{2.12}$$

From (2.11) and (2.12) we have

$$\begin{aligned}&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(s_1,t_1)\bigr)\\ &\le \bigl(\bigl[(s_1-t_0)(t_1-t_0)\bigr]\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr)^{\frac{p_1-1}{p_1}}\\ &\quad\times\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}\right)\\ &\quad\times\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}\right).\end{aligned} \tag{2.13}$$

Applying (1.7) to the term $([(s_1-t_0)(t_1-t_0)][(k_1-t_0)(r_1-t_0)])^{\frac{p_1-1}{p_1}}$, we get the inequality

$$\begin{aligned}
&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)\\
&\quad\le \bigl(h\bigl[(s_1-t_0)(t_1-t_0)\bigr]+h^*\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr)^{\frac{p_1-1}{p_1}}\\
&\qquad\times\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right)\\
&\qquad\times\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right).
\end{aligned}\tag{2.14}$$

Applying Lemma 1.4, we have

$$\begin{aligned}
&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)\\
&\quad\le \bigl(\bigl|h\bigl[(s_1-t_0)(t_1-t_0)\bigr]\bigr|^{\frac{1}{2\beta}}+\bigl|h^*\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr|^{\frac{1}{2\beta}}\bigr)^{\frac{2\alpha(p_1-1)}{p_1}}\\
&\qquad\times\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right)\\
&\qquad\times\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right).
\end{aligned}\tag{2.15}$$

From (2.15) we have

$$\begin{aligned}
&\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\\
&\quad\le\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right)\\
&\qquad\times\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right).
\end{aligned}\tag{2.16}$$

Integrating both sides of (2.16) firstly with respect to r_1 and k_1 and then with respect to s_1 and t_1, we get

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\le\left(\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\Delta s_1\Delta t_1\right)\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\Delta k_1\Delta r_1\right).
\end{aligned}\tag{2.17}$$

From Hölder's inequality with indices p_1 and $\frac{p_1}{p_1-1}$ we have

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^{*}[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\leq\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}}\Delta s_1\Delta t_1\right)^{\frac{p_1-1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\right)^{\frac{p_1}{p_1-1}}\Delta k_1\Delta r_1\right)^{\frac{p_1-1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}\\
&=D_1(p)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p(\xi,\eta)\Phi\left[\frac{f(\xi,\eta)}{p(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q(\xi,\eta)\Psi\left[\frac{g(\xi,\eta)}{q(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}.
\end{aligned}\tag{2.18}$$

Applying Fubini's theorem to (2.18), we obtain

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^{*}[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p-1)}{p}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\leq D_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}(x-s_1)(y-t_1)\left(p(s_1,t_1)\Phi\left[\frac{f(s_1,t_1)}{p(s_1,t_1)}\right]\right)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}(z-k_1)(\omega-r_1)\left(q(k_1,r_1)\Psi\left[\frac{g(k_1,r_1)}{q(k_1,r_1)}\right]\right)^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}.
\end{aligned}$$

From the relations $\sigma(x)\geq x$, $\sigma(y)\geq y$, $\sigma(\omega)\geq\omega$, and $\sigma(z)=z$ we obtain

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^{*}[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\leq D_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\left(p(s_1,t_1)\Phi\left[\frac{f(s_1,t_1)}{p(s_1,t_1)}\right]\right)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\big(\sigma(z)-k_1\big)\big(\sigma(\omega)-r_1\big)\left(q(k_1,r_1)\Psi\left[\frac{g(k_1,r_1)}{q(k_1,r_1)}\right]\right)^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}},
\end{aligned}$$

where

$$\begin{aligned}
D_1(p_1)={}&\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}}\Delta s_1\Delta t_1\right)^{\frac{p_1-1}{p_1}}\\
&\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\right)^{\frac{p_1}{p_1-1}}\Delta k_1\Delta r_1\right)^{\frac{p_1-1}{p_1}}.
\end{aligned}$$

This completes the proof. □

Taking $\mathbb{T}_1 = \mathbb{T}_1 = \mathbb{R}$ in Theorem 2.5, we have $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(\omega) = \omega$, $\sigma(z) = z$, and we get the following result.

Corollary 2.6 *Let $f(s_1,t_1)$ and $g(k_1,r_1)$ be real-valued continuous functions, and let $p(s_1,t_1)$ and $q(k_1,r_1)$ be positive functions. Define*

$$F(s_1,t_1) := \int_0^{s_1}\int_0^{t_1} f(\xi,\eta)\,d\xi\,d\eta, \qquad G(k_1,r_1) := \int_0^{k_1}\int_0^{r_1} g(\xi,\eta)\,d\xi\,d\eta,$$
$$P(s_1,t_1) := \int_0^{s_1}\int_0^{t_1} p(\xi,\eta)\,d\xi\,d\eta, \quad \textit{and} \quad Q(k_1,r_1) := \int_0^{k_1}\int_0^{r_1} q(\xi,\eta)\,d\xi\,d\eta.$$

Then

$$\begin{aligned}
&\int_0^x\int_0^y\left(\int_0^z\int_0^\omega \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[s_1t_1]|^{\frac{1}{2\beta}} + |h^*[k_1r_1]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\,dk_1\,dr_1\right)ds_1\,dt_1\\
&\quad\le D_1^*(p_1)\left(\int_0^x\int_0^y (x-s_1)(y-t_1)\left(p(s_1,t_1)\Phi\left[\frac{f(s_1,t_1)}{p(s_1,t_1)}\right]\right)^{p_1} ds_1\,dt_1\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\int_0^z\int_0^\omega (z-k_1)(\omega-r_1)\left(q(k_1,r_1)\Psi\left[\frac{g(k_1,r_1)}{q(k_1,r_1)}\right]\right)^{p_1} dk_1\,dr_1\right)^{\frac{1}{p_1}},
\end{aligned}$$

where

$$\begin{aligned}
D_1^*(p_1) &= \left(\int_0^x\int_0^y\left(\frac{\Phi(P(s_1,t_1))}{P(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}} ds_1\,dt_1\right)^{\frac{p_1-1}{p_1}}\\
&\quad\times\left(\int_0^z\int_0^\omega\left(\frac{\Psi(Q(k_1,r_1))}{Q(k_1,r_1)}\right)^{\frac{p_1}{p_1-1}} dk_1\,dr_1\right)^{\frac{p_1-1}{p_1}}.
\end{aligned}$$

In the particular case of Theorem 2.5 where $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, we have $\sigma(x) = x+1$, $\sigma(y) = y+1$, $\sigma(\omega) = \omega+1$, $\sigma(z) = z+1$, and we get the following result.

Corollary 2.7 *Let $\{a_{m_1,n_1}\}_{0\le m_1,n_1\le N}$ and $\{b_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ be nonnegative sequences of real numbers, and let $\{p_{m_1,n_1}\}_{0\le m_1,n_1\le N}$ be $\{q_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ positive sequences. Define*

$$A_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} a_{\xi,\eta}, \qquad B_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} b_{\xi,\eta},$$
$$P_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} p_{\xi,\eta} \quad \textit{and} \quad Q_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} q_{\xi,\eta}.$$

Then

$$\begin{aligned}
&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1}\frac{\Phi(A_{s_1,t_1})\Psi(B_{k_1,r_1})}{(|h(s_1t_1)|^{\frac{1}{2\beta}} + |h^*(k_1r_1)|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\right)\\
&\quad\le D^{**}(p_1)\left\{\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\big((m_1+1)-s_1\big)\big((n_1+1)-t_1\big)\left(p_{s_1,t_1}\Phi\left[\frac{a_{s_1,t_1}}{p_{s_1,t_1}}\right]\right)^{p_1}\right\}^{\frac{1}{p_1}}
\end{aligned}$$

$$\times \left\{ \sum_{k_1=1}^{z_1} \sum_{r_1=1}^{\omega_1} \bigl((z_1+1)-k_1\bigr)\bigl((\omega_1+1)-r_1\bigr)\left(q_{k_1,r_1} \Psi\left[\frac{b_{k_1,r_1}}{q_{k_1,r_1}} \right] \right)^{p_1} \right\}^{\frac{1}{p_1}},$$

where

$$D^{**}(p_1) = \left\{ \sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} \left(\frac{\Phi(P_{s_1,t_1})}{P_{s_1,t_1}} \right)^{\frac{p_1}{p_1-1}} \right\}^{\frac{p_1-1}{p_1}} \left\{ \sum_{k_1=1}^{z_1} \sum_{r_1=1}^{\omega_1} \left(\frac{\Psi(Q_{k_1,r_1})}{Q_{k_1,r_1}} \right)^{\frac{p_1}{p_1-1}} \right\}^{\frac{p_1-1}{p_1}}.$$

Corollary 2.8 *Under the assumptions of Theorem 2.5, we have*

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega} \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}} \Delta k_1 \Delta r_1\right)\Delta s_1 \Delta t_1 \\
&\quad \le D_1(p_1)\left\{ h\left(\int_{t_0}^{x}\int_{t_0}^{y} \bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\left(p(s_1,t_1)\Phi\left[\frac{f(s_1,t_1)}{p(s_1,t_1)} \right] \right)^{p_1} \Delta s_1 \Delta t_1 \right) \right. \\
&\qquad \left. + h^*\left(\int_{t_0}^{z}\int_{t_0}^{\omega} \bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(\omega)-r_1\bigr)\left(q(k_1,r_1)\Psi\left[\frac{g(k_1,r_1)}{q(k_1,r_1)} \right] \right)^{p_1} \Delta k_1 \Delta r_1 \right) \right\}^{\frac{1}{p_1}}.
\end{aligned}$$

Proof Using (1.7) in (2.9), we get the desired result. □

Theorem 2.9 *Under the assumptions of Theorem 2.5, define*

$$\begin{aligned}
F(s_1,t_1) &:= \frac{1}{(s_1-t_0)(t_1-t_0)} \int_{t_0}^{s_1}\int_{t_0}^{t_1} f(\xi,\eta)\Delta\xi\Delta\eta, \\
G(k_1,r_1) &:= \frac{1}{(k_1-t_0)(r_1-t_0)} \int_{t_0}^{k_1}\int_{t_0}^{r_1} g(\xi,\eta)\Delta\xi\Delta\eta.
\end{aligned} \tag{2.19}$$

Then, for $(s_1,t_1) \in I_x \times I_y$ and $(k_1,r_1) \in I_z \times I_\omega$, we have

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega} \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}} \Delta k_1 \Delta r_1\right)\Delta s_1 \Delta t_1 \\
&\quad \le K_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y} \bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\bigl(\Phi\bigl(f(s_1,t_1)\bigr)\bigr)^{p_1} \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}} \\
&\qquad \times \left(\int_{t_0}^{z}\int_{t_0}^{\omega} \bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(\omega)-r_1\bigr)\bigl(\Psi\bigl(g(k_1,r_1)\bigr)\bigr)^{p_1} \Delta k_1 \Delta r_1 \right)^{\frac{1}{p_1}},
\end{aligned} \tag{2.20}$$

where

$$K_1(p_1) = \bigl[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\bigr]^{\frac{p_1-1}{p_1}}.$$

Proof From (2.19), using Jensen's inequality, we see that

$$\begin{aligned}
\Phi\bigl(F(s_1,t_1)\bigr) &= \Phi\left(\frac{1}{(s_1-t_0)(t_1-t_0)} \int_{t_0}^{s_1}\int_{t_0}^{t_1} f(\xi,\eta)\Delta\xi\Delta\eta \right) \\
&\le \frac{1}{(s_1-t_0)(t_1-t_0)} \int_{t_0}^{s_1}\int_{t_0}^{t_1} \Phi\bigl(f(\xi,\eta)\bigr)\Delta\xi\Delta\eta.
\end{aligned} \tag{2.21}$$

Similarly,

$$\begin{aligned}\Psi\bigl(G(k_1,r_1)\bigr) &= \Psi\biggl(\frac{1}{(k_1-t_0)(r_1-t_0)}\int_{t_0}^{k_1}\int_{t_0}^{r_1} g(\xi,\eta)\Delta\xi\,\Delta\eta\biggr)\\ &\le \frac{1}{(k_1-t_0)(r_1-t_0)}\int_{t_0}^{k_1}\int_{t_0}^{r_1}\Psi\bigl(g(\xi,\eta)\bigr)\Delta\xi\,\Delta\eta. \qquad (2.22)\end{aligned}$$

By multiplying (2.21) and (2.22) we get

$$\begin{aligned}&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)\\ &\quad\le \frac{1}{(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}\\ &\qquad\times\int_{t_0}^{s_1}\int_{t_0}^{t_1}\Phi\bigl(f(\xi,\eta)\bigr)\Delta\xi\,\Delta\eta\int_{t_0}^{k_1}\int_{t_0}^{r_1}\Psi\bigl(g(\xi,\eta)\bigr)\Delta\xi\,\Delta\eta. \qquad (2.23)\end{aligned}$$

This implies that

$$\begin{aligned}&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le \int_{t_0}^{s_1}\int_{t_0}^{t_1}\Phi\bigl(f(\xi,\eta)\bigr)\Delta\xi\,\Delta\eta\int_{t_0}^{k_1}\int_{t_0}^{r_1}\Psi\bigl(g(\xi,\eta)\bigr)\Delta\xi\,\Delta\eta. \qquad (2.24)\end{aligned}$$

Using Hölder's inequality with indices p_1 and $\frac{p_1}{p_1-1}$, we have

$$\begin{aligned}&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le \bigl[(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\\ &\qquad\times\biggl(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl(\Phi\bigl(f(\xi,\eta)\bigr)\bigr)^{p_1}\Delta\xi\,\Delta\eta\biggr)^{\frac{1}{p_1}}\biggl(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl(\Psi\bigl(g(\xi,\eta)\bigr)\bigr)^{p_1}\Delta\xi\,\Delta\eta\biggr)^{\frac{1}{p_1}}. \qquad (2.25)\end{aligned}$$

Applying Lemma 1.9 to the term $[(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)]^{\frac{p_1-1}{p_1}}$, we get

$$\begin{aligned}&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le \bigl[h\bigl[(s_1-t_0)(t_1-t_0)\bigr]+h^*\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr]^{\frac{p_1-1}{p_1}}\\ &\qquad\times\biggl(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl(\Phi\bigl(f(\xi,\eta)\bigr)\bigr)^{p_1}\Delta\xi\,\Delta\eta\biggr)^{\frac{1}{p_1}}\biggl(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl(\Psi\bigl(g(\xi,\eta)\bigr)\bigr)^{p_1}\Delta\xi\,\Delta\eta\biggr)^{\frac{1}{p_1}}. \qquad (2.26)\end{aligned}$$

Applying Lemma 1.4 to (2.26), we obtain

$$\begin{aligned}&\Phi\bigl(F(s_1,t_1)\bigr)\Psi\bigl(G(k_1,r_1)\bigr)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le \bigl[\bigl|h\bigl[(s_1-t_0)(t_1-t_0)\bigr]\bigr|^{\frac{1}{2\beta}}+\bigl|h^*\bigl[(k_1-t_0)(r_1-t_0)\bigr]\bigr|^{\frac{1}{2\beta}}\bigr]^{\frac{2\alpha(p_1-1)}{p_1}}\\ &\qquad\times\biggl(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl(\Phi\bigl(f(\xi,\eta)\bigr)\bigr)^{p_1}\Delta\xi\,\Delta\eta\biggr)^{\frac{1}{p_1}}\biggl(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl(\Psi\bigl(g(\xi,\eta)\bigr)\bigr)^{p_1}\Delta\xi\,\Delta\eta\biggr)^{\frac{1}{p_1}}. \qquad (2.27)\end{aligned}$$

Dividing both sides of (2.27) by $[|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}}]^{\frac{2\alpha(p_1-1)}{p_1}}$, we get

$$\begin{aligned}&\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{[|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}}]^{\frac{2\alpha(p_1-1)}{p_1}}}\\&\quad\leq\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(\Phi\left(f(\xi,\eta)\right)\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(\Psi\left(g(\xi,\eta)\right)\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}.\end{aligned}\tag{2.28}$$

Integrating both sides of (2.28) firstly with respect to r_1 and k_1 and then with respect to s_1 and t_1 and using Hölder's inequality with indices $\frac{p_1}{p_1-1}$ and p_1, we get

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\quad\leq\left[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\right]^{\frac{p_1-1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(\Phi\left(f(\xi,\eta)\right)\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(\Psi\left(g(\xi,\eta)\right)\right)_1^{p}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}\\&\quad=K_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(\Phi\left(f(\xi,\eta)\right)\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(\Psi\left(g(\xi,\eta)\right)\right)^{p_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}.\end{aligned}\tag{2.29}$$

Applying Fubini's theorem to (2.29), we have

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\quad\leq K_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}(x-s_1)(y-t_1)\left(\Phi\left(f(s_1,t_1)\right)\right)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}(z-k_1)(\omega-r_1)\left(\Psi\left(g(k_1,r_1)\right)\right)^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}}.\end{aligned}$$

From the relations $\sigma(x)\geq x$, $\sigma(\omega)\geq\omega$, and $\sigma(z)\geq z$ we obtain

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\quad\leq K_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\sigma(x)-s_1\right)\left(\sigma(y)-t_1\right)\left(\Phi\left(f(s_1,t_1)\right)\right)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\left(\sigma(z)-k_1\right)\left(\sigma(\omega)-r_1\right)\left(\Psi\left(g(k_1,r_1)\right)\right)^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}},\end{aligned}$$

where

$$K_1(p_1) = \left[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\right]^{\frac{p_1-1}{p_1}}.$$

This completes the proof. □

Taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ and $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(\omega) = \omega$, and $\sigma(z) = z$, by Theorem 2.9 we obtain the following corollaries.

Corollary 2.10 *Let $f(s_1,t_1)$ and $g(k_1,r_1)$ be real-valued continuous functions, and define*

$$F(s_1,t_1) := \frac{1}{s_1t_1}\int_0^{s_1}\int_0^{t_1} f(\xi,\tau)\,d\xi\,d\tau \quad \textit{and} \quad G(k_1,r_1) := \frac{1}{k_1r_1}\int_0^{k_1}\int_0^{r_1} g(\xi,\tau)\,d\xi\,d\tau.$$

Then, for $(s_1,t_1) \in I_x \times I_y$ and $(k_1,r_1) \in I_z \times I_\omega$, we have

$$\begin{aligned}
&\int_0^x\int_0^y\left(\int_0^z\int_0^\omega \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1t_1)(k_1r_1)}{(|h[(s_1t_1)]|^{\frac{1}{2\beta}} + |h^*[(k_1r_1)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\,dk_1\,dr_1\right)ds_1\,dt_1 \\
&\quad \leq K_1^*(p_1)\left(\int_0^x\int_0^y (x-s_1)(y-t_1)\bigl(\Phi\bigl(f(s_1,t_1)\bigr)\bigr)^{p_1}\,ds_1\,dt_1\right)^{\frac{1}{p_1}} \\
&\qquad \times\left(\int_0^z\int_0^\omega (z-k_1)(\omega-r_1)\bigl(\Psi\bigl(g(k_1,r_1)\bigr)\bigr)^{p_1}\,dk_1\,dr_1\right)^{\frac{1}{p_1}},
\end{aligned}$$

where

$$K_1^*(p_1) = \left[(x)(y)(z)(\omega)\right]^{\frac{p_1-1}{p_1}}.$$

Taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorem 2.9, we have $\sigma(x) = x+1$, $\sigma(y) = y+1$, $\sigma(\omega) = \omega+1$, and $\sigma(z) = z+1$, and we get the following result.

Corollary 2.11 *Let $\{a_{m_1,n_1}\}_{0\leq m_1,n_1\leq N}$ and $\{b_{k_1,r_1}\}_{0\leq k_1,r_1\leq N}$ be nonnegative sequences of real numbers, and define*

$$A_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} a_{\xi,\eta}, \quad \textit{and} \quad B_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} b_{\xi,\eta}.$$

Then

$$\begin{aligned}
&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1} \frac{(s_1t_1)(k_1r_1)\Phi(A_{s_1,t_1})\Psi(B_{k_1,r_1})}{(|h(s_1t_1)|^{\frac{1}{2\beta}} + |h^*(k_1r_1)|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\right) \\
&\quad \leq K^{**}(p_1)\left\{\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\bigl((m_1+1)-s_1\bigr)\bigl((n_1+1)-t_1\bigr)\bigl(\Phi(a_{s_1,t_1})\bigr)^{p_1}\right\}^{\frac{1}{p_1}} \\
&\qquad \times\left\{\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1}\bigl((z_1+1)-k_1\bigr)\bigl((\omega_1+1)-r_1\bigr)\bigl(\Psi(b_{k_1,r_1})\bigr)^{p_1}\right\}^{\frac{1}{p_1}},
\end{aligned}$$

where

$$K^{**}(p_1) = (m_1 n_1 z_1 \omega_1)^{\frac{p_1-1}{p_1}}.$$

Corollary 2.12 *Under the assumptions of Theorem 2.9, we have*

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1 \\
&\quad\le K_1(p_1)\biggl\{h\biggl(\int_{t_0}^{x}\int_{t_0}^{y}\bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\bigl(\Phi\bigl(f(s_1,t_1)\bigr)\bigr)^{p_1}\Delta s_1\Delta t_1\biggr) \\
&\qquad+h^*\biggl(\int_{t_0}^{z}\int_{t_0}^{\omega}\bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(\omega)-r_1\bigr)\bigl(\Psi\bigl(g(k_1,r_1)\bigr)\bigr)^{p_1}\Delta k_1\Delta r_1\biggr)\biggr\}^{\frac{1}{p_1}}.
\end{aligned}$$

Proof Using (1.7) in (2.20), we get the desired result. □

Theorem 2.13 *Under the assumptions of Theorem 2.5, assume that*

$$\begin{aligned}
F(s_1,t_1) &:= \frac{1}{P(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)f(\xi,\eta)\Delta\xi\Delta\eta, \\
G(k_1,r_1) &:= \frac{1}{Q(k_1,r_1)}\int_{t_0}^{k_1}\int_{t_0}^{r_1}q(\xi,\eta)g(\xi,\eta)\Delta\xi\Delta\eta.
\end{aligned}\tag{2.30}$$

Then, for $(s_1,t_1)\in I_x\times I_y$ *and* $(k_1,r_1)\in I_z\times I_\omega$, *we have*

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1 \\
&\quad\le H_1(p_1)\biggl(\int_{t_0}^{x}\int_{t_0}^{y}\bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\bigl(p(s_1,t_1)\Phi\bigl(f(s_1,t_1)\bigr)\bigr)^{p_1}\Delta s_1\Delta t_1\biggr)^{\frac{1}{p_1}} \\
&\qquad\times\biggl(\int_{t_0}^{z}\int_{t_0}^{\omega}\bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(\omega)-r_1\bigr)\bigl(q(k_1,r_1)\Psi\bigl(g(k_1,r_1)\bigr)\bigr)^{p_1}\Delta k_1\Delta r_1\biggr)^{\frac{1}{p_1}},
\end{aligned}\tag{2.31}$$

where

$$H_1(p_1) = \bigl[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\bigr]^{\frac{p_1-1}{p_1}}.$$

Proof From (2.30), using Jensen's inequality, we see that

$$\begin{aligned}
\Phi\bigl(F(s_1,t_1)\bigr) &= \Phi\biggl(\frac{1}{P(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)f(\xi,\eta)\Delta\xi\Delta\eta\biggr) \\
&\le \frac{1}{P(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p(\xi,\eta)\Phi\bigl(f(\xi,\eta)\bigr)\Delta\xi\Delta\eta.
\end{aligned}\tag{2.32}$$

Similarly,

$$\begin{aligned}\Psi\big(G(k_1,r_1)\big) &= \Psi\bigg(\frac{1}{Q(k_1,r_1)}\int_{t_0}^{k_1}\int_{t_0}^{r_1} q(\xi,\eta)g(\xi,\eta)\Delta\xi\,\Delta\eta\bigg)\\ &\le \frac{1}{Q(k_1,r_1)}\int_{t_0}^{k_1}\int_{t_0}^{r_1} q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\Delta\xi\,\Delta\eta.\end{aligned} \tag{2.33}$$

By multiplying (2.32) and (2.33) we get

$$\begin{aligned}\Phi\big(F(s_1,t_1)\big)\Psi\big(G(k_1,r_1)\big) \le{}& \frac{1}{P(s_1,t_1)Q(k_1,r_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1} p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\Delta\xi\,\Delta\eta\\ &\times\int_{t_0}^{k_1}\int_{t_0}^{r_1} q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\Delta\xi\,\Delta\eta.\end{aligned} \tag{2.34}$$

This implies that

$$\begin{aligned}&\Phi\big(F(s_1,t_1)\big)\Psi\big(G(k_1,r_1)\big)P(s_1,t_1)Q(k_1,r_1)\\ &\quad\le \int_{t_0}^{s_1}\int_{t_0}^{t_1} p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\Delta\xi\,\Delta\eta\int_{t_0}^{k_1}\int_{t_0}^{r_1} q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\Delta\xi\,\Delta\eta.\end{aligned} \tag{2.35}$$

Using Hölder's inequality with indices p_1 and $\frac{p_1}{p_1-1}$, we obtain

$$\begin{aligned}&\Phi\big(F(s_1,t_1)\big)\Psi\big(G(k_1,r_1)\big)P(s_1,t_1)Q(k_1,r_1)\\ &\quad\le \big[(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\big]^{\frac{p_1-1}{p_1}}\bigg(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big(p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\big)^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}\\ &\qquad\times\bigg(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big(q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\big)^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}.\end{aligned} \tag{2.36}$$

Applying Lemma 1.9 to the term $[(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)]^{\frac{p_1-1}{p_1}}$, we get

$$\begin{aligned}&\Phi\big(F(s_1,t_1)\big)\Psi\big(G(k_1,r_1)\big)P(s_1,t_1)Q(k_1,r_1)\\ &\quad\le \big[h\big[(s_1-t_0)(t_1-t_0)\big]+h^*\big[(k_1-t_0)(r_1-t_0)\big]\big]^{\frac{p_1-1}{p_1}}\\ &\qquad\times\bigg(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big(p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\big)^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}\\ &\qquad\times\bigg(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big(q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\big)^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}.\end{aligned} \tag{2.37}$$

Applying Lemma 1.4 to (2.37), we obtain

$$\begin{aligned}&\Phi\big(F(s_1,t_1)\big)\Psi\big(G(k_1,r_1)\big)P(s_1,t_1)Q(k_1,r_1)\\ &\quad\le \big[\big|h\big[(s_1-t_0)(t_1-t_0)\big]\big|^{\frac{1}{2\beta}}+\big|h^*\big[(k_1-t_0)(r_1-t_0)\big]\big|^{\frac{1}{2\beta}}\big]^{\frac{2\alpha(p_1-1)}{p_1}}\end{aligned}$$

$$\times \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \big(p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \big(q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right)^{\frac{1}{p_1}}. \tag{2.38}$$

Dividing both sides of (2.27) by $[|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}}]^{\frac{2\alpha(p_1-1)}{p_1}}$, we get

$$\begin{aligned} &\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{[|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}}]^{\frac{2\alpha(p_1-1)}{p_1}}} \\ &\quad\le \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \big(p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right)^{\frac{1}{p_1}} \\ &\qquad\times \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \big(q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right)^{\frac{1}{p_1}}. \end{aligned} \tag{2.39}$$

Integrating both sides of (2.39) firstly with respect to r_1 and k_1 and then with respect to s_1 and t_1 and using Hölder's inequality with indices $\frac{p_1}{p_1-1}$ and p_1, we get

$$\begin{aligned} &\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega} \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}} \Delta k_1 \Delta r_1\right) \Delta s_1 \Delta t_1 \\ &\quad\le \big[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\big]^{\frac{p_1-1}{p_1}} \\ &\qquad\times \left(\int_{t_0}^{x} \int_{t_0}^{y} \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \big(p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right) \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}} \\ &\qquad\times \left(\int_{t_0}^{z} \int_{t_0}^{\omega} \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \big(q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right) \Delta k_1 \Delta r_1 \right)^{\frac{1}{p_1}} \\ &\quad= H_1(p_1) \left(\int_{t_0}^{x} \int_{t_0}^{y} \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \big(p(\xi,\eta)\Phi\big(f(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right) \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}} \\ &\qquad\times \left(\int_{t_0}^{z} \int_{t_0}^{\omega} \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \big(q(\xi,\eta)\Psi\big(g(\xi,\eta)\big)\big)^{p_1} \Delta\xi \Delta\eta \right) \Delta k_1 \Delta r_1 \right)^{\frac{1}{p_1}}. \end{aligned} \tag{2.40}$$

Applying Fubini's theorem, we get

$$\begin{aligned} &\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega} \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}} \Delta k_1 \Delta r_1\right) \Delta s_1 \Delta t_1 \\ &\quad\le H_1(p_1) \left(\int_{t_0}^{x} \int_{t_0}^{y} (x-s_1)(y-t_1)\big(p(s_1,t_1)\Phi\big(f(s_1,t_1)\big)\big)^{p_1} \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}} \\ &\qquad\times \left(\int_{t_0}^{z} \int_{t_0}^{\omega} (z-k_1)(\omega-r_1)\big(q(k_1,r_1)\Psi\big(g(k_1,r_1)\big)\big)^{p_1} \Delta k_1 \Delta r_1 \right)^{\frac{1}{p_1}}. \end{aligned}$$

From the relations $\sigma(x) \geq x$, $\sigma(y) \geq y$, $\sigma(\omega) \geq \omega$, and $\sigma(z) \geq z$ we obtain

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}}+|h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1 \\
&\quad\leq H_1(p_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\big(p(s_1,t_1)\Phi\big(f(s_1,t_1)\big)\big)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}} \\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\big(\sigma(z)-k_1\big)\big(\sigma(\omega)-r_1\big)\big(q(k_1,r_1)\Psi\big(g(k_1,r_1)\big)\big)^{p_1}\Delta k_1\Delta r_1\right)^{\frac{1}{p_1}},
\end{aligned}$$

where

$$H_1(p_1) = \big[(x-t_0)(y-t_0)(z-t_0)(\omega-t_0)\big]^{\frac{p_1-1}{p_1}}.$$

This completes the proof. □

Taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 2.9, we have $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(\omega) = \omega$, $\sigma(z) = z$, and we get the following result.

Corollary 2.14 *Let $f(s_1,t_1)$, $g(k_1,r_1)$ be real-valued continuous functions, and let $p(s_1,t_1)$, $q(r_1,k_1)$ be positive functions. Define*

$$\begin{aligned}
F(s_1,t_1) &:= \frac{1}{P(s_1,t_1)}\int_0^{s_1}\int_0^{t_1}p(\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau, \\
P(s_1,t_1) &:= \int_0^{s_1}\int_0^{t_1}p(\xi,\tau)\,d\xi\,d\tau, \\
G(k_1,r_1) &:= \frac{1}{Q(k_1,r_1)}\int_0^{k_1}\int_0^{r_1}q(\xi,\tau)g(\xi,\tau)\,d\xi\,d\tau, \\
Q(k_1,r_1) &:= \int_0^{k_1}\int_0^{r_1}q(\xi,\tau)\,d\xi\,d\tau.
\end{aligned}$$

Then, for $(s_1,t_1) \in I_x \times I_y$ and $(k_1,r_1) \in I_z \times I_\omega$, we have

$$\begin{aligned}
&\int_0^{x}\int_0^{y}\left(\int_0^{z}\int_0^{\omega}\frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{(|h[(s_1t_1)]|^{\frac{1}{2\beta}}+|h^*[(k_1r_1)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\,dk_1\,dr_1\right)ds_1\,dt_1 \\
&\quad\leq H_1^*(p_1)\left(\int_0^{x}\int_0^{y}(x-s_1)(y-t_1)\big(p(s_1,t_1)\Phi\big(f(s_1,t_1)\big)\big)^{p_1}\,ds_1\,dt_1\right)^{\frac{1}{p_1}} \\
&\qquad\times\left(\int_0^{z}\int_0^{\omega}(z-k_1)(\omega-r_1)\big(q(k_1,r_1)\Psi\big(g(k_1,r_1)\big)\big)^{p_1}\,dk_1\,dr_1\right)^{\frac{1}{p_1}}
\end{aligned}$$

where

$$H_1^*(p_1) = \big[(x)(y)(z)(\omega)\big]^{\frac{p_1-1}{p_1}}.$$

Taking $\mathbb{T}_1 = \mathbb{T}_1 = \mathbb{Z}$ in Theorem 2.9, we have $\sigma(x) = x+1$, $\sigma(y) = y+1$, $\sigma(\omega) = \omega+1$, $\sigma(z) = z+1$, and we get the following result.

Corollary 2.15 *Let* $\{a_{m_1,n_1}\}_{0\le m_1,n_1\le N}$ *and* $\{b_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ *be nonnegative sequences of real numbers, and let* $\{p_{m_1,n_1}\}_{0\le m_1,n_1\le N}$ *and* $\{q_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ *be positive sequences. Define*

$$A_{m_1,n_1} = \frac{1}{P_{m_1,n_1}} \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} a_{\xi,\eta}, \qquad B_{k_1,r_1} = \frac{1}{Q_{k_1,r_1}} \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} b_{\xi,\eta},$$

$$P_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} p_{\xi,\eta}, \quad \text{and} \quad Q_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} q_{\xi,\eta}.$$

Then

$$\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1} \frac{P_{m_1,n_1}Q_{k_1,r_1}\Phi(A_{s_1,t_1})\Psi(B_{k_1,r_1})}{(|h(s_1t_1)|^{\frac{1}{2\beta}} + |h^*(k_1r_1)|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\right)$$
$$\le H^{**}(p_1)\left\{\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\big((m_1+1)-s_1\big)\big((n_1+1)-t_1\big)\big(p_{s_1,t_1}\Phi(a_{s_1,t_1})\big)^{p_1}\right\}^{\frac{1}{p_1}}$$
$$\times\left\{\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{\omega_1}\big((z_1+1)-k_1\big)\big((\omega_1+1)-r_1\big)\big(q_{k_1,r_1}\Psi(b_{k_1,r_1})\big)^{p_1}\right\}^{\frac{1}{p_1}},$$

where

$$H^{**}(p_1) = (m_1n_1z_1\omega_1)^{\frac{p_1-1}{p_1}}.$$

Corollary 2.16 *Under the assumptions of Theorem 2.9, we have*

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{\omega} \frac{\Phi(F(s_1,t_1))\Psi(G(k_1,r_1))P(s_1,t_1)Q(k_1,r_1)}{(|h[(s_1-t_0)(t_1-t_0)]|^{\frac{1}{2\beta}} + |h^*[(k_1-t_0)(r_1-t_0)]|^{\frac{1}{2\beta}})^{\frac{2\alpha(p_1-1)}{p_1}}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\le H_1(p_1)\left\{h\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\big(p(s_1,t_1)\Phi\big(f(s_1,t_1)\big)\big)^{p_1}\Delta s_1\Delta t_1\right)\right.$$
$$\left. + h^*\left(\int_{t_0}^{z}\int_{t_0}^{\omega}\big(\sigma(z)-k_1\big)\big(\sigma(\omega)-r_1\big)\big(q(k_1,r_1)\Psi\big(g(k_1,r_1)\big)\big)^{p_1}\Delta k_1\Delta r_1\right)\right\}^{\frac{1}{p_1}}.$$

Proof Using (1.7) in (2.31), we get the desired result.. □

3 Conclusion

In this paper, we established some dynamic Hilbert-type inequalities in two separate variables on time scales by using the Fenchel–Legendre transform. We also applied our inequalities to discrete and continuous calculus to obtain some new inequalities as particular cases.

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