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Fractional isospectral and non-isospectral AKNS hierarchies and their analytic methods for N -fractal solutions with Mittag-Leffler functions

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Abstract

Ablowitz–Kaup–Newell–Segur (AKNS) linear spectral problem gives birth to many important nonlinear mathematical physics equations including nonlocal ones. This paper derives two fractional order AKNS hierarchies which have not been reported in the literature by equipping the AKNS spectral problem and its adjoint equations with local fractional order partial derivative for the first time. One is the space-time fractional order isospectral AKNS (stfisAKNS) hierarchy, three reductions of which generate the fractional order local and nonlocal nonlinear Schrödinger (flnNLS) and modified Kortweg–de Vries (fmKdV) hierarchies as well as reverse- t NLS (frtNLS) hierarchy, and the other is the time-fractional order non-isospectral AKNS (tfnisAKNS) hierarchy. By transforming the stfisAKNS hierarchy into two fractional bilinear forms and reconstructing the potentials from fractional scattering data corresponding to the tfnisAKNS hierarchy, three pairs of uniform formulas of novel N -fractal solutions with Mittag-Leffler functions are obtained through the Hirota bilinear method (HBM) and the inverse scattering transform (IST). Restricted to the Cantor set, some obtained continuous everywhere but nondifferentiable one- and two-fractal solutions are shown by figures directly. More meaningfully, the problems worth exploring of constructing N -fractal solutions of soliton equation hierarchies by HBM and IST are solved, taking stfisAKNS and tfnisAKNS hierarchies as examples, from the point of view of local fractional order derivatives. Furthermore, this paper shows that HBM and IST can be used to construct some N -fractal solutions of other soliton equation hierarchies.

Keywords: Fractional order isospectral AKNS hierarchy; Fractional order non-isospectral AKNS hierarchy; Local fractional order partial derivative; N -fractal solutions with Mittag-Leffler functions; Hirota bilinear method; Inverse scattering transform

1 Introduction

It is well known that AKNS spectral equation [1] is an important linear problem, from which and its associated time evolution equations of eigenfunctions abundant nonlinear partial differential equations (PDEs) [2, 3] have been derived, such as the KdV equation,

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mKdV equation, NLS equation, Burgers equation, sinh-Gordon (sG) equation, loop soliton equations, Harry–Dym (HD) equation, Heisenberg ferromagnet (HF) equation, and the N -wave equations. Besides, from the same AKNS linear problem, isospectral and non-isospectral hierarchies of nonlinear PDEs can be derived. Researchers often called such derived hierarchies the isospectral AKNS (isAKNS) hierarchies and non-isospectral AKNS (nisAKNS) hierarchies, respectively. It was Ablowitz et al. [1] who first solved the isAKNS equations by IST. In 1998, Gesztesy and Ratnaseelan [4] obtained algebro-geometric solutions of the AKNS hierarchy based on elementary algebraic methods. In 2004, Ning et al. [5] solved the nisAKNS hierarchy by IST. In 2008, Yin et al. [6] solved the isAKNS hierarchy by its bilinear form. In 2012, Chen et al. [7] solved the isAKNS hierarchy by HBM. In 2017, Zhang and Gao [8] solved a generalized isAKNS hierarchy by HBM. In 2018, Zhang and Hong [9] solved a generalized isAKNS hierarchy by IST. In 2018, Zhang and Hong [10] solved a generalized nisAKNS hierarchy by IST. Recently, some nonlocal integrable evolution equations [11, 12] have been found from the symmetry reductions of the isAKNS hierarchy. In 2011, Wu and Zhang [13] constructed Hamiltonian structures of a fractional AKNS hierarchy by a generalized Tu formula. In 2020, Gao et al. [14] solved the $(2 + 1)$ -dimensional AKNS equation with conformable derivatives and a perturbation parameter by the sine-Gordon expansion method. However, to the best of our knowledge, there are no reports on HBM, IST, and fractal solutions for fractional AKNS equations.

The aim of this paper is to derive the stfisAKNS hierarchy

$$D_t^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = L^{m\alpha} \begin{pmatrix} -u \\ v \end{pmatrix} \quad (m = 1, 2, \dots), \tag{1}$$

with the fractional order operator

$$L^\alpha = \begin{pmatrix} -D_x^\alpha + 2uI_{x,a}^\alpha v & 2uI_{x,a}^\alpha u \\ -2vI_{x,a}^\alpha v & D_x^\alpha - 2vI_{x,a}^\alpha u \end{pmatrix}, \tag{2}$$

and the tfnisAKNS hierarchy

$$D_t^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = L^m \begin{pmatrix} -xu \\ xv \end{pmatrix} \quad (m = 1, 2, \dots), \tag{3}$$

with the following operator:

$$L = \sigma \partial + 2 \begin{pmatrix} u \\ -v \end{pmatrix} \partial^{-1}(v, u), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

$$\partial^{-1} = \frac{1}{2} \left(\int_{-\infty}^x dx - \int_x^{+\infty} dx \right),$$

and then solve the fractional order AKNS hierarchies (1) and (3) by extending HBM [15] and IST [1], respectively. Here, D_x^α and $I_{x,a}^\alpha$ represent the local fractional order partial derivative operator [16]

$$D_x^\alpha \phi(x, t) = \Gamma(1 + \alpha) \lim_{\varepsilon \rightarrow 0} \frac{\phi(x + \varepsilon, t) - \phi(x, t)}{\varepsilon^\alpha} \quad (0 < \alpha \leq 1) \tag{5}$$

and the local fractional integral operator [16]

$$I_{x,a}^\alpha \varphi(x, t) = \frac{1}{\Gamma(1 + \alpha)} \int_a^x \varphi(\omega, t) (d\omega)^\alpha \quad (0 < \alpha \leq 1) \tag{6}$$

for any nondifferentiable functions $\phi(x, t)$ and $\varphi(x, t)$ defined on a fractal set Ω , respectively. The concept of local fractional derivative was first proposed by Kolwankar and Gangal [17], and it has received continuous developments and extensive applications like [18–25]. In addition, this article will show some obtained fractal solutions for more insights into novel nonlinearities hidden behind the fractional order models.

The rest of this article consists of four parts: Sect. 2 derives the stfisAKNS and tfnisAKNS hierarchies and gives three reductions for the fractional order nonlocal hierarchies of evolution equations; Sect. 3 constructs N -fractal solutions with Mittag-Leffler functions of the stfisAKNS hierarchy by considering two fractional order bilinear forms and shows the obtained one- and two-fractal solutions restricted to the Cantor set; Sect. 4 extends IST for constructing N -fractal solutions with Mittag-Leffler functions of the tfnisAKNS hierarchy and shows the obtained one-fractal solutions restricted to the Cantor set; Sect. 5 concludes this article.

2 Derivations of the fractional order AKNS hierarchies

Based on the Lax scheme [26], this section derives the stfisAKNS and tfnisAKNS hierarchies (1) and (3).

2.1 Fractional order isospectral AKNS hierarchy

Theorem 1 *Suppose*

$$A = I_{x,a}^\alpha (v, u) \begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2} (2i^\alpha k)^m, \tag{7}$$

where i^α is the necessary formal imaginary number unit [16] to connect the relationships between Mittag-Leffler functions and trigonometric functions defined in a fractal set Ω . Then the stfisAKNS hierarchy (1) can be derived from the fractional order zero curvature equation, i.e., the fractional order compatibility condition

$$D_t^\alpha U - D_x^\alpha V + [U, V] = 0, \quad [U, V] \equiv UV - VU \tag{8}$$

of the following fractional order linear spectral problem:

$$D_x^\alpha F = UF, \quad U = \begin{pmatrix} -i^\alpha k & u \\ v & i^\alpha k \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{9}$$

and its associated time-fractional order evolution equation

$$D_t^\alpha F = VF, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{10}$$

Here, $u = u(x, t)$ and $v = v(x, t)$ and their local fractional order derivatives with respect to x and t are all nondifferentiable functions, $i^\alpha k$ is the spectral parameter being independent

of x and t , while A, B , and C are all undetermined local fractional differentiable functions of x, t, u, v , and k .

Proof On the one hand, from Eq. (9) we obtain the α -order local fractional derivative with respect to t :

$$D_t^\alpha(D_x^\alpha F) = (D_t^\alpha U)F + U(D_t^\alpha F). \tag{11}$$

On the other hand, the local fractional derivative of Eq. (10) with respect to x gives

$$D_x^\alpha(D_t^\alpha F) = (D_x^\alpha V)F + V(D_x^\alpha F). \tag{12}$$

Since u, v, A, B , and C are all local fractional differentiable functions, F is local fractional continuous (see Definition 1.2 of [25]), and then we have $D_x^\alpha(D_t^\alpha F) = D_t^\alpha(D_x^\alpha F)$ from Theorem 3 in [25]. Thus, Eqs. (9)–(12) lead to

$$(D_t^\alpha V + UV)F = (D_x^\alpha V + VU)F, \tag{13}$$

which is namely Eq. (8) by using the arbitrariness of F .

Substituting the matrices U and V of Eqs. (9) and (10) into Eq. (8) yields

$$D_x^\alpha A = uC - vB, \tag{14}$$

$$D_t^\alpha u = D_x^\alpha B + 2i^\alpha kB + 2uA, \quad D_t^\alpha v = D_x^\alpha C - 2i^\alpha kC - 2vA. \tag{15}$$

In view of Eq. (7), we rewrite Eqs. (14) and (15) as

$$D_t^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = L^\alpha \begin{pmatrix} -B \\ C \end{pmatrix} - 2i^\alpha k \begin{pmatrix} -B \\ C \end{pmatrix} + (2i^\alpha k)^m \begin{pmatrix} -u \\ v \end{pmatrix}. \tag{16}$$

Further supposing

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{l=1}^m \begin{pmatrix} -b_l \\ c_l \end{pmatrix} (2i^\alpha k)^{m-l} \tag{17}$$

and substituting it into Eq. (16), then comparing the coefficients of the same powers of $2i^\alpha k$ in Eq. (17), we have

$$(2i^\alpha k)^0 : D_{t_m}^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = L^\alpha \begin{pmatrix} -b_m \\ c_m \end{pmatrix}, \tag{18}$$

$$(2i^\alpha k)^{m+1-l} : \begin{pmatrix} -b_l \\ c_l \end{pmatrix} = L^\alpha \begin{pmatrix} -b_{l-1} \\ c_{l-1} \end{pmatrix} \quad (l = 2, \dots, m), \tag{19}$$

$$(2i^\alpha k)^m : \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -u \\ v \end{pmatrix}. \tag{20}$$

Equations (19) and (20) give

$$\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = L^{(m-1)\alpha} \begin{pmatrix} -u \\ v \end{pmatrix}. \tag{21}$$

Finally, the substitution of Eq. (21) into Eq. (18) leads to the stfisAKNS hierarchy (1). \square

2.2 Fractional order non-isospectral AKNS hierarchy

Theorem 2 *Let the spectral parameter ik satisfy*

$$D_t^\alpha(ik) = \frac{1}{2}(2ik)^m, \tag{22}$$

and suppose

$$A = \partial^{-1}(v, u) \begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}(2ik)^m x, \tag{23}$$

then the time-fractional order zero curvature equation

$$D_t^\alpha U - V_x + [U, V] = 0, \tag{24}$$

i.e., the time-fractional order compatibility condition of the following linear spectral problem:

$$F_x = UF, \quad U = \begin{pmatrix} -ik & u \\ v & ik \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{25}$$

and its associated time-fractional order evolution Eq. (10) generate the tfnisAKNS hierarchy (3). Here, $u = u(x, t)$, $v = v(x, t)$ and their integer order derivatives with respect to x are all smooth functions, but $u = u(x, t)$, $v = v(x, t)$ and their fractional order derivatives with respect to t are all nondifferentiable functions.

Proof In a similar way to the proof of Theorem 1, with the help of the matrices U and V of Eqs. (25) and (10), we convert Eq. (24) into

$$A_x = uC - vB - \frac{1}{2}(2ik)^m, \tag{26}$$

$$D_t^\alpha u = B_x + 2ikB + 2uA, \quad D_t^\alpha v = C_x - 2ikC - 2vA, \tag{27}$$

which can be written as

$$D_t^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = L_\alpha \begin{pmatrix} -B \\ C \end{pmatrix} - 2ik \begin{pmatrix} -B \\ C \end{pmatrix} + (2ik)^m \begin{pmatrix} -xu \\ xv \end{pmatrix} \tag{28}$$

by using Eq. (23). Introducing [3]

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{l=1}^m \begin{pmatrix} -b_l \\ c_l \end{pmatrix} (2ik)^{m-l} \tag{29}$$

and substituting it into Eq. (28), then comparing the coefficients of the same powers of $2ik$ in Eq. (28) yield

$$(2ik)^0 : D_t^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} -b_m \\ c_m \end{pmatrix}, \tag{30}$$

$$(2ik)^{m+1-l} : \begin{pmatrix} -b_l \\ c_l \end{pmatrix} = L \begin{pmatrix} -b_{l-1} \\ c_{l-1} \end{pmatrix} \quad (l = 2, \dots, m), \tag{31}$$

$$(2ik)^m : \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -xu \\ xv \end{pmatrix}. \tag{32}$$

Then we have

$$\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = L^{m-1} \begin{pmatrix} -xu \\ xv \end{pmatrix}, \tag{33}$$

and hence we reach the tfnisAKNS hierarchy (3). □

2.3 Nonlocal and reverse- t reductions of stfisAKNS hierarchy

Conveniently, we rewrite the stfisAKNS hierarchy (1) as the following equivalent form:

$$D_{t_{m+1}}^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = K_m^\alpha = \begin{pmatrix} K_{1,m}^\alpha \\ K_{2,m}^\alpha \end{pmatrix} = L_\alpha D_{t_m}^\alpha \begin{pmatrix} u \\ v \end{pmatrix} \quad (m = 1, 2, \dots) \tag{34}$$

by introducing the variables $t_1 = x, t_2, t_3, \dots$ and treating u and v as two infinite-dimensional functions of these variables.

As special cases of Eq. (34), we would like to give three reductions. The first one is the flnNLS hierarchy

$$i^\alpha u_{t_{2s}}^\alpha = K_{2s}^{\alpha, flnNLS} = -K_{1,2s}^\alpha \tag{36} \quad (s = 1, 2, \dots) \tag{35}$$

reduced from Eq. (34) with transformation

$$v(x, t) = \delta u^*(\sigma x, t), \quad \delta, \sigma = \pm 1, t \rightarrow i^\alpha t, t \in \mathbb{R}, \tag{36}$$

the representative of which is the flnNLS equation

$$i^\alpha D_{t_2}^\alpha u = K_2^{\alpha, flnNLS} = D_x^{2\alpha} u - 2\delta u^2 u^*(\delta x, t). \tag{37}$$

The second one is the frtNLS hierarchy

$$u_{t_{2s}}^\alpha = K_{2s}^{\alpha, frtNLS} = K_{1,2s}^\alpha \tag{39} \quad (s = 1, 2, \dots) \tag{38}$$

reduced from Eq. (34) with transformation

$$v(x, t) = \delta u(\sigma x, -t), \quad \delta, \sigma = \pm 1, \tag{39}$$

one representative of which is the frtNLS equation

$$D_{t_2}^\alpha u = K_2^{\alpha,NLS} = -D_x^{2\alpha} u + 2\delta u^2 u(\delta x, -t). \tag{40}$$

And the last one is the fmKdV hierarchy

$$u_{t_{2s+1}}^\alpha = K_{2s+1}^{\alpha,fmKdV} = K_{1,2s+1}^\alpha \tag{41}$$

reduced from Eq. (34) with transformation

$$v(x, t) = \delta u(\sigma x, \sigma t), \quad \delta, \sigma = \pm 1, \tag{42}$$

one representative of which is the fmKdV equation

$$D_{t_3}^\alpha u = K_3^{\alpha,fmKdV} = D_x^{3\alpha} u - 6\delta u(D_x^\alpha u)u(\delta x, \sigma t). \tag{43}$$

3 HBM for N-fractal solutions of the stfiAKNS hierarchy

To construct N -fractal solutions, we first derive the fractional bilinear forms of the stfi-AKNS hierarchy (34). We then employ two reduced fractional bilinear forms to construct N -fractal solutions.

3.1 Fractional bilinear forms

Theorem 3 *Letting*

$$u = \frac{g}{f}, \quad v = \frac{h}{f}, \tag{44}$$

then the stfiAKNS hierarchy (34) can be bilinearized as

$$(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha)g \cdot f = fgI_{x,0}^\alpha \left\{ D_{t_m}^\alpha \left[\frac{1}{f^2} (H_x^{2\alpha} f \cdot f + 2gh) \right] \right\}, \tag{45}$$

$$(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha)h \cdot f = -fhI_{x,0}^\alpha \left\{ D_{t_m}^\alpha \left[\frac{1}{f^2} (H_x^{2\alpha} f \cdot f + 2gh) \right] \right\}, \tag{46}$$

where $H_{t_{m+1}}^\alpha$, $H_{t_m}^\alpha$, and H_x^α are local fractional versions [27] of the Hirota bilinear operator [15].

Proof We write the component forms of Eq. (34) as

$$D_{t_{m+1}}^\alpha u + D_{t_m,x}^{2\alpha} u = 2uI_{x,0}^\alpha [D_{t_m}^\alpha (uv)], \tag{47}$$

$$D_{t_{m+1}}^\alpha v - D_{t_m,x}^{2\alpha} v = -2vI_{x,0}^\alpha [D_{t_m}^\alpha (uv)]. \tag{48}$$

In view of Eq. (44), we have

$$D_{t_{m+1}}^\alpha u = \frac{D_{t_{m+1}}^\alpha g D_x^\alpha f - g D_{t_{m+1}}^\alpha f}{f^2}, \tag{49}$$

$$D_{t_m, x}^{2\alpha} u = \frac{(D_{t_{m+1}, x}^{2\alpha} g)f - D_{t_{m+1}}^\alpha g D_x^\alpha f - D_x^\alpha g D_{t_{m+1}}^\alpha f - g D_{t_{m+1}, x}^{2\alpha} f}{f^2} + \frac{2g D_x^\alpha f D_{t_{m+1}}^\alpha f}{f^3}, \tag{50}$$

$$2u I_{x,0}^\alpha [D_{t_m}^\alpha (uv)] = \frac{2g}{f} I_{x,0}^\alpha D_{t_m}^\alpha \frac{gh}{f^2}. \tag{51}$$

Using Eqs. (49)–(51), we convert the left- and right-hand sides of Eq. (47) into

$$\begin{aligned} \text{LHS} &= \frac{D_{t_{m+1}}^\alpha g D_x^\alpha f - g D_{t_{m+1}}^\alpha f}{f^2} \\ &+ \frac{(D_{t_{m+1}, x}^{2\alpha} g)f - D_{t_{m+1}}^\alpha g D_x^\alpha f - D_x^\alpha g D_{t_{m+1}}^\alpha f + g D_{t_{m+1}, x}^{2\alpha} f}{f^2}, \end{aligned} \tag{52}$$

$$\begin{aligned} \text{RHS} &= \frac{2g D_{t_{m+1}, x}^{2\alpha} f}{f^2} - \frac{2g D_x^\alpha f D_{t_{m+1}}^\alpha f}{f^3} + \frac{2g}{f} I_{x,0}^\alpha D_{t_m}^\alpha \frac{gh}{f^2} \\ &= \frac{2g}{f} I_{x,0}^\alpha \left\{ D_{t_m}^\alpha \left[\frac{(D_x^{2\alpha} f)f - (D_x^\alpha f)^2 + gh}{f^2} \right] \right\}. \end{aligned} \tag{53}$$

With the help of Eqs. (52) and (53), we can rewrite Eq. (47) as Eq. (45) by employing the fractional order Hirota bilinear operators. In a similar way we reach Eq. (48). Thus, we finish the proof. \square

Generally, if we construct N -fractal solutions of the stfisANKS hierarchy (34) directly by using Eqs. (45) and (46), the calculation is relatively complex and difficult. Inspired by the work [7], this section employs two reductions of the fractional bilinear forms (45) and (46) to construct fractional N -fractal solutions. The one adopts the constraint $H_x^{2\alpha} f \cdot f + 2gh = 0$, and the other weakens this constraint by letting $H_x^{2\alpha} f \cdot f + 2gh = a^2 f^2$, here a is a constant.

3.2 First reduced bilinear forms and N -fractal solutions

We set

$$H_x^{2\alpha} f \cdot f + 2gh = 0, \tag{54}$$

then the fractional order bilinear forms (45) and (46) become

$$(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha)g \cdot f = 0, \tag{55}$$

$$(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha)h \cdot f = 0. \tag{56}$$

For the fractional one-fractal solutions, we assume that

$$f = 1 + \varepsilon^2 f^{(2)} + \varepsilon^4 f^{(4)} + \dots + \varepsilon^{2j} f^{(2j)} + \dots, \tag{57}$$

$$g = \varepsilon^1 g^{(1)} + \varepsilon^3 g^{(3)} + \dots + \varepsilon^{2j+1} g^{(2j+1)} + \dots, \tag{58}$$

$$h = \varepsilon^1 h^{(1)} + \varepsilon^3 h^{(3)} + \dots + \varepsilon^{2j+1} h^{(2j+1)} + \dots, \tag{59}$$

and substitute Eqs. (57)–(59) into Eqs. (54)–(56). Collecting the coefficients of the same order of ε yields a system of fractional differential equations as follows:

$$D_{t_{m+1}}^\alpha g^{(1)} + D_{t_m, x}^{2\alpha} g^{(1)} = 0, \tag{60}$$

$$D_{t_{m+1}}^\alpha h^{(1)} - D_{t_m, x}^\alpha h^{(1)} = 0, \tag{61}$$

$$D_x^{2\alpha} f^{(2)} + g^{(1)} h^{(1)} = 0, \tag{62}$$

$$(D_{t_{m+1}}^\alpha + D_{t_m, x}^\alpha) g^{(3)} = -(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) g^{(1)} f^{(2)}, \tag{63}$$

$$(D_{t_{m+1}}^\alpha - D_{t_m, x}^\alpha) h^{(3)} = -(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) h^{(1)} f^{(2)}, \tag{64}$$

$$2D_x^{2\alpha} f^{(4)} = -H_x^{2\alpha} f^{(2)} f^{(2)} - 2(g^{(1)} h^{(3)} + g^{(3)} h^{(1)}), \tag{65}$$

$$(D_{t_{m+1}}^\alpha + D_{t_m, x}^\alpha) g^{(5)} = -(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) (g^{(1)} f^{(4)} + g^{(3)} f^{(2)}), \tag{66}$$

$$(D_{t_{m+1}}^\alpha - D_{t_m, x}^\alpha) h^{(5)} = -(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) (h^{(1)} f^{(4)} + h^{(3)} f^{(2)}), \tag{67}$$

$$2D_x^{2\alpha} f^{(6)} = -H_x^{2\alpha} f^{(2)} f^{(4)} - 2(g^{(1)} h^{(5)} + g^{(3)} h^{(3)} + g^{(5)} h^{(1)}), \tag{68}$$

⋮

and so forth. Letting

$$g^{(1)} = E_\alpha(\xi_1^\alpha), \quad \xi_1 = k_1 x + \sum_{m=1}^\infty w_{1,m+1} t_{m+1} + \xi_1^{(0)}, \tag{69}$$

$$h^{(1)} = E_\alpha(\eta_1^\alpha), \quad \eta_1 = p_1 x + \sum_{m=1}^\infty q_{1,m+1} t_{m+1} + \eta_1^{(0)}, \tag{70}$$

from Eqs. (60) and (61) we have

$$w_{1,m+1}^\alpha = -k_1^\alpha w_{1,m}^\alpha, \quad w_{1,1}^\alpha = k_1^\alpha, \tag{71}$$

$$q_{1,m+1}^\alpha = p_1^\alpha q_{1,m}^\alpha, \quad q_{1,1}^\alpha = p_1^\alpha. \tag{72}$$

In view of Eq. (62), we suppose that

$$f^{(2)} = E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \theta_{13}^\alpha), \tag{73}$$

where θ_{13}^α is a constant determined later and the Mittag-Leffler function is defined on a fractal set Ω [16]:

$$E_\alpha(\mu^\alpha) = \sum_{k=0}^\infty \frac{\mu^{k\alpha}}{\Gamma(1 + k\alpha)}. \tag{74}$$

Then Eqs. (69), (70), and (73) hint

$$E_\alpha(\theta_{13}^\alpha) = -\frac{1}{(k_1 + k_2)^2}. \tag{75}$$

Substituting Eqs. (69)–(74) into Eqs. (63)–(65) and setting

$$g^{(3)} = h^{(3)} = f^{(4)} = \dots = 0, \tag{76}$$

we can see that Eq. (61) and the equations behind all hold. In this case, we write

$$f_1 = 1 + E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \theta_{13}^\alpha), \quad g_1 = E_\alpha(\xi_1^\alpha), \quad h_1 = E_\alpha(\eta_1^\alpha), \tag{77}$$

and hence we obtain the fractional one-fractal solutions of the stfisAKNS hierarchy (34):

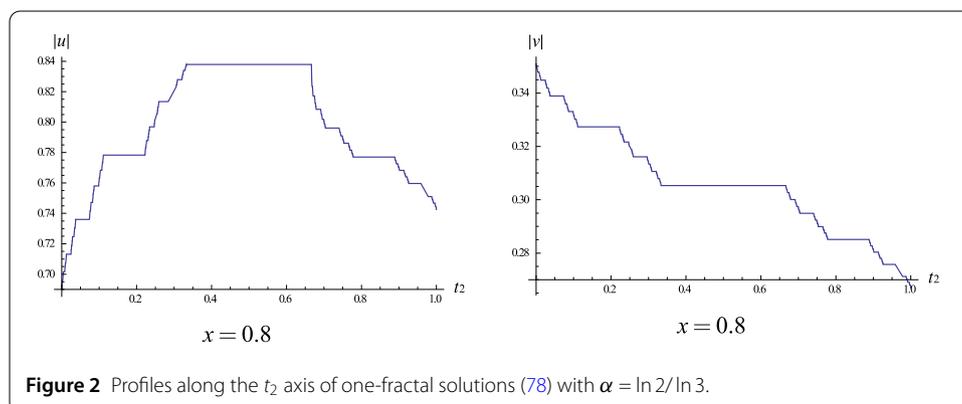
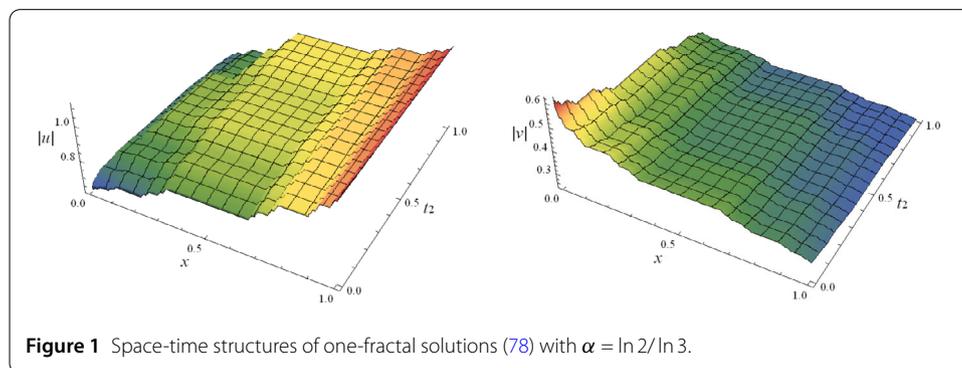
$$u = \frac{E_\alpha(\xi_1^\alpha)}{1 + E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \theta_{13}^\alpha)}, \quad v = \frac{E_\alpha(\eta_1^\alpha)}{1 + E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \theta_{13}^\alpha)}. \tag{78}$$

The one-fractal solutions (78) restricted to the Cantor set are shown in Figs. 1–3, where the parameters $k_1 = i$ and $p_1 = 0.5$.

For the two-fractal solutions, we assume that

$$g^{(1)} = E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha), \quad \xi_i = k_i x + \sum_{m=1}^{\infty} w_{i,m+1} t_{m+1} + \xi_i^{(0)} \quad (i = 1, 2), \tag{79}$$

$$h^{(1)} = E_\alpha(\eta_1^\alpha) + E_\alpha(\eta_2^\alpha), \quad \eta_i = p_i x + \sum_{m=1}^{\infty} q_{i,m+1} t_{m+1} + \eta_i^{(0)} \quad (i = 1, 2), \tag{80}$$



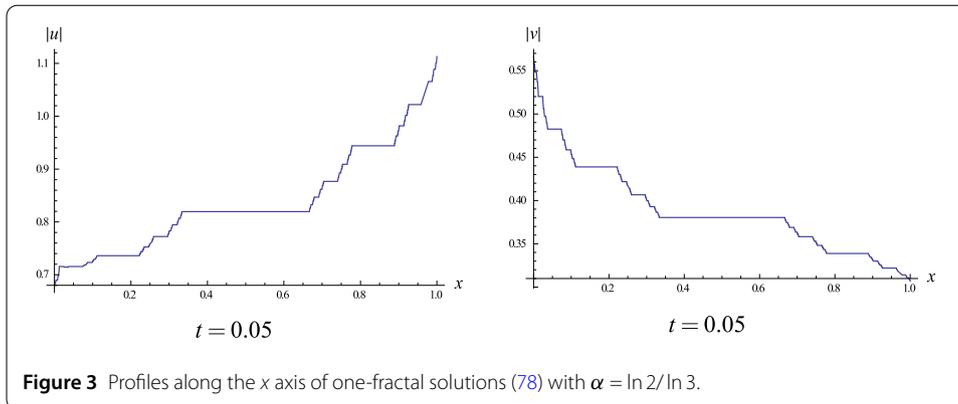


Figure 3 Profiles along the x axis of one-fractional solutions (78) with $\alpha = \ln 2 / \ln 3$.

and substitute Eqs. (79) and (80) into Eqs. (60) and (61). Then the resulting equations show

$$w_{2,m+1}^\alpha = -k_2^\alpha w_{2,m}^\alpha, \quad w_{2,1}^\alpha = k_2^\alpha, \tag{81}$$

$$q_{2,m+1}^\alpha = p_2^\alpha q_{2,m}^\alpha, \quad q_{2,1}^\alpha = p_2^\alpha. \tag{82}$$

In view of Eq. (57), we further let

$$f^{(2)} = E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \theta_{13}^\alpha) + E_\alpha(\xi_1^\alpha + \eta_2^\alpha + \theta_{14}^\alpha) + E_\alpha(\xi_2^\alpha + \eta_1^\alpha + \theta_{23}^\alpha) + E_\alpha(\xi_2^\alpha + \eta_2^\alpha + \theta_{24}^\alpha), \tag{83}$$

where θ_{14}^α , θ_{23}^α , and θ_{24}^α are constants. Substituting Eq. (83) into Eq. (62) yields

$$E_\alpha(\theta_{14}^\alpha) = -\frac{1}{(k_1 + l_2)^2}, \quad E_\alpha(\theta_{23}^\alpha) = -\frac{1}{(k_2 + l_1)^2}, \quad E_\alpha(\theta_{24}^\alpha) = -\frac{1}{(k_2 + l_2)^2}. \tag{84}$$

Taking into consideration Eq. (63), we select

$$g^{(3)} = E_\alpha(\xi_1^\alpha + \xi_2^\alpha + \eta_1^\alpha + \theta_{12}^\alpha + \theta_{13}^\alpha + \theta_{23}^\alpha) + E_\alpha(\xi_1^\alpha + \xi_2^\alpha + \eta_2^\alpha + \theta_{12}^\alpha + \theta_{14}^\alpha + \theta_{24}^\alpha), \tag{85}$$

where θ_{12}^α is a constant. We then obtain from Eq. (63)

$$E_\alpha(\theta_{12}^\alpha) = -(k_1 - k_2)^2. \tag{86}$$

In view of Eqs. (64) and (65), we set

$$h^{(3)} = E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \eta_2^\alpha + \theta_{13}^\alpha + \theta_{14}^\alpha + \theta_{34}^\alpha) + E_\alpha(\xi_2^\alpha + \eta_1^\alpha + \eta_2^\alpha + \theta_{23}^\alpha + \theta_{24}^\alpha + \theta_{34}^\alpha), \tag{87}$$

$$f^{(4)} = E_\alpha(\xi_1^\alpha + \xi_2^\alpha + \eta_1^\alpha + \eta_2^\alpha + \theta_{12}^\alpha + \theta_{13}^\alpha + \theta_{14}^\alpha + \theta_{23}^\alpha + \theta_{24}^\alpha + \theta_{34}^\alpha), \tag{88}$$

and then obtain

$$E_\alpha(\theta_{34}^\alpha) = -(p_1 - p_2)^2. \tag{89}$$

Substituting Eqs. (86)–(89) into Eq. (66), we can see that if

$$g^{(5)} = h^{(5)} = f^{(6)} = \dots = 0, \tag{90}$$

then Eq. (67) and those equations behind all hold. In this case, we obtain two-fractal solutions of the stfisAKNS hierarchy (34):

$$u = \frac{g_2}{f_2}, \quad v = \frac{h_2}{f_2}, \tag{91}$$

where

$$g_2 = E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha) + E_\alpha(\xi_1^\alpha + \xi_2^\alpha + \eta_1^\alpha + \theta_{12}^\alpha + \theta_{13}^\alpha + \theta_{23}^\alpha) + E_\alpha(\xi_1^\alpha + \xi_2^\alpha + \eta_2^\alpha + \theta_{12}^\alpha + \theta_{14}^\alpha + \theta_{24}^\alpha), \tag{92}$$

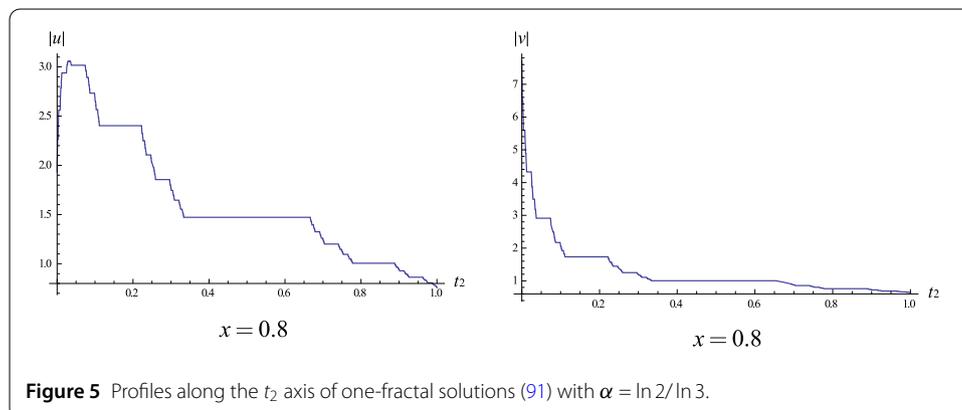
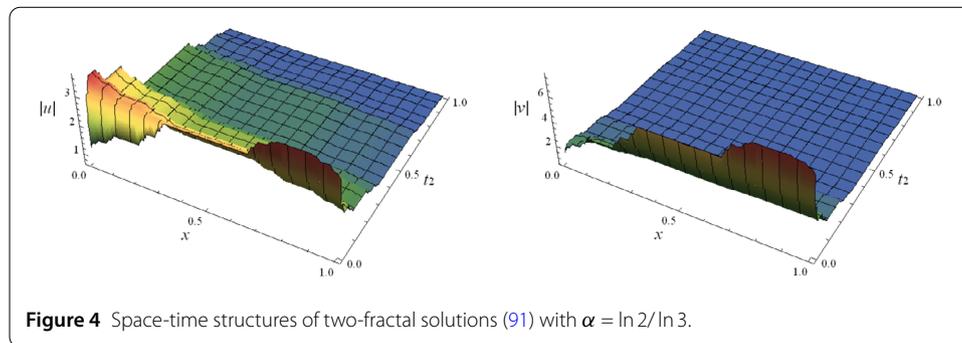
$$h_2 = E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha) + E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \eta_2^\alpha + \theta_{13}^\alpha + \theta_{14}^\alpha + \theta_{34}^\alpha) + E_\alpha(\xi_2^\alpha + \eta_1^\alpha + \eta_2^\alpha + \theta_{23}^\alpha + \theta_{24}^\alpha + \theta_{34}^\alpha), \tag{93}$$

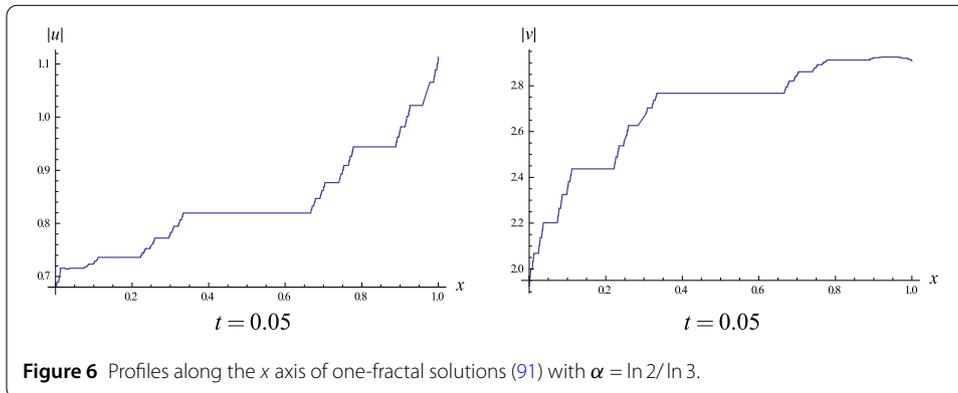
$$f_2 = 1 + E_\alpha(\xi_1^\alpha + \eta_1^\alpha + \theta_{13}^\alpha) + E_\alpha(\xi_1^\alpha + \eta_2^\alpha + \theta_{14}^\alpha) + E_\alpha(\xi_2^\alpha + \eta_1^\alpha + \theta_{23}^\alpha) + E_\alpha(\xi_2^\alpha + \eta_2^\alpha + \theta_{24}^\alpha) + E_\alpha(\xi_1^\alpha + \xi_2^\alpha + \eta_1^\alpha + \eta_2^\alpha + \theta_{12}^\alpha + \theta_{13}^\alpha + \theta_{14}^\alpha + \theta_{23}^\alpha + \theta_{24}^\alpha + \theta_{34}^\alpha), \tag{94}$$

$$E_\alpha(\theta_{12}^\alpha) = -(k_1 - k_2)^2, \quad E_\alpha(\theta_{34}^\alpha) = -(p_1 - p_2)^2, \tag{95}$$

$$E_\alpha[\theta_{i(j+2)}^\alpha] = -\frac{1}{(k_i + p_j)^2} \quad (i, j = 1, 2).$$

In Figs. 4–6, the two-fractal solutions (91) restricted to the Cantor set are shown, where $k_1 = i^\alpha$, $k_2 = 3i^\alpha$, $p_1 = 0.01$, $p_2 = 3$, and $\alpha = \ln 2 / \ln 3$.





Proceeding with a similar manipulation, we can obtain three-fractional solutions and induce the uniform formulas of the N -fractional solutions of the stfisAKNS hierarchy (34):

$$u = \frac{g_N}{f_N}, \quad v = \frac{h_N}{f_N}, \tag{96}$$

with

$$g_N = \sum_{\mu=0,1} Z_2(\mu) E_\alpha \left(\sum_{i=1}^{2N} \mu_i \xi_i + \sum_{1 \leq i < j}^{2N} \mu_i \mu_j \theta_{ij} \right), \tag{97}$$

$$h_N = \sum_{\mu=0,1} Z_3(\mu) E_\alpha \left(\sum_{i=1}^{2N} \mu_i \xi_i + \sum_{1 \leq i < j}^{2N} \mu_i \mu_j \theta_{ij} \right), \tag{98}$$

$$f_N = \sum_{\mu=0,1} Z_1(\mu) E_\alpha \left(\sum_{i=1}^{2N} \mu_i \xi_i + \sum_{1 \leq i < j}^{2N} \mu_i \mu_j \theta_{ij} \right), \tag{99}$$

$$\xi_i = k_i x + \sum_{m=1}^{\infty} w_{i,m+1} t_{m+1} + \xi_i^{(0)}, \quad w_{i,m+1} = -k_i w_{i,m}, \quad w_{i,1} = k_i, \tag{100}$$

$$\eta_i = p_i x + \sum_{m=1}^{\infty} q_{i,m+1} t_{m+1} + \eta_i^{(0)}, \quad q_{i,m+1} = p_i q_{i,m}, \tag{101}$$

$$q_{i,1} = p_i, \quad \xi_{N+i}^\alpha = \eta_i^\alpha \quad (i = 1, 2, \dots, N),$$

$$E_\alpha(\theta_{ij}) = -(k_i - k_j)^2, \quad E_\alpha[\theta_{(i+N)(j+N)}] = -(p_i - p_j)^2 \quad (i < j = 2, 3, \dots, N), \tag{102}$$

$$E_\alpha[\theta_{i(j+N)}] = -\frac{1}{(k_i + p_j)^2} \quad (i, j = 1, 2, \dots, N), \tag{103}$$

where the summation $\sum_{\mu=0,1}$ refers to all possible combinations of each $\mu_i = 0, 1$ for $i = 1, 2, \dots, 2N$, $Z_1(\mu)$, $Z_2(\mu)$, and $Z_3(\mu)$ denote that when we select all the possible combinations $\mu_i = 0, 1$ ($i = 1, 2, \dots, 2N$) the following conditions hold, respectively:

$$\sum_{i=1}^N \mu_i = \sum_{i=1}^N \mu_{N+i}, \quad \sum_{i=1}^N \mu_i = \sum_{i=1}^N \mu_{N+i} + 1, \quad \sum_{i=1}^N \mu_i + 1 = \sum_{i=1}^N \mu_{N+i}. \tag{104}$$

3.3 Second reduced bilinear forms and N -fractal solutions

Without loss of generality, we suppose

$$H_x^{2\alpha} f \cdot f + 2gh = a^2 f^2, \tag{105}$$

then the fractional order bilinear forms (45) and (46) become

$$(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) g \cdot f = c f g, \tag{106}$$

$$(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) h \cdot f = -c f h, \tag{107}$$

where $c = a^2$ is a non-zero constant. Further letting

$$f = \tilde{f}, \quad g = E_\alpha(ct_{m+1}^\alpha) \tilde{g}, \quad h = E_\alpha(-ct_{m+1}^\alpha) \tilde{h}, \tag{108}$$

we transform Eqs. (105)–(107) into

$$H_x^{2\alpha} \tilde{f} \cdot \tilde{f} = -2\tilde{g}\tilde{h} + a^2 \tilde{f}^2, \tag{109}$$

$$(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) \tilde{g} \cdot \tilde{f} = 0, \tag{110}$$

$$(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) \tilde{h} \cdot \tilde{f} = 0. \tag{111}$$

In what follows, we employ Eqs. (109)–(111) to construct N -fractal solutions of the stfisAKNS hierarchy (34). To construct one-fractal solutions, we suppose that

$$\tilde{f} = 1 + \varepsilon \tilde{f}^{(1)} + \varepsilon^2 \tilde{f}^{(2)} + \dots + \varepsilon^j \tilde{f}^{(j)} + \dots, \tag{112}$$

$$\tilde{g} = \tilde{g}^{(0)} + \varepsilon \tilde{g}^{(1)} + \varepsilon^2 \tilde{g}^{(2)} + \dots + \varepsilon^j \tilde{g}^{(j)} + \dots, \tag{113}$$

$$\tilde{h} = \tilde{h}^{(0)} + \varepsilon \tilde{h}^{(1)} + \varepsilon^2 \tilde{h}^{(2)} + \dots + \varepsilon^j \tilde{h}^{(j)} + \dots. \tag{114}$$

Substituting Eqs. (112)–(114) into Eqs. (109)–(110) and collecting the coefficients of the same order of ε yield a system of fractional differential equations, the first several equations of which are as follows:

$$2\tilde{g}^{(0)}\tilde{h}^{(0)} = a^2, \tag{115}$$

$$D_{t_{m+1}}^\alpha \tilde{g}^{(0)} + D_{x,t_m}^{2\alpha} \tilde{g}^{(0)} = 0, \tag{116}$$

$$D_{t_{m+1}}^\alpha \tilde{h}^{(0)} + D_{x,t_m}^{2\alpha} \tilde{h}^{(0)} = 0, \tag{117}$$

$$D_x^{2\alpha} \tilde{f}^{(1)} = -\tilde{g}^{(0)}\tilde{h}^{(1)} - \tilde{g}^{(1)}\tilde{h}^{(0)} + a^2 \tilde{f}^{(1)}, \tag{118}$$

$$D_{t_{m+1}}^\alpha \tilde{g}^{(1)} + D_{x,t_m}^{2\alpha} \tilde{g}^{(1)} = -(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) \tilde{g}^{(0)} \cdot \tilde{f}^{(1)}, \tag{119}$$

$$D_{t_{m+1}}^\alpha \tilde{h}^{(1)} - D_{x,t_m}^{2\alpha} \tilde{h}^{(1)} = -(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) \tilde{h}^{(0)} \cdot \tilde{f}^{(1)}, \tag{120}$$

$$D_x^{2\alpha} \tilde{f}^{(2)} = -\frac{1}{2} H_x^{2\alpha} \tilde{f}^{(1)} \cdot \tilde{f}^{(1)} - \tilde{g}^{(0)}\tilde{h}^{(2)} - \tilde{g}^{(1)}\tilde{h}^{(1)} - \tilde{g}^{(2)}\tilde{h}^{(0)} + a^2 \left[\tilde{f}^{(2)} + \frac{1}{2} (\tilde{f}^{(1)})^2 \right], \tag{121}$$

$$D_{t_{m+1}}^\alpha \tilde{g}^{(2)} + D_{x,t_m}^{2\alpha} \tilde{g}^{(2)} = -(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) (\tilde{g}^{(0)} \cdot \tilde{f}^{(2)} + \tilde{g}^{(1)} \cdot \tilde{f}^{(1)}), \tag{122}$$

$$D_{t_{m+1}}^\alpha \tilde{h}^{(2)} + D_{x,t_m}^{2\alpha} \tilde{h}^{(2)} = -(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) (\tilde{h}^{(0)} \cdot \tilde{f}^{(2)} + \tilde{h}^{(1)} \cdot \tilde{f}^{(1)}), \tag{123}$$

$$D_x^\alpha \tilde{f}^{(3)} = -H_x^{2\alpha} \tilde{f}^{(1)} \cdot \tilde{f}^{(2)} - \tilde{g}^{(0)} \tilde{h}^{(3)} - \tilde{g}^{(1)} \tilde{h}^{(2)} - \tilde{g}^{(2)} \tilde{h}^{(1)} - \tilde{g}^{(3)} \tilde{h}^{(0)} + a^2 (\tilde{f}^{(3)} + \tilde{f}^{(1)} \tilde{f}^{(2)}), \tag{124}$$

$$D_{x,t_m}^\alpha \tilde{g}^{(3)} + D_{x,t_m}^{2\alpha} \tilde{g}^{(3)} = -(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha) (\tilde{g}^{(0)} \cdot \tilde{f}^{(3)} + \tilde{g}^{(1)} \cdot \tilde{f}^{(2)} + \tilde{g}^{(2)} \cdot \tilde{f}^{(1)}), \tag{125}$$

$$D_{t_{m+1}}^\alpha \tilde{h}^{(3)} + D_{x,t_m}^{2\alpha} \tilde{h}^{(3)} = -(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha) (\tilde{h}^{(0)} \cdot \tilde{f}^{(3)} + \tilde{h}^{(1)} \cdot \tilde{f}^{(2)} + \tilde{h}^{(2)} \cdot \tilde{f}^{(1)}). \tag{126}$$

Setting

$$\tilde{g}^{(0)} = \tilde{h}^{(0)} = \frac{a}{\sqrt{2}} \tag{127}$$

and then substituting Eq. (127) into Eqs. (118)–(120), we have

$$\tilde{f}^{(1)} = 2E_\alpha(\xi_1^\alpha), \quad \tilde{g}^{(1)} = \sqrt{2}aE_\alpha(\xi_1^\alpha + 2\vartheta_1^\alpha), \quad \tilde{h}^{(1)} = \sqrt{2}aE_\alpha(\xi_1^\alpha - 2\vartheta_1^\alpha), \tag{128}$$

$$\xi_1 = k_1 x + \sum_{m=1}^\infty w_{1,m+1} t_{m+1} + \xi_1^{(0)}, \tag{129}$$

$$w_{1,1}^{2\alpha} = k_1^{2\alpha} = -2a^2 \sinh_\alpha^2 \vartheta_1^\alpha, \quad w_{1,m+1}^{2\alpha} = -2a^2 w_{1,m}^{2\alpha} \cosh_\alpha^2 \vartheta_1^\alpha,$$

where the hyperbolic functions $\sinh_\alpha \mu^\alpha$ and $\cosh_\alpha \mu^\alpha$ are defined on the fractal set Ω [16]:

$$\sinh_\alpha \mu^\alpha = \frac{E_\alpha(\mu^\alpha) - E_\alpha(-\mu^\alpha)}{2}, \quad \cosh_\alpha \mu^\alpha = \frac{E_\alpha(\mu^\alpha) + E_\alpha(-\mu^\alpha)}{2}. \tag{130}$$

If we set $\tilde{g}^{(2)} = \tilde{g}^{(3)} = \tilde{h}^{(2)} = \tilde{h}^{(3)} = \tilde{f}^{(2)} = \tilde{f}^{(3)} = \dots = 0$, we can see that Eqs. (128) and (129) satisfy Eqs. (118)–(126) and the other equations not explicitly written in the system of fractional differential equations. In this case, we write

$$\begin{aligned} \tilde{f}_1 &= 1 + 2E_\alpha(\xi_1^\alpha), & \tilde{g}_1 &= \frac{a[1 + 2E_\alpha(\xi_1^\alpha + 2\vartheta_1^\alpha)]}{\sqrt{2}}, \\ \tilde{h}_1 &= \frac{a[1 + 2E_\alpha(\xi_1^\alpha - 2\vartheta_1^\alpha)]}{\sqrt{2}}, \end{aligned} \tag{131}$$

and then we obtain one-fractal solutions of the stfisAKNS hierarchy (34):

$$u = \frac{aE_\alpha(ct_{m+1}^\alpha)[1 + 2E_\alpha(\xi_1^\alpha + 2\vartheta_1^\alpha)]}{\sqrt{2}(1 + 2E_\alpha(\xi_1^\alpha))}, \quad v = \frac{aE_\alpha(-ct_{m+1}^\alpha)[1 + 2E_\alpha(\xi_1^\alpha - 2\vartheta_1^\alpha)]}{\sqrt{2}(1 + 2E_\alpha(\xi_1^\alpha))}, \tag{132}$$

by using Eqs. (108), (112)–(114), (128), (129), and (131).

In order to construct two-fractal solutions, we let

$$\tilde{f}^{(1)} = 2(E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha)), \tag{133}$$

$$\tilde{g}^{(1)} = \sqrt{2}a[E_\alpha(\xi_1^\alpha + 2\vartheta_1^\alpha) + E_\alpha(\xi_2^\alpha + 2\vartheta_2^\alpha)], \tag{134}$$

$$\tilde{h}^{(1)} = \sqrt{2}a[E_\alpha(\xi_1^\alpha - 2\vartheta_1^\alpha) + E_\alpha(\xi_2^\alpha - 2\vartheta_2^\alpha)], \tag{135}$$

where

$$\xi_i = k_i x + \sum_{m=1}^\infty w_{i,m+1} t_{m+1} + \xi_i^{(0)}, \quad w_{i,1}^{2\alpha} = k_i^{2\alpha} = -2a^2 \sinh_\alpha^2 \vartheta_i^\alpha, \tag{136}$$

$$w_{i,m+1}^{2\alpha} = -2a^2 w_{i,m}^{2\alpha} \cosh_\alpha^2 \vartheta_i^\alpha \quad (i = 1, 2),$$

and suppose

$$(H_{t_{m+1}}^\alpha + H_x^\alpha H_{t_m}^\alpha)(\tilde{g}^{(1)} \cdot \tilde{f}^{(2)} + \tilde{g}^{(2)} \cdot \tilde{f}^{(1)}) = 0, \tag{137}$$

$$(H_{t_{m+1}}^\alpha - H_x^\alpha H_{t_m}^\alpha)(\tilde{h}^{(1)} \cdot \tilde{f}^{(2)} + \tilde{h}^{(2)} \cdot \tilde{f}^{(1)}) = 0. \tag{138}$$

$$H_x^{2\alpha} \tilde{f}^{(1)} \cdot \tilde{f}^{(2)} + \tilde{g}^{(1)} \tilde{h}^{(2)} + \tilde{g}^{(2)} \tilde{h}^{(1)} - a^2 \tilde{f}^{(1)} \tilde{f}^{(2)} = 0. \tag{139}$$

We then obtain from Eqs. (121)–(123) and Eqs. (136)–(138)

$$\tilde{f}^{(2)} = 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha + A_{12}^\alpha), \tag{140}$$

$$\tilde{g}^{(2)} = 2\sqrt{2}aE_\alpha(\xi_1^\alpha + \xi_2^\alpha + 2\vartheta_1^\alpha + 2\vartheta_2^\alpha + A_{12}^\alpha), \tag{141}$$

$$\tilde{h}^{(2)} = 2\sqrt{2}aE_\alpha(\xi_1^\alpha + \xi_2^\alpha - 2\vartheta_1^\alpha - 2\vartheta_2^\alpha + A_{12}^\alpha). \tag{142}$$

Substituting Eqs. (133)–(135) and Eqs. (140)–(142) into Eqs. (137)–(139) yields

$$\tilde{f}^{(3)} = \tilde{g}^{(3)} = \tilde{h}^{(3)} = \tilde{f}^{(4)} = \tilde{g}^{(4)} = \tilde{h}^{(4)} = \dots = 0. \tag{143}$$

Thus, from Eqs. (105)–(107) we have

$$f_2 = 1 + 2(E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha)) + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha + A_{12}^\alpha), \tag{144}$$

$$g_2 = \frac{aE_\alpha(ct_{m+1}^\alpha)\{1 + 2[E_\alpha(\xi_1^\alpha + 2\vartheta_1^\alpha) + E_\alpha(\xi_2^\alpha + 2\vartheta_2^\alpha)] + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha + 2\vartheta_1^\alpha + 2\vartheta_2^\alpha + A_{12}^\alpha)\}}{\sqrt{2}}, \tag{145}$$

$$h_2 = \frac{aE_\alpha(-ct_{m+1}^\alpha)\{1 + 2[E_\alpha(\xi_1^\alpha - 2\vartheta_1^\alpha) + E_\alpha(\xi_2^\alpha - 2\vartheta_2^\alpha)] + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha - 2\vartheta_1^\alpha - 2\vartheta_2^\alpha + A_{12}^\alpha)\}}{\sqrt{2}}, \tag{146}$$

and hence we obtain two-fractal solutions of the stfisAKNS hierarchy (34):

$$u = \frac{aE_\alpha(ct_{m+1}^\alpha)\{1 + 2[E_\alpha(\xi_1^\alpha + 2\vartheta_1^\alpha) + E_\alpha(\xi_2^\alpha + 2\vartheta_2^\alpha)] + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha + 2\vartheta_1^\alpha + 2\vartheta_2^\alpha + A_{12}^\alpha)\}}{\sqrt{2}[1 + 2(E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha)) + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha + A_{12}^\alpha)]}, \tag{147}$$

$$v = \frac{aE_\alpha(-ct_{m+1}^\alpha)\{1 + 2[E_\alpha(\xi_1^\alpha - 2\vartheta_1^\alpha) + E_\alpha(\xi_2^\alpha - 2\vartheta_2^\alpha)] + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha - 2\vartheta_1^\alpha - 2\vartheta_2^\alpha + A_{12}^\alpha)\}}{\sqrt{2}[1 + 2(E_\alpha(\xi_1^\alpha) + E_\alpha(\xi_2^\alpha)) + 4E_\alpha(\xi_1^\alpha + \xi_2^\alpha + A_{12}^\alpha)]}, \tag{148}$$

where

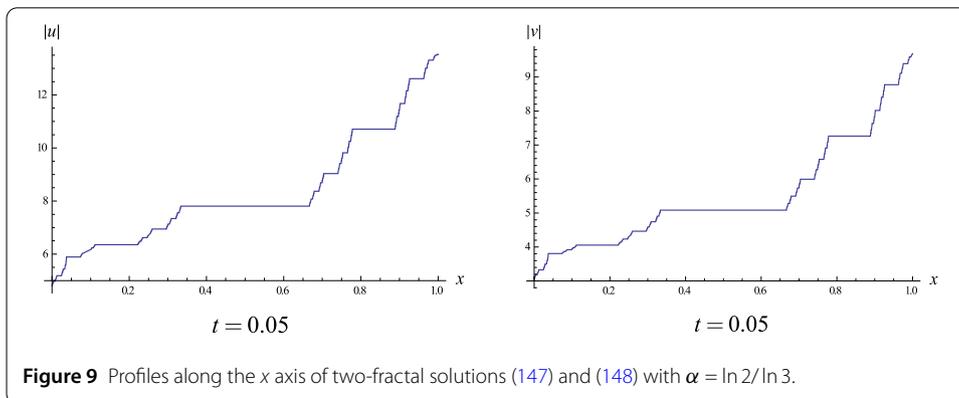
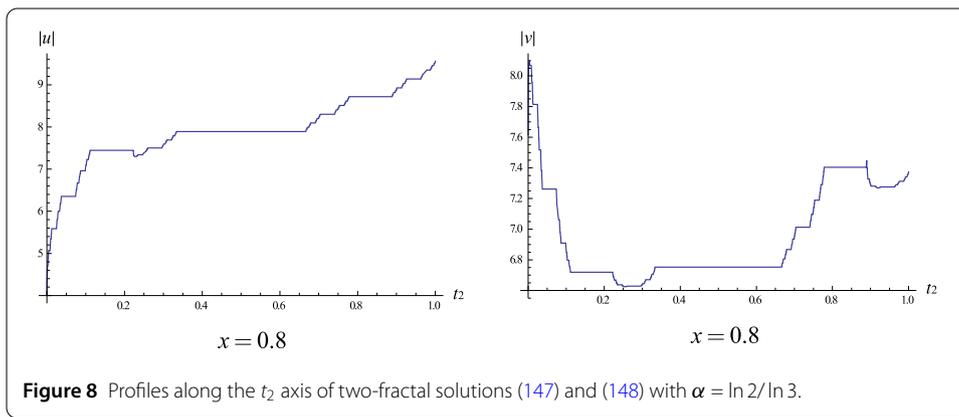
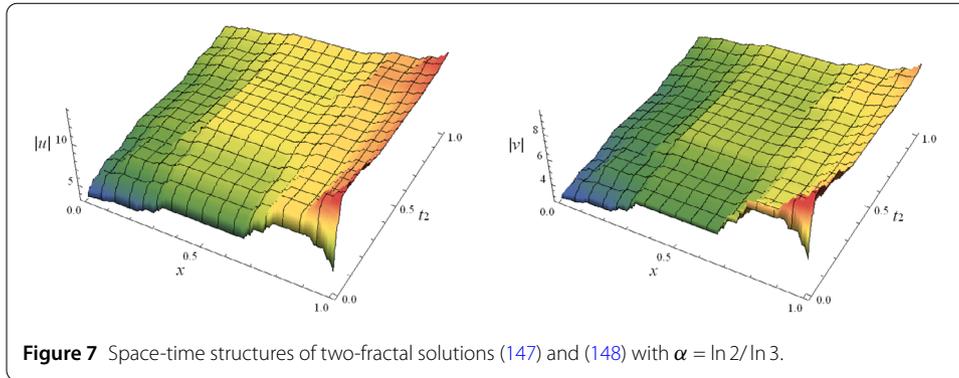
$$E_\alpha(A_{12}^\alpha) = \frac{\sinh_\alpha^2 \frac{\vartheta_1^\alpha - \vartheta_2^\alpha}{2}}{\sinh_\alpha^2 \frac{\vartheta_1^\alpha + \vartheta_2^\alpha}{2}}. \tag{149}$$

Selecting $c = 2.5$, $k_1 = 1$, $k_2 = -4$, $\vartheta_1 = 1$, $\vartheta_2 = 2$, and $\alpha = \ln 2 / \ln 3$, we show the two-fractal solutions (147) and (148) in Figs. 7–9.

Similarly, we can obtain three-fractal and then determine the uniform formulas (96) for the N -fractal solutions of the stfisAKNS hierarchy (34) by

$$f_N = \sum_{\mu=0,1} E_\alpha \left[\sum_{i=1}^N \mu_i (\xi_i^\alpha + \ln_\alpha 2) + \sum_{1 \leq i < l}^N \mu_i \mu_l A_{il}^\alpha \right], \tag{150}$$

$$g_N = \frac{aE_\alpha(ct_{m+1}^\alpha) \sum_{\mu=0,1} E_\alpha \left[\sum_{i=1}^N \mu_i (\xi_i^\alpha + 2\vartheta_i^\alpha + \ln_\alpha 2) + \sum_{1 \leq i < l}^N \mu_i \mu_l A_{il}^\alpha \right]}{\sqrt{2}}, \tag{151}$$



$$h_N = \frac{aE_\alpha(-ct_{m+1}^\alpha) \sum_{\mu=0,1} E_\alpha[\sum_{i=1}^N \mu_i (\xi_i^\alpha - 2\vartheta_i^\alpha + \ln_\alpha 2) + \sum_{1 \leq i < l \leq N} \mu_i \mu_l A_{il}^\alpha]}{\sqrt{2}}, \tag{152}$$

$$\xi_i = k_i x + \sum_{m=1}^\infty w_{i,m+1} t_{m+1} + \xi_i^{(0)}, \tag{153}$$

$$w_{i,1}^{2\alpha} = k_i^{2\alpha} = -2a^2 \sinh_\alpha^2 \vartheta_i^\alpha, \quad w_{i,m+1}^{2\alpha} = -2a^2 w_{i,m}^{2\alpha} \cosh_\alpha^2 \vartheta_i^\alpha \quad (i = 1, 2, \dots, N), \tag{154}$$

$$e^{A_{il}^\alpha} = \frac{\sinh^2 \frac{\vartheta_i^\alpha - \vartheta_l^\alpha}{2}}{\sinh^2 \frac{\vartheta_i^\alpha + \vartheta_l^\alpha}{2}} \quad (1 \leq i < l \leq N). \tag{155}$$

4 IST for N-fractal solutions of the tfnisAKNS hierarchy

Since the space derivative of the linear spectral problem (25) is integer order, all the existing results [2, 3] about the spectral problem (25) are valid for the tfnisAKNS hierarchy (3). The main objective of this section is to determine the time-dependence of scattering data by using the associated time-fractional order evolution Eq. (10).

4.1 Time-dependence of scattering data

Theorem 4 *Suppose that the potentials u and v of the linear spectral problem (25) equipped with the non-isospectral parameter ik satisfying Eq. (22) develop according to the tfnisAKNS hierarchy (3), then the scattering data*

$$\left\{ \kappa_j(t), c_j(t), R(t, k) = \frac{\gamma(t, k)}{\delta(t, k)}, j = 1, 2, \dots, N \right\}, \tag{156}$$

$$\left\{ \bar{\kappa}_l(t), \bar{c}_l(t), \bar{R}(t, k) = \frac{\bar{\gamma}(t, k)}{\bar{\delta}(t, k)}, l = 1, 2, \dots, \bar{N} \right\} \tag{157}$$

have the following time-dependence:

$$D_t^\alpha \kappa_j = -\frac{i}{2}(2i\kappa_j)^m, \quad c_j^2(t) = c_j^2(0)E_\alpha(mI_{0,t}^\alpha(2i\kappa_j)^{m-1}), \tag{158}$$

$$\delta(t, k) = \delta(0, k), \quad \gamma(t, k) = \gamma(0, k), \tag{159}$$

$$D_t^\alpha \bar{\kappa}_l = -\frac{i}{2}(2i\bar{\kappa}_l)^m, \quad \bar{c}_l^2(t) = \bar{c}_l^2(0)E_\alpha(-mI_{0,t}^\alpha(2i\bar{\kappa}_l)^{m-1}), \tag{160}$$

$$\bar{\delta}(t, k) = \bar{\delta}(0, k), \quad \bar{\gamma}(t, k) = \bar{\gamma}(0, k), \tag{161}$$

where $c_j^2(0), \bar{c}_l^2(0), R(0, k) = \beta(0, k)/\alpha(0, k)$, and $\bar{R}(0, k) = \bar{\beta}(0, k)/\bar{\alpha}(0, k)$ are the corresponding scattering data of the linear spectral problem (25) in the case of $(u(x, 0), v(x, 0))^T$.

Proof Since $F(x, k)$ solves Eq. (25), $P(x, k) = D_t^\alpha F(x, k) - NF(x, k)$ is a solution of Eq. (25). So, $P(x, k)$ can be expressed by $F(x, k)$ and $\tilde{F}(x, k)$, which is another solution of Eq. (25) but independent of $F(x, k)$, i.e., there exist two functions $\mu(t, k)$ and $\tau(t, k)$ such that

$$D_t^\alpha F(x, k) - VF(x, k) = \mu(t, k)F(x, k) + \tau(t, k)\tilde{F}(x, k). \tag{162}$$

We next consider the first case, i.e., discrete spectral $k = \kappa_j$ ($\text{Im } \kappa_j > 0$). It is easy to see that $F(x, \kappa_j)$ decays exponentially, while $\tilde{F}(x, \kappa_j)$ increases as $x \rightarrow +\infty$. Thus, we have $\tau(t, k) = 0$ and simplify Eq. (162) as

$$D_t^\alpha F(x, \kappa_j) - VF(x, \kappa_j) = \mu(t, \kappa_j)F(x, \kappa_j). \tag{163}$$

The left-multiplication on Eq. (163) by the inner product $(F_2(x, \kappa_j), F_1(x, \kappa_j))$ yields

$$D_t^\alpha F_1(x, \kappa_j)F_2(x, \kappa_j) - [CF_1^2(x, \kappa_j) + BF_2^2(x, \kappa_j)] = 2\mu(t, \kappa_j)F_1(x, \kappa_j)F_2(x, \kappa_j). \tag{164}$$

Since $F(x, \kappa_j)$ is a normalized eigenfunction and

$$2 \int_{-\infty}^{\infty} c_j^2 F_1(x, \kappa_j)F_2(x, \kappa_j) dx = 1, \tag{165}$$

we can rewrite Eq. (164) as

$$\mu(t, \kappa_j) = -c_j^2 \left((F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, (B, C)^T \right). \tag{166}$$

Making use of Eq. (25), we have

$$\left(F_1(x, \kappa_j) F_2(x, \kappa_j) \right)_x = u(x) F_2^2(x, \kappa_j) + v(x) F_1^2(x, \kappa_j). \tag{167}$$

The integration of Eq. (167) with respect to x from $-\infty$ to $+\infty$ gives

$$\int_{-\infty}^{\infty} \left(u(x) F_2^2(x, \kappa_j) + v(x) F_1^2(x, \kappa_j) \right) dx = \int_{-\infty}^{\infty} \left(F_1(x, \kappa_j) F_2(x, \kappa_j) \right)_x dx = 0. \tag{168}$$

At the same time, we rewrite Eq. (29) as

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{l=1}^m (2ik)^{m-l} \bar{L}^{l-1} \begin{pmatrix} xu \\ xv \end{pmatrix}, \quad \bar{L} = \sigma \partial - 2 \begin{pmatrix} u \\ v \end{pmatrix} \partial^{-1} (-v, u). \tag{169}$$

Then with the help of the conjugation operator \bar{L} [3]

$$\bar{L}^* = -\sigma \partial + 2 \begin{pmatrix} -v \\ u \end{pmatrix} \partial^{-1} (u, v) \tag{170}$$

and the results

$$\begin{aligned} & \left(\bar{L}^{*l-1} (F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, \begin{pmatrix} xu \\ xv \end{pmatrix} \right) \\ &= (2i\kappa_j(t))^{l-1} \left((F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, \begin{pmatrix} xu \\ xv \end{pmatrix} \right), \end{aligned} \tag{171}$$

$$\left((F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, \begin{pmatrix} xu \\ xv \end{pmatrix} \right) = \int_{-\infty}^{\infty} x [F_1(x, \kappa_j) F_2(x, \kappa_j)]_x dx = -\frac{1}{2c_j^2}, \tag{172}$$

we gain from Eq. (167)

$$\begin{aligned} \mu(t, \kappa_j) &= -c_j^2 \left((F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, (B, C)^T \right) \\ &= -c_j^2 \left((F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, \sum_{l=1}^m (2ik)^{m-l} \bar{L}^{l-1} \begin{pmatrix} xu \\ xv \end{pmatrix} \right) \\ &= -c_j^2 \sum_{l=1}^m (2i\kappa_j)^{m-l} \left((F_2^2(x, \kappa_j), F_1^2(x, \kappa_j))^T, \bar{L}^{l-1} \begin{pmatrix} xu \\ xv \end{pmatrix} \right) \\ &= \frac{1}{2} m (2i\kappa_j)^{m-1}. \end{aligned} \tag{173}$$

Thus, Eq. (163) becomes

$$D_t^\alpha F(x, \kappa_j) - VF(x, \kappa_j) = \frac{1}{2} m (2i\kappa_j)^{m-1} F(x, \kappa_j). \tag{174}$$

Noting that when $x \rightarrow +\infty$,

$$V \rightarrow \begin{pmatrix} -\frac{1}{2}(2i\kappa_j)^m x & 0 \\ 0 & \frac{1}{2}(2i\kappa_j)^m x \end{pmatrix}, \tag{175}$$

$$F(x, \kappa_j) \rightarrow c_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\kappa_j x}, \tag{176}$$

$$D_t^\alpha F(x, \kappa_j) \rightarrow (D_t^\alpha c_j) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\kappa_j x} + i(D_t^\alpha \kappa_j) x c_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\kappa_j x},$$

from Eq. (174) we then have

$$iD_t^\alpha \kappa_j - \frac{1}{2}(2i\kappa_j)^m = 0, \quad D_t^\alpha c_j - \frac{1}{2}m(2i\kappa_j)^{m-1}c_j = 0. \tag{177}$$

Similarly, the following fractional order equations can be obtained:

$$iD_t^\alpha \bar{\kappa}_l - \frac{1}{2}(2i\bar{\kappa}_l)^m = 0, \quad D_t^\alpha \bar{c}_l + \frac{1}{2}m(2i\bar{\kappa}_l)^{m-1}\bar{c}_l = 0. \tag{178}$$

We further consider the second case, i.e., real continuous spectral k . We select a solution $G(x, k)$ of Eq. (25), then

$$Q(x, k) = D_t^\alpha G(x, k) - NG(x, k) \tag{179}$$

solves Eq. (25). Therefore, there exists a pair of linearly independent fundamental solutions $G(x, k)$ and $\bar{G}(x, k)$ and two determined functions $\omega(t, k)$ and $\vartheta(t, k)$ such that

$$G_t(x, k) - VG(x, k) = \omega(t, k)G(x, k) + \vartheta(t, k)\bar{G}(x, k). \tag{180}$$

In view of the asymptotic properties

$$\begin{aligned} D_t^\alpha G(x, k) &\rightarrow -i(D_t^\alpha k)x \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, & G(x, k) &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \\ \bar{G}(x, k) &\rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{ikx}, \end{aligned} \tag{181}$$

as $x \rightarrow -\infty$, from Eqs. (180) and (181) we have

$$\vartheta(t, k) = 0, \quad \omega(t, k) = 0. \tag{182}$$

The substitution of the Jost relationship $G(x, k) = \gamma(t, k)\bar{F}(x, k) + \delta(t, k)F(x, k)$ into Eq. (180) yields

$$D_t^\alpha (\gamma(t, k)\bar{F}(x, k) + \delta(t, k)F(x, k)) - V(\gamma(t, k)\bar{F}(x, k) + \delta(t, k)F(x, k)) = 0. \tag{183}$$

Letting $x \rightarrow +\infty$ and using

$$F(x, k) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \bar{F}(x, k) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \tag{184}$$

then Eq. (183) hints

$$D_t^\alpha \gamma(t, k) = 0, \quad D_t^\alpha \delta(t, k) = 0. \tag{185}$$

Similarly, we have

$$D_t^\alpha \bar{\gamma}(t, k) = 0, \quad D_t^\alpha \bar{\delta}(t, k) = 0. \tag{186}$$

Finally, from Eqs. (177), (178), (185), and (186) we can obtain Eqs. (158)–(161). We finish the proof. \square

4.2 Exact solutions

Theorem 5 *The tfnisAKNS hierarchy (3) has the following exact solutions:*

$$u = -2K_1(x, x, t), \quad v = \frac{K_{2x}(x, x, t)}{K_1(x, x, t)}, \tag{187}$$

where $K(x, y, t) = (K_1(x, y, t), K_2(x, y, t))^T$ satisfies the Gel'fand–Levitan–Marchenko (GLM) integral equation

$$\begin{aligned} K(x, y, t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x + y, t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_x^\infty F(z + x, t) \bar{F}(z + y, t) dz \\ + \int_x^\infty K(x, s, t) \int_x^\infty F(z + s, t) \bar{F}(z + y, t) dz ds = 0, \end{aligned} \tag{188}$$

while

$$F(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty R(t, k) e^{ikx} dk + \sum_{j=1}^N \bar{c}_j^2 e^{ik_j x}, \tag{189}$$

$$\bar{F}(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{R}(t, k) e^{-ikx} dk - \sum_{j=1}^{\bar{N}} \bar{c}_j^2 e^{-i\bar{k}_j x} \tag{190}$$

are determined by the scattering data (158)–(160).

Proof The proof is similar to the integer case [3], and the essential difference is that Theorem 5 has the different scattering data (158)–(160). We omit the proof here for simplification. \square

4.3 N-fractal solutions

Theorem 6 *Suppose that*

$$M(x, t) = I + P(x, t)P^T(x, t), \quad P(x, t) = \begin{pmatrix} c_j(t)\bar{c}_m(t) \\ \kappa_j - \bar{\kappa}_m \end{pmatrix}_{\bar{N} \times N} e^{i(\kappa_j - \bar{\kappa}_m)x}, \tag{191}$$

$$\bar{z} = (\bar{c}_1(t)e^{-i\bar{\kappa}_1 x}, \bar{c}_2(t)e^{-i\bar{\kappa}_2 x}, \dots, \bar{c}_n(t)e^{-i\bar{\kappa}_N x})^T \tag{192}$$

are determined by the scattering data (158)–(160) and $M^{-1}(x, t)$ exists, while I is an $\bar{N} \times \bar{N}$ unit matrix, then the tfnisAKNS hierarchy (3) has N -fractal solutions

$$u = 2 \operatorname{tr}(M^{-1}(x, t)\bar{z}(x, t)\bar{z}^T(x, t)), \tag{193}$$

$$v = -\frac{\frac{\partial}{\partial x} \operatorname{tr}(M^{-1}(x, t)P(x, t)\frac{\partial}{\partial x} P^T(x, t))}{\operatorname{tr}(M^{-1}(x, t)\bar{z}(x, t)\bar{z}^T(x, t))}, \tag{194}$$

where $\operatorname{tr}(\cdot)$ represents the trace of a given matrix.

Proof Considering the reflectionless case, i.e., $R(t, k) = \bar{R}(t, k) = 0$, from Eq. (189) we have

$$\int_x^\infty F_d(t, s+z)\bar{F}_d(t, z+y) dz = -\sum_{j=1}^N \sum_{m=1}^{\bar{N}} \frac{ic_j^2(t)\bar{c}_m^2(t)}{\kappa_j - \bar{\kappa}_m} e^{\kappa_j(x+s) - i\bar{\kappa}_m(x+y)}. \tag{195}$$

Further suppose that the components of $K(x, y, t) = (K_1(x, y, t), K_2(x, y, t))^T$ can be expressed by

$$K_1(x, y, t) = \sum_{p=1}^{\bar{N}} \bar{c}_p(t)g_p(x, t)e^{-i\bar{\kappa}_p y}, \quad K_2(x, y, t) = \sum_{p=1}^{\bar{N}} \bar{c}_p(t)h_p(x, t)e^{-i\bar{\kappa}_p y}, \tag{196}$$

then we can convert Eq. (188) into a set of algebraic equations

$$g_m(x, t) + \bar{c}_m(t)e^{-i\bar{\kappa}_m x} + \sum_{j=1}^N \sum_{p=1}^{\bar{N}} \frac{c_j^2(t)\bar{c}_m(t)\bar{c}_p(t)}{(\kappa_j - \bar{\kappa}_m)(\kappa_j - \bar{\kappa}_p)} e^{i(2\kappa_j - \bar{\kappa}_m - \bar{\kappa}_p)x} g_p(x, t) = 0, \tag{197}$$

$$h_m(x, t) - \sum_{j=1}^N \frac{c_j^2(t)\bar{c}_m(t)}{\kappa_j - \bar{\kappa}_m} e^{i(2\kappa_j - \bar{\kappa}_m)x} + \sum_{j=1}^N \sum_{p=1}^{\bar{N}} \frac{c_j^2(t)\bar{c}_m(t)\bar{c}_p(t)}{(\kappa_j - \bar{\kappa}_m)(\kappa_j - \bar{\kappa}_p)} e^{i(2\kappa_j - \bar{\kappa}_m - \bar{\kappa}_p)x} h_p(x, t) = 0 \tag{198}$$

for $m = 1, 2, \dots, \bar{N}$. Introducing vectors

$$g(x, t) = (g_1(x, t), g_2(x, t), \dots, g_{\bar{N}}(x, t))^T, \tag{199}$$

$$h(x, t) = (h_1(x, t), h_2(x, t), \dots, h_{\bar{N}}(x, t))^T, \tag{200}$$

$$z = (c_1(t)e^{i\kappa_1 x}, c_2(t)e^{i\kappa_2 x}, \dots, c_n(t)e^{i\kappa_N x})^T, \tag{201}$$

we can rewrite the above set of algebraic equations as

$$M(x, t)g(x, t) = -\bar{N}(x, t), \quad M(t, x)h(t, x) = iP(t, x)z(t, x), \tag{202}$$

namely

$$g(x, t) = -M^{-1}(x, t)z(x, t), \quad h(x, t) = iM^{-1}(x, t)P(x, t)z(x, t). \tag{203}$$

Substituting Eq. (203) into Eq. (196) yields

$$K_1(x, y, t) = -\text{tr}(M^{-1}(x, t)\bar{z}(x, t)\bar{z}^T(y, t)), \tag{204}$$

$$K_2(x, y, t) = i\text{tr}(M^{-1}(x, t)P(x, t)z(x, t)\bar{z}^T(y, t)). \tag{205}$$

Then we reach Eqs. (193) and (194) by means of Eqs. (187), (204), and (205). The proof is complete. \square

Particularly, when $N = \bar{N} = 1$, Eqs. (193) and (194) give one-fractional solutions

$$u = \frac{2\bar{c}_1^2(0)e^{-2i\bar{\kappa}_1x}E_\alpha(-mI_{t,0}^\alpha(2i\bar{\kappa}_j)^{m-1})}{1 + \frac{c_1^2(0)\bar{c}_1^2(0)}{(\kappa_1-\bar{\kappa}_1)^2}e^{2i(\kappa_1-\bar{\kappa}_1)x}E_\alpha(mI_{t,0}^\alpha((2i\kappa_j)^{m-1} - (2i\bar{\kappa}_j)^{m-1}))}, \tag{206}$$

$$v = \frac{2c_1^2(0)e^{2i\kappa_1x}E_\alpha(mI_{t,0}^\alpha(2i\kappa_j)^{m-1})}{1 + \frac{c_1^2(0)\bar{c}_1^2(0)}{(\kappa_1-\bar{\kappa}_1)^2}e^{2i(\kappa_1-\bar{\kappa}_1)x}E_\alpha(mI_{t,0}^\alpha((2i\kappa_j)^{m-1} - (2i\bar{\kappa}_j)^{m-1}))}, \tag{207}$$

where κ_1 and $\bar{\kappa}_1$ are determined by

$$D_t^\alpha \kappa_1 = -\frac{i}{2}(2i\kappa_1)^m, \quad D_t^\alpha \bar{\kappa}_1 = -\frac{i}{2}(2i\bar{\kappa}_1)^m. \tag{208}$$

To be more specific, given $m = 2$, the second member equations of the tfnisAKNS hierarchy (3) read

$$D_t^\alpha \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2u_x - xu_{xx} + 2u\partial^{-1}(uv) + 2xu^2v \\ 2v_x + xv_{xx} - 2u\partial^{-1}(uv) - 2xuv^2 \end{pmatrix}, \tag{209}$$

which have one-fractional solutions

$$u = \frac{2\bar{c}_1^2(0)e^{-2i\bar{\kappa}_1x}E_\alpha(-\frac{i}{2}\ln[\Gamma(1+\alpha) - 2i\bar{\kappa}_1(0)t^\alpha])}{1 + \frac{c_1^2(0)\bar{c}_1^2(0)}{(\kappa_1-\bar{\kappa}_1)^2}e^{2i(\kappa_1-\bar{\kappa}_1)x}E_\alpha(\frac{i}{2}\ln\frac{[\Gamma(1+\alpha)-2i\kappa_1(0)t^\alpha]}{[\Gamma(1+\alpha)-2i\bar{\kappa}_1(0)t^\alpha]})}, \tag{210}$$

$$v = \frac{2c_1^2(0)e^{2i\kappa_1x}E_\alpha(\frac{i}{2}\ln[\Gamma(1+\alpha) - 2i\kappa_1(0)t^\alpha])}{1 + \frac{c_1^2(0)\bar{c}_1^2(0)}{(\kappa_1-\bar{\kappa}_1)^2}e^{2i(\kappa_1-\bar{\kappa}_1)x}E_\alpha(\frac{i}{2}\ln\frac{[\Gamma(1+\alpha)-2i\kappa_1(0)t^\alpha]}{[\Gamma(1+\alpha)-2i\bar{\kappa}_1(0)t^\alpha]})}, \tag{211}$$

where

$$\kappa_1 = \frac{i\kappa_1(0)\Gamma(1+\alpha)}{2\kappa_1(0)t^\alpha + i\Gamma(1+\alpha)}, \quad \bar{\kappa}_1 = \frac{i\bar{\kappa}_1(0)\Gamma(1+\alpha)}{2\bar{\kappa}_1(0)t^\alpha + i\Gamma(1+\alpha)}. \tag{212}$$

Figures 10–12 show the spatial structures and profiles of one-fractional solutions (206) and (207) restricted to the Cantor set by selecting $\kappa_1(0) = 1$, $\bar{\kappa}_1(0) = 1.8$, $c_1(0) = -1$, $\bar{c}_1(0) = 9$, and $\alpha = \ln 2 / \ln 3$.

5 Conclusions

Though there are many results for both the important AKNS linear spectral and the non-linear AKNS partial differential systems, the local fractional versions of them and N -fractional solutions of the AKNS systems are still blank. HBM and IST are two very important

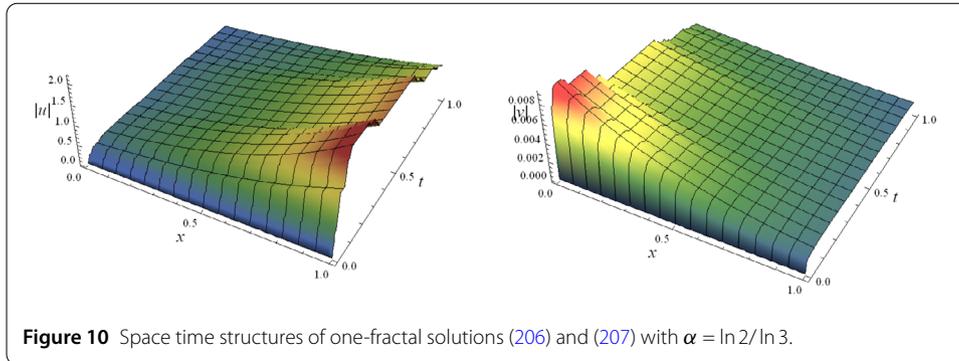


Figure 10 Space time structures of one-fractional solutions (206) and (207) with $\alpha = \ln 2 / \ln 3$.

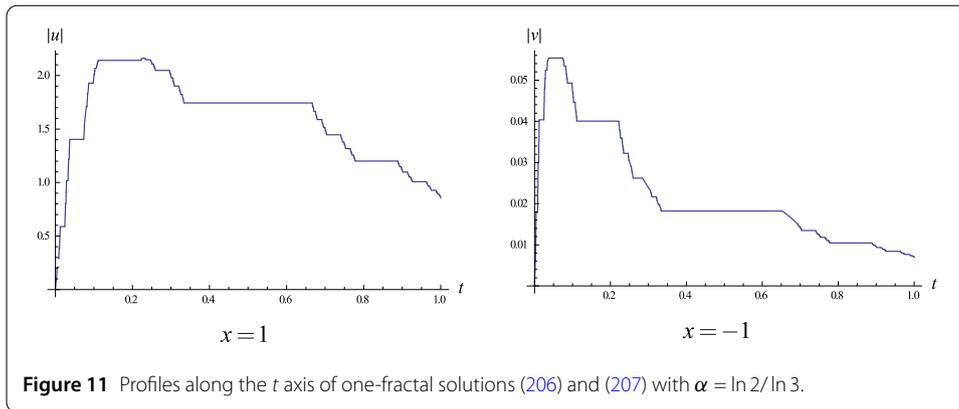


Figure 11 Profiles along the t axis of one-fractional solutions (206) and (207) with $\alpha = \ln 2 / \ln 3$.

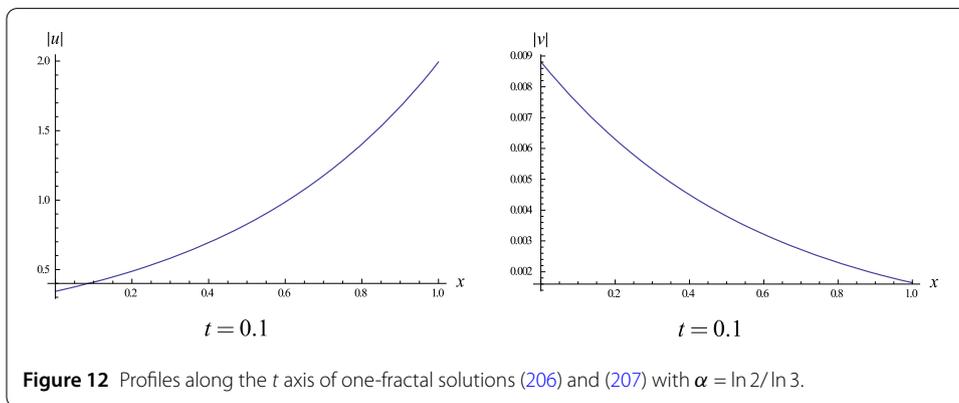


Figure 12 Profiles along the t axis of one-fractional solutions (206) and (207) with $\alpha = \ln 2 / \ln 3$.

analytical methods for solving soliton equations in the field of nonlinear sciences, which are extended to the stfisAKNS and tfnisAKNS hierarchies by introducing the fractional order bilinear operators [27] and fractional order scattering data (see [24] for our preliminary work). Compared with the existing literature, the work of this paper is novel. For the comparisons, we would like to point out that when $\alpha = 1$, both the derived stfisAKNS and tfnisAKNS hierarchies and the obtained N -fractal solutions become the known hierarchies [3, 7] and their corresponding exact solutions [3, 5, 7]. All the derived fractional AKNS hierarchies and the obtained N -fractal solutions benefit from the local fractional order partial derivatives with graceful properties [16] and the existing results [3] about the spectral problem (25) which are valid for the tfnisAKNS hierarchy (3).

As shown in Figs. 1–12, the obtained N -fractal solutions restricted to the Cantor set are continuous everywhere but nondifferentiable. This is essentially different from the existing classical soliton solutions. Though the obtained one-fractal solutions (78), (206), and (207) and two-fractal solutions (91), (147), and (148) are shown by figures, the features of one-fractal solutions and two-fractal solutions and the differences between them could not be concluded from them. On the one hand, these fractal solutions are limited to the Cantor set; on the other hand, they may be related to the properties of the fractional AKNS hierarchies and the parameters selected for plotting the figures.

This paper could extend IST to the tfnisAKNS hierarchy (3) but not the space-time fractional one. It is because of the fact that if the space derivative of the spectral problem (25) is fractional order, then the corresponding analysis is complex. At the same time, extending HBM to the tfnisAKNS hierarchy (3) is worth exploring. Integrable couplings [28–32] include the original integrable systems as sub-systems. Extending HBM and IST to the integrable couplings and their fractional order generalizations is worthy of study. At the same time, the practical application of fractional calculus in medicine, for example the conformable fractional mathematical model [33], is worth further research and exploration. With the developments [34–39] of fractional calculus, more and more theories and methods will be extended to fractional order nonlinear differential equations. Among them, two efficient methods called q -homotopy analysis transform method and fractional natural decomposition method used by Gao et al. [40] for numerical solution of fractional Benney–Lin equation deserve attention.

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Abbreviations

AKNS, Ablowitz–Kaup–Newell–Segur; HBM, Hirota bilinear method; IST, Inverse scattering transform; isAKNS, isospectral AKNS; nisAKNS, non-isospectral AKNS; stfisAKNS, space-time fractional order isospectral AKNS; tfnisAKNS, time-fractional order non-isospectral AKNS; flnNLS, fractional order local and nonlocal nonlinear Schrödinger; frtNLS, fractional order reverse- t NLS; fmKdV, fractional order modified KdV.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have made equal contributions to this paper and they have discussed the results, reviewed and approved the present version of the manuscript.

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