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Solving fuzzy $(1 + n)$-dimensional Burgers' equation

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Abstract

In this paper, we study the comparison of fuzzy differential transform method (FDTM), fuzzy Adomian decomposition method (FADM), fuzzy homotopy perturbation method (FHPM), and fuzzy reduced differential transform method (FRDTM) to obtain the solutions of fuzzy $(1 + n)$-dimensional Burgers' equation under gH-differentiability. We have investigated many new results to solve the above problem, and the methods have been implemented. The four illustrative numerical examples are presented to demonstrate the effectiveness of the proposed methods and also to demonstrate the efficiency and simplicity of the ways they were developed and derived. The results also show that the methods are powerful mathematical tools for solving fuzzy $(1 + n)$-dimensional Burgers' equation.

MSC: 35R13; 65M55; 65H20

Keywords: Fuzzy numbers; Fuzzy-valued functions; gH-differentiability; FDTM, FADM, FHPM, and FRDTM; Fuzzy $(1 + n)$-dimensional Burgers equation

1 Introduction

In 1978, the concept of fuzzy differential equations (FDEs) was originally introduced by Kandel and Byatt [52]. FDEs play an important role within several areas; see [21, 61, 63–66, 69]. First-order linear FDEs or systems according to various interpretations are searched for in many papers [20, 27]. There are only a few works like [56, 59] in which fuzzy, intuitive numbers are used in differential equations. Recently, several authors used numerical and analytical methods for solving fuzzy differential and integral equations; for example, see [3, 6, 18, 22, 24, 25, 35–37, 50, 60, 67, 71].

In the literature, the study of FDEs has several interpretations. First is based on the idea of the Hukuhara derivative [33, 75, 84]. Fractal theory is the theoretical basis for the fractal space-time [26, 47] El Naschieis E-infinity theory [31] and life science [85] also. Lately, various authors have presented the fractal calculus [2, 34, 48, 49].

In 1986, the notion of differential transform method (DTM) was introduced for the first time by Zhou in [89], this method adopts an analytical solution in the form of a polynomial, which is different from the traditional higher-order Taylor formula method. Recently, several researchers used DTM to solve FDEs; see [8, 32, 62, 70, 76].

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Keskin and Oturanc in [54] proposed the concept of the reduced differential transform method (RDTM), defining a set of transformation rules to overcome the complicated complex calculations of traditional DTM. Recently, some authors used the method to solve many equations, for example, see [11, 23, 53, 73, 83, 88].

The Adomian decomposition method (ADM), which was primarily introduced by Adomian [1], is a semi-numerical technique for solving linear-nonlinear differential equations by generating a functional series solution in a very efficient manner. Several researchers have already used this method in their work; see [4, 5, 19, 30, 71, 74].

The classical perturbation methods have different limitations and are strongly invalid for nonlinear equations. To overcome the shortcomings, many new technologies have appeared in the open literature; see [10, 16, 40, 42–44, 57]. The homotopy perturbation method (HPM) is a new analytical method that was initially introduced by He [41, 45, 46] to solve linear-nonlinear differential equations. Applications of homotopy theory have recently appeared for different scientists, and the homotopy theory has become a powerful mathematical tool when it is successfully combined with perturbation theory; see [12, 51, 55, 68, 72, 77–79, 81, 82].

This paper is organized as follows. In Sect. 2, some basic definitions, remarks, and theorems that will be used are given. In Sect. 3, we present an analytical solution for the fuzzy $(1 + n)$-dimensional Burgers' equation under gH-differentiability by using FDTM, FADM, FHPM, and FRDTM. In Sect. 4, the applied fuzzy $(1 + n)$-dimensional Burgers' equation is developed, derived, and illustrated by four numerical examples. Finally, a conclusion is drawn in Sect. 5.

2 Preliminaries

In this section, there are various definitions for the concept of fuzzy numbers, fuzzy-valued functions, and fuzzy derivatives as follows.

Definition 2.1 ([28]) Fuzzy numbers are a fuzzy set like $\tilde{u} : R \to I = [0,1]$ which satisfies the requirements:

(a) $\tilde{u}$ is upper semicontinuous,

(b) $\tilde{u}(x) = 0$ outside some interval [c,d],

(c) there are real numbers a, b such that $c \le a \le b \le d$ and

- $\tilde{u}(x)$ is monotonic increasing on [c,a],
- $\tilde{u}(x)$ is monotonic decreasing on [b,d],
- $\tilde{u}(x) = 1, a \le x \le b$.

Definition 2.2 ([33]) Let $\tilde{u} \in E^1$ and $[\tilde{u}]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$. Then the following conditions are satisfied:

(1) $\underline{u}_\alpha$ is a bounded left continuous nondecreasing function on (0,1).

(2) $\overline{u}_\alpha$ is a bounded left continuous nonincreasing function on (0,1).

(3) $\underline{u}_\alpha$ and $\overline{u}(\alpha)$ are right continuous at $\alpha = 0$.

(4) $\underline{u}_1 \le \overline{u}_1$.

Conversely, if the pair of functions $a(\alpha)$ and $b(\alpha)$ satisfy conditions (1)–(4), then there exists unique $\tilde{u}_\alpha \in E^1$ such that $[\tilde{u}]_\alpha = [a(\alpha), b(\alpha)]$ for each $\alpha \in [0,1]$.

Define $D : E^1 \times E^1 \to R_+ \cup \{0\}$ by

$$D(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \max\left\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\right\},$$

where $[\tilde{u}]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $[\tilde{v}]_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$. $D(\tilde{u}, \tilde{v})$ is called the distance between fuzzy numbers $\tilde{u}$ and $\tilde{v}$. Using the results in [29], we know that

(a) (E^1, D) is a complete metric space;

(b) $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$;

(c) $D(k \cdot \tilde{u}, k \cdot \tilde{v}) = |k| . D(\tilde{u}, \tilde{v}), k \in R$, where $\tilde{u}, \tilde{v}, \tilde{w} \in E^1$.

Definition 2.3 ([15]) As discussed above, fuzzy numbers may be transformed into an interval through an α-level approach. So, for any arbitrary fuzzy number $\tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)]$, $\tilde{y} = [\underline{y}(\alpha), \overline{y}(\alpha)]$ and scalar k, we get the interval-based fuzzy arithmetic as follows:

(a) $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(\alpha) = \underline{y}(\alpha)$ and $\overline{x}(\alpha) = \overline{y}(\alpha)$,

(b) $\tilde{x} \oplus \tilde{y} = [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)]$,

(c) $\tilde{x} \otimes \tilde{y} = [\min(S), \max(S)]$, where $S = \{\underline{x}(\alpha)\underline{y}(\alpha), \underline{x}(\alpha)\overline{y}(\alpha), \overline{x}(\alpha)\underline{y}(\alpha), \overline{x}(\alpha)\overline{y}(\alpha)\}$,

(d) $k \odot \tilde{x} = \begin{cases} [k\overline{x}(\alpha), k\underline{x}(\alpha)], & k < 0, \\ [k\underline{x}(\alpha), k\overline{x}(\alpha)], & k \geq 0. \end{cases}$

Remark 2.4 Let $\tilde{x}, \tilde{y}$ be fuzzy numbers, and $\tilde{x} \geq 0$ (it means $\underline{x}(r) \geq 0$ and $\overline{x}(r) \geq 0$). Then

$$(\tilde{x} \odot \tilde{y})(r) = [\underline{x}(r)\underline{y}(r), \overline{x}(r)\overline{y}(r)].$$

Definition 2.5 ([86, 87]) For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1, \tilde{u} = [\underline{u}_\alpha, \overline{u}_\alpha], \tilde{v} = [\underline{v}_\alpha, \overline{v}_\alpha]$, the quantity $D(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}$ is the distance between $\tilde{u}$ and $\tilde{v}$, and also the following properties hold:

(a) (E^1, D) is a complete metric space,

(b) $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v}, \tilde{w} \in E^1$,

(c) $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E^1$,

(d) $D(\tilde{u} \oplus \tilde{v}, \tilde{0}) \leq D(\tilde{u}, \tilde{0}) + D(\tilde{v}, \tilde{0}), \forall \tilde{u}, \tilde{v} \in E^1$,

(e) $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v} \in E^1, k \in R$,

(f) $D(k_1 \odot \tilde{u}, k_2 \odot \tilde{u}) = |k_1 - k_2| D(\tilde{u}, \tilde{0}), \forall \tilde{u} \in E^1, k_1, k_2 \in R$, with $k_1 \cdot k_2 \geq 0$.

Recall the definition of Hukuhara difference in [14]. Let $\tilde{u}, \tilde{v} \in E^1$. The Hukuhara difference has been presented as a set $\tilde{w}$ for which $\tilde{u} \ominus \tilde{v} = \tilde{w} \Leftrightarrow \tilde{u} = \tilde{v} \oplus \tilde{w}$. The H-difference is unique, but it does not always exist (a necessary condition for $\tilde{u} \ominus \tilde{v}$ to exist is that $\tilde{u}$ contains a translate $\{c\} \oplus \tilde{v}$of$\tilde{v}$).

Definition 2.6 ([14, 87]) The generalized Hukuhara difference of two fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$ is defined as follows:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \quad \Leftrightarrow \quad \begin{cases} \text{(i)} & \tilde{u} = \tilde{v} \oplus \tilde{w} \quad \text{or} \\ \text{(ii)} & \tilde{v} = \tilde{u} \oplus (-\tilde{w}). \end{cases} \tag{1}$$

In terms of the α-levels, we get $[\tilde{u} \ominus_{gH} \tilde{v}] = [\min\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}, \max\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}]$, and if the H-difference exists, then $\tilde{u} \ominus \tilde{v} = \tilde{u} \ominus_{gH} \tilde{v}$; the conditions for existence of $\tilde{w} = \tilde{u} \ominus_{gH} \tilde{v} \in E^1$ are:

$$\text{Case (i)} \quad \begin{cases} \underline{w}_\alpha = \underline{u}_\alpha - \underline{v}_\alpha \text{ and } \overline{w}_\alpha = \overline{u}_\alpha - \overline{v}_\alpha, \forall \alpha \in [0,1], \\ \text{with } \underline{w}_\alpha \text{ increasing}, \overline{w}_\alpha \text{ decreasing}, \underline{w}_\alpha \leq \overline{w}_\alpha. \end{cases} \tag{2}$$

$$\text{Case (ii)} \quad \begin{cases} \underline{w}_\alpha = \overline{u}_\alpha - \overline{v}_\alpha \text{ and } \overline{w}_\alpha = \underline{u}_\alpha - \underline{v}_\alpha, \forall \alpha \in [0,1], \\ \text{with } \underline{w}_\alpha \text{ increasing}, \overline{w}_\alpha \text{ decreasing}, \underline{w}_\alpha \leq \overline{w}_\alpha. \end{cases} \tag{3}$$

It is easy to show that (i) and (ii) are both valid if and only if $\tilde{w}$ is a crisp number.

Definition 2.7 ([13]) Let $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We say that $\tilde{u}$ is strongly generalized Hukuhara differentiable on (x_0, t) (gH-differentiable for short) if there exists an element $\frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)} \in E^1$ such that

(i) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h \to 0+} = \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}_{gH}\Big|_{(x_0,t)},$$

or

(ii) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h \to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}_{gH}\Big|_{(x_0,t)},$$

or

(iii) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h \to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}_{gH}\Big|_{(x_0,t)},$$

or

(iv) for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h \to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}_{gH}\Big|_{(x_0,t)}.$$

Definition 2.8 ([36]) Let $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We define the nth-order derivative of $\tilde{u}$ as follows: we say that $\tilde{u}$ is strongly generalized Hukuhara differentiable of the nth-order at (x_0, t) (gH-differentiable for short) if there exist elements $\frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)} \in E^1$, $\forall s = 1, 2, \dots, n$,

(i) for all $h > 0$ sufficiently small,
$\exists \tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t), \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{h}$$
$$= \lim_{h \to 0+} = \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}_{gH}\Big|_{(x_0,t)},$$

or

(ii) for all $h > 0$ sufficiently small,
$\exists \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t), \tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t)}{-h}$$

$$= \lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s} \bigg|_{gH} \bigg|_{(x_0, t)},$$

or

(iii) for all $h > 0$ sufficiently small,

$\exists \tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t), \tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{h}$$

$$= \lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s} \bigg|_{gH} \bigg|_{(x_0, t)},$$

or

(iv) for all $h > 0$ sufficiently small,

$\exists \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t), \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)$ and the limits

$$\lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t)}{-h}$$

$$= \lim_{h \to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s} \bigg|_{gH} \bigg|_{(x_0, t)}.$$

Definition 2.9 ([8]) Let $\tilde{u}(x, t) : D \to E^1$ be a function and set $\tilde{u}(x, t) = ((\underline{u}(x, t)(\alpha), \overline{u}(x, t)(\alpha))$ for each $\alpha \in [0, 1]$. Then

(1) If $\tilde{u}$ is gH-differentiable in the first form (i), then $(\underline{u}(x, t)(\alpha)$ and $\overline{u}(x, t)(\alpha)$ are differentiable functions and

$$\left[\frac{\partial \tilde{u}}{\partial x} \right]_\alpha = \left[\frac{\partial \underline{u}(x, t)(\alpha)}{\partial x}, \frac{\partial \overline{u}(x, t)(\alpha)}{\partial x} \right]; \tag{4}$$

(2) If $\tilde{u}$ is gH-differentiable in the second form (ii), then $(\overline{u}(x, t)(\alpha)$ and $\underline{u}(x, t)(\alpha)$ are differentiable functions and

$$\left[\frac{\partial \tilde{u}}{\partial x} \right]_\alpha = \left[\frac{\partial \overline{u}(x, t)(\alpha)}{\partial x}, \frac{\partial \underline{u}(x, t)(\alpha)}{\partial x} \right]. \tag{5}$$

Assumption 2.10 We only discuss the case of $\tilde{u} \geq 0$ satisfying $\frac{\partial \tilde{u}}{\partial t} \geq 0$. In order to simplify our results presentation, we only consider the following case:

$$\left[\tilde{u} \odot \frac{\partial \tilde{u}}{\partial t} \right]_\alpha = \left[\underline{u} \frac{\partial \underline{u}}{\partial t}, \overline{u} \frac{\partial \overline{u}}{\partial t} \right]_\alpha. \tag{6}$$

Definition 2.11 ([9, 87]) A fuzzy-number-valued function $\tilde{f} : [a, b] \to E^1$ is said to be continuous at $t_0 \in [a, b]$ if, for each $\varepsilon > 0$, there is $\delta > 0$ such that $D(\tilde{f}(t), \tilde{f}(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta$. If $\tilde{f}$ is continuous for each $t \in [a, b]$, then we say that $\tilde{f}$ is fuzzy continuous on $[a, b]$.

Definition 2.12 ([9]) A fuzzy-valued function $\tilde{f} : [a, b] \to E^1$ is said to bounded iff there is $M \geq 0$ such that $D(\tilde{f}(t), 0) = \|\tilde{f}(u)\| \leq M$ for all $t \in [a, b]$. Equivalently we get $\chi_{-M} \leq \tilde{f}(x) \leq \chi_M, \forall x \in [a, b]$.

Definition 2.13 ([7]) A fuzzy-valued function $\tilde{f}$ of two variables is *a* rule that assigns to each ordered pair of real numbers (x, t) in a set D a unique fuzzy number denoted by $\tilde{f}(x, t)$. The set D is the domain of $\tilde{f}$, and its range is the set of values taken by $\tilde{f}$, i.e., $\{\tilde{f}(x,t)|(x,t) \in D\}$.

The parametric representation of the fuzzy-valued function $\tilde{f} : D \to E^1$ is expressed by $\tilde{f}(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$ for all $(x,t) \in D$ and $\alpha \in [0,1]$.

Definition 2.14 ([80]) A fuzzy-valued function $\tilde{f} : [a,b] \to E^1$ is said to satisfy the condition (gH) on $[a,b]$ if, for any $x_1 < x_2 \in [a,b]$, there exists $\tilde{u} \in E^1$ such that $\tilde{f}(x_2) = \tilde{f}(x_1) + \tilde{u}$. We call $\tilde{u}$ the gH-difference of $\tilde{f}(x_2)$ and $\tilde{f}(x_1)$, denoted by $\tilde{f}(x_2) \ominus_{gH} \tilde{f}(x_1)$.

3 Fuzzy $(1 + n)$-dimensional Burgers' equation

In this section, we analyze fuzzy $(1 + n)$-dimensional Burgers' equation under gH-differentiability by using some methods. Let us take the following fuzzy $(1+n)$-dimensional Burgers equation:

$$\frac{\partial \tilde{u}}{\partial t} = \alpha_1^* \frac{\partial^2 \tilde{u}}{\partial x_1^2} \oplus \alpha_2^* \frac{\partial^2 \tilde{u}}{\partial x_2^2} \oplus \alpha_3^* \frac{\partial^2 \tilde{u}}{\partial x_3^2} \oplus \cdots \oplus \alpha_n^* \frac{\partial^2 \tilde{u}}{\partial x_n^2} \oplus \beta \tilde{u} \odot \frac{\partial \tilde{u}}{\partial x_1}, \tag{7}$$

with the initial condition

$$\tilde{u}(x_1, x_2, x_3, \ldots, x_n, 0) = \tilde{u}_0(x_1, x_2, x_3, \ldots, x_n), \tag{8}$$

where α_i^*, $i = 1, 2, 3, 4, \ldots, n$, and β are positive constants.

Example 3.1 We consider the following shows that $\tilde{u}^2$ does not always satisfy $\tilde{u}^2 \geq \tilde{0}$.

$$\tilde{u}(s) = \begin{cases} (s+1), & s \in [-1, 0], \\ \frac{1}{2}(2-s), & s \in (0, 2], \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

and $\underline{u}(\alpha) = \alpha - 1, \overline{u}(\alpha) = 2 - 2\alpha$ for any $\alpha \in [0,1]$. Then

$$\left[\tilde{u}^2 \right](\alpha) = [\tilde{u} \odot \tilde{u}](\alpha) = \left[(\alpha - 1)(2 - 2\alpha), (2 - 2\alpha)^2 \right],$$

$[\underline{\tilde{u}^2}](\alpha) \leq 0$ for any $\alpha \in [0,1]$.

3.1 Fuzzy differential transform method

Assume that D denotes a fuzzy DTM operator with D^{-1} the inverse fuzzy DTM operator.

Definition 3.1 ([82]) If $\tilde{u}(X_m, t; \alpha) = [\underline{u}(X_m, t; \alpha), \overline{u}(X_m, t; \alpha)]$ is analytic in the domain Ψ, then its fuzzy $(n + 1)$-dimensional DTM can be obtained as follows:

$$\underline{U}(J_m, j_{m+1}; \alpha) = \left(\frac{1}{J_m! j_{m+1}!} \right) \frac{\partial^{J_m + j_{m+1}}}{\partial X_m^{J_m} \partial t^{j_{m+1}}} \underline{u}(X_m, t; \alpha) \Bigg|_{X_m = 0, t = 0}, \quad 0 \leq \alpha \leq 1, \tag{10}$$

and

$$\overline{U}(J_m, j_{m+1}; \alpha) = \left(\frac{1}{J_m! j_{m+1}!} \right) \frac{\partial^{J_m + j_{m+1}}}{\partial X_m^{J_m} \partial t^{j_{m+1}}} \overline{u}(X_m, t; \alpha) \Big|_{X_m=0, t=0}, \quad 0 \le \alpha \le 1, \tag{11}$$

where $\sum_{J_i=0}^{\infty} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty}, X_m = x_1, x_2, \ldots, x_n$, for $J_m = j_1, j_2, \ldots, j_n$, then

$$\underline{u}(X_m, t; \alpha) = \sum_{J_m=0}^{\infty} \sum_{j_{m+1}=0}^{\infty} \underline{U}(J_m, j_{m+1}; \alpha),$$

$$X_m^{J_m} t^{j_{m+1}} = D^{-1}\big[\underline{U}(J_m, j_{m+1}; \alpha)\big], \tag{12}$$

and

$$\overline{u}(X_m, t; \alpha) = \sum_{J_m=0}^{\infty} \sum_{j_{m+1}=0}^{\infty} \underline{U}(J_m, j_{m+1}; \alpha),$$

$$X_m^{J_m} t^{j_{m+1}} = D^{-1}\big[\overline{U}(J_m, j_{m+1}; \alpha)\big]. \tag{13}$$

Definition 3.2 ([82]) If $\tilde{u}(X_m, t; \alpha) = D^{-1}[\underline{U}(J_m, j_{m+1}; \alpha), \overline{U}(J_m, j_{m+1}; \alpha)]$, $\tilde{v}(X_m, t; \alpha) = D^{-1}[\underline{V}(J_m, j_{m+1}; \alpha), \overline{V}(J_m, j_{m+1}; \alpha)]$, and $\otimes$ denotes convolution, then the basic operations of a fuzzy $(n + 1)$-dimensional DTM are represented as follows:

1.

$$D\big[\underline{u}(X_m, t; \alpha)\underline{v}(X_m, t; \alpha)\big]$$

$$= \underline{U}(J_m, j_{m+1}; \alpha) \otimes \underline{V}(J_m, j_{m+1}; \alpha)$$

$$= \sum_{A_m=0}^{J_m} \sum_{a_{n+1}=0}^{j_{n+1}} \underline{U}(J_m - A_m, j_{m+1} - a_{m+1}; \alpha)\underline{V}(A_m, a_{m+1}; \alpha), \tag{14}$$

and

$$D\big[\overline{u}(X_m, t; \alpha)\overline{v}(X_m, t; \alpha)\big]$$

$$= \overline{U}(J_m, j_{m+1}; \alpha) \otimes \overline{V}(J_m, j_{m+1}; \alpha)$$

$$= \sum_{A_m=0}^{J_m} \sum_{a_{n+1}=0}^{j_{n+1}} \overline{U}(J_m - A_m, j_{m+1} - a_{m+1}; \alpha)\overline{V}(A_m, a_{m+1}; \alpha), \tag{15}$$

where $A_m = a_1, a_2, a_3, \ldots, a_n$.

2.

$$D\big[\alpha^*\underline{u}(X_m, t; \alpha) \pm \beta\underline{v}(X_m, t; \alpha)\big] = \alpha^*\underline{U}(J_m, j_{n+1}; \alpha) \pm \beta\underline{V}(J_m, j_{n+1}; \alpha),$$

$$D\big[\alpha^*\overline{u}(X_m, t; \alpha) \pm \beta\overline{v}(X_m, t; \alpha)\big] = \alpha^*\overline{U}(J_m, j_{n+1}; \alpha) \pm \beta\overline{V}(J_m, j_{n+1}; \alpha).$$

3.

$$D\left[\frac{\partial^{R_m + r_{m+1}}}{\partial X_m^{R_m} \partial t^{r_{n+1}}} \underline{u}(X_m, t; \alpha) \right]$$

$$= (j_1 + 1)(j_1 + 2)\cdots(j_1 + r_1)(j_2 + 1)$$
$$\times (j_2 + 2)\cdots(j_2 + r_2)\cdots(j_{m+1} + 1)(j_{m+1} + 2)\cdots(j_{m+1} + r_{m+1})$$
$$\times \underline{U}(j_1 + r_1, j_2 + r_2, \ldots, j_{m+1} + r_{m+1})(\alpha),$$

and

$$D\left[\frac{\partial^{R_m + r_{m+1}}}{\partial X_m^{R_m} \partial t^{r_{n+1}}} \overline{u}(X_m, t; \alpha)\right]$$
$$= (j_1 + 1)(j_1 + 2)\cdots(j_1 + r_1)(j_2 + 1)$$
$$\times (j_2 + 2)\cdots(j_2 + r_2)\cdots(j_{m+1} + 1)(j_{m+1} + 2)\cdots(j_{m+1} + r_{m+1})$$
$$\times \overline{U}(j_1 + r_1, j_2 + r_2, \ldots, j_{m+1} + r_{m+1})(\alpha),$$

where $R_m = r_1, r_2, \ldots, r_n$.

3.2 Fuzzy Adomian decomposition method

We propose the fuzzy linear operators with these inverse operators

$$\mathcal{L}_t = \frac{\partial}{\partial t}, \qquad \mathcal{L}_{x_i x_i} = \frac{\partial^2}{\partial x_i^2}, \quad i = 1, 2, \ldots, n;$$
$$\mathcal{L}_t^{-1} = \int_0^t (\cdot)\, d\tau; \qquad \mathcal{L}_{x_i x_i}^{-1} = \int_0^{x_i}\int_0^{x_i} (\cdot)\, d\tau\, d\tau, \quad i = 1, 2, \ldots, n.$$

Taking these notations, equation (7) becomes

$$\mathcal{L}_t(\underline{u}) = \left(\sum_{i=1}^n \alpha_i^* \mathcal{L}_{x_i x_i}(\underline{u})\right) + \gamma\underline{u}\frac{\partial\underline{u}}{\partial x_1}, \tag{16}$$

$$\mathcal{L}_t(\overline{u}) = \left(\sum_{i=1}^n \alpha_i^* \mathcal{L}_{x_i x_i}(\overline{u})\right) + \gamma\overline{u}\frac{\partial\overline{u}}{\partial x_1}. \tag{17}$$

Apply the inverse operators $\mathcal{L}_t^{-1}$ to (16) and (17) with initial condition (8) as follows:

$$\underline{u}(X_m, t; \alpha) = \left[\underline{u}_0(X_m, t; \alpha) + \mathcal{L}_t^{-1}\left(\sum_{i=1}^n \alpha_i^* \mathcal{L}_{x_i x_i}\big(\underline{u}(X_m, t; \alpha)\big)\right)\right.$$
$$\left. + \beta\mathcal{L}_t^{-1}\left(\underline{u}\frac{\partial\underline{u}}{\partial x_1}\right)(\alpha)\right], \tag{18}$$

$$\overline{u}(X_m, t; \alpha) = \left[\underline{u}_0(X_m, t; \alpha) + \mathcal{L}_t^{-1}\left(\sum_{i=1}^n \alpha_i^* \mathcal{L}_{x_i x_i}\big(\overline{u}(X_m, t; \alpha)\big)\right)\right.$$
$$\left. + \beta\mathcal{L}_t^{-1}\left(\overline{u}\frac{\partial\overline{u}}{\partial x_1}\right)(\alpha)\right], \tag{19}$$

where $X_m = x_1, x_2, x_3, \ldots, x_n$ for $\alpha \in [0, 1]$.

A fuzzy ADM consists of demonstrating a solution

$$\tilde{u}(X_m, t; \alpha) = \left[\underline{u}(X_m, t; \alpha), \overline{u}(X_m, t; \alpha)\right]$$

using a fuzzy decomposition series as follows:

$$\underline{u}(X_m, t; \alpha) = \sum_{q=0}^{\infty} \underline{u}_q(X_m, t; \alpha), \tag{20}$$

$$\overline{u}(X_m, t; \alpha) = \sum_{q=0}^{\infty} \overline{u}_q(X_m, t; \alpha). \tag{21}$$

Fuzzy nonlinear term

$$\tilde{u} \odot \frac{\partial \tilde{u}}{\partial x_1} = \left[\min(S), \max(S)\right], \tag{22}$$

where

$$S = \left\{ \underline{u}(\alpha) \frac{\partial \underline{u}(\alpha)}{\partial x_1}, \underline{u}(\alpha) \frac{\partial \overline{u}(\alpha)}{\partial x_1}, \overline{u}(\alpha) \frac{\partial \underline{u}(\alpha)}{\partial x_1}, \overline{u}(\alpha) \frac{\partial \overline{u}(\alpha)}{\partial x_1} \right\} \tag{23}$$

appeared to use a series of the fuzzy Adomian polynomials as

$$\underline{u}(\alpha) \frac{\partial \underline{u}(\alpha)}{\partial x_1} = \sum_{q=0}^{\infty} \underline{A}_q(X_m, t; \alpha), \tag{24}$$

$$\overline{u}(\alpha) \frac{\partial \overline{u}(\alpha)}{\partial x_1} = \sum_{q=0}^{\infty} \overline{A}_q(X_m, t; \alpha). \tag{25}$$

Component

$$\tilde{u}_q(X_m, t; \alpha) = \left[\underline{u}_q(X_m, t; \alpha), \overline{u}_q(X_m, t; \alpha)\right],$$

the solution $\tilde{u}(X_m, t; \alpha)$ is specified in an iterative manner. Substituting the fuzzy decomposition string (20) – (25) for $\tilde{u}(X_m, t; \alpha)$ in equations (18) and (19), we get

$$\sum_{q=0}^{\infty} \underline{u}_q(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha)$$
$$+ \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\left(\underline{u}(\alpha)\right)\right) \tag{26}$$
$$+ \beta \mathcal{L}_t^{-1}\left(\sum_{q=0}^{\infty} \underline{A}_q(X_m, t; \alpha)\right)$$

and

$$\sum_{q=0}^{\infty} \overline{u}_q(X_m, t; \alpha) = \overline{u}_0(X_m, t; \alpha)$$

$$+ \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\left(\overline{u}(\alpha)\right)\right) \tag{27}$$

$$+ \beta \mathcal{L}_t^{-1}\left(\sum_{q=0}^{\infty} \overline{A}_q(X_m, t; \alpha)\right).$$

Applying the fuzzy ADM to the zeroth component

$$\tilde{u}_0(X_m, t; \alpha) = \left[\underline{u}_0(X_m, t; \alpha)\right), \overline{u}_0(X_m, t; \alpha)\right],$$

the surviving components of $\tilde{u}(X_m, t; \alpha) = [\underline{u}(X_m, t; \alpha), \overline{u}(X_m, t; \alpha)]$ are specified in such a frequent way that

$$\underline{u}_0(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha), \tag{28}$$

$$\underline{u}_{j+1}(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\left(\underline{u}(\alpha)\right)\right) + \beta \mathcal{L}_t^{-1}(\underline{A}_j)(\alpha), \quad j \geq 0, \tag{29}$$

and

$$\overline{u}_0(X_m, t; \alpha) = \overline{u}_0(X_m, t; \alpha), \tag{30}$$

$$\overline{u}_{j+1}(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\left(\overline{u}(\alpha)\right)\right) + \beta \mathcal{L}_t^{-1}(\overline{A}_j)(\alpha), \quad j \geq 0. \tag{31}$$

The fuzzy Adomian polynomials for a fuzzy nonlinear term $\tilde{u} \odot \frac{\partial \tilde{u}}{\partial x_1} = [\min(S), \max(S)]$, where $S = \{\underline{u}(\alpha)\frac{\partial \underline{u}(\alpha)}{\partial x_1}, \underline{u}(\alpha)\frac{\partial \overline{u}(\alpha)}{\partial x_1}, \overline{u}(\alpha)\frac{\partial \underline{u}(\alpha)}{\partial x_1}, \overline{u}(\alpha)\frac{\partial \overline{u}(\alpha)}{\partial x_1}\}$ is derived from the recursive formula as follows:

$$\underline{A}_j(\alpha) = \frac{1}{j!}\frac{d^j}{d\lambda^j}\left[\left(\sum_{i=0}^{\infty} \lambda^i \underline{u}_i(\alpha)\right)\left(\sum_{i=0}^{\infty} \lambda^i \underline{u}_i(\alpha)\right)\right]_{\lambda=0}, \quad j = 0, 1, 2, 3, \ldots, \tag{32}$$

$$\overline{A}_j(\alpha) = \frac{1}{j!}\frac{d^j}{d\lambda^j}\left[\left(\sum_{i=0}^{\infty} \lambda^i \overline{u}_i(\alpha)\right)\left(\sum_{i=0}^{\infty} \lambda^i \overline{u}_i(\alpha)\right)\right]_{\lambda=0}, \quad j = 0, 1, 2, 3, \ldots. \tag{33}$$

The fuzzy Adomian polynomials are defined as follows:

$$
\begin{aligned}
\underline{A}_0(\alpha) &= \underline{u}_0\frac{\partial \underline{u}_0(\alpha)}{\partial x_1}, \\
\underline{A}_1(\alpha) &= \underline{u}_1\frac{\partial \underline{u}_0(\alpha)}{\partial x_1} + \underline{u}_0\frac{\partial \underline{u}_1(\alpha)}{\partial x_1}, \\
\underline{A}_2(\alpha) &= \underline{u}_2\frac{\partial \underline{u}_0(\alpha)}{\partial x_1} + \underline{u}_1\frac{\partial \underline{u}_1(\alpha)}{\partial x_1} + \underline{u}_0\frac{\partial \underline{u}_2(\alpha)}{\partial x_1}, \\
\underline{A}_3(\alpha) &= \underline{u}_3\frac{\partial \underline{u}_0(\alpha)}{\partial x_1} + \underline{u}_2\frac{\partial \underline{u}_1(\alpha)}{\partial x_1} + \underline{u}_1\frac{\partial \underline{u}_2(\alpha)}{\partial x_1} + \underline{u}_0\frac{\partial \underline{u}_3(\alpha)}{\partial x_1},
\end{aligned} \tag{34}
$$

and

$$\overline{A}_0(\alpha) = \overline{u}_0 \frac{\partial \overline{u}_0(\alpha)}{\partial x_1},$$

$$\overline{A}_1(\alpha) = \overline{u}_1 \frac{\partial \overline{u}_0(\alpha)}{\partial x_1} + \overline{u}_0 \frac{\partial \overline{u}_1(\alpha)}{\partial x_1},$$

$$\overline{A}_2(\alpha) = \overline{u}_2 \frac{\partial \overline{u}_0(\alpha)}{\partial x_1} + \overline{u}_1 \frac{\partial \overline{u}_1(\alpha)}{\partial x_1} + \overline{u}_0 \frac{\partial \overline{u}_2(\alpha)}{\partial x_1}, \tag{35}$$

$$\overline{A}_3(\alpha) = \overline{u}_3 \frac{\partial \overline{u}_0(\alpha)}{\partial x_1} + \overline{u}_2 \frac{\partial \overline{u}_1(\alpha)}{\partial x_1} + \overline{u}_1 \frac{\partial \overline{u}_2(\alpha)}{\partial x_1} + \overline{u}_0 \frac{\partial \overline{u}_3(\alpha)}{\partial x_1}.$$

Applying (29) and (31) for fuzzy Adomian polynomials $\tilde{A}_j(\alpha) = [\underline{A}_j(\alpha), \overline{A}_j(\alpha)]$, we obtain

$$\underline{u}_0(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha), \tag{36}$$

$$\underline{u}_1(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\big(\underline{u}_0(\alpha)\big)\right) + \gamma \mathcal{L}_t^{-1}(\underline{A}_0)(\alpha), \tag{37}$$

$$\underline{u}_2(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\big(\underline{u}_1(\alpha)\big)\right) + \gamma \mathcal{L}_t^{-1}(\underline{A}_1)(\alpha), \tag{38}$$

$$\underline{u}_3(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i, x_i}\big(\underline{u}_2(\alpha)\big)\right) + \gamma \mathcal{L}_t^{-1}(\underline{A}_2)(\alpha), \tag{39}$$

$$\vdots$$

and

$$\overline{u}_0(X_m, t; \alpha) = \overline{u}_0(X_m, t; \alpha), \tag{40}$$

$$\overline{u}_1(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\big(\overline{u}_0(\alpha)\big)\right) + \gamma \mathcal{L}_t^{-1}(\overline{A}_0)(\alpha), \tag{41}$$

$$\overline{u}_2(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i x_i}\big(\overline{u}_1(\alpha)\big)\right) + \gamma \mathcal{L}_t^{-1}(\overline{A}_1)(\alpha), \tag{42}$$

$$\overline{u}_3(X_m, t; \alpha) = \mathcal{L}_t^{-1}\left(\sum_{i=1}^{n} \alpha_i^* \mathcal{L}_{x_i, x_i}\big(\overline{u}_2(\alpha)\big)\right) + \gamma \mathcal{L}_t^{-1}(\overline{A}_2)(\alpha), \tag{43}$$

$$\vdots$$

The qth term, $\tilde{u}_q(\alpha) = [\underline{u}_q(\alpha), \overline{u}_q(\alpha)]$ can be specified from

$$\underline{u}_q(X_m, t; \alpha) = \sum_{j=0}^{q-1} \underline{u}_j(X_m, t; \alpha), \tag{44}$$

$$\overline{u}_q(X_m, t; \alpha) = \sum_{j=0}^{q-1} \overline{u}_j(X_m, t; \alpha). \tag{45}$$

3.2.1 Convergence analysis of fuzzy ADM

According to [38, 39, 58], we present the convergence analysis of fuzzy ADM for the general fuzzy operator equation given by

$$\mathcal{L}\big(\tilde{u}(x,t;\alpha)\big) + \mathcal{R}\big(\tilde{u}(x,t;\alpha)\big) + \mathcal{N}\big(\tilde{u}(x,t;\alpha)\big) = \tilde{g}(x,t;\alpha), \quad 0 \le \alpha \le 1, \tag{46}$$

where $\tilde{g}(x,t;\alpha)$ is given in $\mathcal{H}'$. Suppose that $\mathcal{T}$ is an operator defined by $\mathcal{T}\tilde{u}(x,t;\alpha) = -\mathcal{R}\tilde{u}(x,t;\alpha) - \mathcal{N}\tilde{u}(x,t;\alpha)$.

We consider the Hilbert space $\mathcal{H} = \mathcal{L}^2((\alpha^*, \beta^*) \times [0, \mathcal{T}])$ defined by the set of applications as follows:

$$\tilde{u}(x,t;\alpha) : (\alpha^*, \beta^*) \times [0, \mathcal{T}] \longrightarrow \mathcal{R}$$

with

$$\int_{(\alpha^*,\beta^*) \times [0,T]} \tilde{u}(x,t;\alpha)\, dx\, dt < +\infty, \tag{47}$$

where $\tilde{u}(x,t;\alpha) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$.

Theorem 3.3 *Assume that* $\mathcal{T}\tilde{u}(x,t;\alpha) = -\mathcal{R}\tilde{u}(x,t;\alpha) - \mathcal{N}\tilde{u}(x,t;\alpha)$ *is semi-continuous* (*i.e., the restriction of* $(-\mathcal{R} - \mathcal{N})$) *to the segments of* $\mathcal{H}$ *is continuous, in* $\mathcal{H}'$ *weak*) *and satisfies the hypotheses* $\mathcal{H}_1, \mathcal{H}_2$ *as follows*:

- $[\mathcal{H}_1] : (\mathcal{T}\tilde{u}(x,t;\alpha) - \mathcal{T}\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)) \ge K\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\|^2, \kappa > 0, \forall \tilde{u}, \tilde{v} \in \mathcal{H}.$
- $[\mathcal{H}_2] : \forall M > 0, \exists D(M) > 0\, such\, that, for \|\tilde{u}\| \le M, \|\tilde{v}\| \le M, \tilde{u}, \tilde{v} \in \mathcal{H}, we\, obtain$
  $\Rightarrow (\mathcal{T}\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha), w(x,t;\alpha)) \le D(M)\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\| \|\tilde{w}(x,t;\alpha)\|, \forall \tilde{w} \in \mathcal{H}.$

For every $\tilde{g}(x,t;\alpha) \in \mathcal{H}'$, *the fuzzy nonlinear functional equation* (46) *admits a unique solution* $\tilde{u}(x,t;\alpha) \in \mathcal{H}$. *Furthermore, if the solution* $\tilde{u}(x,t;\alpha)$ *can be assimilated as a series* $\tilde{u}(x,t;\alpha) = \sum_{n=0}^{\infty} \tilde{u}_n(x,t;\alpha)\lambda^n$, *then the fuzzy ADM diagram corresponding to the functional equation under study converges strongly to* $\tilde{u}(x,t;\alpha) \in \mathcal{H}$, *which is the unique solution to the functional equation.*

Proof The operator $\mathcal{A}$ defined by $\mathcal{A}\tilde{u}(x,t;\alpha) = -\mathcal{L}\tilde{u}(x,t;\alpha) - \mathcal{T}\tilde{u}(x,t;\alpha)$, where $\tilde{u}(x,t;\alpha) \in \mathcal{H}$, satisfies the conditions:

$(\mathcal{H}_1)$

$$\big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big)$$
$$= (-L\big(\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big)$$
$$+ \big(T\tilde{u}(x,t;\alpha) - T\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big), \quad 0 \le \alpha \le 1$$
$$\Rightarrow \quad \big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big)$$
$$= (-L\big(\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)$$
$$+ \big(T\tilde{u}(x,t;\alpha) - T\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big), \quad 0 \le \alpha \le 1.$$

According to $(\mathcal{H}_1)$,

$$\big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big)$$

$$\geq \big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|^2 + K\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|^2$$

$$= (1+K)\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|^2, \quad 0 \leq \alpha \leq 1$$

$$\Rightarrow \quad \big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big) \geq C\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|^2,$$

where

$$C = 1 + k; \quad C > 0.$$

$(\mathcal{H}_2)$

$$\big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big)$$

$$= \big(-L\big(\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big)$$

$$+ \big(T\tilde{u}(x,t;\alpha) - T\tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big), \quad 0 \leq \alpha \leq 1$$

$$\Rightarrow \quad \big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big)$$

$$= \big(-L\big(\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big)$$

$$+ \big(T\tilde{u}(x,t;\alpha) - T\tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big), \quad 0 \leq \alpha \leq 1.$$

According to Schwarz's inequality and $(\mathcal{H}_2)$, we obtain

$$\big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big)\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|\big\|\tilde{w}(x,t;\alpha)\big\|$$

$$+ C(M)\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|\big\|\tilde{w}(x,t;\alpha)\big\|$$

$$= \big\|1 + C(M)\big\|\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|\big\|\tilde{w}(x,t;\alpha)\big\|, \tag{48}$$

and therefore

$$\big(A\tilde{u}(x,t;\alpha) - A\tilde{v}(x,t;\alpha), \tilde{w}(x,t;\alpha)\big) \leq D(M)\big\|\tilde{u}(x,t;\alpha) - \tilde{v}(x,t;\alpha)\big\|\big\|\tilde{w}(x,t;\alpha)\big\|, \tag{49}$$

where

$$D(M) = 1 + C(M),$$

which completes the proof. □

3.3 Fuzzy homotopy perturbation method

We establish the general nonlinear fuzzy differential equation as follows:

$$\tilde{A}(\tilde{u}) = \tilde{f}(r), \quad r \in \Psi, \tag{50}$$

where $\tilde{f}(r;\alpha) = [\underline{f}(r;\alpha), \overline{f}(r;\alpha)] \in E^1$, and we shall define:

- $\tilde{A}(\tilde{u})$ is a fuzzy differential operator, which means that $\underline{A(\tilde{u})}$ and $\overline{A(\tilde{u})}$ are differential operators,
- $\underline{\tilde{A}(\tilde{u})}(\alpha) = \underline{f}(r; \alpha)$ and $\overline{\tilde{A}(\tilde{u})}(\alpha) = \overline{f}(r; \alpha)$ for any $\alpha \in [0, 1]$,

with the boundary condition

$$B\left(\tilde{u}, \frac{\partial \tilde{u}}{\partial r} \right) = \tilde{0}, \quad r \in \partial \Psi, \tag{51}$$

where B is a boundary operator and $\partial \Psi$ is boundary of the domain Ψ. The fuzzy operator $\tilde{A}$ can be discordant into two parts $\mathcal{L}$ and $\mathcal{N}$, where $\mathcal{L}$ is a fuzzy linear operator, while $\mathcal{N}$ is a fuzzy nonlinear operator. From equation (50), we obtain

$$\mathcal{L}(\underline{u})(\alpha) + \mathcal{N}(\underline{u})(\alpha) - \underline{f}(r, \alpha) = \tilde{0}, \tag{52}$$

$$\mathcal{L}(\overline{u})(\alpha) + \mathcal{N}(\overline{u})(\alpha) - \overline{f}(r, \alpha) = \tilde{0}. \tag{53}$$

We construct a fuzzy homotopy applying a fuzzy homotopy technique as follows:

$$\tilde{v}(r, p; \alpha) = \left[\underline{v}(r, p; \alpha), \overline{v}(r, p; \alpha) \right] : \Psi \times [0, 1] \to R, \quad \text{which satisfies}$$

$$\left. \begin{aligned} H\big(\underline{v}(\alpha), p\big) &= (1 - p)\big[\mathcal{L}(\underline{v})(\alpha) - \mathcal{L}\big(\underline{u}_0(\alpha)\big) \big] + p\big[\underline{A(v)}(\alpha) - \underline{f}(r; \alpha) \big], \\ H\big(\overline{v}(\alpha), p\big) &= (1 - p)\big[\mathcal{L}(\overline{v})(\alpha) - \mathcal{L}\big(\overline{u}_0(\alpha)\big) \big] + p\big[\overline{A(v)}(\alpha) - \overline{f}(r; \alpha) \big], \end{aligned} \right\},$$

$$r \in \Psi, p \in [0, 1], \tag{54}$$

or

$$\left. \begin{aligned} H\big(\underline{v}(\alpha), p\big) &= \mathcal{L}(\underline{v})(\alpha) - \mathcal{L}\big(\underline{u}_0(\alpha)\big) + p\mathcal{L}\big(\underline{u}_0(\alpha)\big) + p\big[\mathcal{N}(\underline{v})(\alpha) - \underline{f}(r; \alpha) \big] \\ H\big(\overline{v}(\alpha), p\big) &= \mathcal{L}(\overline{v})(\alpha) - \mathcal{L}\big(\overline{u}_0(\alpha)\big) + p\mathcal{L}\big(\overline{u}_0(\alpha)\big) + p\big[\mathcal{N}(\overline{v})(\alpha) - \overline{f}(r; \alpha) \big], \end{aligned} \right\},$$

$$r \in \Psi, p \in [0, 1], \tag{55}$$

for $p \in [0, 1]$ is the included parameter, $\tilde{u}_0(\alpha) = [\underline{u}_0(\alpha), \overline{u}_0(\alpha)]$ is the initial approximation of equation (50) that satisfies the boundary conditions. Clearly, looking (54) and (55), we obtain

$$\begin{aligned} H\big(\underline{v}(\alpha), 0\big) &= \big[\mathcal{L}(\underline{v})(\alpha) - \mathcal{L}\big(\underline{u}_0(\alpha)\big) \big] = \tilde{0}, \\ H\big(\underline{v}(\alpha), 1\big) &= \big[\underline{A(v)}(\alpha) - \underline{f}(r, \alpha) \big] = \tilde{0}, \end{aligned} \tag{56}$$

and

$$\begin{aligned} H\big(\overline{v}(\alpha), 0\big) &= \big[\mathcal{L}(\overline{v})(\alpha) - \mathcal{L}\big(\overline{u}_0(\alpha)\big) \big] = \tilde{0}, \\ H\big(\overline{v}(\alpha), 1\big) &= \big[\overline{A(v)}(\alpha) - \overline{f}(r, \alpha) \big] = \tilde{0}. \end{aligned} \tag{57}$$

The process of changing p from zero to unit is that $\tilde{v}(r, p; \alpha) = [\underline{v}(r, p; \alpha), \overline{v}(r, p; \alpha)]$ from $\tilde{u}_0(r; \alpha) = [\underline{u}_0(r; \alpha), \overline{u}_0(r; \alpha)]$ to $\tilde{u}(r; \alpha) = [\underline{u}(r; \alpha), \overline{u}(r; \alpha)]$. In topology, this is called disfigurement, $\mathcal{L}(\tilde{v})(\alpha) = [\mathcal{L}(\underline{v})(\alpha), \mathcal{L}(\overline{v})(\alpha)]$ and $\underline{A(v)}(\alpha) - \underline{f}(r; \alpha), \overline{A(v)}(\alpha) - \overline{f}(r; \alpha)$ are called fuzzy

homotopy. A fuzzy HPM uses the fuzzy homotopy parameter p as an extending parameter, we get

$$\underline{v}(\alpha) = \sum_{n=0}^{\infty} p^n \underline{v}_n(\alpha), \tag{58}$$

$$\overline{v}(\alpha) = \sum_{n=0}^{\infty} p^n \overline{v}_n(\alpha). \tag{59}$$

$p \to 1$ produces the approximate solution of equation (50) as follows:

$$\underline{u}(\alpha) = \lim_{p \to 1} \underline{v}(\alpha) = \sum_{n=0}^{\infty} \underline{v}_n(\alpha), \tag{60}$$

$$\overline{u}(\alpha) = \lim_{p \to 1} \overline{v}(\alpha) = \sum_{n=0}^{\infty} \overline{v}_n(\alpha). \tag{61}$$

Comparison of p similar powers gives various order solutions. We know that series (60) and (61) represent convergence in most cases. Yet the convergence rate is dependent on the fuzzy nonlinear operator $\mathcal{N}(\tilde{v})$. We will also look at the same opinions in [46] about $\mathcal{N}(\tilde{v})$ in a fuzzy environment as follows:

- A second fuzzy derivative of $\mathcal{N}(\tilde{v})$ with respect to $\tilde{v}$ should be small as the parameter p may be relatively large.
- A norm of $\mathcal{L}^{-1} \frac{\partial \mathcal{N}}{\partial \tilde{v}}$ should be smaller than one so that the series converges.

Applying a fuzzy HPM to (7) and using a fuzzy HPM, we are building a fuzzy simple homotopy as follows:

$$\frac{\partial \underline{u}(\alpha)}{\partial t} + p\left(\left[-\vartheta_m \frac{\partial^2 \underline{u}(\alpha)}{\partial X_m^2} \right] - \beta \underline{u} \frac{\partial \underline{u}(\alpha)}{\partial x_1} \right) = \tilde{0}, \tag{62}$$

$$\frac{\partial \overline{u}(\alpha)}{\partial t} + p\left(\left[-\vartheta_m \frac{\partial^2 \overline{u}(\alpha)}{\partial X_m^2} \right] - \beta \overline{u} \frac{\partial \overline{u}(\alpha)}{\partial x_1} \right) = \tilde{0}, \tag{63}$$

where $\frac{\partial^2}{\partial X_m^2} = \frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2}$, $X_m = x_1, x_2, \ldots, x_n$ for $\vartheta_m = \vartheta_1, \vartheta_2, \ldots, \vartheta_n$.

Subject to

$$\underline{u}(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha),$$

$$\overline{u}(X_m, 0)(\alpha) = \overline{u}_0(X_m)(\alpha),$$

let the solution be as follows:

$$\underline{u}(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha) + p\underline{u}_1(X_m, t; \alpha) + p^2\underline{u}_2(X_m, t; \alpha) + \cdots, \tag{64}$$

$$\underline{u}(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha) + p\underline{u}_1(X_m, t; \alpha) + p^2\underline{u}_2(X_m, t; \alpha) + \cdots. \tag{65}$$

Taking (65) into (62) with the terms equating and powers comparable to p, we obtain

$$p^0 : \frac{\partial \underline{u}_0(\alpha)}{\partial t} = \tilde{0}, \tag{66}$$

$$p^1 : \frac{\partial \underline{u}_1(\alpha)}{\partial t} = \left[\vartheta_m \frac{\partial^2 \underline{u}_0(\alpha)}{\partial X_m^2} \right] + \beta \underline{u} \frac{\partial \underline{u}_0(\alpha)}{\partial x_1}, \tag{67}$$

$$p^2 : \frac{\partial \underline{u}_2(\alpha)}{\partial t} = \left[\vartheta_m \frac{\partial^2 \underline{u}_1(\alpha)}{\partial X_m^2} \right] + \beta \left[\underline{u}_0 \frac{\partial \underline{u}_1(\alpha)}{\partial x_1} + \underline{u}_1 \frac{\partial \underline{u}_0(\alpha)}{\partial x_1} \right], \tag{68}$$

$$\vdots$$

and

$$p^0 : \frac{\partial \overline{u}_0(\alpha)}{\partial t} = \tilde{0}, \tag{69}$$

$$p^1 : \frac{\partial \overline{u}_1(\alpha)}{\partial t} = \left[\vartheta_m \frac{\partial^2 \overline{u}_0(\alpha)}{\partial X_m^2} \right] + \beta \overline{u} \frac{\partial \overline{u}_0(\alpha)}{\partial x_1}, \tag{70}$$

$$p^2 : \frac{\partial \overline{u}_2(\alpha)}{\partial t} = \left[\vartheta_m \frac{\partial^2 \overline{u}_1(\alpha)}{\partial X_m^2} \right] + \beta \left[\overline{u}_0 \frac{\partial \overline{u}_1(\alpha)}{\partial x_1} + \overline{u}_1 \frac{\partial \overline{u}_0(\alpha)}{\partial x_1} \right], \tag{71}$$

$$\vdots$$

Using (66) into (71), we obtain the fuzzy-valued function $\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \dots, \tilde{u}_n$. Consequently, keeping (64) and (65) and allowing $p = 1$, we get the approximate solution of (7) as follows:

$$\underline{u}(X_m, t; \alpha) = \underline{u}_0(X_m, t; \alpha) + \underline{u}_1(X_m, t; \alpha) + \underline{u}_2(X_m, t; \alpha) + \cdots, \tag{72}$$

$$\overline{u}(X_m, t; \alpha) = \overline{u}_0(X_m, t; \alpha) + \overline{u}_1(X_m, t; \alpha) + \overline{u}_2(X_m, t; \alpha) + \cdots. \tag{73}$$

3.3.1 Convergence analysis of fuzzy HPM

According to [17], we can write equation (55) in the following forms:

$$\mathcal{L}\big(\big[\underline{V}(x, t; \alpha) \big] \big) = \mathcal{L}\big(\big[\underline{U}_0(x, t; \alpha) \big] \big) + p\big[\underline{f}(r; \alpha) - \mathcal{N}\big(\big[\underline{V}(x, t; \alpha) \big] \big) - \mathcal{L}\big[\underline{U}_0(x, t; \alpha) \big] \big], \tag{74}$$

$$\mathcal{L}\big(\big[\overline{V}(x, t; \alpha) \big] \big) = \mathcal{L}\big(\big[\overline{U}_0(x, t; \alpha) \big] \big) + p\big[\overline{f}(r; \alpha) - \mathcal{N}\big(\big[\overline{V}(x, t; \alpha) \big] \big) - \mathcal{L}\big[\overline{U}_0(x, t; \alpha) \big] \big], \tag{75}$$

where $\alpha \in [0, 1]$.

Applying the inverse operator $\mathcal{L}^{-1}$ to both sides of (74) and (75), we get

$$\big[\underline{V}(x, t; \alpha) \big] = \big[\underline{U}_0(x, t; \alpha) \big] + p\big[\mathcal{L}^{-1} \underline{f}(r; \alpha) - \mathcal{L}^{-1} \mathcal{N}\big(\big[\underline{V}(x, t; \alpha) \big] \big) - \big[\underline{U}_0(x, t; \alpha) \big] \big], \tag{76}$$

$$\big[\overline{V}(x, t; \alpha) \big] = \big[\overline{U}_0(x, t; \alpha) \big] + p\big[\mathcal{L}^{-1} \overline{f}(r; \alpha) - \mathcal{L}^{-1} \mathcal{N}\big(\big[\overline{V}(x, t; \alpha) \big] \big) - \big[\overline{U}_0(x, t; \alpha) \big] \big]. \tag{77}$$

Let

$$\underline{V}(x, t; \alpha) = \sum_{i=0}^{\infty} p^i \underline{V}_i(x, t; \alpha), \tag{78}$$

$$\overline{V}(x, t; \alpha) = \sum_{i=0}^{\infty} p^i \overline{V}_i(x, t; \alpha), \tag{79}$$

substituting (78) and (79) into the right-hand side of (76) and (77), we obtain

$$\big[\underline{V}(x, t; \alpha) \big]$$

$$= \left[\underline{U}_0(x,t;\alpha)\right] + p\left[\mathcal{L}^{-1}\underline{f}(r;\alpha) - \left(\mathcal{L}^{-1}\mathcal{N}\right)\left[\sum_{i=0}^{\infty}p^i\underline{V}_i(x,t;\alpha)\right] - \left[\underline{U}_0(x,t;\alpha)\right]\right],$$

$$\left[\overline{V}(x,t;\alpha)\right]$$

$$= \left[\overline{U}_0(x,t;\alpha)\right] + p\left[\mathcal{L}^{-1}\overline{f}(r;\alpha) - \left(\mathcal{L}^{-1}\mathcal{N}\right)\left[\sum_{i=0}^{\infty}p^i\overline{V}_i(x,t;\alpha)\right] - \left[\overline{U}_0(x,t;\alpha)\right]\right].$$

If $p \to 1$, the exact solution may be obtained by

$$\underline{U}(x,t;\alpha) = \lim_{p\to 1}\underline{V}(x,t;\alpha)$$

$$= \mathcal{L}^{-1}\left(\underline{f}(r;\alpha)\right) - \left(\mathcal{L}^{-1}\mathcal{N}\right)\left[\sum_{i=0}^{\infty}\underline{v}_i(x,t;\alpha)\right]$$

$$= \mathcal{L}^{-1}\left(\underline{f}(r;\alpha)\right) - \left[\sum_{i=0}^{\infty}\left(\mathcal{L}^{-1}\mathcal{N}\right)\underline{v}_i(x,t;\alpha)\right]$$

and

$$\overline{U}(x,t;\alpha) = \lim_{p\to 1}\overline{V}(x,t;\alpha)$$

$$= \mathcal{L}^{-1}\left(\overline{f}(r;\alpha)\right) - \left(\mathcal{L}^{-1}\mathcal{N}\right)\left[\sum_{i=0}^{\infty}\overline{v}_i(x,t;\alpha)\right]$$

$$= \mathcal{L}^{-1}\left(\overline{f}(r;\alpha)\right) - \left[\sum_{i=0}^{\infty}\left(\mathcal{L}^{-1}\mathcal{N}\right)\overline{v}_i(x,t;\alpha)\right].$$

Theorem 3.4 *Let $\tilde{X}(\alpha) = [\underline{X}(\alpha), \overline{X}(\alpha)]$ and $\tilde{Y}(\alpha) = [\underline{Y}(\alpha), \overline{Y}(\alpha)]$ be Banach spaces and N : $\tilde{X}(\alpha) \to \tilde{Y}(\alpha)$ be a contractive nonlinear mapping, that is,*

$$\forall \underline{w}(\alpha), \underline{w}^*(\alpha) \in \underline{X}(\alpha);$$

$$\left\|\mathcal{N}(\underline{w})(\alpha) - \mathcal{N}(\underline{w}^*)(\alpha)\right\| \leq \xi\left\|(\underline{w})(\alpha) - (\underline{w}^*)(\alpha)\right\|, \quad 0 \leq \xi \leq 1, \alpha \in [0,1],$$

$$\forall \overline{w}(\alpha), \overline{w}^*(\alpha) \in \overline{X}(\alpha);$$

$$\left\|\mathcal{N}(\overline{w})(\alpha) - \mathcal{N}(\overline{w}^*)(\alpha)\right\| \leq \xi\left\|(\overline{w})(\alpha) - (\overline{w}^*)(\alpha)\right\|, \quad 0 \leq \xi \leq 1, \alpha \in [0,1].$$

According to Banach's fixed point theorem, $\mathcal{N}$ has a unique fixed point $\tilde{u}(\alpha) = [\underline{u}(\alpha), \overline{u}(\alpha)]$, that is, $\mathcal{N}(\underline{u})(\alpha) = (\underline{u})(\alpha)$ and $\mathcal{N}(\overline{u})(\alpha) = (\overline{u})(\alpha)$. Assume that the sequence generated by using fuzzy HPM can be written as follows:

$$\underline{W}_n(x,t;\alpha) = \mathcal{N}\left(\underline{W}_{n-1}(x,t;\alpha)\right), \qquad \underline{W}_{n-1}(x,t;\alpha) = \sum_{i=0}^{n-1}\underline{w}_i(x,t;\alpha), \quad n = 1,2,3,\dots,$$

$$\overline{W}_n(x,t;\alpha) = \mathcal{N}\left(\overline{W}_{n-1}(x,t;\alpha)\right), \qquad \overline{W}_{n-1}(x,t;\alpha) = \sum_{i=0}^{n-1}\overline{w}_i(x,t;\alpha), \quad n = 1,2,3,\dots,$$

assume that

$$\underline{W}_0(x,t;\alpha) = \underline{w}_0(x,t;\alpha) \in B_r\left(\underline{w}(x,t;\alpha)\right),$$

$$\overline{W}_0(x,t;\alpha) = \overline{w}_0(x,t;\alpha) \in B_r\big(\overline{w}(x,t;\alpha)\big),$$

where

$$B_r\big(\underline{w}(x,t;\alpha)\big) = \big\{\underline{w}^*(x,t;\alpha) \in \underline{X} \,|\| \underline{w}^*(x,t;\alpha) - \underline{w}(x,t;\alpha) < r\big\},$$

$$B_r\big(\overline{w}(x,t;\alpha)\big) = \big\{\overline{w}^*(x,t;\alpha) \in \overline{X} \,|\| \overline{w}^*(x,t;\alpha) - \overline{w}(x,t;\alpha) < r\big\},$$

then we obtain

   (a) $\tilde{W}_n(x,t;\alpha) \in B_r(\tilde{w}(x,t;\alpha))$,

   (b) $\lim_{n\to\infty} \tilde{W}_n(x,t;\alpha) = \tilde{w}(x,t;\alpha)$.

Proof (a) By inductive approach, for $n = 1$, where $\alpha \in [0,1]$, we obtain

$$\big\|\underline{W}_1(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\| = \big\|\mathcal{N}\big(\underline{W}_0(x,t;\alpha)\big) - \mathcal{N}\big(\underline{w}(x,t;\alpha)\big)\big\|$$

$$\leq \xi \big\|\underline{w}_0(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\|,$$

$$\big\|\overline{W}_1(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\| = \big\|\mathcal{N}\big(\overline{W}_0(x,t;\alpha)\big) - \mathcal{N}\big(\overline{w}(x,t;\alpha)\big)\big\|$$

$$\leq \xi \big\|\overline{w}_0(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\|.$$

Assume that

$$\big\|\underline{W}_{n-1}(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\| \leq \xi^{n-1}\big\|\underline{w}_0(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\|,$$

$$\big\|\overline{W}_{n-1}(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\| \leq \xi^{n-1}\big\|\overline{w}_0(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\|,$$

by induction hypothesis, then

$$\big\|\underline{W}_n(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\| = \big\|\mathcal{N}\big(\underline{W}_{n-1}(x,t;\alpha)\big) - \mathcal{N}\big(\underline{w}(x,t;\alpha)\big)\big\|$$

$$\leq \xi \big\|\underline{W}_{n-1}(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\|$$

$$\leq \xi^n \big\|\underline{w}_0(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\|,$$

$$\big\|\overline{W}_n(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\| = \big\|\mathcal{N}\big(\overline{W}_{n-1}(x,t;\alpha)\big) - \mathcal{N}\big(\overline{w}(x,t;\alpha)\big)\big\|$$

$$\leq \xi \big\|\overline{W}_{n-1}(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\|$$

$$\leq \xi^n \big\|\overline{w}_0(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\|.$$

Using (a), we get

$$\big\|\underline{W}_n(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\| \leq \xi^n \big\|\underline{w}_0(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\| \leq \xi^n r < r$$

$$\Rightarrow \quad \underline{W}_n(x,t;\alpha) \in B_r\underline{w}(x,t;\alpha),$$

$$\big\|\overline{W}_n(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\| \leq \xi^n \big\|\overline{w}_0(x,t;\alpha) - \overline{w}(x,t;\alpha)\big\| \leq \xi^n r < r$$

$$\Rightarrow \quad \overline{W}_n(x,t;\alpha) \in B_r\overline{w}(x,t;\alpha).$$

(b). Because of

$$\big\|\underline{W}_n(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\| \leq \xi^n \big\|\underline{w}_0(x,t;\alpha) - \underline{w}(x,t;\alpha)\big\|,$$

$$\left\| \overline{W}_n(x,t;\alpha) - \overline{w}(x,t;\alpha) \right\| \leq \xi^n \left\| \overline{w}_0(x,t;\alpha) - \overline{w}(x,t;\alpha) \right\|,$$

and

$$\lim_{n\to\infty} \xi^n = \tilde{0},$$

$$\lim_{n\to\infty} \left\| \underline{W}_n(x,t;\alpha) - \underline{w}(x,t;\alpha) \right\| = \tilde{0},$$

$$\lim_{n\to\infty} \left\| \overline{W}_n(x,t;\alpha) - \overline{w}(x,t;\alpha) \right\| = \tilde{0},$$

that is,

$$\lim_{n\to\infty} \underline{W}_n(x,t;\alpha) = \underline{w}(x,t;\alpha),$$

$$\lim_{n\to\infty} \overline{W}_n(x,t;\alpha) = \overline{w}(x,t;\alpha). \qquad \qquad \square$$

3.4 Fuzzy reduced differential transform method

We present a fuzzy-valued function $\tilde{u}(X_m,t;\alpha) = [\underline{u}(X_m,t;\alpha), \overline{u}(X_m,t;\alpha)]$ of $(n+1)$-variables and assume that it can be represented as a product of $(n+1)$ single-variable fuzzy-valued function

$$\underline{u}(X_m,t;\alpha) = \left[\underline{\mathcal{F}}_i^*(x_i)\underline{F}_m(t) \right](\alpha), \qquad (80)$$

$$\overline{u}(X_m,t;\alpha) = \left[\overline{\mathcal{F}}_i^*(x_i)\overline{F}_m(t) \right](\alpha), \qquad (81)$$

where

$$\underline{\mathcal{F}}_m^*(x_i) = \underline{F}_1(x_1)\underline{F}_2(x_2)\cdots\underline{F}_n(x_n),$$

$$\overline{\mathcal{F}}_m^*(x_i) = \overline{F}_1(x_1)\overline{F}_2(x_2)\cdots\overline{F}_n(x_n).$$

Based on the one-dimensional fuzzy DTM properties, a fuzzy-valued function $\tilde{u}(X_m,t;\alpha)$ as

$$\underline{u}(X_m,t;\alpha) = \sum_{J_m=0}^{\infty} \underline{\mathcal{F}}_m^*(J_m)X_i^{j_i}(\alpha) \sum_{j_{m+1}=0}^{\infty} \underline{F}_m(j_m)t^{j_{m+1}}(\alpha)$$

$$= \sum_{J_m=0}^{\infty}\sum_{j_{m+1}=0}^{\infty} \underline{U}(J_m,j_{i+1};\alpha)X_m^{J_m}t^{j_{i+1}}, \qquad (82)$$

and

$$\overline{u}(X_m,t;\alpha) = \sum_{J_m=0}^{\infty} \overline{\mathcal{F}}_i^*(J_m)X_i^{j_i}(\alpha) \sum_{j_{m+1}=0}^{\infty} \overline{F}_m(j_m)t^{j_{m+1}}(\alpha)$$

$$= \sum_{J_m=0}^{\infty}\sum_{j_{m+1}=0}^{\infty} \overline{U}(J_m,j_{i+1};\alpha)X_i^{J_m}t^{j_{i+1}}, \qquad (83)$$

where $\sum_{J_m=0}^{\infty} = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}$ and

$$\tilde{U}(J_i,j_{m+1};\alpha) = \left[\underline{U}(J_i,j_{m+1};\alpha), \overline{U}(J_i,j_{m+1};\alpha) \right],$$

for

$$\underline{U}(J_i, j_{m+1}; \alpha) = \left[\underline{F}_1(j_1) \times \underline{F}_2(j_2) \times \cdots \times \underline{F}_n(j_n) \times \underline{E}_m(j_{m+1})\right](\alpha),$$

$$\overline{U}(J_n, j_{m+1}; \alpha) = \left[\overline{F}_1(j_1) \times \overline{F}_2(j_2) \times \cdots \times \overline{F}_n(j_n) \times F_m(j_{m+1})\right](\alpha),$$

recalled spectrum $\tilde{u}(X_m, t; \alpha)$.

Suppose that R_D, R_D^{-1} denote a fuzzy RDTM operator and an inverse operator, respectively.

Definition 3.5 ([11]) If $\tilde{u}(X_m, t; \alpha)$ is continuously differentiable with respect to space with time in the domain of interest, then it is a fuzzy-valued function of the spectrum

$$R_D\left[\underline{u}(X_m, t; \alpha)\right] \approx \underline{U}_j(X_m; \alpha) = \frac{1}{j!}\left[\frac{\partial^j}{\partial t^j}\underline{u}(X_m, t; \alpha)\right]_{t=t_0}, \tag{84}$$

$$R_D\left[\overline{u}(X_m, t; \alpha)\right] \approx \overline{U}_j(X_m; \alpha) = \frac{1}{j!}\left[\frac{\partial^j}{\partial t^j}\overline{u}(X_m, t; \alpha)\right]_{t=t_0}. \tag{85}$$

A fuzzy reduced differential function of $\tilde{u}(X_m, t; \alpha)$. The lowercase $\tilde{u}(X_m, t; \alpha)$ assimilates the original fuzzy-valued function, while the uppercase $\tilde{U}_j(X_m; \alpha)$ stands for the fuzzy RDTM fuzzy-valued function.

Note that the relation shown in (84) and (85) is the Poisson series form of the input term $\tilde{u}(X_m, t; \alpha)$ with respect to the variables X_m, t, to request N, using variable weights $\tilde{U}_j(X_m; \alpha)$.

The fuzzy differential inverse reduced transform of

$$\tilde{U}_j(X_m; \alpha) = \left[\underline{U}_j(X_m; \alpha), \overline{U}_j(X_m; \alpha)\right]$$

is given as follows:

$$R_D^{-1}\left[\underline{U}_j(X_m; \alpha)\right] \approx \underline{u}(X_m, t; \alpha) = \sum_{j=0}^{\infty} \underline{U}_j(X_m; \alpha)(t - t_0)^j, \tag{86}$$

$$R_D^{-1}\left[\overline{U}_j(X_m; \alpha)\right] \approx \overline{u}(X_m, t; \alpha) = \sum_{j=0}^{\infty} \overline{U}_j(X_m; \alpha)(t - t_0)^j. \tag{87}$$

Taking (87) into (84), we obtain

$$\underline{u}(X_m, t; \alpha) = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\frac{\partial^j}{\partial t^j}\underline{u}(X_m, t; \alpha)\right]_{t=t_0}(t - t_0)^j, \tag{88}$$

$$\overline{u}(X_m, t; \alpha) = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\frac{\partial^j}{\partial t^j}\overline{u}(X_m, t; \alpha)\right]_{t=t_0}(t - t_0)^j. \tag{89}$$

Definition 3.6 ([11]) If $\tilde{u}(X_m, t; \alpha) = R_D^{-1}[\tilde{U}_j(X_m; \alpha)]$, $\tilde{v}(X_m, t; \alpha) = R_D^{-1}[\tilde{V}_j(X_m; \alpha)]$ and the convolution $\otimes$ denotes the fuzzy $(n + 1)$-dimensional RDTM version of multiplication, then the basic operations of the fuzzy $(n + 1)$-dimensional RDTM are:

1.

$$R_D\big[\underline{u}(X_m,t;\alpha)\underline{v}(X_m,t;\alpha)\big]$$

$$= \underline{U}_j(X_m;\alpha) \otimes \underline{V}_j(X_m;\alpha)$$

$$= \sum_{r=0}^{j} \underline{U}_j(X_m;\alpha)\underline{V}_{j-r}(X_m;\alpha), \tag{90}$$

$$R_D\big[\overline{u}(X_m,t;\alpha)\overline{v}(X_m;\alpha)\big]$$

$$= \overline{U}_j(X_m,t;\alpha) \otimes \overline{V}_j(X_m;\alpha)$$

$$= \sum_{r=0}^{j} \overline{U}_j(X_m;\alpha)\overline{V}_{j-r}(X_m;\alpha). \tag{91}$$

2.

$$R_D\big[\alpha^*\underline{u}(X_m,t;\alpha) \pm \beta\underline{v}(X_m,t;\alpha)\big]$$

$$= \alpha^*\underline{U}_j(X_m;\alpha) \pm \beta\underline{V}_j(X_m;\alpha), \tag{92}$$

$$R_D\big[\alpha^*\overline{u}(X_m;\alpha) \pm \beta\overline{v}(X_m,t;\alpha)\big]$$

$$= \alpha^*\overline{U}_j(X_m;\alpha) \pm \beta\overline{V}_j(X_m;\alpha). \tag{93}$$

3.

$$R_D\left(\frac{\partial^r}{\partial x_i^r}\underline{u}(X_m,t;\alpha)\right)$$

$$= \frac{\partial^r}{\partial x_i^r}\underline{U}_j(X_m;\alpha), \quad i = 1,2,3,\ldots,n; r = 1,2,3,4,\ldots \tag{94}$$

$$R_D\left(\frac{\partial^r}{\partial x_i^r}\overline{u}(X_m,t;\alpha)\right)$$

$$= \frac{\partial^r}{\partial x_i^r}\overline{U}_j(X_m;\alpha), \quad i = 1,2,3,\ldots,n; r = 1,2,3,4,\ldots \tag{95}$$

4.

$$R_D\left(\frac{\partial^r}{\partial t^r}\underline{u}(X_m,t;\alpha)\right) = (j+1)(j+2)(j+3)\cdots(j+r)\underline{U}_{j+r}(X_m;\alpha)$$

$$= \frac{(j+r)!}{j!}\underline{U}_{j+r}(X_m;\alpha), \quad r = 1,2,3,4,\ldots \tag{96}$$

$$R_D\left(\frac{\partial^r}{\partial t^r}\overline{u}(X_m,t;\alpha)\right) = (j+1)(j+2)(j+3)\cdots(j+r)\overline{U}_{j+r}(X_m;\alpha)$$

$$= \frac{(j+r)!}{j!}\overline{U}_{j+r}(X_m;\alpha), \quad r = 1,2,3,4,\ldots \tag{97}$$

5.

$$RD\left[\frac{\partial^{\mathcal{R}_m^*+r_{m+1}}}{\partial X_m^{\mathcal{R}_m^*}\partial t^{r_{m+1}}}\underline{u}(X_m,t;\alpha)\right]$$

$$= \frac{(j + r_{m+1})!}{j!} \frac{\partial^{\mathcal{R}_m^* + r_{m+1}}}{\partial X_m^{\mathcal{R}_m^*}} \underline{U}_{j+r_{m+1}}(X_m; \alpha), \tag{98}$$

$$RD\left[\frac{\partial^{\mathcal{R}_m^* + r_{m+1}}}{\partial X_m^{R_m} \partial t^{r_{m+1}}} \underline{u}(X_m, t; \alpha) \right]$$

$$= \frac{(j + r_{m+1})!}{j!} \frac{\partial^{\mathcal{R}_m^* + r_{m+1}}}{\partial X_m^{\mathcal{R}_m^*}} \underline{U}_{j+r_{m+1}}(X_m; \alpha). \tag{99}$$

6.

$$R_D\left(X_m^{a_m} t^{am} \right) = X_m^{a_m} \delta(j_m - a_m)$$

$$= \begin{cases} X_m^{a_m}, & j_m = a_m, \\ 0 & \text{otherwise}, \end{cases} \tag{100}$$

where $X_m^{a_m} = x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_n^{a_n}$, for $\mathcal{R}_m^* = r_1, r_2, \ldots, r_n$.

4 Numerical examples

In this section, we applying some numerical techniques for solving the fuzzy $(1 + n)$-dimensional Burgers'equation under gH-differentiability. In the ending, Four examples are provided.

Example 4.1 We consider the following fuzzy $(1 + 1)$-dimensional Burgers' equation:

$$\frac{\partial \tilde{u}}{\partial t} \oplus \tilde{u} \odot \frac{\partial \tilde{u}}{\partial x} = \frac{\partial^2 \tilde{u}}{\partial x^2}, \tag{101}$$

with the initial condition

$$\tilde{u}(x, 0) = \tilde{u}_0(x) = \tilde{k}^n \odot 2x, \tag{102}$$

where $\tilde{u} = \tilde{u}(x, t)$ is a fuzzy-valued function satisfying $\tilde{u}(x, t) \geq \tilde{0}$. Above $\tilde{k}^n \in E^1$, $n = 1, 2, 3, \ldots$, a fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} 5(s - 0.2), & s \in [0.2, 0.4], \\ 5(0.6 - s), & s \in (0.4, 0.6], \\ 0 & s \notin [0.2, 0.6], \end{cases} \tag{103}$$

and $[\underline{\tilde{k}^n}](\alpha) = (0.2 + 0.2\alpha)^n$, $[\overline{\tilde{k}^n}](\alpha) = (0.6 - 0.2\alpha)^n$.

The parametric form of (101) is

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \frac{\partial \underline{u}}{\partial x} = \frac{\partial^2 \underline{u}}{\partial x^2}, \tag{104}$$

$$\frac{\partial \overline{u}}{\partial t} + \overline{u} \frac{\partial \overline{u}}{\partial x} = \frac{\partial^2 \overline{u}}{\partial x^2}, \tag{105}$$

for $\alpha \in [0, 1]$, where $\underline{u}$ stands for $\underline{u}(x, t; \alpha)$, similar to $\overline{u}$.

Case [A]. Fuzzy differential transform method:

Applying the FDTM to equations (104) and (105) yields

$$(j_2 + 1)\underline{U}(j_1, j_2 + 1) + \sum_{a_1=0}^{j_1} \sum_{a_2=0}^{j_2} (j_1 + 1 - a_1)\underline{U}(j_1 + 1 - a_1, a_2)\underline{U}(a_1, j_2 - a_2)(\alpha)$$

$$= (j_1 + 2)(j_1 + 1)\underline{U}(j_1 + 2, j_2)(\alpha) \tag{106}$$

and

$$(j_2 + 1)\overline{U}(j_1, j_2 + 1) + \sum_{a_1=0}^{j_1} \sum_{a_2=0}^{j_2} (j_1 + 1 - a_1)\overline{U}(j_1 + 1 - a_1, a_2)\overline{U}(a_1, j_2 - a_2)(\alpha)$$

$$= (j_1 + 2)(j_1 + 1)\overline{U}(j_1 + 2, j_2)(\alpha). \tag{107}$$

Using initial condition (102), we have

$$\underline{u}(j_1, 0)(\alpha) = \sum_{j_1=0}^{\infty} \underline{U}(j_1, 0)(\alpha)x^{j_1} = (0.2 + 0.2\alpha)^n \cdot 2x, \tag{108}$$

$$\overline{u}(j_1, 0)(\alpha) = \sum_{j_1=0}^{\infty} \overline{U}(j_1, 0)(\alpha)x^{j_1} = (0.6 - 0.2\alpha)^n \cdot 2x, \tag{109}$$

where

$$\underline{U}(j_1, 0)(\alpha) = \begin{cases} 2(0.2 + 0.2\alpha)^n & \text{if } j_1 = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{110}$$

$$\overline{U}(j_1, 0)(\alpha) = \begin{cases} 2(0.6 - 0.2\alpha)^n & \text{if } j_1 = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{111}$$

Taking (110) and (111) into equations (106) and (107), we have a fuzzy-valued function of

$$\tilde{U}(j_1, j_2)(\alpha) = \left[\underline{U}(j_1, j_2)(\alpha), \overline{U}(j_1, j_2)(\alpha)\right]$$

as follows:

$$\underline{U}(j_1, 1)(\alpha) = \begin{cases} -4(0.2 + 0.2\alpha)^n & \text{if } j_1 = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{112}$$

$$\underline{U}(j_1, 2)(\alpha) = \begin{cases} 8(0.2 + 0.2\alpha)^n & \text{if } j_1 = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{113}$$

$$\vdots$$

and

$$\overline{U}(j_1, 1)(\alpha) = \begin{cases} -4(0.6 - 0.2\alpha)^n & \text{if } j_1 = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{114}$$

$$\overline{U}(j_1, 2)(\alpha) = \begin{cases} 8(0.6 - 0.2\alpha)^n & \text{if } j_1 = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{115}$$

$$\vdots$$

From (12) and (13), we obtain

$$\underline{u}(x, t)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \underline{U}(j_1, j_2)(\alpha) x^{j_1} t^{j_2} = \sum_{j_2=0}^{\infty} \underline{U}(1, j_2)(\alpha)$$

$$= (0.2 + 0.2\alpha)^n \big[(2x) + (-4x)t + (8x)t^2 + (-16x)t^3 + \cdots \big],$$

$$\overline{u}(x, t)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \overline{U}(j_1, j_2)(\alpha) x^{j_1} t^{j_2} = \sum_{j_2=0}^{\infty} \overline{U}(1, j_2)(\alpha),$$

$$= (0.6 - 0.2\alpha)^n \big[(2x) + (-4x)t + (8x)t^2 + (-16x)t^3 + \cdots \big].$$

The exact solution is given as follows:

$$\underline{u}(x, t)(\alpha) = (0.2 + 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0,$$

$$\overline{u}(x, t)(\alpha) = (0.6 - 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0.$$

Case [B]. Fuzzy Adomian decomposition method:

Applying (32) into (35), we obtain

$$\left. \begin{aligned} \underline{u}_0(x, t)(\alpha) &= (0.2 + 0.2\alpha)^n 2x, \\ \underline{u}_1(x, t)(\alpha) &= -(0.2 + 0.2\alpha)^n 4xt, \\ \underline{u}_2(x, t)(\alpha) &= (0.2 + 0.2\alpha)^n 8xt^2, \\ \underline{u}_3(x, t)(\alpha) &= -(0.2 + 0.2\alpha)^n 16xt^3, \\ \underline{u}_4(x, t)(\alpha) &= (0.2 + 0.2\alpha)^n 32t^4, \\ \vdots \end{aligned} \right\} \tag{116}$$

and

$$
\left.
\begin{aligned}
&\overline{u}_0(x,t)(\alpha) = (0.6 - 0.2\alpha)^n 2x, \\
&\overline{u}_1(x,t)(\alpha) = -(0.6 - 0.2\alpha)^n 4xt, \\
&\overline{u}_2(x,t)(\alpha) = (0.6 - 0.2\alpha)^n 8xt^2, \\
&\overline{u}_3(x,t)(\alpha) = -(0.6 - 0.2\alpha)^n 16xt^3, \\
&\overline{u}_4(x,t)(\alpha) = (0.6 - 0.2\alpha)^n 32t^4, \\
&\vdots
\end{aligned}
\right\}
\tag{117}
$$

Using the fuzzy decomposition series, we have

$$
\underline{u}(x,t)(\alpha) = \sum_{j=0}^{\infty} \underline{u}_j(x,t)(\alpha) = \left[\underline{u}_0(x,t)(\alpha) + \underline{u}_1(x,t)(\alpha) + \underline{u}_2(x,t)(\alpha) + \cdots \right]
$$

$$
= (0.2 + 0.2\alpha)^n \left[(2x) + (-4x)t + (8x)t^2 + (-16x)t^3 + \cdots \right],
$$

$$
\overline{u}(x,t)(\alpha) = \sum_{j=0}^{\infty} \overline{u}_j(x,t)(\alpha) = \left[\overline{u}_0(x,t)(\alpha) + \overline{u}_1(x,t)(\alpha) + \overline{u}_2(x,t)(\alpha) + \cdots \right]
$$

$$
= (0.6 - 0.2\alpha)^n \left[(2x) + (-4x)t + (8x)t^2 + (-16x)t^3 + \cdots \right].
$$

The exact solution is given in a closed form as follows:

$$
\underline{u}(x,t)(\alpha) = (0.2 + 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0,
$$

$$
\overline{u}(x,t)(\alpha) = (0.6 - 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0.
$$

Case [D]. Fuzzy homotopy perturbation method:
Applying the FHPM to equations (104) and (105), we get

$$
\left[\frac{\partial \underline{u}(\alpha)}{\partial t} + p \left(-\frac{\partial^2 \underline{u}(\alpha)}{\partial x^2} + \underline{u}\frac{\partial \underline{u}(\alpha)}{\partial x} \right) \right] = \tilde{0},
\tag{118}
$$

$$
\left[\frac{\partial \overline{u}(\alpha)}{\partial t} + p \left(-\frac{\partial^2 \overline{u}(\alpha)}{\partial x^2} + \overline{u}\frac{\partial \overline{u}(\alpha)}{\partial x} \right) \right] = \tilde{0}.
\tag{119}
$$

Applying the fuzzy homotopy parameter to extend a solution, we have

$$
\underline{u}(x,t)(\alpha) = \left[\underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + p^3\underline{u}_3 + \cdots \right](\alpha),
\tag{120}
$$

$$
\overline{u}(x,t)(\alpha) = \left[\overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + p^3\overline{u}_3 + \cdots \right](\alpha).
\tag{121}
$$

Using $p = 1$ in (120) and (121), we get

$$
\underline{u}(x,t)(\alpha) = \left[\underline{u}_0 + \underline{u}_1 + \underline{u}_2 + \underline{u}_3 + \cdots \right](\alpha),
\tag{122}
$$

$$
\overline{u}(x,t)(\alpha) = \left[\overline{u}_0 + \overline{u}_1 + \overline{u}_2 + \overline{u}_3 + \cdots \right](\alpha).
\tag{123}
$$

Using (121) into (118), with the terminology equating and powers similar to p, we obtain a series of fuzzy linear equations:

$$p^0 : \frac{\partial \underline{u}_0}{\partial t}(\alpha) = 0, \tag{124}$$

$$p^1 : \frac{\partial \underline{u}_1}{\partial t}(\alpha) = \frac{\partial^2 \underline{u}_0}{\partial x^2}(\alpha) - \underline{u}_0 \frac{\partial \underline{u}_0}{\partial x}(\alpha), \tag{125}$$

$$p^2 : \frac{\partial \underline{u}_2}{\partial t}(\alpha) = \frac{\partial^2 \underline{u}_1}{\partial x^2}(\alpha) - \left[u_0 \frac{\partial \underline{u}_1}{\partial x} + \underline{u}_1 \frac{\partial \underline{u}_0}{\partial x} \right](\alpha), \tag{126}$$

and

$$p^0 : \frac{\partial \overline{u}_0}{\partial t}(\alpha) = 0, \tag{127}$$

$$p^1 : \frac{\partial \overline{u}_1}{\partial t}(\alpha) = \frac{\partial^2 \overline{u}_0}{\partial x^2}(\alpha) - u_0 \frac{\partial \overline{u}_0}{\partial x}(\alpha), \tag{128}$$

$$p^2 : \frac{\partial \overline{u}_2}{\partial t}(\alpha) = \frac{\partial^2 \overline{u}_1}{\partial x^2}(\alpha) - \left[u_0 \frac{\partial \overline{u}_1}{\partial x} + \overline{u}_1 \frac{\partial \overline{u}_0}{\partial x} \right](\alpha). \tag{129}$$

According to initial condition (102) and the solution of (124) and (127), we obtain

$$\underline{u}_0(x, \alpha) = \underline{u}(x, 0)(\alpha) = (0.2 + 0.2\alpha)^n \cdot 2x, \tag{130}$$

$$\overline{u}_0(x, \alpha) = \overline{u}(x, 0)(\alpha) = (0.6 - 0.2\alpha)^n \cdot 2x. \tag{131}$$

Substituting $\underline{u}_0, \overline{u}_0$ from (130) and (131) in equations (125) and (128), we obtain the solution to (125) and (128) as follows:

$$\underline{u}_1(x, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \underline{u}_0}{\partial x^2}(\alpha) - \underline{u}_0 \frac{\partial \underline{u}_0}{\partial x}(\alpha) \right) dt = (0.2 + 0.2\alpha)^n (-2x)t, \tag{132}$$

$$\overline{u}_1(x, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \overline{u}_0}{\partial x^2}(\alpha) - \overline{u}_0 \frac{\partial \overline{u}_0}{\partial x}(\alpha) \right) dt = (0.6 - 0.2\alpha)^n (-2x)t. \tag{133}$$

From $\underline{u}_0, \overline{u}_0$ and $\underline{u}_1, \overline{u}_1$ in (126) and (129), we obtain the solution to (126) and (129) as follows:

$$\underline{u}_2(x, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \underline{u}_1}{\partial x^2}(\alpha) - \left[\overline{u}_0 \frac{\partial^2 \underline{u}_1}{\partial x}(\alpha) + \overline{u}_1 \frac{\partial^2 \underline{u}_0}{\partial x}(\alpha) \right] \right) dt$$

$$= (0.2 + 0.2\alpha)^n (4x)t^2, \tag{134}$$

$$\overline{u}_2(x, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \overline{u}_1}{\partial x^2}(\alpha) - \left[\overline{u}_0 \frac{\partial^2 \overline{u}_1}{\partial x}(\alpha) + \overline{u}_1 \frac{\partial^2 \overline{u}_0}{\partial x}(\alpha) \right] \right) dt$$

$$= (0.6 - 0.2\alpha)^n (4x)t^2. \tag{135}$$

Applying (122) and (123), the approximate solution of (104) and (105), we get

$$\underline{u}(x, t)(\alpha) = (0.2 + 0.2\alpha)^n \big[(2x) + (-2x)t + (2x)t^2 + (-8x)t^3 + \cdots \big],$$

$$\overline{u}(x, t)(\alpha) = (0.6 - 0.2\alpha)^n \big[(2x) + (-2x)t + (2x)t^2 + (-8x)t^3 + \cdots \big].$$

The exact solution can be obtained as follows:

$$\underline{u}(x,t)(\alpha) = (0.2 + 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0, 0 \leq \alpha \leq 1,$$

$$\overline{u}(x,t)(\alpha) = (0.6 - 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0, 0 \leq \alpha \leq 1.$$

Case [D]. Fuzzy reduced differential transform method:

Applying the fuzzy RDTM to equations (104) and (105), we get

$$(j + 1)\underline{U}_{j+1}(\alpha) = -\sum_{r=0}^{j} \underline{U}_r \frac{\partial}{\partial x} \underline{U}_{j-r}(\alpha) + \frac{\partial^2}{\partial x^2} \underline{U}_j(\alpha), \tag{136}$$

$$(j + 1)\overline{U}_{j+1}(\alpha) = -\sum_{r=0}^{j} \overline{U}_r \frac{\partial}{\partial x} \overline{U}_{j-r}(\alpha) + \frac{\partial^2}{\partial x^2} \overline{U}_j(\alpha). \tag{137}$$

Using initial condition (102), we obtain

$$\underline{U}_0(\alpha) = (0.2 + 0.2\alpha)^n \cdot 2x, \tag{138}$$

$$\overline{U}_0(\alpha) = (0.6 - 0.2\alpha)^n \cdot 2x. \tag{139}$$

Taking (140) into (137), we obtain $\tilde{U}_j(\alpha) = [\underline{U}_j(\alpha), \overline{U}_j(\alpha)]$ fuzzy-valued function respectively:

$$\underline{U}_1(\alpha) = (0.2 + 0.2\alpha)^n (-4x), \qquad \underline{U}_2(\alpha) = (0.2 + 0.2\alpha)^n (8x),$$
$$\underline{U}_3(\alpha) = (0.2 + 0.2\alpha)^n (-16x), \ldots \tag{140}$$

$$\overline{U}_1(\alpha) = (0.6 - 0.2\alpha)^n (-4x), \qquad \overline{U}_2(\alpha) = (0.6 - 0.2\alpha)^n (8x),$$
$$\overline{U}_3(\alpha) = (0.6 - 0.2\alpha)^n (-16x), \ldots \tag{141}$$

According to the fuzzy differential inverse transformation, we have

$$\underline{u}(x,y,t)(\alpha) = \sum_{j=0}^{\infty} \underline{U}_j t^j = (0.2 + 0.2\alpha)^n \left[(2x) + (-4x)t + (8x)t^2 + (-16)t^3 + \cdots \right] \tag{142}$$

$$\overline{u}(x,y,t)(\alpha) = \sum_{j=0}^{\infty} \overline{U}_j t^j = (0.6 - 0.2\alpha)^n \left[(2x) + (-4x)t + (8x)t^2 + (-16)t^3 + \cdots \right]. \tag{143}$$

The exact solution is obtained as follows:

$$\underline{u}(x,t,\alpha) = (0.2 + 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0, 0 \leq \alpha \leq 1,$$

$$\overline{u}(x,t,\alpha) = (0.6 - 0.2\alpha)^n \left[\frac{2x}{1 + 2t} \right], \quad \text{provided } x \neq 0, 0 \leq \alpha \leq 1.$$

Example 4.2 We consider the following fuzzy $(1 + 2)$-dimensional Burgers' equation:

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} \oplus \frac{\partial^2 \tilde{u}}{\partial y^2} \oplus \tilde{u} \odot \frac{\partial \tilde{u}}{\partial x}, \tag{144}$$

subject to the initial condition

$$\tilde{u}(x, y, 0) = \tilde{u}_0(x, y) = \tilde{k}^n \odot (x + y), \tag{145}$$

where $\tilde{u} = \tilde{u}(x, y, t)$ is a fuzzy-valued function satisfying $\tilde{u}(x, y, t) \geq \tilde{0}$. Above $\tilde{k}^n \in E^1$, $n = 1, 2, 3, \dots$, a fuzzy number is

$$\tilde{k}(s) = \begin{cases} 10(s - 0.1), & s \in [0.1, 0.2], \\ 10(0.3 - s), & s \in (0.2, 0.3], \\ 0, & s \notin [0.1, 0.3], \end{cases} \tag{146}$$

and $[\underline{\tilde{k}^n}](\alpha) = (0.1 + 0.1\alpha)^n$, $[\overline{\tilde{k}^n}](\alpha) = (0.3 - 0.1\alpha)^n$.

The parametric form of (144) is

$$\frac{\partial \underline{u}}{\partial t} = \frac{\partial^2 \underline{u}}{\partial x^2} + \frac{\partial^2 \underline{u}}{\partial y^2} + \underline{u}\frac{\partial \underline{u}}{\partial x}, \tag{147}$$

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \overline{u}\frac{\partial \overline{u}}{\partial x}, \tag{148}$$

for $\alpha \in [0, 1]$, where $\underline{u}$ stands for $\underline{u}(x, y, t)(\alpha)$, similar to $\overline{u}$.

Case [A]. Fuzzy differential transform method

Applying the FDTM to (147) and (148), we obtain

$$(j_3 + 1)\underline{U}(j_1, j_2, j_3 + 1)(\alpha)$$
$$= (j_1 + 2)(j_1 + 1)\underline{U}(j_1 + 2, j_2, j_3)(\alpha) + (j_2 + 2)(j_2 + 1)$$
$$\times \underline{U}(j_1, j_2 + 2, j_3)(\alpha) + \sum_{a_1=0}^{j_1}\sum_{a_2=0}^{j_2}\sum_{a_3=0}^{j_3}(j_1 + 1 - a_1)\underline{U}(j_1 + 1 - a_1, a_2, a_3) \tag{149}$$
$$\times \underline{U}(a_1, j_2 - a_2, j_3 - a_3)(\alpha)$$

and

$$(j_3 + 1)\overline{U}(j_1, j_2, j_3 + 1)(\alpha)$$
$$= (j_1 + 2)(j_1 + 1)\overline{U}(j_1 + 2, j_2, j_3)(\alpha) + (j_2 + 2)(j_2 + 1)$$
$$\times \overline{U}(j_1, j_2 + 2, j_3)(\alpha) + \sum_{a_1=0}^{j_1}\sum_{a_2=0}^{j_2}\sum_{a_3=0}^{j_3}(j_1 + 1 - a_1)\overline{U}(j_1 + 1 - a_1, a_2, a_3) \tag{150}$$
$$\times \overline{U}(a_1, j_2 - a_2, j_3 - a_3)(\alpha),$$

Using initial condition (145), we get

$$\underline{u}(x, y, 0)(\alpha) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\underline{U}(j_1, j_2, 0)(\alpha)x^{j_1}y^{j_2} = (0.1 + 0.1\alpha)^n(x + y), \tag{151}$$

$$\overline{u}(x, y, 0)(\alpha) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\overline{U}(j_1, j_2, 0)(\alpha)x^{j_1}y^{j_2} = (0.3 - 0.1\alpha)^n(x + y), \tag{152}$$

where

$$\underline{U}(j_1, j_2, 0)(\alpha) = \begin{cases} (0.1 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \\ 0 & \text{otherwise}, \end{cases} \tag{153}$$

$$\overline{U}(j_1, j_2, 0)(\alpha) = \begin{cases} (0.3 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \\ 0 & \text{otherwise}. \end{cases} \tag{154}$$

From (154) into (149), we get a fuzzy-valued function

$$\tilde{U}(j_1, j_2, j_3)(\alpha) = \big[\underline{U}(j_1, j_2, j_3)(\alpha), \overline{U}(j_1, j_2, j_3)(\alpha) \big],$$

as follows:

$$\underline{U}(j_1, j_2, 1)(\alpha) = \begin{cases} (0.1 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \\ 0 & \text{otherwise}, \end{cases} \tag{155}$$

$$\underline{U}(j_1, j_2, 2)(\alpha) = \begin{cases} (0.1 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \\ 0 & \text{otherwise} \end{cases} \tag{156}$$

and

$$\overline{U}(j_1, j_2, 1)(\alpha) = \begin{cases} (0.3 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \\ 0 & \text{otherwise}, \end{cases} \tag{157}$$

$$\overline{U}(j_1, j_2, 2)(\alpha) = \begin{cases} (0.3 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \\ 0 & \text{otherwise}. \end{cases} \tag{158}$$

Taking (12) and (13), we get

$$\underline{u}(x, y, t)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \underline{U}(j_1, j_2, j_3)(\alpha) x^{j_1} y^{j_2} t^{j_3}$$

$$= (0.1 + 0.1\alpha)^n \big[(x + y) + (x + y)t + (x + y)t^2 + \cdots \big],$$

$$\overline{u}(x, y, t)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \overline{U}(j_1, j_2, j_3)(\alpha) x^{j_1} y^{j_2} t^{j_3}$$

$$= (0.3 - 0.1\alpha)^n \big[(x + y) + (x + y)t + (x + y)t^2 + \cdots \big].$$

The exact solution can be obtained as follows:

$$\underline{u}(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n \left[\frac{x + y}{1 - t} \right] \quad \text{provided } 0 \leq t < 1, x \text{ or } y \neq 0, 0 \leq \alpha \leq 1,$$

$$\overline{u}(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n \left[\frac{x + y}{1 - t} \right], \quad \text{provided } 0 \leq t < 1, x \text{ or } y \neq 0, 0 \leq \alpha \leq 1.$$

Case [B]. Fuzzy Adomian decomposition method

Applying equations ([32]) into ([35]), we obtain

$$
\left.
\begin{aligned}
&\underline{u}_0(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n(x + y), \\
&\underline{u}_1(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n(x + y)t, \\
&\underline{u}_2(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n(x + y)t^2, \\
&\underline{u}_3(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n(x + y)t^3, \\
&\vdots
\end{aligned}
\right\}
\tag{159}
$$

and

$$
\left.
\begin{aligned}
&\overline{u}_0(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n(x + y), \\
&\overline{u}_1(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n(x + y)t, \\
&\overline{u}_2(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n(x + y)t^2, \\
&\overline{u}_3(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n(x + y)t^3, \\
&\vdots
\end{aligned}
\right\}
\tag{160}
$$

Using the fuzzy decomposition series, we obtain

$$
\begin{aligned}
\underline{u}(x, y, t)(\alpha) &= \sum_{j=0}^{\infty} \underline{u}_j(x, y, t)(\alpha) = \left[\underline{u}_0(x, y, t)(\alpha) + \underline{u}_1(x, y, t)(\alpha) + \underline{u}_2(x, y, t)(\alpha) + \cdots\right], \\
&= (0.1 + 0.1\alpha)^n\left[(x + y) + (x + y)t + (x + y)t^2 + (x + y)t^3 + \cdots\right], \\
\overline{u}(x, y, t)(\alpha) &= \sum_{j=0}^{\infty} \overline{u}_j(x, y, t)(\alpha) = \left[\overline{u}_0(x, y, t)(\alpha) + \overline{u}_1(x, y, t)(\alpha) + \overline{u}_2(x, y, t)(\alpha) + \cdots\right], \\
&= (0.3 - 0.1\alpha)^n\left[(x + y) + (x + y)t + (x + y)t^2 + (x + y)t^3 + \cdots\right].
\end{aligned}
$$

The exact solution is obtained as follows:

$$
\underline{u}(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n\left[\frac{x + y}{1 - t}\right], \quad \text{provided } 0 \leq t < 1, x \text{ or } y \neq 0, 0 \leq \alpha \leq 1,
$$

$$
\overline{u}(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n\left[\frac{x + y}{1 - t}\right], \quad \text{provided } 0 \leq t < 1, x \text{ or } y \neq 0, 0 \leq \alpha \leq 1.
$$

Case [C]. Fuzzy homotopy perturbation method

Applying the fuzzy HPM, we structure the simple homotopy as follows:

$$
\left[\frac{\partial \underline{u}(\alpha)}{\partial t} + p\left(-\frac{\partial^2 \underline{u}(\alpha)}{\partial x^2} - \frac{\partial^2 \underline{u}(\alpha)}{\partial y^2} - \underline{u}\frac{\partial \underline{u}(\alpha)}{\partial x}\right)\right] = \tilde{0},
\tag{161}
$$

$$
\left[\frac{\partial \overline{u}(\alpha)}{\partial t} + p\left(-\frac{\partial^2 \overline{u}(\alpha)}{\partial x^2} - \frac{\partial^2 \overline{u}(\alpha)}{\partial y^2} - \overline{u}\frac{\partial \overline{u}(\alpha)}{\partial x}\right)\right] = \tilde{0}.
\tag{162}
$$

Let the solution have the following form:

$$\underline{u}(x,y,t)(\alpha) = \left[\underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + p^3\underline{u}_3 + \cdots\right](\alpha), \tag{163}$$

$$\overline{u}(x,y,t)(\alpha) = \left[\overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + p^3\overline{u}_3 + \cdots\right](\alpha). \tag{164}$$

According to $p = 1$ in equations (163) and (164), we get the approximate solution as follows:

$$\underline{u}(x,y,t)(\alpha) = [\underline{u}_0 + \underline{u}_1 + \underline{u}_2 + \underline{u}_3 + \cdots](\alpha), \tag{165}$$

$$\overline{u}(x,y,t)(\alpha) = [\overline{u}_0 + \overline{u}_1 + \overline{u}_2 + \overline{u}_3 + \cdots](\alpha). \tag{166}$$

Taking (163) and (164) into (161) and (162), and the terminology equating with powers similar to p, we get

$$p^0 : \frac{\partial \underline{u}_0}{\partial t}(\alpha) = 0, \tag{167}$$

$$p^1 : \frac{\partial \underline{u}_0}{\partial t}(\alpha) = \frac{\partial^2 \underline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial y^2}(\alpha) + \underline{u}_0 \frac{\partial \underline{u}_0}{\partial x_1}(\alpha), \tag{168}$$

$$p^2 : \frac{\partial \underline{u}_2}{\partial t}(\alpha) = \frac{\partial^2 \underline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial y^2}(\alpha) + \left[u_0 \frac{\partial \underline{u}_1}{\partial x} + \underline{u}_1 \frac{\partial \underline{u}_0}{\partial x}\right](\alpha), \tag{169}$$

and

$$p^0 : \frac{\partial \overline{u}_0}{\partial t}(\alpha) = 0, \tag{170}$$

$$p^1 : \frac{\partial \overline{u}_0}{\partial t}(\alpha) = \frac{\partial^2 \overline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial y^2}(\alpha) + u_0 \frac{\partial \overline{u}_0}{\partial x_1}(\alpha), \tag{171}$$

$$p^2 : \frac{\partial \overline{u}_2}{\partial t}(\alpha) = \frac{\partial^2 \overline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial y^2}(\alpha) + \left[u_0 \frac{\partial \overline{u}_1}{\partial x} + \overline{u}_1 \frac{\partial \overline{u}_0}{\partial x}\right](\alpha). \tag{172}$$

Taking (145) and a solution of (167) and (170), we get

$$\underline{u}_0(x,y,t)(\alpha) = \underline{u}(x,y,0)(\alpha) = (0.1 + 0.1\alpha)^n(x + y), \tag{173}$$

$$\overline{u}_0(x,y,t)(\alpha) = \overline{u}(x,y,0)(\alpha) = (0.3 - 0.1\alpha)^n(x + y). \tag{174}$$

From the solution of (168) and (171), we have

$$\underline{u}_1(x,y,t)(\alpha) = \int_0^t \left(\frac{\partial^2 \underline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial y^2}(\alpha) + \underline{u}_0 \frac{\partial \underline{u}_0}{\partial x}(\alpha)\right) dt = (0.1 + 0.1\alpha)^n(x + y)t, \tag{175}$$

$$\overline{u}_1(x,y,t)(\alpha) = \int_0^t \left(\frac{\partial^2 \overline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial y^2}(\alpha) + \overline{u}_0 \frac{\partial \overline{u}_0}{\partial x}(\alpha)\right) dt = (0.3 - 0.1\alpha)^n(x + y)t. \tag{176}$$

We obtain the solution of (169) and (172) as follows:

$$\underline{u}_2(x,y,t)(\alpha) = \int_0^t \left(\frac{\partial^2 \underline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial y^2}(\alpha) + \left[\overline{u}_0 \frac{\partial \underline{u}_1}{\partial x}(\alpha) + \underline{u}_1 \frac{\partial \underline{u}_0}{\partial x}(\alpha)\right]\right) dt$$

$$= (0.1 + 0.1\alpha)^n(x + y)t^2, \tag{177}$$

$$\vdots$$

and

$$\overline{u}_2(x, y, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \overline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial y^2}(\alpha) + \left[\overline{u}_0 \frac{\partial \overline{u}_1}{\partial x}(\alpha) + \overline{u}_1 \frac{\partial \overline{u}_0}{\partial x}(\alpha) \right] \right) dt$$

$$= (0.3 - 0.1\alpha)^n(x + y)t^2, \tag{178}$$

$$\vdots$$

Using (165) and (166), the approximate solution of (147) and (148), we obtain

$$\underline{u}(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n \big[(x + y) + (x + y)t + (x + y)t^2 + \cdots \big], \quad 0 \le \alpha \le 1,$$

$$\overline{u}(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n \big[(x + y) + (x + y)t + (x + y)t^2 + \cdots \big], \quad 0 \le \alpha \le 1.$$

The exact solution is obtained as follows:

$$\underline{u}(x, y, t)(\alpha) = (0.1 + 0.1\alpha)^n \left[\frac{x + y}{1 - t} \right], \quad \text{provided } 0 \le t < 1, x \text{ or } y \ne 0, 0 \le \alpha \le 1,$$

$$\overline{u}(x, y, t)(\alpha) = (0.3 - 0.1\alpha)^n \left[\frac{x + y}{1 - t} \right], \quad \text{provided } 0 \le t < 1, x \text{ or } y \ne 0, 0 \le \alpha \le 1.$$

Case [D]. Fuzzy reduced differential transform method
Applying the fuzzy RDTM to equations (147) and (148) yields

$$(j + 1)\underline{U}_{j+1}(\alpha) = \frac{\partial^2}{\partial x^2}\underline{U}_j(\alpha) + \frac{\partial^2}{\partial y^2}\underline{U}_j(\alpha) + \sum_{r=0}^{j} \underline{U}_r \frac{\partial}{\partial x}\underline{U}_{j-r}(\alpha) \tag{179}$$

$$(j + 1)\overline{U}_{j+1}(\alpha) = \frac{\partial^2}{\partial x^2}\overline{U}_j(\alpha) + \frac{\partial^2}{\partial y^2}\overline{U}_j(\alpha) + \sum_{r=0}^{j} \overline{U}_r \frac{\partial}{\partial x}\overline{U}_{j-r}(\alpha). \tag{180}$$

Using initial condition (145), we get

$$\underline{U}_0(\alpha) = (0.1 + 0.1\alpha)^n(x + y), \tag{181}$$

$$\overline{U}_0(\alpha) = (0.3 - 0.1\alpha)^n(x + y). \tag{182}$$

Taking (183) into (180), we get $\tilde{U}_0(\alpha) = [\underline{U}_0(\alpha), \overline{U}_0(\alpha)]$ fuzzy-valued function, respectively:

$$\underline{U}_1(\alpha) = (0.1 + 0.1\alpha)^n(x + y),$$

$$\underline{U}_2(\alpha) = (0.1 + 0.1\alpha)^n(x + y), \dots, \underline{U}_n(\alpha) = (0.1 + 0.1\alpha)^n(x + y), \tag{183}$$

$$\overline{U}_1(\alpha) = (0.3 - 0.1\alpha)^n(x + y),$$

$$\overline{U}_2(\alpha) = (0.3 - 0.1\alpha)^n(x + y), \dots, \overline{U}_n(\alpha) = (0.3 - 0.1\alpha)^n(x + y). \tag{184}$$

Applying the differential inverse transformation, we have

$$\underline{u}(x,y,t;\alpha) = \sum_{j=0}^{\infty} \underline{U}_{j=0}t^j = (0.1 + 0.1\alpha)^n\big[(x+y) + (x+y)t + \cdots + (x+y)t^n \cdots\big], \qquad (185)$$

$$\overline{u}(x,y,t;\alpha) = \sum_{j=0}^{\infty} \overline{U}_{j=0}t^j = (0.3 - 0.1\alpha)^n\big[(x+y) + (x+y)t + \cdots + (x+y)t^n \cdots\big]. \qquad (186)$$

The exact solution can be obtained as follows:

$$\underline{u}(x,y,t;\alpha) = (0.1 + 0.1\alpha)^n\left[\frac{x+y}{1-t}\right], \quad \text{provided } 0 \le t < 1, x \text{ or } y \ne 0, 0 \le \alpha \le 1.$$

$$\overline{u}(x,y,t;\alpha) = (0.3 - 0.1\alpha)^n\left[\frac{x+y}{1-t}\right], \quad \text{provided } 0 \le t < 1, x \text{ or } y \ne 0, 0 \le \alpha \le 1.$$

Example 4.3 We consider the following fuzzy $(1+3)$-dimensional Burgers' equation:

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} \oplus \frac{\partial^2 \tilde{u}}{\partial y^2} \oplus \frac{\partial^2 \tilde{u}}{\partial z^2} \oplus \tilde{u} \odot \frac{\partial \tilde{u}}{\partial x}, \qquad (187)$$

subject to the initial condition

$$\tilde{u}(x,y,z,0) = \tilde{u}_0(x,y,z) = \tilde{k}^n \odot (x + y + z), \qquad (188)$$

where $\tilde{u} = \tilde{u}(x,y,z,t)$ is a fuzzy-valued function satisfying $\tilde{u}(x,y,z,t) \ge \tilde{0}$. Above $\tilde{k}^n \in E^1$, $n = 1,2,3,\ldots$, a fuzzy number is defined by

$$\tilde{k}_1(s) = \begin{cases} 10(s - 0.4), & s \in [0.4, 0.5], \\ 10(0.6 - s), & s \in (0.5, 0.6], \\ 0, & s \notin [0.4, 0.6], \end{cases} \qquad (189)$$

and $[\underline{\tilde{k}^n}](\alpha) = (0.4 + 0.1\alpha)^n, [\overline{\tilde{k}^n}](\alpha) = (0.6 - 0.1\alpha)^n$.

The parametric form of (187) is

$$\frac{\partial \underline{u}}{\partial t} = \frac{\partial^2 \underline{u}}{\partial x^2} + \frac{\partial^2 \underline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2} + \underline{u}\frac{\partial \underline{u}}{\partial x}, \qquad (190)$$

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2} + \overline{u}\frac{\partial \overline{u}}{\partial x}, \qquad (191)$$

for $\alpha \in [0,1]$, where $\underline{u}$ stands for $\underline{u}(x,y,z,t)(\alpha)$, similar to $\overline{u}$.

Case (A). *Fuzzy differential transform method*:

Applying the FDTM to equations (190) and (191), we get

$$
\begin{aligned}
(j_4 + 1)\underline{U}&(j_1, j_2, j_3, j_4 + 1)(\alpha) \\
= \Bigg[&(j_1 + 2)(j_1 + 1)\underline{U}(j_1 + 2, j_2, j_3, j_4)(\alpha) + (j_2 + 2)(j_2 + 1) \\
&\times \underline{U}(j_1, j_2 + 2, j_3, j_4)(\alpha) + (j_3 + 2)(j_3 + 1)\underline{U}(j_1, j_2, j_3 + 2, j_4)(\alpha) \\
&+ \sum_{a_1=0}^{j_1} \sum_{a_2=0,}^{j_2} \sum_{a_3=0}^{j_3} \sum_{a_4=0}^{j_4} (j_1 + 1 - a_1) \\
&\times \underline{U}(j_1 + 1 - a_1, a_2, a_3, a_4)(\alpha)\underline{U}(a_1, j_2 - a_2, j_3 - a_3, j_4 - a_4)(\alpha) \Bigg],
\end{aligned}
\tag{192}
$$

and

$$
\begin{aligned}
(j_4 + 1)\overline{U}&(j_1, j_2, j_3, j_4 + 1)(\alpha) \\
= \Bigg[&(j_1 + 2)(j_1 + 1)\overline{U}(j_1 + 2, j_2, j_3, j_4)(\alpha) + (j_2 + 2)(j_2 + 1) \\
&\times \overline{U}(j_1, j_2 + 2, j_3, j_4)(\alpha) + (j_3 + 2)(j_3 + 1)\overline{U}(j_1, j_2, j_3 + 2, j_4)(\alpha) \\
&+ \sum_{a_1=0}^{j_1} \sum_{a_2=0,}^{j_2} \sum_{a_3=0}^{j_3} \sum_{a_4=0}^{j_4} (j_1 + 1 - a_1) \\
&\times \overline{U}(j_1 + 1 - a_1, a_2, a_3, a_4)(\alpha)\overline{U}(a_1, j_2 - a_2, j_3 - a_3, j_4 - a_4)(\alpha) \Bigg].
\end{aligned}
\tag{193}
$$

Using initial condition (188), we obtain

$$
\underline{u}(x, y, z, t, 0)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{0j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \underline{U}(j_1, j_2, j_3, 0)(\alpha)x^{j_1} y^{j_2} z^{j_3} = (0.4 + 0.1\alpha)^n[x + y + z], \tag{194}
$$

$$
\overline{u}(x, y, z, t, 0)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{0j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \overline{U}(j_1, j_2, j_3, 0)(\alpha)x^{j_1} y^{j_2} z^{j_3} = (0.6 - 0.1\alpha)^n[x + y + z], \tag{195}
$$

where

$$
\underline{U}(j_1, j_2, j_3, 0)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise,} \end{cases} \tag{196}
$$

$$
\overline{U}(j_1, j_2, j_3, 0)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases} \tag{197}
$$

Taking (196) and (197) into equations (192) and (193), we get

$$
\tilde{U}(j_1, j_2, j_3, j_4)(\alpha) = \big[\underline{U}(j_1, j_2, j_3, j_4)(\alpha), \overline{U}(j_1, j_2, j_3, j_4)(\alpha) \big],
$$

as follows:

$$\underline{U}(j_1, j_2, j_3, 1)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$\underline{U}(j_1, j_2, j_3, 2)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$\underline{U}(j_1, j_2, j_3, 3)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{U}(j_1, j_2, j_3, 1)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$\overline{U}(j_1, j_2, j_3, 2)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$\overline{U}(j_1, j_2, j_3, 3)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, 3, \\ 0 & \text{otherwise}. \end{cases}$$

Using (12) and (13), we obtain

$$\underline{u}(x, y, z, t)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \underline{U}(j_1, j_2, j_3, j_4)(\alpha) x^{j_1} y^{j_2} z^{j_3} t^{j_4}$$

$$= (0.4 + 0.1\alpha)^n \big[(x + y + z) + (x + y + z)t + (x + y + z)t^2 + \cdots \big],$$

$$\overline{u}(x, y, z, t)(\alpha) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \overline{U}(j_1, j_2, j_3, j_4)(\alpha) x^{j_1} y^{j_2} z^{j_3} t^{j_4}$$

$$= (0.6 - 0.1\alpha)^n \big[(x + y + z) + (x + y + z)t + (x + y + z)t^2 + \cdots \big].$$

The exact solution can be obtained as follows:

$$\underline{u}(x, y, z, t)(\alpha) = (0.4 + 0.1\alpha)^n \left[\frac{(x + y + z)}{1 - t} \right],$$

provided $0 \leq t < 1, x$ or y or $z \neq 0, 0 \leq \alpha \leq 1,$

$$\overline{u}(x, y, z, t)(\alpha) = (0.6 - 0.1\alpha)^n \left[\frac{(x + y + z)}{1 - t} \right],$$

provided $0 \leq t < 1, x$ or y or $z \neq 0, 0 \leq \alpha \leq 1.$

Case (B). Fuzzy Adomian decomposition method:

Applying equations (32) into (35), we obtain

$$
\left.\begin{aligned}
&\underline{u}_0(x,y,z,t)(\alpha) = (0.4 + 0.1\alpha)^n (x + y + z), \\
&\underline{u}_1(x,y,z,t)(\alpha) = (0.4 + 0.1\alpha)^n (x + y + z)t, \\
&\underline{u}_2(x,y,z,t)(\alpha) = (0.4 + 0.1\alpha)^n (x + y + z)t^2, \\
&\underline{u}_3(x,y,z,t)(\alpha) = (0.4 + 0.1\alpha)^n (x + y + z)t^3, \\
&\underline{u}_4(x,y,z,t)(\alpha) = (0.4 + 0.1\alpha)^n (x + y + z)t^4, \\
&\;\vdots
\end{aligned}\right\}
\tag{198}
$$

and

$$
\left.\begin{aligned}
&\overline{u}_0(x,y,z,t)(\alpha) = (0.6 - 0.1\alpha)^n (x + y + z), \\
&\overline{u}_1(x,y,z,t)(\alpha) = (0.6 - 0.1\alpha)^n (x + y + z)t, \\
&\overline{u}_2(x,y,z,t)(\alpha) = (0.6 - 0.1\alpha)^n (x + y + z)t^2, \\
&\overline{u}_3(x,y,z,t)(\alpha) = (0.6 - 0.1\alpha)^n (x + y + z)t^3, \\
&\overline{u}_4(x,y,z,t)(\alpha) = (0.6 - 0.1\alpha)^n (x + y + z)t^4, \\
&\;\vdots
\end{aligned}\right\}
\tag{199}
$$

Taking the fuzzy decomposition series, we have

$$
\begin{aligned}
\underline{u}(x,y,z,t)(\alpha) &= \sum_{j=0}^{\infty} \underline{u}_j(x,y,z,t)(\alpha) \\
&= \left[\underline{u}_0(x,y,z,t)(\alpha) + \underline{u}_1(x,y,z,t)(\alpha) + \underline{u}_2(x,y,z,t)(\alpha) + \cdots \right] \\
&= (0.4 + 0.1\alpha)^n \big[(x + y + z) + (x + y + z)t + (x + y + z)t^2 \\
&\quad + (x + y + z)t^3 + \cdots \big], \\
\overline{u}(x,y,z,t)(\alpha) &= \sum_{j=0}^{\infty} \overline{u}_k(x,y,z,t)(\alpha) \\
&= \left[\overline{u}_0(x,y,z,t)(\alpha) + \overline{u}_1(x,y,z,t)(\alpha) + \overline{u}_2(x,y,z,t)(\alpha) + \cdots \right] \\
&= (0.6 - 0.1\alpha)^n \big[(x + y + z) + (x + y + z)t + (x + y + z)t^2 \\
&\quad + (x + y + z)t^3 + \cdots \big].
\end{aligned}
$$

The exact solution can be obtained as follows:

$$
\underline{u}(x,y,z,t)(\alpha) = (0.4 + 0.1\alpha)^n \left[\frac{x + y + z}{1 - t}\right],
$$

provided $0 \le t < 1, x$ or y or $z \ne 0, 0 \le \alpha \le 1,$

$$
\overline{u}(x,y,z,t)(\alpha) = (0.6 - 0.1\alpha)^n \left[\frac{x + y + z}{1 - t}\right],
$$

provided $0 \le t < 1, x$ or y or $z \ne 0, 0 \le \alpha \le 1$.

Case (C). Fuzzy homotopy perturbation method:

Applying a fuzzy HPM, construct a simple homotopy as follows:

$$\left[\frac{\partial \underline{u}}{\partial t} + p\left(-\frac{\partial^2 \underline{u}}{\partial x^2} - \frac{\partial^2 \underline{u}}{\partial y^2} - \frac{\partial^2 \underline{u}}{\partial z^2} - \underline{u}\frac{\partial \underline{u}}{\partial x} \right) \right](\alpha) = \tilde{0}, \tag{200}$$

$$\left[\frac{\partial \overline{u}}{\partial t} + p\left(-\frac{\partial^2 \overline{u}}{\partial x^2} - \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial^2 \overline{u}}{\partial z^2} - \overline{u}\frac{\partial \overline{u}}{\partial x} \right) \right](\alpha) = \tilde{0}. \tag{201}$$

Assume that the solution contains the following:

$$\underline{u}(x,y,z,t)(\alpha) = \left[\underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + p^3\underline{u}_3 + p^4\underline{u}_4 + \cdots \right](\alpha), \tag{202}$$

$$\overline{u}(x,y,z,t)(\alpha) = \left[\overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + p^3\overline{u}_3 + p^4\overline{u}_4 + \cdots \right](\alpha). \tag{203}$$

Using $p = 1$ in equations (202) and (203), we obtain

$$\underline{u}(x,y,z,t)(\alpha) = [\underline{u}_0 + \underline{u}_1 + \underline{u}_2 + \underline{u}_3 + \underline{u}_4 + \cdots](\alpha), \tag{204}$$

$$\overline{u}(x,y,z,t)(\alpha) = [\overline{u}_0 + \overline{u}_1 + \overline{u}_2 + \overline{u}_3 + \overline{u}_4 + \cdots](\alpha). \tag{205}$$

Using (203) into (200) and the terminology equating with powers similar to p, we obtain

$$p^0 : \frac{\partial \underline{u}_0}{\partial t}(\alpha) = 0, \tag{206}$$

$$p^1 : \frac{\partial \underline{u}_1}{\partial t}(\alpha) = \left[\frac{\partial^2 \underline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial y^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial z^2}(\alpha) + \underline{u}_0\frac{\partial \underline{u}_0}{\partial x}(\alpha) \right], \tag{207}$$

$$p^2 : \frac{\partial \underline{u}_2}{\partial t}(\alpha) = \left[\frac{\partial^2 \underline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial y^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial z^2}(\alpha) + \left(u_0\frac{\partial \underline{u}_1}{\partial x}(\alpha) + \underline{u}_1\frac{\partial \underline{u}_0}{\partial x}(\alpha) \right) \right], \tag{208}$$

and

$$p^0 : \frac{\partial \overline{u}_0}{\partial t}(\alpha) = 0, \tag{209}$$

$$p^1 : \frac{\partial \overline{u}_1}{\partial t}(\alpha) = \left[\frac{\partial^2 \overline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial y^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial z^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial z^2}(\alpha) + u_0\frac{\partial \overline{u}_0}{\partial x}(\alpha) \right], \tag{210}$$

$$p^2 : \frac{\partial \overline{u}_2}{\partial t}(\alpha) = \left[\frac{\partial^2 \overline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial y^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial z^2}(\alpha) + \left(u_0\frac{\partial \overline{u}_1}{\partial x}(\alpha) + \overline{u}_1\frac{\partial \overline{u}_0}{\partial x}(\alpha) \right) \right]. \tag{211}$$

Taking initial condition (188) and the solution of (206) and (209), we get

$$\underline{u}_0(x,y,z)(\alpha) = \underline{u}(x,y,z,0)(\alpha) = (0.4 + 0.1\alpha)^n(x + y + z), \tag{212}$$

$$\overline{u}_0(x,y,z)(\alpha) = \overline{u}(x,y,z,0)(\alpha) = (0.6 - 0.1\alpha)^n(x + y + z). \tag{213}$$

From the solution of (207) and (210), we get

$$\underline{u}_1(x,y,z,t)(\alpha) = \int_0^t \left(\frac{\partial^2 \underline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial y^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial z^2}(\alpha) + \underline{u}_0\frac{\partial^2 \underline{u}_0}{\partial x}(\alpha) \right) dt$$

$$= (0.4 + 0.1\alpha)^n(x + y + z)t, \tag{214}$$

$$\overline{u}_1(x, y, z, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \overline{u}_0}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial y^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial z^2}(\alpha) + \overline{u}_0 \frac{\partial^2 \overline{u}_0}{\partial x}(\alpha) \right) dt$$

$$= (0.6 - 0.1\alpha)^n(x + y + z)t. \tag{215}$$

We obtain the solution of (208) and (211) as follows:

$$\underline{u}_2(x, y, z, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \underline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial y^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial z^2}(\alpha) \right.$$

$$\left. + \left[\underline{u}_0 \frac{\partial \underline{u}_1}{\partial x}(\alpha) + \underline{u}_1 \frac{\partial \underline{u}_0}{\partial x}(\alpha) \right] \right) dt$$

$$= (0.4 + 0.1\alpha)^n(x + y + z)t^2, \tag{216}$$

$$\vdots$$

and

$$\overline{u}_2(x, y, z, t)(\alpha) = \int_0^t \left(\frac{\partial^2 \overline{u}_1}{\partial x^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial y^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial z^2}(\alpha) \right.$$

$$\left. + \left[\overline{u}_0 \frac{\partial \overline{u}_1}{\partial x}(\alpha) + \overline{u}_1 \frac{\partial \overline{u}_0}{\partial x}(\alpha) \right] \right) dt$$

$$= (0.6 - 0.1\alpha)^n(x + y + z)t^2, \tag{217}$$

$$\vdots$$

Using (204) and (205), the approximate solution of (190) and (191) is

$$\underline{u}(x, y, z, t)(\alpha) = (0.4 + 0.1\alpha)^n \left[(x + y + z) + (x + y + z)t + (x + y + z)t^2 + \cdots \right],$$

$$\overline{u}(x, y, z, t)(\alpha) = (0.6 - 0.1\alpha)^n \left[(x + y + z) + (x + y + z)t + (x + y + z)t^2 + \cdots \right].$$

The exact solution can be obtained as follows:

$$\underline{u}(x, y, z, t)(\alpha) = (0.4 + 0.1\alpha)^n \left[\frac{x + y + z}{1 - t} \right],$$

provided $0 \le t < 1, x$ or y or $z \ne 0, 0 \le \alpha \le 1,$

$$\overline{u}(x, y, z, t)(\alpha) = (0.6 - 0.1\alpha)^n \left[\frac{x + y + z}{1 - t} \right],$$

provided $0 \le t < 1, x$ or y or $z \ne 0, 0 \le \alpha \le 1.$

Case [D]. Fuzzy reduced differential transform method
Taking a fuzzy RDTM to (190) and (191) yields

$$(j + 1)\underline{U}_{j+1}(\alpha) = \frac{\partial^2}{\partial x^2}\underline{U}_j(\alpha) + \frac{\partial^2}{\partial y^2}\underline{U}_j(\alpha) + \frac{\partial^2}{\partial z^2}\underline{U}_j(\alpha) + \sum_{r=0}^{j} \underline{U}_r \frac{\partial}{\partial x}\underline{U}_{j+r}(\alpha), \tag{218}$$

$$(j + 1)\overline{U}_{j+1}(\alpha) = \frac{\partial^2}{\partial x^2}\overline{U}_j(\alpha) + \frac{\partial^2}{\partial y^2}\overline{U}_j(\alpha) + \frac{\partial^2}{\partial z^2}\overline{U}_j(\alpha) + \sum_{r=0}^{j}\overline{U}_r\frac{\partial}{\partial x}\overline{U}_{j+r}(\alpha).$$ (219)

Using initial condition (188), we obtain

$$\underline{U}_0(\alpha) = (0.4 + 0.1\alpha)^n(x + y + z),$$ (220)

$$\overline{U}_0(\alpha) = (0.6 - 0.1\alpha)^n(x + y + z).$$ (221)

According to (222) into (219), we have $\tilde{U}_j(\alpha)$ fuzzy-valued function, respectively:

$$\underline{U}_1(\alpha) = (0.4 + 0.1\alpha)^n(x + y + z), \qquad \underline{U}_2(\alpha) = (0.4 + 0.1\alpha)^n(x + y + z), \ldots,$$

$$\underline{U}_n(\alpha) = (0.4 + 0.1\alpha)^n(x + y + z),$$ (222)

$$\overline{U}_1(\alpha) = (0.6 - 0.1\alpha)^n(x + y + z), \qquad \overline{U}_2(\alpha) = (0.6 - 0.1\alpha)^n(x + y + z), \ldots,$$

$$\overline{U}_n(\alpha) = (0.6 - 0.1\alpha)^n(x + y + z).$$ (223)

Applying the fuzzy differential inverse transformation, we have

$$\begin{aligned}
\underline{u}(x, y, z, t)(\alpha) &= \sum_{j=0}^{\infty}\underline{U}_j t^j \\
&= (0.4 + 0.1\alpha)^n\big[(x + y + z) + (x + y + z)t \\
&\quad + (x + y + z)t^2 + \cdots + (x + y + z)t^n + \cdots\big],
\end{aligned}$$ (224)

$$\begin{aligned}
\overline{u}(x, y, z, t)(\alpha) &= \sum_{j=0}^{\infty}\overline{U}_j t^j \\
&= (0.6 - 0.1\alpha)^n\big[(x + y + z) + (x + y + z)t \\
&\quad + (x + y + z)t^2 + \cdots + (x + y + z)t^n + \cdots\big].
\end{aligned}$$ (225)

The exact solution is obtained as follows:

$$\underline{u}(x, y, z, t)(\alpha) = (0.4 + 0.1\alpha)^n\left[\frac{x + y + z}{1 - t}\right],$$

provided $0 \le t < 1, x$ or y or $z \ne 0, 0 \le \alpha \le 1$

$$\overline{u}(x, y, z, t)(\alpha) = (0.6 - 0.1\alpha)^n\left[\frac{x + y + z}{1 - t}\right],$$

provided $0 \le t < 1, x$ or y or $z \ne 0, 0 \le \alpha \le 1$.

Example 4.4 We consider the following fuzzy $(1 + n)$-dimensional Burgers' equation:

$$\frac{\partial\tilde{u}}{\partial t} = \left[\frac{\partial^2\tilde{u}}{\partial x_1^2} \oplus \frac{\partial^2\tilde{u}}{\partial x_2^2} \oplus \frac{\partial^2\tilde{u}}{\partial x_3^2} \oplus \cdots \oplus \frac{\partial^2\tilde{u}}{\partial x_n^2}\right] \oplus \tilde{u} \odot \frac{\partial\tilde{u}}{\partial x_1}$$ (226)

with the initial condition

$$\tilde{u}(x_1, x_2, x_3, \ldots, x_n, 0) = \tilde{u}_0(x_1, x_2, x_3, \ldots, x_n)$$

$$= \tilde{k}^n \oplus (x_1 + x_2 + x_3 + \cdots + x_n) \tag{227}$$

for $\tilde{u} = \tilde{u}(x_1, x_2, x_3, \ldots, x_n, t)$ is a fuzzy-valued function satisfying $\tilde{u}(x_1, x_2, x_3, \ldots, x_n, t) \geq \tilde{0}$. Above $\tilde{k}^n \in E^1$, $n = 1, 2, 3, \ldots$, a fuzzy number is

$$\tilde{k}_1(s) = \begin{cases} 10(s - 0.4), & s \in [0.4, 0.5], \\ 10(0.6 - s), & s \in (0.5, 0.6], \\ 0, & s \notin [0.4, 0.6], \end{cases} \tag{228}$$

and $[\underline{\tilde{k}^n}](\alpha) = (0.4 + 0.1\alpha)^n$, $[\overline{\tilde{k}^n}](\alpha) = (0.6 - 0.1\alpha)^n$.

The parametric form of (226) is

$$\frac{\partial \underline{u}}{\partial t} = \left[\frac{\partial^2 \underline{u}}{\partial x_1^2} + \frac{\partial^2 \underline{u}}{\partial x_2^2} + \frac{\partial^2 \underline{u}}{\partial x_3^2} + \cdots + \frac{\partial^2 \underline{u}}{\partial x_n^2} \right] + \underline{u} \frac{\partial \underline{u}}{\partial x_1}, \tag{229}$$

$$\frac{\partial \overline{u}}{\partial t} = \left[\frac{\partial^2 \overline{u}}{\partial x_1^2} + \frac{\partial^2 \overline{u}}{\partial x_2^2} + \frac{\partial^2 \overline{u}}{\partial x_3^2} + \cdots + \frac{\partial^2 \overline{u}}{\partial x_n^2} \right] + \overline{u} \frac{\partial \overline{u}}{\partial x_1}, \tag{230}$$

for $\alpha \in [0, 1]$, where $\underline{u}$ stands for $\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha)$, similar to $\overline{u}$.

Case (A). Fuzzy differential transform method:

Applying the FDTM to (229) and (230), we get

$$(j_{n+1} + 1)\underline{U}(j_1, j_2, \ldots, j_n, j_{n+1} + 1)(\alpha)$$

$$= (j_1 + 2)(j_1 + 1)\underline{U}(j_1 + 2, j_2, j_3, \ldots, j_{n+1})(\alpha) + (j_2 + 2)(j_2 + 1)$$

$$\times \underline{U}(j_1, j_2 + 2, j_3, \ldots, j_{n+1}) + \cdots + (j_n + 2)(j_n + 1)\underline{U}(j_1, j_2, j_3, \ldots, j_n + 2, j_{n+1})(\alpha)$$

$$+ \sum_{a_1=0}^{j_1} \sum_{a_2=0}^{j_2} \cdots \sum_{a_n=0}^{j_n} \sum_{a_{n+1}=0}^{j_{n+1}} (j_1 + 1 - a_1)\underline{U}(j_1 + 1 - a_1, a_2, a_3, \ldots, a_n, a_{n+1})(\alpha)$$

$$\times \underline{U}(a_1, j_2 - a_2, j_3 - a_3, \ldots, j_n - a_n, j_{n+1} - a_{n+1})(\alpha), \tag{231}$$

and

$$(j_{n+1} + 1)\overline{U}(j_1, j_2, \ldots, j_n, j_{n+1} + 1)(\alpha)$$

$$= (j_1 + 2)(j_1 + 1)\overline{U}(j_1 + 2, j_2, j_3, \ldots, j_{n+1})(\alpha) + (j_2 + 2)(j_2 + 1)$$

$$\times \overline{U}(j_1, j_2 + 2, j_3, \ldots, j_{n+1}) + \cdots + (j_n + 2)(j_n + 1)\overline{U}(j_1, j_2, j_3, \ldots, j_n + 2, j_{n+1})(\alpha)$$

$$+ \sum_{a_1=0}^{j_1} \sum_{a_2=0}^{j_2} \cdots \sum_{a_n=0}^{j_n} \sum_{a_{n+1}=0}^{j_{n+1}} (j_1 + 1 - a_1)\overline{U}(j_1 + 1 - a_1, a_2, a_3, \ldots, a_n, a_{n+1})(\alpha)$$

$$\times \overline{U}(a_1, j_2 - a_2, j_3 - a_3, \ldots, j_n - a_n, j_{n+1} - a_{n+1})(\alpha). \tag{232}$$

Using initial condition (227), we get

$$\underline{u}(x_1, x_2, \ldots, x_n, 0)(\alpha) = \left[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \underline{U}(j_1, j_2, j_3, \ldots, j_n, 0)(\alpha) \right],$$

$$\bigl(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}\bigr)(\alpha) = (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n), \tag{233}$$

and

$$\overline{u}(x_1, x_2, \ldots, x_n, 0)(\alpha) = \left[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \overline{U}(j_1, j_2, j_3, \ldots, j_n, 0)(\alpha) \right],$$

$$\bigl(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}\bigr)(\alpha) = (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n), \tag{234}$$

where

$$\underline{U}(j_1, j_2, j_3, \ldots, j_n, 0)(\alpha)$$

$$= \begin{cases} (0.4 + 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases} \tag{235}$$

$$\overline{U}(j_1, j_2, j_3, \ldots, j_n, 0)(\alpha)$$

$$= \begin{cases} (0.6 - 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{236}$$

Taking (235) into (236), we get

$$\tilde{U}(j_1, j_2, \ldots, j_n, j_{n+1})(\alpha) = \bigl[\underline{U}(j_1, j_2, \ldots, j_n, j_{n+1})(\alpha), \overline{U}(j_1, j_2, \ldots, j_n, j_{n+1})(\alpha)\bigr],$$

as follows:

$$\underline{U}(j_1, j_2, j_3, \ldots, j_n, 1)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\underline{U}(j_1, j_2, j_3, \ldots, j_n, 2)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\underline{U}(j_1, j_2, j_3, \ldots, j_n, 3)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\underline{U}(j_1, j_2, j_3, \ldots, j_n, n)(\alpha) = \begin{cases} (0.4 + 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{U}(j_1, j_2, j_3, \ldots, j_n, 1)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{U}(j_1, j_2, j_3, \ldots, j_n, 2)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{U}(j_1, j_2, j_3, \ldots, j_n, 3)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{U}(j_1, j_2, j_3, \ldots, j_n, n)(\alpha) = \begin{cases} (0.6 - 0.1\alpha)^n + 1 & \text{if } j_i = 1, j_{j^*} = 0, i \neq j^*; i, j^* = 1, 2, \ldots, n. \\ 0 & \text{otherwise.} \end{cases}$$

According to (12) and (13), we obtain

$$\underline{u}(x_1, x_2, \ldots, x_n, t)(\alpha) = \left[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \sum_{j_{n+1}=0}^{\infty} \underline{U}(j_1, j_2, \ldots, j_n, j_{n+1})(\alpha) \right],$$

$$\left(x_1^{j_1} x_2^{j_2} x_3^{j_3} \cdots x_n^{j_n} t^{j_{n+1}} \right)(\alpha)$$

$$= (0.4 + 0.1\alpha)^n + \left[(x_1 + x_2 + x_3 + \cdots + x_n) \right.$$

$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t,$$

$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t^2$$

$$\left. + (x_1 + x_2 + x_3 + \cdots + x_n)t^3 + \cdots \right],$$

and

$$\overline{u}(x_1, x_2, \ldots, x_n, t)(\alpha) = \left[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \sum_{j_{n+1}=0}^{\infty} \overline{U}(j_1, j_2, \ldots, j_n, j_{n+1})(\alpha) \right],$$

$$\left(x_1^{j_1} x_2^{j_2} x_3^{j_3} \cdots x_n^{j_n} t^{j_{n+1}} \right)(\alpha)$$

$$= (0.6 - 0.1\alpha)^n + \left[(x_1 + x_2 + x_3 + \cdots + x_n) \right.$$

$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t,$$

$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t^2$$

$$\left. + (x_1 + x_2 + x_3 + \cdots + x_n)t^3 + \cdots \right].$$

The exact solution can be obtained as follows:

$$\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = (0.4 + 0.1\alpha)^n + \left[\frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{1 - t} \right],$$

provided $0 \leq t < 1, 0 \leq \alpha \leq 1$

$$\overline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = (0.6 - 0.1\alpha)^n + \left[\frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{1 - t} \right],$$

provided $0 \leq t < 1, 0 \leq \alpha \leq 1$.

Case (B). Fuzzy Adomian decomposition method:

Applying (32) into (35), we obtain

$$
\left.\begin{aligned}
\underline{u}_0(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.4+0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n),\\
\underline{u}_1(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.4+0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t,\\
\underline{u}_2(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.4+0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t^2,\\
\underline{u}_3(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.4+0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t^3,\\
\underline{u}_4(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.4+0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t^4,
\end{aligned}\right\} \tag{237}
$$

and

$$
\left.\begin{aligned}
\overline{u}_0(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.6-0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n),\\
\overline{u}_1(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.6-0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t,\\
\overline{u}_2(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.6-0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t^2,\\
\overline{u}_3(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.6-0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t^3,\\
\overline{u}_4(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) &= (0.6-0.1\alpha)^n + (x_1+x_2+x_3+\cdots+x_n)t^4.
\end{aligned}\right\} \tag{238}
$$

Using the fuzzy decomposition series, we have

$$
\begin{aligned}
&\underline{u}(x_1,x_2,x_3,\ldots,x_n,t)(\alpha)\\
&= \left[\sum_{j=0}^{\infty} \underline{u}_j(x_1,x_2,x_3,\ldots,x_n,t)(\alpha)\right]\\
&= \left[\underline{u}_0(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) + \underline{u}_1(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) + \cdots\right]\\
&= (0.4+0.1\alpha)^n + \big[(x_1,x_2,x_3+\cdots+x_n) + (x_1+x_2+x_3+\cdots+x_n)t\\
&\quad + (x_1+x_2+x_3+\cdots+x_n)t^2 + (x_1+x_2+x_3+\cdots+x_n)t^3 + \cdots\big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{u}(x_1,x_2,x_3,\ldots,x_n,t)(\alpha)\\
&= \left[\sum_{j=0}^{\infty} \overline{u}_j(x_1,x_2,x_3,\ldots,x_n,t)(\alpha)\right]\\
&= \left[\overline{u}_0(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) + \overline{u}_1(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) + \cdots\right]\\
&= (0.6-0.1\alpha)^n + \big[(x_1,x_2,x_3+\cdots+x_n) + (x_1+x_2+x_3+\cdots+x_n)t\\
&\quad + (x_1+x_2+x_3+\cdots+x_n)t^2 + (x_1+x_2+x_3+\cdots+x_n)t^3 + \cdots\big].
\end{aligned}
$$

The exact solution is obtained as follows:

$$
\underline{u}(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) = (0.4+0.1\alpha)^n + \left[\frac{(x_1+x_2+x_3+\cdots+x_n)}{1-t}\right],
$$

provided $0 \le t < 1, 0 \le \alpha \le 1$,

$$
\overline{u}(x_1,x_2,x_3,\ldots,x_n,t)(\alpha) = (0.6-0.1\alpha)^n + \left[\frac{(x_1+x_2+x_3+\cdots+x_n)}{1-t}\right],
$$

provided $0 \le t < 1, 0 \le \alpha \le 1$.

Case (C). Fuzzy homotopy perturbation method:

According to the fuzzy HPM equations (229) and (230), we get

$$\left[\frac{\partial \underline{u}(\alpha)}{\partial t} + p\left(\left(-\frac{\partial^2 \underline{u}(\alpha)}{\partial x_1^2} - \frac{\partial^2 \underline{u}(\alpha)}{\partial x_2^2} - \frac{\partial^2 \underline{u}(\alpha)}{\partial x_3^2} - \cdots - \frac{\partial \underline{u}(\alpha)}{\partial x_n} \right) - u\frac{\partial \underline{u}(\alpha)}{\partial x_1} \right) \right] = \tilde{0}, \quad (239)$$

$$\left[\frac{\partial \overline{u}(\alpha)}{\partial t} + p\left(\left(-\frac{\partial^2 \overline{u}(\alpha)}{\partial x_1^2} - \frac{\partial^2 \overline{u}(\alpha)}{\partial x_2^2} - \frac{\partial^2 \overline{u}(\alpha)}{\partial x_3^2} - \cdots - \frac{\partial \overline{u}(\alpha)}{\partial x_n} \right) - u\frac{\partial \overline{u}(\alpha)}{\partial x_1} \right) \right] = \tilde{0}. \quad (240)$$

Taking a fuzzy HPM with the fuzzy homotopy parameter p extends the solution as follows:

$$\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = \left[\underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + p^3\underline{u}_3 + \cdots \right](\alpha), \quad (241)$$

$$\overline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = \left[\overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + p^3\overline{u}_3 + \cdots \right](\alpha). \quad (242)$$

Applying $p = 1$ in equations (241) and (242) with the approximate solution as follows:

$$\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = [\underline{u}_0 + \underline{u}_1 + \underline{u}_2 + \overline{u}_3 + \cdots](\alpha), \quad (243)$$

$$\overline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = [\overline{u}_0 + \overline{u}_1 + \overline{u}_2 + \overline{u}_3 + \cdots](\alpha). \quad (244)$$

Taking (242) into (239), with the terminology equating and powers similar to p, we get

$$p^0 : \frac{\partial \underline{u}_0}{\partial t}(\alpha) = 0, \quad (245)$$

$$p^1 : \frac{\partial \underline{u}_1}{\partial t}(\alpha) = \left[\frac{\partial^2 \underline{u}_0}{\partial x_1^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial x_2^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial x_3^2}(\alpha) + \cdots + \frac{\partial^2 \underline{u}_1}{\partial x_n^2}(\alpha) \right] + \underline{u}_0\frac{\partial \underline{u}_0}{\partial x_1}(\alpha), \quad (246)$$

$$p^2 : \frac{\partial \underline{u}_2}{\partial t}(\alpha) = \left[\frac{\partial^2 \underline{u}_1}{\partial x_1^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial x_2^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial x_3^2}(\alpha) + \cdots + \frac{\partial^2 \underline{u}_1}{\partial x_n^2}(\alpha) \right]$$
$$+ \left[\underline{u}_0\frac{\partial \underline{u}_1}{\partial x_1}(\alpha) + \underline{u}_1\frac{\partial \underline{u}_0}{\partial x_1}(\alpha) \right], \quad (247)$$

and

$$p^0 : \frac{\partial \overline{u}_0}{\partial t}(\alpha) = 0, \quad (248)$$

$$p^1 : \frac{\partial \overline{u}_1}{\partial t}(\alpha) = \left[\frac{\partial^2 \overline{u}_0}{\partial x_1^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial x_2^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial x_3^2}(\alpha) + \cdots + \frac{\partial^2 \overline{u}_1}{\partial x_n^2}(\alpha) \right] + \overline{u}_0\frac{\partial \overline{u}_0}{\partial x_1}(\alpha), \quad (249)$$

$$p^2 : \frac{\partial \overline{u}_2}{\partial t}(\alpha) = \left[\frac{\partial^2 \overline{u}_1}{\partial x_1^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial x_2^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial x_3^2}(\alpha) + \cdots + \frac{\partial^2 \overline{u}_1}{\partial x_n^2}(\alpha) \right]$$
$$+ \left[\overline{u}_0\frac{\partial \overline{u}_1}{\partial x_1}(\alpha) + \overline{u}_1\frac{\partial \overline{u}_0}{\partial x_1}(\alpha) \right]. \quad (250)$$

Applying the solution to (245) and (248) with initial condition (227) gives

$$\underline{u}_0(x_1, x_2, x_3, \ldots, x_n)(\alpha) = \underline{u}(x_1, x_2, x_3, \ldots, x_n, 0)(\alpha)$$
$$= (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + x_4, \ldots, x_n), \quad (251)$$

$$\overline{u}_0(x_1, x_2, x_3, \ldots, x_n)(\alpha) = \overline{u}(x_1, x_2, x_3, \ldots, x_n, 0)(\alpha)$$

$$= (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + x_4, \dots, x_n). \tag{252}$$

The solution to (246) and (249) is as follows:

$$\underline{u}_1(x_1, x_2, x_3, \dots, x_n, t)(\alpha) = \int_0^t \left[\left(\frac{\partial^2 \underline{u}_0}{\partial x_1^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial x_2^2}(\alpha) + \frac{\partial^2 \underline{u}_0}{\partial x_3^2}(\alpha) \right. \right.$$
$$\left. \left. + \cdots + \frac{\partial^2 \underline{u}_0}{\partial x_n^2}(\alpha) \right) + \underline{u}_0 \frac{\partial^2 \underline{u}_1}{\partial x_1}(\alpha) \right] dt$$
$$= (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n)t, \tag{253}$$

$$\overline{u}_1(x_1, x_2, x_3, \dots, x_n, t)(\alpha) = \int_0^t \left[\left(\frac{\partial^2 \overline{u}_0}{\partial x_1^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial x_2^2}(\alpha) + \frac{\partial^2 \overline{u}_0}{\partial x_3^2}(\alpha) \right. \right.$$
$$\left. \left. + \cdots + \frac{\partial^2 \overline{u}_0}{\partial x_n}(\alpha) \right) + \overline{u}_0 \frac{\partial^2 \overline{u}_1}{\partial x_1}(\alpha) \right] dt$$
$$= (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n)t. \tag{254}$$

We obtain the solution to (247) and (250) as

$$\underline{u}_2(x_1, x_2, x_3, \dots, x_n, t)(\alpha) = \int_0^t \left(\left[\frac{\partial^2 \underline{u}_1}{\partial x_1^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial x_2^2}(\alpha) + \frac{\partial^2 \underline{u}_1}{\partial x_3^2}(\alpha) \right. \right.$$
$$\left. + \cdots + \frac{\partial^2 \underline{u}_1}{\partial x_n^2}(\alpha) \right] + \left[u_0 \frac{\partial \underline{u}_1}{\partial x_1}(\alpha) + \underline{u}_1 \frac{\partial u_0}{\partial x_1}(\alpha) \right] \right)$$
$$= (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n)t^2, \tag{255}$$
$$\vdots$$

and

$$\overline{u}_2(x_1, x_2, x_3, \dots, x_n, t)(\alpha) = \int_0^t \left(\left[\frac{\partial^2 \overline{u}_1}{\partial x_1^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial x_2^2}(\alpha) + \frac{\partial^2 \overline{u}_1}{\partial x_3^2}(\alpha) + \cdots + \frac{\partial^2 \overline{u}_1}{\partial x_n^2}(\alpha) \right] \right.$$
$$\left. + \left[u_0 \frac{\partial \overline{u}_1}{\partial x_1}(\alpha) + \underline{u}_1 \frac{\partial \overline{u}_0}{\partial x_1}(\alpha) \right] \right)$$
$$= (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n)t^2, \tag{256}$$
$$\vdots$$

Applying (243) and (244), the approximate solution of (229) and (230), we have

$$\underline{u}(x_1 + x_2 + x_3 + \cdots + x_n)(\alpha) = (0.4 + 0.1\alpha)^n + \left[(x_1 + x_2 + x_3 + \cdots + x_n) \right.$$
$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t$$
$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t^2$$
$$\left. + (x_1 + x_2 + x_3 + \cdots + x_n)t^3 + \cdots \right] \tag{257}$$

and

$$\overline{u}(x_1 + x_2 + x_3 + \cdots + x_n)(\alpha) = (0.6 - 0.1\alpha)^n + \Big[(x_1 + x_2 + x_3 + \cdots + x_n)$$
$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t$$
$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t^2$$
$$+ (x_1 + x_2 + x_3 + \cdots + x_n)t^3 + \cdots \Big]. \tag{258}$$

The exact solution is obtained as follows:

$$\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = (0.4 + 0.1\alpha)^n + \left[\frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{1 - t}\right],$$

provided $0 \le t < 1, 0 \le \alpha \le 1$,

$$\overline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = (0.6 - 0.1\alpha)^n + \left[\frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{1 - t}\right],$$

provided $0 \le t < 1, 0 \le \alpha \le 1$.

Case [D]. Fuzzy reduced differential transform method
Applying the fuzzy RDTM to (229) and (230) yields

$$(j + 1)\underline{U}_{j+1}(\alpha) = \frac{\partial^2}{\partial x_1^2}\underline{U}_j(\alpha) + \frac{\partial^2}{\partial x_2^2}\underline{U}_j(\alpha) + \frac{\partial^2}{\partial x_3^2}\underline{U}_j(\alpha)$$
$$+ \cdots + \frac{\partial^2}{\partial x_n^2}\underline{U}_j(\alpha) + \sum_{r=0}^{j}\underline{U}_r\frac{\partial}{\partial x_1}\underline{U}_{j-r}(\alpha), \tag{259}$$

$$(j + 1)\overline{U}_{j+1}(\alpha) = \frac{\partial^2}{\partial x_1^2}\overline{U}_j(\alpha) + \frac{\partial^2}{\partial x_2^2}\overline{U}_j(\alpha) + \frac{\partial^2}{\partial x_3^2}\overline{U}_j(\alpha)$$
$$+ \cdots + \frac{\partial^2}{\partial x_n^2}\overline{U}_j(\alpha) + \sum_{r=0}^{j}\overline{U}_r\frac{\partial}{\partial x_1}\overline{U}_{j-r}(\alpha). \tag{260}$$

Using initial condition (227), we get

$$\underline{U}_0(\alpha) = (0.4 + 0.1\alpha)^n + (x_1 + x_2 + \cdots + x_n), \tag{261}$$
$$\overline{U}_0(\alpha) = (0.6 - 0.1\alpha)^n + (x_1 + x_2 + \cdots + x_n). \tag{262}$$

According to (262) into (259), we can get $\tilde{U}_j(\alpha)$ fuzzy-valued function, respectively:

$$\underline{U}_1(\alpha) = (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n),$$
$$\underline{U}_2(\alpha) = (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n),$$
$$\vdots$$
$$\underline{U}_n(\alpha) = (0.4 + 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n),$$

and

$$\overline{U}_1(\alpha) = (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n),$$

$$\overline{U}_2(\alpha) = (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n),$$

$$\vdots$$

$$\overline{U}_n(\alpha) = (0.6 - 0.1\alpha)^n + (x_1 + x_2 + x_3 + \cdots + x_n).$$

Taking the fuzzy differential inverse transformation, we have

$$\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha)$$

$$= \sum_{j=0}^{\infty} \underline{U}_j t^j$$

$$= (0.4 + 0.1\alpha)^n + \left[(x_1 + x_2 + x_3 + \cdots + x_n) + (x_1 + x_2 + x_3 + \cdots + x_n)t \right.$$

$$\left. + (x_1 + x_2 + x_3 + \cdots + x_n)t^2 + \cdots \right] \qquad (263)$$

$$\overline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha)$$

$$= \sum_{j=0}^{\infty} \overline{U}_j t^j$$

$$= (0.6 - 0.1\alpha)^n + \left[(x_1 + x_2 + x_3 + \cdots + x_n) + (x_1 + x_2 + x_3 + \cdots + x_n)t \right.$$

$$\left. + (x_1 + x_2 + x_3 + \cdots + x_n)t^2 + \cdots \right]. \qquad (264)$$

The exact solutions can be obtained as follows:

$$\underline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = (0.4 + 0.1\alpha)^n + \left[\frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{1 - t} \right],$$

$$\text{provided } 0 \le t < 1, x_1, x_2, x_3, \ldots, x_n \ne 0,$$

$$\overline{u}(x_1, x_2, x_3, \ldots, x_n, t)(\alpha) = (0.6 - 0.1\alpha)^n + \left[\frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{1 - t} \right],$$

$$\text{provided } 0 \le t < 1, x_1, x_2, x_3, \ldots, x_n \ne 0.$$

5 Conclusions

In this paper, we studied the comparison of fuzzy differential transform method (FDTM), fuzzy Adomian decomposition method (FADM), fuzzy homotopy perturbation method (FHPM), and fuzzy reduced differential transform method (FRDTM), which have been successfully applied to the solutions of fuzzy $(1 + n)$-dimensional Burgers' equation under gH-differentiability. We have investigated many new results to solve the mentioned problem and applied the methods. The results obtained show that the methods are powerful mathematical tools for solving fuzzy $(1 + n)$-dimensional Burgers' equation. Finally, we presented some numerical examples and figures to illustrate the results of this work.
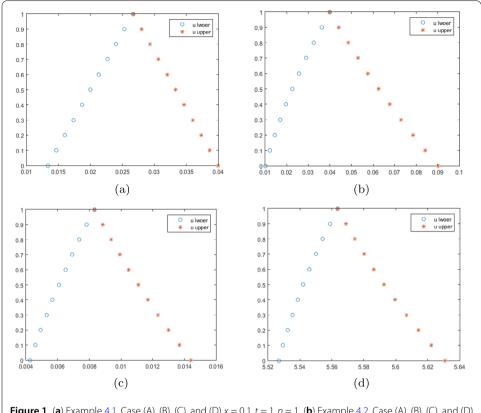
Figure 1 (**a**) Example 4.1, Case (A), (B), (C), and (D) $x = 0.1, t = 1, n = 1$, (**b**) Example 4.2, Case (A), (B), (C), and (D) $x = 0.001, y = 1, t = 0, n = 2$, (**c**) Example 4.3, Case (A), (B), (C), and (D) $x = 0.02, y = 0, z = 0, t = 0.7, n = 3$, (**d**) Example 4.4, Case (A), (B), (C), and (D) (Let $x_1 = 0, x_2 = 0.1, x_3 = 0.2, x_4 = 0.3$, $x_5 = 0.4, x_6 = 0.5, x_7 = 0.6, x_8 = 0.7, x_9 = 0.8, x_{10} = 0.9, x_{11} = 1$), $t = 0.0002, n = 4$

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
In this paper, all authors contributed equally. The authors read and approved the manuscript.

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