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On solutions of fractional multi-term sequential problems via some special categories of functions and (AEP)-property

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Abstract

The main intention of this article is that new techniques of existence theory are used to derive some required criteria pertinent to a given fractional multi-term problem and its inclusion version. In such an approach, we do our research on a fractional integral equation corresponding to the mentioned BVPs. In more precise words, by virtue of this integral equation, we construct new operators which belong to a special category of functions named α -admissible and α - ψ -contraction maps coupled with operators having (AEP)-property. Next, by considering some new properties on the existing Banach space having properties (B) and (C_α) , our argument for ensuring the existence of solutions is completed. In addition, we also add two simulative examples to review our findings by a numerical view.

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1 Introduction

As is well known, fractional calculus (FC), for the sake of its higher accuracy than that of the integer one, is an essential topic that is considered as a strong tool in description of natural laws in many branches of science including electrical networks, rheology, biology, dynamical systems, biophysics coupled with many mathematical modelings formulated by a vast diversity of fractional operators; review for details [1–8].

Among theoretical concepts and methods, the theory of the existence of solution on the large domain of different fractional constructions including differential equations and inclusions has gained the attention of many mathematicians and relevant researchers. Most of them have focused on applying Caputo, Riemann–Liouville (RL), Hadamard, and many other derivation operators to illustrate the underlying fractional differential equations. Along with these structures, we observe different published articles recently in which the existence of solutions is derived for interesting categories of fractional local or nonlocal, multi-term, multi-point, multi-strip, multi-order fractional differential equations; see [9–21].

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In 2018, Tariboon et al. [22] discussed a new category of separated sequential BVP in the context of mixed Hadamard and Caputo operators as follows:

$$\begin{cases} {}^C\mathcal{D}_m^{\sigma_1^*} [{}^H\mathcal{D}_m^{\sigma_2^*} v](z) = \Phi^*(z, v(z)) & (z \in \mathbb{J}^* := (m, n)); \\ a_1 v(m) + a_2 {}^H\mathcal{D}_m^{\sigma_2^*} v(m) = 0, \\ a_3 v(n) + a_4 {}^H\mathcal{D}_m^{\sigma_2^*} v(n) = 0, \end{cases}$$

where ${}^C\mathcal{D}_m^{\sigma_1^*}$ and ${}^H\mathcal{D}_m^{\sigma_2^*}$ stand for two different kinds of derivatives named Caputo and Hadamard of order $\sigma_1^*, \sigma_2^* \in (0, 1]$, $a_1, a_2, a_3, a_4 \in \mathbb{R}$ and $\Phi^* : \mathbb{J}^* \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. In that article, some standard methods pertinent to fixed point theory are used for obtaining the existence results.

In 2019, Mahmudov and Al-Khateeb [23] studied a novel sequential system of coupled BVPs with nonseparated conditions as follows:

$$\begin{cases} ({}^C\mathcal{D}_{0^+}^{\sigma_1^*} + \mu {}^C\mathcal{D}_{0^+}^{\sigma_1^*-1})v(z) = \Phi_1^*(z, v(z), w(z)) & (z \in \mathbb{J}^* := [0, K]), \\ ({}^C\mathcal{D}_{0^+}^{\sigma_2^*} + \mu {}^C\mathcal{D}_{0^+}^{\sigma_2^*-1})w(z) = \Phi_2^*(z, v(z), w(z)) & (z \in \mathbb{J}^* := [0, K]), \\ a_1 v(\eta) + a_2 v(K) = \int_0^K h_1(v(\tau)) d\tau, & a_1 w(\eta) + a_2 w(K) = \int_0^K h_2(w(\tau)) d\tau, \\ b_1 v'(\eta) + b_2 v'(K) = \int_0^K \Upsilon_1(v(\tau)) d\tau, & b_1 w'(\eta) + b_2 w'(K) = \int_0^K \Upsilon_2(w(\tau)) d\tau, \end{cases}$$

in which a derivative operator ${}^C\mathcal{D}_{0^+}^j$ is of Caputo type with $j \in \{\sigma_1^*, \sigma_2^*\} \subseteq (1, 2]$, $\mu \in \mathbb{R}_{>0}$, and $a_l, b_l \in \mathbb{R}$ with $l \in \{1, 2\}$. Stability analysis and existence notions are two main goals of that work which are investigated by means of fixed point methods.

In 2020, Mohammadi et al. [24] studied a specific kind of three-point sequential inclusion BVP which has the following form:

$$\begin{cases} \zeta_1 ({}^C\mathcal{D}_{0^+}^{\sigma^*} + \zeta_2 {}^C\mathcal{D}_{0^+}^{\sigma^*-1})\mu(z) \in \mathfrak{T}_*(z, \mu(z)) & (z \in \mathbb{J}^* := [0, 1]), \\ \mu(0) = 0, & \mu'(0) + \mu''(0) = 0, & \mu(1) + {}^{\mathcal{R}\mathcal{L}}\mathcal{I}_{0^+}^\gamma \mu(\xi) = 0, \end{cases}$$

where $\sigma^* \in (2, 3]$, $\xi \in (0, 1)$, $\zeta_1, \zeta_2 > 0$, and ${}^{\mathcal{R}\mathcal{L}}\mathcal{I}_{0^+}^\gamma$ denotes the Riemann–Liouville integral of order $\gamma > 0$. With the help of a new analytical process, the authors explored desired outcomes in relation to solutions of a non-hybrid structure.

With the help of ideas of the above articles, we investigate and discuss new existence techniques and methods to ensure the existence of solutions for a new multi-term problem illustrated as follows:

$$\begin{cases} {}^C\mathcal{D}_0^{\sigma^*} [\alpha_1 {}^C\mathcal{D}_0^2 v](z) = \Phi^*(z, v(z), {}^C\mathcal{D}_0^1 v(z)) & (z \in \mathbb{J} := [0, 1], \alpha_1^* \in \mathbb{R}^+); \\ v(0) = 0, \\ a_1^C\mathcal{D}_0^1 v(0) + a_2 {}^C\mathcal{D}_0^{\delta^*} v(1) = 0, \\ a_3^C\mathcal{D}_0^2 v(0) + a_4 {}^{\mathcal{R}\mathcal{L}}\mathcal{I}_0^{\theta^*} v(1) = 0, \end{cases} \tag{1}$$

where ${}^{\mathcal{R}\mathcal{L}}\mathcal{I}_0^{(\cdot)}$ and ${}^C\mathcal{D}_0^{(\cdot)}$ denote integration and derivation operators in the sense of Riemann–Liouville and Caputo, respectively, such that $\theta^* > 0$, $0 < \sigma^* < 1$, $1 < \delta^* < 2$. Besides, we have taken $a_1, a_2, a_3, a_4 \in \mathbb{R}^+$ and $\Phi^* : \mathbb{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as a continuous map.

Also along with the above problem, we derive the existence of solution for the following inclusion version which takes a form as follows:

$$\begin{cases} {}^C\mathcal{D}_0^{\sigma^*}[\alpha_1^* {}^C\mathcal{D}_0^2 v](z) \in \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z)) & (z \in \mathbb{J} := [0, 1], \alpha_1^* \in \mathbb{R}^+); \\ v(0) = 0, \\ a_1^C {}^C\mathcal{D}_0^1 v(0) + a_1^C {}^C\mathcal{D}_0^{\delta^*} v(1) = 0, \\ a_3^C {}^C\mathcal{D}_0^2 v(0) + a_4^{\mathcal{RL}} \mathcal{I}_0^{\theta^*} v(1) = 0, \end{cases} \tag{2}$$

where we consider \mathfrak{H} as a multifunction on the product space $\mathbb{J} \times \mathbb{R}^2$ containing different properties which will be introduced later.

By studying a wide range of published articles pertinent to the existence and uniqueness notions in the context of fractional boundary value problems, we see that many authors usually utilize some standard methods based on the famous fixed point techniques to derive desired results in relation to the existence of solutions. The novelty of our work in the current research is that we introduce a new construction of two fractional multi-term BVPs having integral conditions and then build new operators which belong to a new class of specific functions. Two main functions belonging to this class are α -admissible maps and α - ψ -contractive maps. Here, by using these functions on a space having properties (B) and (C_α) , we derive the existence results for both suggested BVPs (1) and (2). Besides, by virtue of the (AEP)-property for the obtained multifunction in the proof and with the help of the endpoint notion, we derive another criterion of the existence of solutions. As you will see, our techniques used for supposed problems (1) and (2) have been done in limited works, and for the first time, we apply these methods on a multi-term structures simultaneously.

The arrangement of the notions of this research are as follows: Sect. 2 is devoted to reviewing some primitive notions. In Sect. 3, by means of α - ψ -contractive functions, we prove our existence results for (1). Then, in Sect. 4, we derive other criteria stating the solution's existence for the inclusion BVP (2) by considering the extended category of α - ψ -contractive functions on multifunctions. Two numerical examples are also given to confirm findings in Sect. 5. At last, we summarize our method and specify new directions for the future works in Sect. 6.

2 Preliminaries

At this moment, we collect and review several auxiliary and primitive concepts in the context of our methods used in this research. As you know, the concepts of the Riemann–Liouville operator and the Caputo one have an important role in fractional calculus, we recall some properties of them here.

Definition 2.1 ([25, 26]) The Riemann–Liouville integral of a map $v : [0, +\infty) \rightarrow \mathbb{R}$ of order $\sigma^* > 0$ is defined by

$${}^{\mathcal{RL}}\mathcal{I}_0^{\sigma^*} v(z) = \int_0^z \frac{(z - q)^{\sigma^*-1}}{\Gamma(\sigma^*)} v(q) dq$$

if the integral exists.

Definition 2.2 ([25, 26]) Let $n = [\sigma^*] + 1$. For a continuous map $v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, the Riemann–Liouville derivative is defined by

$${}^{\mathcal{RL}}\mathcal{D}_0^{\sigma^*} v(z) = \left(\frac{d}{dz}\right)^n \int_0^z \frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)} v(q) dq$$

if the integral exists.

Definition 2.3 ([25, 26]) Let $n = [\sigma^*] + 1$. For a real absolutely continuous function v on $\mathbb{R}^{\geq 0}$, the Caputo derivative is defined by

$${}^{\mathcal{C}}\mathcal{D}_0^{\sigma^*} v(z) = \int_0^z \frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)} v^{(n)}(q) dq$$

such that the integral exists.

Proposition 2.4 ([27]) Let $n - 1 < \sigma^* < n$. Then, for every $v \in C^{n-1}(\mathbb{J})$, we have

$${}^{\mathcal{RL}}\mathcal{I}_0^{\sigma^*} ({}^{\mathcal{C}}\mathcal{D}_0^{\sigma^*} v)(z) = v(z) + c_0^* + c_1^* z + c_2^* z^2 + \dots + c_{n-1}^* z^{n-1}$$

for some $c_0^*, c_1^*, \dots, c_{n-1}^* \in \mathbb{R}$.

Assuming $(\mathcal{M}, \|\cdot\|)$ as a normed space, we mean by $\mathcal{P}_{CL}(\mathcal{M})$, $\mathcal{P}_{BN}(\mathcal{M})$, $\mathcal{P}_{CP}(\mathcal{M})$, and $\mathcal{P}_{CV}(\mathcal{M})$ the collection of all closed, bounded, compact, and convex subsets of \mathcal{M} , respectively.

Definition 2.5 ([28]) Consider $v : \mathbb{R} \rightarrow \mathbb{R}$ as a real-valued function and ν as a multifunction.

- (i) \mathfrak{H} is u.s.c on \mathcal{M} if $\mathfrak{H}(v^*) \in \mathcal{P}_{CL}(\mathcal{M})$ for any $v^* \in \mathcal{M}$ and also a neighborhood \mathfrak{N}_0^* of v^* exists subject to $\mathfrak{H}(\mathfrak{N}_0^*) \subseteq \mathbb{V}$ for $\mathbb{V} \subseteq \mathcal{M}$ in which \mathbb{V} is an arbitrary open set.
- (ii) A real-valued map $v : \mathbb{R} \rightarrow \mathbb{R}$ is upper semi-continuous such that $\limsup_{n \rightarrow \infty} v(\zeta_n) \leq v(\zeta)$ for each $\{\zeta_n\}_{n \geq 1}$ with $\zeta_n \rightarrow \zeta$.

Definition 2.6 ([28]) A metric attributed to Pompeiu–Hausdorff $\mathcal{H}_d : (\mathcal{P}(\mathcal{M}))^2 \rightarrow \mathbb{R} \cup \{\infty\}$ is defined as

$$\mathcal{H}_d(\mathcal{Q}_1^*, \mathcal{Q}_2^*) = \max \left\{ \sup_{q_1^* \in \mathcal{Q}_1^*} d(q_1^*, \mathcal{Q}_2^*), \sup_{q_2^* \in \mathcal{Q}_2^*} d(\mathcal{Q}_1^*, q_2^*) \right\},$$

where d is considered the metric of \mathcal{M} and also $d(\mathcal{Q}_1^*, q_2^*) = \inf_{q_1^* \in \mathcal{Q}_1^*} d(q_1^*, q_2^*)$ and $d(q_1^*, \mathcal{Q}_2^*) = \inf_{q_2^* \in \mathcal{Q}_2^*} d(q_1^*, q_2^*)$.

Definition 2.7 ([28]) For $\mathfrak{H} : \mathcal{M} \rightarrow \mathcal{P}_{CL}(\mathcal{M})$ and $v_1, v_2 \in \mathcal{M}$, let the following inequality hold:

$$\mathcal{H}_d(\mathfrak{H}(v_1), \mathfrak{H}(v_2)) \leq \ell d(v_1, v_2).$$

Then \mathfrak{H} is said to be: (H1) a Lipschitz map if $\ell > 0$; (H2) a contraction if $0 < \ell < 1$.

Definition 2.8 ([28])

- (i) $\mathfrak{H} : \mathbb{J} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is Caratheodory if $z \mapsto \mathfrak{H}(z, v)$ is measurable for any $v \in \mathbb{R}$ and $v \mapsto \mathfrak{H}(a, v)$ is u.s.c for a.e. $z \in \mathbb{J}$.
- (ii) A Caratheodory multifunction $\mathfrak{H} : \mathbb{J} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is L^1 -Caratheodory if, for any $\zeta > 0, \kappa_\zeta \in L^1(\mathbb{J}, \mathbb{R}_+)$ exists subject to $\|\mathfrak{H}(z, v)\| = \sup_{\omega \in \mathfrak{H}(z, v)} |\omega| \leq \kappa_\zeta(z)$ for all $|v| \leq \zeta$ and for almost all $z \in \mathbb{J}$.

Definition 2.9 ([28]) Consider \mathcal{M} as a metric space. Then

- (i) $v \in \mathcal{M}$ is an endpoint of $\mathfrak{H} : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M})$ if $\mathfrak{H}v = \{v\}$.
- (ii) \mathfrak{H} has an approximate endpoint property (or (AEP)-property) if

$$\inf_{w \in \mathcal{M}} \sup_{v \in \mathfrak{H}w} d(w, v) = 0.$$

To meet the argumentative purposes of this paper, we shall apply a specific set of functions and properties pertinent to them. For the first time, Samet, Vetro, and Vetro [29] constructed such a category of mappings in 2012.

Definition 2.10 ([29]) Consider all nondecreasing maps $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ belonging to class Π such that, for all $z > 0, \sum_{j=1}^\infty \psi^j(z) < \infty$ and $\psi(z) < z$. Let $\Phi^* : \mathcal{M} \rightarrow \mathcal{M}$ and $\alpha : \mathcal{M}^2 \rightarrow \mathbb{R}_{\geq 0}$. Then

- (i) Φ^* is an α - ψ -contraction if, for $v_1, v_2 \in \mathcal{M}$,

$$\alpha(v_1, v_2)d(\Phi^*v_1, \Phi^*v_2) \leq \psi(d(v_1, v_2)).$$

- (ii) Φ^* is α -admissible if $\alpha(v_1, v_2) \geq 1$ gives $\alpha(\Phi^*v_1, \Phi^*v_2) \geq 1$.
- (iii) \mathcal{M} has the property (B) if, for every sequence $\{v_n\}_{n \geq 1}$ of \mathcal{M} with $\alpha(v_n, v_{n+1}) \geq 1$ and $v_n \rightarrow v$, the inequality $\alpha(v_n, v) \geq 1$ is satisfied for all $n \geq 1$.

The second category of special maps is an extension of the previous one which was constructed by Mohammadi et al. one year later [30].

Definition 2.11 ([30]) Consider all nondecreasing maps $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ belonging to class Π such that, for all $z > 0, \sum_{j=1}^\infty \psi^j(z) < \infty$ and $\psi(z) < z$. Let $\mathfrak{H} : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M})$ and $\alpha : \mathcal{M}^2 \rightarrow \mathbb{R}_{\geq 0}$. Then

- (i) $\mathfrak{H} : \mathcal{M} \rightarrow \mathcal{P}_{CL, BN}(\mathcal{M})$ is an α - ψ -contraction if, for each $v_1, v_2 \in \mathcal{M}$,

$$\alpha(v_1, v_2)\mathcal{H}_d(\mathfrak{H}v_1, \mathfrak{H}v_2) \leq \psi(d(v_1, v_2)).$$

- (ii) \mathfrak{H} is α -admissible if, for each $v_1 \in \mathcal{M}$ and $v_2 \in \mathfrak{H}v_1$, the inequality $\alpha(v_1, v_2) \geq 1$ gives $\alpha(v_2, v_3) \geq 1$ for each $v_3 \in \mathfrak{H}v_2$.
- (iii) \mathcal{M} has the property (C_α) if, for every sequence $\{v_n\}_{n \geq 1}$ of \mathcal{M} with $v_n \rightarrow v$ and $\alpha(v_n, v_{n+1}) \geq 1$, a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ exists such that $\alpha(v_{n_k}, v) \geq 1$ for all $k \in \mathbb{N}$.

Next theorems are considered as the basis of our arguments until the end of this research.

Theorem 2.12 ([29]) *Consider the complete metric space (\mathcal{M}, d) , $\psi \in \Pi$, $\alpha : \mathcal{M}^2 \rightarrow \mathbb{R}$ and $\Phi^* : \mathcal{M} \rightarrow \mathcal{M}$. Assume that:*

- (i) Φ^* is α -admissible and α - ψ -contraction;
- (ii) $\alpha(v_0, \Phi^*v_0) \geq 1$ for some $v_0 \in \mathcal{M}$;
- (iii) \mathcal{M} has the property (B).

In that case, Φ^ has a fixed point.*

Theorem 2.13 ([31]) *Assuming \mathcal{M} as a Banach space, let $\mathbb{A} \neq \emptyset$ belong to $\mathcal{P}_{CL,BN,CV}(\mathcal{M})$. For \mathfrak{T}_1 and \mathfrak{T}_2 defined on \mathbb{A} , the following assertions hold:*

- (i) $\mathfrak{T}_1v + \mathfrak{T}_2v' \in \mathbb{A}$ for $v, v' \in \mathbb{A}$;
- (ii) \mathfrak{T}_1 is compact-continuous;
- (iii) \mathfrak{T}_2 is a contraction.

Then $v_ \in \mathbb{A}$ exists such that $v_* = \mathfrak{T}_1v_* + \mathfrak{T}_2v_*$.*

Theorem 2.14 ([30]) *Consider the complete metric space (\mathcal{M}, d) , $\psi \in \Pi$, $\alpha : \mathcal{M}^2 \rightarrow \mathbb{R}_{\geq 0}$, and $\mathfrak{H} : \mathcal{M} \rightarrow \mathcal{P}_{CL,BN}(\mathcal{M})$. Assume that:*

- (i) \mathfrak{H} is α -admissible and α - ψ -contraction;
- (ii) $\alpha(v_0, v_1) \geq 1$ for some $v_0 \in \mathcal{M}$ and $v_1 \in \mathfrak{H}v_0$;
- (iii) \mathcal{M} has the property (C_α) .

In that case, \mathfrak{H} has a fixed point.

Theorem 2.15 ([28]) *Consider (\mathcal{M}, d) as a complete metric space. Assume that:*

- (i) $\psi \in \Pi$ is u.s.c coupled with $\liminf_{z \rightarrow \infty} (z - \psi(z)) > 0$ for $z > 0$;
- (ii) $\mathfrak{H} : \mathcal{M} \rightarrow \mathcal{P}_{CL,BN}(\mathcal{M})$ satisfies the property $\mathcal{H}_d(\mathfrak{H}v_1, \mathfrak{H}v_2) \leq \psi(d(v_1, v_2))$ for $v_1, v_2 \in \mathcal{M}$.

Then \mathfrak{H} has a unique endpoint if and only if \mathfrak{H} has the (AEP)-property.

3 Existence criteria for problem (1)

In the present situation of the current research, we start to follow the essential deductions on the existence notion for possible solutions of problem (1) through existing non-linear techniques in the theory of fixed points of an assumed operator. To begin the desired approach, the set $\mathcal{M} = \{v(z) : v(z), {}^C\mathcal{D}_0^1v(z) \in C(\mathbb{J}, \mathbb{R})\}$ regarded as a Banach space is equipped with $\|v\|_{\mathcal{M}} = \sup_{z \in \mathbb{J}} |v(z)| + \sup_{z \in \mathbb{J}} |{}^C\mathcal{D}_0^1v(z)|$ for all $v \in \mathcal{M}$. In the next lemma, the solution of the supposed problem (1) is exhibited in the framework of an integral equation which will be useful for our subsequent arguments.

Lemma 3.1 *Let $\theta^* > 0$, $0 < \sigma^* < 1$, $1 < \delta^* < 2$, $a_1, a_2, a_3, a_4 \in \mathbb{R}^+$, and $\phi \in C(\mathbb{J}, \mathbb{R})$. In that case, the solution of the fractional BVP which takes the structure*

$$\begin{cases} {}^C\mathcal{D}_0^{\sigma^*} [\alpha_1^* {}^C\mathcal{D}_0^2]v(z) = \phi(z) & (z \in \mathbb{J}, \alpha_1^* \in \mathbb{R}^+); \\ v(0) = 0, \\ a_1^C\mathcal{D}_0^1v(0) + a_2^C\mathcal{D}_0^{\delta^*}v(1) = 0, \\ a_3^C\mathcal{D}_0^2v(0) + a_4^{\mathcal{RL}}\mathcal{I}_0^{\theta^*}v(1) = 0 \end{cases} \tag{3}$$

is illustrated as

$$\begin{aligned}
 v(z) = & \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \phi(\tau) \, d\tau + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau \\
 & + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau, \tag{4}
 \end{aligned}$$

where

$$\begin{aligned}
 \nabla_1 = \frac{-a_2}{a_1}, \quad \Delta_1 = & \left[2a_3 + \frac{2\nabla_1 a_4}{\Gamma(3 - \delta^*) \Gamma(2 + \theta^*)} + \frac{2a_4}{\Gamma(3 + \theta^*)} \right], \\
 \nabla_2 = \frac{-a_4}{\Delta_1}, \quad \nabla_3 = \frac{-\nabla_1 a_4}{\Delta_1 \Gamma(2 + \theta^*)}, \quad \nabla_4 = & \frac{2\nabla_1 \nabla_2}{\Gamma(3 - \delta^*)}, \\
 \nabla_5 = \frac{2\nabla_1 \nabla_3}{\Gamma(3 - \delta^*)}, \quad \nabla_6 = & \nabla_5 + \nabla_1 \tag{5}
 \end{aligned}$$

are nonzero constants.

Proof Let $v(z)$ be a solution of (3). With due attention to Proposition 2.4, three constants $c_0, c_1, c_2 \in \mathbb{R}$ exist such that

$$v(z) = \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \phi(\tau) \, d\tau + \frac{c_0}{\alpha_1^*} + \frac{c_1 z}{\alpha_1^*} + \frac{c_2 z^2}{\alpha_1^*}. \tag{6}$$

Using $v(0) = 0$, we have $c_0 = 0$, so (6) becomes

$$v(z) = \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \phi(\tau) \, d\tau + \frac{c_1 z}{\alpha_1^*} + \frac{c_2 z^2}{\alpha_1^*}.$$

Now, we compute

$$\begin{aligned}
 {}^C \mathcal{D}_0^1 v(z) = & \frac{1}{\alpha_1^* \Gamma(\sigma^* + 1)} \int_0^z (z - \tau)^{\sigma^*} \phi(\tau) \, d\tau + \frac{c_1}{\alpha_1^*} + \frac{2c_2 z}{\alpha_1^*}, \\
 {}^C \mathcal{D}_0^2 v(z) = & \frac{1}{\alpha_1^* \Gamma(\sigma^*)} \int_0^z (z - \tau)^{\sigma^* - 1} \phi(\tau) \, d\tau + \frac{2c_2}{\alpha_1^*}, \\
 {}^C \mathcal{D}_0^{\delta^*} v(z) = & \frac{1}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^z (z - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau + \frac{2c_2 z^{2 - \delta^*}}{\alpha_1^* \Gamma(3 - \delta^*)} \quad (1 < \delta^* < 2)
 \end{aligned}$$

and

$${}^{\mathcal{R}\mathcal{L}} \mathcal{J}_0^{\theta^*} v(z) = \frac{1}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^z (z - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau + \frac{c_1 z^{1 + \theta^*}}{\alpha_1^* \Gamma(2 + \theta^*)} + \frac{2c_2 z^{2 + \theta^*}}{\alpha_1^* \Gamma(3 + \theta^*)}.$$

By using the second boundary condition $a_1^C \mathcal{D}_0^1 v(0) + a_2^C \mathcal{D}_0^{\delta^*} v(1) = 0$, this gives

$$a_1 c_1 + \frac{a_2}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau + \frac{2a_2 c_2}{\Gamma(3 - \delta^*)} = 0. \tag{7}$$

By using the third boundary condition $a_3^C \mathcal{D}_0^2 v(0) + a_4^{\mathcal{R}\mathcal{L}} \mathcal{J}_0^{\theta^*} v(1) = 0$, we get that

$$2a_3 c_2 + \frac{a_4}{\Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau + \frac{a_4 c_1}{\Gamma(2 + \theta^*)} + \frac{2a_4 c_2}{\Gamma(3 + \theta^*)} = 0. \tag{8}$$

From (7), we have

$$c_1 = \frac{-a_2}{a_1} \left[\frac{1}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau + \frac{2c_2}{\Gamma(3 - \delta^*)} \right],$$

and hence, by the above notations displayed in (5), we get

$$c_1 = \nabla_1 \left[\frac{1}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau + \frac{2c_2}{\Gamma(3 - \delta^*)} \right]. \tag{9}$$

Inserting the obtained value for c_1 into (8), we reach

$$\begin{aligned} c_2 & \left[2a_3 + \frac{2\nabla_1 a_4}{\Gamma(3 - \delta^*)\Gamma(2 + \theta^*)} + \frac{2a_4}{\Gamma(3 + \theta^*)} \right] \\ & = -\frac{a_4}{\Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau \\ & \quad - \frac{\nabla_1 a_4}{\Gamma(\sigma^* - \delta^* + 2)\Gamma(2 + \theta^*)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau, \end{aligned}$$

and by (5) this turns into

$$\begin{aligned} c_2 & = \frac{-a_4}{\Delta_1 \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau \\ & \quad + \frac{-a_4 \nabla_1}{\Delta_1 \Gamma(\sigma^* - \delta^* + 2)\Gamma(2 + \theta^*)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau, \end{aligned}$$

and so again by (5) it becomes

$$\begin{aligned} c_2 & = \frac{\nabla_2}{\Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau \\ & \quad + \frac{\nabla_3}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau. \end{aligned}$$

On the other hand, putting the value of c_2 in (9), we try to find c_1 as

$$\begin{aligned} c_1 & = \frac{2\nabla_1 \nabla_2}{\Gamma(3 - \delta^*)\Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau \\ & \quad + \frac{2\nabla_1 \nabla_3}{\Gamma(\sigma^* - \delta^* + 2)\Gamma(3 - \delta^*)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau \\ & \quad + \frac{\nabla_1}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau, \end{aligned}$$

which implies

$$\begin{aligned} c_1 & = \frac{\nabla_4}{\Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau \\ & \quad + \frac{\nabla_5}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau \\ & \quad + \frac{\nabla_1}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau, \end{aligned}$$

and so

$$c_1 = \frac{\nabla_4}{\Gamma(\theta^* + \sigma^* + 2)} \int_0^1 (1 - \tau)^{\theta^* + \sigma^* + 1} \phi(\tau) \, d\tau + \frac{\nabla_5 + \nabla_1}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau.$$

Finally, in view of notations (5), we have

$$c_1 = \frac{\nabla_4}{\Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau + \frac{\nabla_6}{\Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau.$$

At last, we put values of c_0, c_1, c_2 computed by the above procedures in (6), and we reach

$$v(z) = \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \phi(\tau) \, d\tau + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \phi(\tau) \, d\tau + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \phi(\tau) \, d\tau.$$

This completes our proof about the structure of solutions for BVP (3). □

Inspired by the previous lemma, we now assume an operator $\mathfrak{T} : \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$(\mathfrak{T}v)(z) = \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Phi^*(\tau, v(\tau), {}^C \mathfrak{D}_0^1 v(\tau)) \, d\tau + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Phi^*(\tau, v(\tau), {}^C \mathfrak{D}_0^1 v(\tau)) \, d\tau + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Phi^*(\tau, v(\tau), {}^C \mathfrak{D}_0^1 v(\tau)) \, d\tau.$$

It is to be noted that v_0 is regarded as a solution for supposed BVP (1) iff v_0 is a fixed point for the newly-defined map \mathfrak{T} . Set:

$$\begin{aligned} \Omega_1^* &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)}, \\ \Omega_2^* &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} + \frac{|\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{|\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)}, \\ \Omega_3^* &= \frac{\|\xi\| |\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{\|\xi\| |\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)}, \\ \Omega_4^* &= \frac{\|\xi\| |\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{\|\xi\| |\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)}. \end{aligned} \tag{10}$$

We are at this moment ready to express and verify the first theorem pertinent to the existence conditions of solutions for BVP (1).

Theorem 3.2 *Let $\psi \in \Pi$ and $\chi^* : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a map along with a continuous map $\Phi^* : \mathbb{J} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$. Suppose that:*

- (i) *for all $v_1, v'_1, v_2, v'_2 \in \mathcal{M}$ with $\chi^*((v_1(z), v_2(z)), (v'_1(z), v'_2(z))) \geq 0$ and $\lambda^* = \frac{1}{\Omega_1^* + \Omega_2^*}$, we have*

$$|\Phi^*(z, v_1(z), v_2(z)) - \Phi^*(z, v'_1(z), v'_2(z))| \leq \lambda^* \psi(|v_1 - v'_1| + |v_2 - v'_2|);$$

- (ii) *there exists $v_0 \in \mathcal{M}$ such that, for all $z \in \mathbb{J}$,*

$$\chi^*((v_0(z), {}^C\mathcal{D}_0^1 v_0(z)), (\mathfrak{I}v_0(z), {}^C\mathcal{D}_0^1 \mathfrak{I}v_0(z))) \geq 0,$$

and also

$$\chi^*((v_1(z), {}^C\mathcal{D}_0^1 v_1(z)), (v_2(z), {}^C\mathcal{D}_0^1 v_2(z))) \geq 0$$

implies

$$\chi^*((\mathfrak{I}v_1(z), {}^C\mathcal{D}_0^1 \mathfrak{I}v_1(z)), (\mathfrak{I}v_2(z), {}^C\mathcal{D}_0^1 \mathfrak{I}v_2(z))) \geq 0$$

for all $z \in \mathbb{J}$ and $v_1, v_2 \in \mathcal{M}$;

- (iii) *for an arbitrary sequence $\{v_n\}_{n \geq 1} \subseteq \mathcal{M}$ with $v_n \rightarrow v$ and*

$$\chi^*((v_n(z), {}^C\mathcal{D}_0^1 v_n(z)), (v_{n+1}(z), {}^C\mathcal{D}_0^1 \mathfrak{I}v_{n+1}(z))) \geq 0$$

for any $n \in \mathbb{N}$ and $z \in \mathbb{J}$, the following inequality holds:

$$\chi^*((v_n(z), {}^C\mathcal{D}_0^1 v_n(z)), (v(z), {}^C\mathcal{D}_0^1 \mathfrak{I}v(z))) \geq 0.$$

Then the given BVP (1) has at least one solution.

Proof Let $v_1, v_2 \in \mathcal{M}$ subject to $\chi^*((v_1(z), {}^C\mathcal{D}_0^1 v_1(z)), (v_2(z), {}^C\mathcal{D}_0^1 v_2(z))) \geq 0$ for every $z \in \mathbb{J}$. In that case, one can estimate

$$\begin{aligned} & |\mathfrak{I}v_1(z) - \mathfrak{I}v_2(z)| \\ & \leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} |\Phi^*(\tau, v_1(\tau), {}^C\mathcal{D}_0^1 v_1(\tau)) - \Phi^*(\tau, v_2(\tau), {}^C\mathcal{D}_0^1 v_2(\tau))| d\tau \\ & \quad + \frac{|\nabla_4 z + \nabla_2 z^2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \\ & \quad \times \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} |\Phi^*(\tau, v_1(\tau), {}^C\mathcal{D}_0^1 v_1(\tau)) - \Phi^*(\tau, v_2(\tau), {}^C\mathcal{D}_0^1 v_2(\tau))| d\tau \\ & \quad + \frac{|\nabla_6 z + \nabla_3 z^2|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \\ & \quad \times \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} |\Phi^*(\tau, v_1(\tau), {}^C\mathcal{D}_0^1 v_1(\tau)) - \Phi^*(\tau, v_2(\tau), {}^C\mathcal{D}_0^1 v_2(\tau))| d\tau \\ & \leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} \lambda^* \psi(|v_1(\tau) - v_2(\tau)| + |{}^C\mathcal{D}_0^1 v_1(\tau) - {}^C\mathcal{D}_0^1 v_2(\tau)|) \end{aligned}$$

$$\begin{aligned}
 & + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} \lambda^* \psi (|v_1(\tau) - v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau) - {}^C \mathcal{D}_0^1 v_2(\tau)|) \\
 & + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \lambda^* \psi (|v_1(\tau) - v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau) - {}^C \mathcal{D}_0^1 v_2(\tau)|) \\
 \leq & \frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} \lambda^* \psi (\|v_1 - v_2\|) + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} \lambda^* \psi (\|v_1 - v_2\|) \\
 & + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \lambda^* \psi (\|v_1 - v_2\|) \\
 = & \lambda^* \Omega_1^* \psi (\|v_1 - v_2\|).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 |{}^C \mathcal{D}_0^1 \mathfrak{T} v_1(z) - {}^C \mathcal{D}_0^1 \mathfrak{T} v_2(z)| \leq & \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \lambda^* \psi (\|v_1 - v_2\|) \\
 & + \frac{|\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} \lambda^* \psi (\|v_1 - v_2\|) \\
 & + \frac{|\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \lambda^* \psi (\|v_1 - v_2\|) \\
 = & \lambda^* \Omega_2^* \psi (\|v_1 - v_2\|).
 \end{aligned}$$

Hence, $|\mathfrak{T} v_1(z) - \mathfrak{T} v_2(z)| \leq (\Omega_1^* + \Omega_2^*) \lambda^* \psi (\|v_1 - v_2\|) = \psi(v_1 - v_2)$. We now build α on $\mathcal{M} \times \mathcal{M}$ by

$$\alpha(v_1, v_2) = \begin{cases} 1 & \text{if } \chi^*((v_1(z), {}^C \mathcal{D}_0^1 v_1(z)), (v_2(z), {}^C \mathcal{D}_0^1 v_2(z))) \geq 0; \\ 0 & \text{otherwise} \end{cases}$$

for all $v_1, v_2 \in \mathcal{M}$. Then we have $\alpha(v_1, v_2)d(\mathfrak{T} v_1, \mathfrak{T} v_2) \leq \psi(d(v_1, v_2))$ for all $v_1, v_2 \in \mathcal{M}$. From this, we see that \mathfrak{T} is an $\alpha - \psi$ -contraction. Moreover, one can easily observe that it is α -admissible and $\alpha(v_0, \mathfrak{T} v_0) \geq 1$ by the definition of χ^* .

Now, in this situation, for all n , we suppose that $\{v_n\}$ is a sequence which belongs to \mathcal{M} with $v_n \rightarrow v$ and $\alpha(v_n, v_{n+1}) \geq 1$. In view of the structure of α ,

$$\chi^*((v_n(z), {}^C \mathcal{D}_0^1 v_n(z)), (v_{n+1}(z), {}^C \mathcal{D}_0^1 \mathfrak{T} v_{n+1}(z))) \geq 0.$$

Thus, by the hypothesis, we get

$$\chi^*((v_n(z), {}^C \mathcal{D}_0^1 v_n(z)), (v(z), {}^C \mathcal{D}_0^1 \mathfrak{T} v(z))) \geq 0.$$

It validates that $\alpha(v_n, v) \geq 1$. Therefore, Banach space \mathcal{M} has the property (B). From Theorem 2.12, we find that \mathfrak{T} has $v^* \in \mathcal{M}$ as a fixed point, and in consequence, the supposed BVP (1) contains a solution. The argument is finished. \square

By following the argument, we focus on our intention to find other existence requirements of solutions for BVP (1) by means of another tool based on the fixed point.

Theorem 3.3 Consider a continuous function $\Phi^* : \mathbb{J} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$. Suppose that:

(iv) A continuous function ξ is formulated on the closed interval \mathbb{J} such that

$$|\Phi^*(z, v_1, v_2) - \Phi^*(z, v'_1, v'_2)| \leq \xi(z) \left(\sum_{k=1}^2 |v_k - v'_k| \right)$$

for all $z \in \mathbb{J}$ and $v_1, v_2, v'_1, v'_2 \in \mathcal{M}$;

(v) There are a continuous function $\eta^* : \mathbb{J} \rightarrow \mathbb{R}^+$ and a nondecreasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$|\Phi^*(z, v_1, v_2)| \leq \eta^*(z) \psi \left(\sum_{k=1}^2 |v_k| \right)$$

for all $z \in \mathbb{J}$ and $v_1, v_2 \in \mathcal{M}$.

Then BVP (1) has at least one solution whenever

$$K^* = \Omega_3^* + \Omega_4^* < 1$$

by terms of constants Ω_3^* and Ω_4^* given by (10).

Proof Let $\|\eta^*\| = \sup_{z \in \mathbb{J}} |\eta^*(z)|$ and put $m^* = \sup_{v \in \mathcal{M}} \psi(\|v\|)$, and assume that there is $\epsilon > 0$ such that

$$\epsilon \geq m^* \|\eta^*\| \{ \Omega_1^* + \Omega_2^* \},$$

where Ω_i^* s are given in (10). We define a set $\mathcal{A}_\epsilon = \{v \in \mathcal{M} : \|v\| \leq \epsilon\}$, and it is easy to verify that $\mathcal{A}_\epsilon \neq \emptyset$ belongs to $\mathcal{P}_{CL}(\mathcal{M})$, $\mathcal{P}_{CV}(\mathcal{M})$, and $\mathcal{P}_{BD}(\mathcal{M})$ simultaneously. Now we define two fractional operators \mathfrak{T}^1 and \mathfrak{T}^2 on a set \mathcal{A}_ϵ by

$$\mathfrak{T}^1 v(z) = \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau)) \, d\tau \tag{11}$$

and

$$\begin{aligned} \mathfrak{T}^2 v(z) &= \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau)) \, d\tau \\ &\quad + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau)) \, d\tau. \end{aligned} \tag{12}$$

For all $z \in \mathbb{J}$ and for $v_1, v_2 \in \mathcal{A}_\epsilon$, we have

$$\begin{aligned} &|\mathfrak{T}^1 v_1(z) + \mathfrak{T}^2 v_2(z)| \\ &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \eta^*(\tau) \psi(|v_1(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau)|) \, d\tau \\ &\quad + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\theta^* + \sigma^* + 2)} \int_0^1 (1 - \tau)^{\theta^* + \sigma^* + 1} \eta^*(\tau) \psi(|v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau \\ &\quad + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \eta^*(\tau) \psi(|v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau \end{aligned}$$

$$\begin{aligned} &\leq m^* \|\eta^*\| \left[\frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \right] \\ &= m^* \|\eta^*\| \Omega_1^*. \end{aligned}$$

Also

$$\begin{aligned} &|{}^C \mathcal{D}_0^1 \mathfrak{T}^1 v_1(z) + {}^C \mathcal{D}_0^1 \mathfrak{T}^2 v_2(z)| \\ &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 1)} \int_0^z (z - \tau)^{\sigma^*} \eta^*(\tau) \psi(|v_1(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau)|) \, d\tau \\ &\quad + \frac{|\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \eta^*(\tau) \psi(|v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau \\ &\quad + \frac{|\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \eta^*(\tau) \psi(|v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau \\ &\leq m^* \|\eta^*\| \left[\frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} + \frac{|\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{|\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \right] \\ &= m^* \|\eta^*\| \Omega_2^*. \end{aligned}$$

Hence $\|\mathfrak{T}^1 v_1(z) + \mathfrak{T}^2 v_2(z)\| \leq \epsilon$, which implies that $\mathfrak{T}^1 v_1(z) + \mathfrak{T}^2 v_2(z) \in \mathcal{A}_\epsilon$. On the other hand, since Φ^* is continuous, this ensures that \mathfrak{T}^1 will be continuous. Moreover, along with these, we compute

$$\begin{aligned} |\mathfrak{T}^1 v(z)| &\leq \frac{1}{\alpha_1^* \Gamma(2 + \sigma^*)} \int_0^z (z - \tau)^{\sigma^* + 1} |\Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau))| \, d\tau \\ &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} \|\eta^*\| m^* \end{aligned}$$

and

$$\begin{aligned} |{}^C \mathcal{D}_0^1 \mathfrak{T}^1 v(z)| &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 1)} \int_0^z (z - \tau)^{\sigma^*} |\Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau))| \, d\tau \\ &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \|\eta^*\| m^* \end{aligned}$$

for all $v \in \mathcal{A}_\epsilon$. Hence,

$$\|\mathfrak{T}^1 v(z)\| \leq \left(\frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} + \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \right) \|\eta^*\| m^*,$$

which guarantees that \mathfrak{T}^1 is uniformly bounded on \mathcal{A}_ϵ . Now, in the next step, we show that the operator \mathfrak{T}^1 is compact on \mathcal{A}_ϵ , for this we assume that $z_1, z_2 \in \mathbb{J}$ with $z_2 > z_1$. Thus we have

$$\begin{aligned} &|\mathfrak{T}^1 v(z_2) - \mathfrak{T}^1 v(z_1)| \\ &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^{z_1} [(z_2 - \tau)^{\sigma^* + 1} - (z_1 - \tau)^{\sigma^* + 1}] |\Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau))| \, d\tau \\ &\quad + \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_{z_1}^{z_2} (z_2 - \tau)^{\sigma^* + 1} |\Phi^*(\tau, v(\tau), {}^C \mathcal{D}_0^1 v(\tau))| \, d\tau \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^{z_1} [(z_2 - \tau)^{\sigma^*+1} - (z_1 - \tau)^{\sigma^*+1}] \eta^*(\tau) \psi(|v(\tau)| + |{}^C \mathcal{D}_0^1 v(\tau)|) \, d\tau \\ &\quad + \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_{z_1}^{z_2} (z_2 - \tau)^{\sigma^*+1} \eta^*(\tau) \psi(|v(\tau)| + |{}^C \mathcal{D}_0^1 v(\tau)|) \, d\tau \\ &\leq \frac{\|\eta^*\| m^*}{\alpha_1^* \Gamma(\sigma^* + 3)} [-(z_2 - z_1)^{\sigma^*+2} + (z_2 - z_1)^{\sigma^*+2} + z_2^{\sigma^*+2} - z_1^{\sigma^*+2}]. \end{aligned}$$

So $|\mathfrak{T}^1 v(z_2) - \mathfrak{T}^1 v(z_1)| \rightarrow 0$ when $z_1 \rightarrow z_2$. Also, in a similar manner,

$$\begin{aligned} &|{}^C \mathcal{D}_0^1 \mathfrak{T}^1 v(z_2) - {}^C \mathcal{D}_0^1 \mathfrak{T}^1 v(z_1)| \\ &\leq \frac{\|\eta^*\| m^*}{\alpha_1^* \Gamma(\sigma^* + 2)} [-(z_2 - z_1)^{\sigma^*+1} + (z_2 - z_1)^{\sigma^*+1} + z_2^{\sigma^*+1} - z_1^{\sigma^*+1}], \end{aligned}$$

which implies that $|{}^C \mathcal{D}_0^1 \mathfrak{T}^1 v(z_2) - {}^C \mathcal{D}_0^1 \mathfrak{T}^1 v(z_1)| \rightarrow 0$ as $z_1 \rightarrow z_2$; hence $\|\mathfrak{T}^1 v(z_2) - \mathfrak{T}^1 v(z_1)\| \rightarrow 0$ as $z_1 \rightarrow z_2$. Thus \mathfrak{T}^1 is equicontinuous and \mathfrak{T}^1 is relatively compact on \mathcal{A}_ϵ . Now, by the Arzela–Ascoli theorem, it is compact. Finally, we prove that \mathfrak{T}^2 is a contraction. For $v_1, v_2 \in \mathcal{A}_\epsilon$, we get

$$\begin{aligned} &|\mathfrak{T}^2 v_1(z) - \mathfrak{T}^2 v_2(z)| \\ &\leq \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \\ &\quad \times \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \xi(\tau) (|v_1(\tau) - v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau) - {}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau \\ &\quad + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \\ &\quad \times \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \xi(\tau) (|v_1(\tau) - v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau) - {}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau, \end{aligned}$$

also

$$\begin{aligned} &|{}^C \mathcal{D}_0^1 \mathfrak{T}^2 v_1(z) - {}^C \mathcal{D}_0^1 \mathfrak{T}^2 v_2(z)| \\ &\leq \frac{|\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \\ &\quad \times \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \xi(\tau) (|v_1(\tau) - v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau) - {}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau \\ &\quad + \frac{|\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \\ &\quad \times \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \xi(\tau) (|v_1(\tau) - v_2(\tau)| + |{}^C \mathcal{D}_0^1 v_1(\tau) - {}^C \mathcal{D}_0^1 v_2(\tau)|) \, d\tau. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \sup_{z \in \mathbb{J}} |\mathfrak{T}^2 v_1(z) - \mathfrak{T}^2 v_2(z)| &\leq \left[\frac{\|\xi\| |\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{\|\xi\| |\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \right] \|v_1 - v_2\|, \\ \sup_{z \in \mathbb{J}} |{}^C \mathcal{D}_0^1 \mathfrak{T}^2 v_1(z) - {}^C \mathcal{D}_0^1 \mathfrak{T}^2 v_2(z)| &\leq \left[\frac{\|\xi\| |\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{\|\xi\| |\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \right] \|v_1 - v_2\|, \end{aligned}$$

where $\|\xi\| = \sup_{z \in \mathbb{J}} |\xi(z)|$. Hence, $\|\mathfrak{I}^2 v_1(z) - \mathfrak{I}^2 v_2(z)\| \leq (\Omega_3^* + \Omega_4^*) \|v_1 - v_2\|$, which implies that

$$\|\mathfrak{I}^2 v_1(z) - \mathfrak{I}^2 v_2(z)\| \leq K^* \|v_1 - v_2\|.$$

Thus, \mathfrak{I}^2 is a contraction on \mathcal{A}_ϵ with constant $K^* = \Omega_3^* + \Omega_4^* < 1$. So, from Theorem 2.13, BVP (1) has at least one solution and the proof is completed. \square

4 Existence criteria for problem (2)

In this part, we derive the existence of solutions to the inclusion BVP (2). The function $v \in C(\mathbb{J}, \mathcal{M})$ is called the solution of problem (2) when it satisfies the boundary conditions and there is $\Upsilon \in L^1(\mathbb{J})$ such that $\Upsilon(z) \in \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z))$ for almost all $z \in \mathbb{J}$ and

$$\begin{aligned} v(z) = & \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon(\tau) \, d\tau \\ & + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon(\tau) \, d\tau \\ & + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$. For each $v \in \mathcal{M}$, we define the set of selections of the operator \mathfrak{H} as

$$\mathfrak{S}_{\mathfrak{H}, v} = \{ \Upsilon \in L^1(\mathbb{J}) : \Upsilon(z) \in \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z)) \text{ for almost all } z \in \mathbb{J} \}.$$

Moreover, define the operator $\mathfrak{N} : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M})$ by

$$\mathfrak{N}(v) = \{ p \in \mathcal{M} : \text{there exists } \Upsilon \in \mathfrak{S}_{\mathfrak{H}, v} \text{ such that } p(z) = \pi(z) \, \forall z \in \mathbb{J} \}, \tag{13}$$

where

$$\begin{aligned} \pi(z) = & \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon(\tau) \, d\tau \\ & + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon(\tau) \, d\tau \\ & + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon(\tau) \, d\tau. \end{aligned}$$

Theorem 4.1 *Let $\mathfrak{H} : \mathbb{J} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{P}_{CP}(\mathcal{M})$ be a multifunction. Assume that:*

- (vi) *The multifunction \mathfrak{H} is integrable and bounded and $\mathfrak{H}(\cdot, v_1, v_2) : \mathbb{J} \rightarrow \mathcal{P}_{CP}(\mathcal{M})$ is measurable for $v_1, v_2 \in \mathcal{M}$;*
- (vii) *There exist $\omega \in C(\mathbb{J}, [0, \infty))$ and $\psi \in \Pi$ such that*

$$\mathcal{H}_d(\mathfrak{H}(z, v_1, v_2), \mathfrak{H}(z, \acute{v}_1, \acute{v}_2)) \leq \frac{\omega(z)\lambda^*}{\|\omega\|} \psi \left(\sum_{k=1}^2 |v_k - \acute{v}_k| \right)$$

for all $z \in \mathbb{J}$ and $v_1, v_2, \acute{v}_1, \acute{v}_2 \in \mathcal{M}$, where $\lambda^* = \frac{1}{\Omega_1^* + \Omega_2^*}$;

(viii) There is $\chi^* : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\chi^*((v_1, v_2), (\dot{v}_1, \dot{v}_2)) \geq 0$ for all $v_k, \dot{v}_k \in \mathcal{M}$ ($k = 1, 2$);

(ix) If $\{v_n\}$ is a sequence in \mathcal{M} with $v_n \rightarrow v$ and

$$\chi^*((v_n(z), {}^C\mathcal{D}_0^1 v_n(z)), (v_{n+1}(z), {}^C\mathcal{D}_0^1 v_{n+1}(z))) \geq 0$$

for all $z \in \mathbb{J}$ and $n \geq 1$, then there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that

$$\chi^*((v_{n_j}(z), {}^C\mathcal{D}_0^1 v_{n_j}(z)), (v(z), {}^C\mathcal{D}_0^1 v(z))) \geq 0$$

for $z \in \mathbb{J}$ and $j \geq 1$;

(x) There exist $v_0 \in \mathcal{M}$ and $p \in \mathfrak{N}(v_0)$ such that

$$\chi^*((v_0(z), {}^C\mathcal{D}_0^1 v_0(z)), (p(z), {}^C\mathcal{D}_0^1 p(z))) \geq 0 \quad (z \in \mathbb{J}),$$

in which $\mathfrak{N} : \mathcal{M} \rightarrow P(\mathcal{M})$ is defined by (13);

(xi) For any $v \in \mathcal{M}$ and $p \in \mathfrak{N}(v)$ with

$$\chi^*((v(z), {}^C\mathcal{D}_0^1 v(z)), (p(z), {}^C\mathcal{D}_0^1 p(z))) \geq 0,$$

there exists a member $w^* \in \mathfrak{N}(v)$ such that

$$\chi^*((p(z), {}^C\mathcal{D}_0^1 p(z)), (w^*(z), {}^C\mathcal{D}_0^1 w^*(z))) \geq 0$$

for all $z \in \mathbb{J}$.

In that case, the inclusion problem (2) has at least one solution.

Proof Obviously, the fixed point of $\mathfrak{N} : \mathcal{M} \rightarrow P(\mathcal{M})$ is the solution of BVP (2). Since the multi-valued map $z \rightarrow \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z))$ is closed-valued and measurable for all $v \in \mathcal{M}$, so \mathfrak{H} has a measurable selection and $\mathfrak{S}_{\mathfrak{H}, v}$ is nonempty. We have to prove that $\mathfrak{N}(v)$ is closed in \mathcal{M} for $v \in \mathcal{M}$. Take $\{v_n\}$ in $\mathfrak{N}(v)$ that $v_n \rightarrow v$. For each n , $\Upsilon_n \in \mathfrak{S}_{\mathfrak{H}, v}$ is chosen such that

$$\begin{aligned} v_n(z) &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon_n(\tau) \, d\tau \\ &+ \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon_n(\tau) \, d\tau \\ &+ \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon_n(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$. Since \mathfrak{H} has compact values, we define a subsequence of $\{\Upsilon_n\}$ (again by the same notation) which converges to $\Upsilon \in L^1([0, 1])$. Hence, $\Upsilon \in \mathfrak{S}_{\mathfrak{H}, v}$ and

$$\begin{aligned} v_n(z) \rightarrow v(z) &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon(\tau) \, d\tau \\ &+ \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon(\tau) \, d\tau \\ &+ \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$, which gives $v \in \mathfrak{N}(v)$ and \mathfrak{N} is closed-valued. As \mathfrak{H} is compact-valued, it is a simple task to affirm the boundedness of $\mathfrak{N}(v)$ for arbitrary $v \in \mathcal{M}$. We have to prove that \mathfrak{N} is an $\alpha - \psi$ -contraction. For such a goal, we define

$$\alpha(v, \acute{v}) = \begin{cases} 1 & \text{if } \chi^*((v(z), {}^C\mathcal{D}_0^1 v(z)), (\acute{v}(z), {}^C\mathcal{D}_0^1 \acute{v}(z))); \\ 0 & \text{otherwise} \end{cases}$$

for all $v, \acute{v} \in X$. Let $v, \acute{v} \in X$ and $h_1^* \in \mathfrak{N}(\acute{v})$ and choose $\Upsilon_1 \in \mathfrak{S}_{\mathfrak{H}, \acute{v}}$ such that

$$\begin{aligned} h_1^*(z) &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon_1(\tau) \, d\tau \\ &\quad + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon_1(\tau) \, d\tau \\ &\quad + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon_1(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$. We estimate

$$\mathcal{H}_a(\mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z)), \mathfrak{H}(z, \acute{v}(z), {}^C\mathcal{D}_0^1 \acute{v}(z))) \leq \frac{\omega(z)\lambda^*}{\|\omega\|} \psi(|v_1 - \acute{v}_1| - |{}^C\mathcal{D}_0^1 \acute{v} - {}^C\mathcal{D}_0^1 v|)$$

for all $v, \acute{v} \in \mathcal{M}$ with $\chi^*((v(z), {}^C\mathcal{D}_0^1 v(z)), (\acute{v}(z), {}^C\mathcal{D}_0^1 \acute{v}(z))) \geq 0$ for almost all $z \in \mathbb{J}$. Thus there exists $\pi \in \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z))$ such that

$$|\Upsilon_1(z) - \pi| \leq \frac{\omega(z)\lambda^*}{\|\omega\|} \psi(|v_1 - \acute{v}_1| - |{}^C\mathcal{D}_0^1 \acute{v} - {}^C\mathcal{D}_0^1 v|).$$

Now let $\mathfrak{B}^* : [0, 1] \rightarrow \mathcal{P}(\mathcal{M})$ be a multi-valued map as

$$\mathfrak{B}^*(z) = \left\{ \pi \in \mathcal{M} : |\Upsilon_1(z) - \pi| \leq \frac{\omega(z)\lambda^*}{\|\omega\|} \psi(|v_1 - \acute{v}_1| - |{}^C\mathcal{D}_0^1 \acute{v} - {}^C\mathcal{D}_0^1 v|) \right\}$$

for all $z \in \mathbb{J}$. As Υ_1 and $\zeta = \frac{\omega\lambda^*}{\|\omega\|} \psi(|v_1 - \acute{v}_1| - |{}^C\mathcal{D}_0^1 \acute{v} - {}^C\mathcal{D}_0^1 v|)$ are measurable, the multi-valued function $\mathfrak{B}^*(\cdot) \cap \mathfrak{H}(\cdot, v(\cdot), {}^C\mathcal{D}_0^1 v(\cdot))$ is too. Now let $\Upsilon_2 \in \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z))$ provided

$$|\Upsilon_1(z) - \Upsilon_2(z)| \leq \frac{\omega(z)\lambda^*}{\|\omega\|} \psi(|v_1 - \acute{v}_1| - |{}^C\mathcal{D}_0^1 \acute{v} - {}^C\mathcal{D}_0^1 v|)$$

for all $z \in \mathbb{J}$. Let us define $h_2^* \in \mathfrak{N}(z)$ by

$$\begin{aligned} h_2^*(z) &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon_2(\tau) \, d\tau \\ &\quad + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon_2(\tau) \, d\tau \\ &\quad + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon_2(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$. Let $\sup_{z \in \mathbb{J}} |\omega(z)| = \|\omega\|$. Then

$$\begin{aligned} |h_1^* - h_2^*| &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} |\Upsilon_1(\tau) - \Upsilon_2(\tau)| \, d\tau \\ &\quad + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} |\Upsilon_1(\tau) - \Upsilon_2(\tau)| \, d\tau \\ &\quad + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} |\Upsilon_1(\tau) - \Upsilon_2(\tau)| \, d\tau \\ &\leq \frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} \|\omega\| \psi(\|v - \hat{v}\|) \frac{\lambda^*}{\|\omega\|} \\ &\quad + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} \|\omega\| \psi(\|v - \hat{v}\|) \frac{\lambda^*}{\|\omega\|} \\ &\quad + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \|\omega\| \psi(\|v - \hat{v}\|) \frac{\lambda^*}{\|\omega\|} \\ &= \left[\frac{1}{\alpha_1^* \Gamma(\sigma^* + 3)} + \frac{|\nabla_4 + \nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{|\nabla_6 + \nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \right] \\ &\quad \times \|\omega\| \psi(\|v - \hat{v}\|) \frac{\lambda^*}{\|\omega\|} \\ &= \Omega_1^* \|\omega\| \psi(\|v - \hat{v}\|) \frac{\lambda^*}{\|\omega\|} \\ &= \Omega_1^* \lambda^* \psi(\|v - \hat{v}\|). \end{aligned}$$

Also we have

$$\begin{aligned} |{}^c \mathcal{D}_0^1 h_1^* - {}^c \mathcal{D}_0^1 h_2^*| &\leq \left[\frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} + \frac{|\nabla_4 + 2\nabla_2|}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 3)} + \frac{|\nabla_6 + 2\nabla_3|}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 3)} \right] \|\omega\| \psi(\|v - \hat{v}\|) \frac{\lambda^*}{\|\omega\|} \\ &= \Omega_2^* \lambda^* \psi(\|v - \hat{v}\|) \end{aligned}$$

for all $z \in \mathbb{J}$. Hence

$$\begin{aligned} \|h_1^* - h_2^*\| &= \sup_{z \in \mathbb{J}} |h_1^*(z) - h_2^*(z)| + \sup_{z \in \mathbb{J}} |{}^c \mathcal{D}_0^1 h_1^*(z) - {}^c \mathcal{D}_0^1 h_2^*(z)| \\ &\leq \lambda^* \psi(\|v - \hat{v}\|) (\Omega_1^* + \Omega_2^*) \\ &= \psi(\|v - \hat{v}\|). \end{aligned}$$

Thus $\alpha(v, \hat{v}) \mathcal{H}_d(\mathfrak{N}(v), \mathfrak{N}(\hat{v})) \leq \psi(\|v - \hat{v}\|)$ holds for all $v, \hat{v} \in \mathcal{M}$, which implies that \mathfrak{N} is an $\alpha - \psi$ -contraction. Now, let $v \in \mathcal{M}$ and $\hat{v} \in \mathfrak{N}(v)$ be two functions such that $\alpha(v, \hat{v}) \geq 1$. In this case, $\chi^*((v(z), {}^c \mathcal{D}_0^1 v(z)), (\hat{v}(z), {}^c \mathcal{D}_0^1 \hat{v}(z))) \geq 0$; so, $\pi \in \mathfrak{N}(\hat{v})$ exists such that $\chi^*((\hat{v}(z), {}^c \mathcal{D}_0^1 \hat{v}(z)), (\pi(z), {}^c \mathcal{D}_0^1 \pi(z))) \geq 0$. It follows from this that $\alpha(\hat{v}, \pi) \geq 1$, which means that the operator \mathfrak{N} is α -admissible. Now, suppose that $v_0 \in \mathcal{M}$ and $\hat{v} \in \mathfrak{N}(v_0)$ are such that $\chi^*((v_0(z), {}^c \mathcal{D}_0^1 v_0(z)), (\hat{v}(z), {}^c \mathcal{D}_0^1 \hat{v}(z))) \geq 0$ for any $z \in \mathbb{J}$. Subsequently, we have

$\alpha(v_0, \acute{v}) \geq 1$. Consider $\{v_n\} \subseteq \mathcal{M}$ coupled with $v_n \rightarrow v$ and $\alpha(v_n, v_{n+1}) \geq 1$. Then we get

$$\chi^*((v_n(z), {}^C\mathcal{D}_0^1 v_n(z)), (v_{n+1}(z), {}^C\mathcal{D}_0^1 v_{n+1}(z))) \geq 0.$$

By using hypothesis (ix), there is a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that

$$\chi^*((v_{n_j}(z), {}^C\mathcal{D}_0^1 v_{n_j}(z)), (v(z), {}^C\mathcal{D}_0^1 v(z))) \geq 0 \quad (\forall z \in \mathbb{J}).$$

Thus $\alpha(v_{n_j}, v) \geq 1$ ($\forall j$); that is, \mathcal{M} has the property C_α . Theorem 2.14 guarantees that \mathfrak{N} has a fixed point which is the solution of the inclusion BVP (2). \square

Theorem 4.2 Consider the multifunction $\mathfrak{H} : \mathbb{J} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M})$. Assume that:

- (xii) $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is u.s.c nondecreasing map with $\liminf_{z \rightarrow \infty} (z - \psi(z)) > 0$, $\psi(z) < z$ for all $z > 0$;
- (xiii) The operator $\mathfrak{H} : \mathbb{J} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{P}_{CP}(\mathcal{M})$ is integrable bounded and $\mathfrak{H}(\cdot, v'_1, v'_2) : \mathbb{J} \rightarrow \mathcal{P}_{CP}(\mathcal{M})$ is measurable for all $v_1, v_2 \in \mathcal{M}$;
- (xiv) There is $\omega \in C(\mathbb{J}, [0, \infty))$ with

$$\mathcal{H}_d(\mathfrak{H}(z, v_1, v_2) - \mathfrak{H}(z, \acute{v}_1, \acute{v}_2)) \leq \omega(z)\lambda^* \psi \left(\sum_{k=1}^2 |v_k - \acute{v}_k| \right)$$

for all $v_k, \acute{v}_k \in \mathcal{M}$ ($k = 1, 2$), where $\lambda^* = \frac{1}{\Omega_1^* + \Omega_2^*}$;

- (xv) \mathfrak{N} has the (AEP)-property.

Then the inclusion BVP (2) has a solution.

Proof We have to prove that $\mathfrak{N} : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M})$ includes endpoints. Firstly, we must prove that $\mathfrak{N}(v)$ is closed for every $v \in \mathcal{M}$. Since the mapping $z \rightarrow \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z))$ is closed-valued and measurable for $v \in \mathcal{M}$, thus it has a measurable selection and $\mathfrak{S}_{\mathfrak{H}, v}^* \neq \emptyset$. By applying the same deduction done in the proof of Theorem 4.1, one may simply verify that $\mathfrak{N}(v)$ is closed. Also, $\mathfrak{N}(v)$ is bounded because of the compactness of \mathfrak{H} . Finally, it is simple to prove that $\mathcal{H}_d(\mathfrak{N}(v), \mathfrak{N}(\pi)) \leq \psi(\|v - \pi\|)$ holds. Suppose that $v, \pi \in \mathcal{M}$ and $\acute{h}_1^* \in \mathfrak{N}(\pi)$. Select $\Upsilon_1 \in \mathfrak{S}_{\mathfrak{H}, \pi}^*$ such that

$$\begin{aligned} \acute{h}_1^*(z) &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon_1(\tau) \, d\tau \\ &+ \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon_1(\tau) \, d\tau \\ &+ \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon_1(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$. As

$$\mathcal{H}_d(\mathfrak{H}(z, v, {}^C\mathcal{D}_0^1 v) - \mathfrak{H}(z, \pi(z), {}^C\mathcal{D}_0^1 \pi(z))) \leq \omega(z)\lambda^* \psi (|v - \pi| + |{}^C\mathcal{D}_0^1 v - {}^C\mathcal{D}_0^1 \pi|)$$

for all $z \in \mathbb{J}$, there exists $\varpi^* \in \mathfrak{H}(z, v(z), {}^C\mathcal{D}_0^1 v(z))$ such that

$$|\Upsilon_1(z) - \varpi^*| \leq \omega(z)\lambda^* \psi (|v(z) - \pi(z)| + |{}^C\mathcal{D}_0^1 v(z) - {}^C\mathcal{D}_0^1 \pi(z)|)$$

for all $z \in \mathbb{J}$. Consider the multi-valued map $\mathfrak{D}^* : \mathbb{J} \rightarrow \mathcal{P}(\mathcal{M})$ defined by

$$\mathfrak{D}^*(z) = \{ \varpi^* \in \mathcal{M} : |\Upsilon_1(z) - \varpi^*| \leq \omega(z) \lambda^* \psi(|\nu(z) - \pi(z)| - |{}^C\mathfrak{D}_0^1 \nu(z) - {}^C\mathfrak{D}_0^1 \pi(z)|) \}.$$

By the measurability of Υ_1 and $\varphi^* = \omega \lambda^* \psi(|\nu - \pi| - |{}^C\mathfrak{D}_0^1 \nu - {}^C\mathfrak{D}_0^1 \pi|)$, it is obvious that the multifunction $\mathfrak{D}^*(\cdot) \cap \mathfrak{H}(\cdot, \nu(\cdot), {}^C\mathfrak{D}_0^1 \nu(\cdot))$ will be also measurable. Now we take $\Upsilon_2 \in \mathfrak{H}(z, \nu(z), {}^C\mathfrak{D}_0^1 \nu(z))$ such that

$$|\Upsilon_1(z) - \Upsilon_2(z)| \leq \omega(z) \frac{1}{\Omega_1^* + \Omega_2^*} \psi(|\nu - \pi| - |{}^C\mathfrak{D}_0^1 \nu - {}^C\mathfrak{D}_0^1 \pi|)$$

for all $z \in \mathbb{J}$. Select $h_2^* \in \mathfrak{N}(\nu)$ such that

$$\begin{aligned} h_2^*(z) &= \frac{1}{\alpha_1^* \Gamma(\sigma^* + 2)} \int_0^z (z - \tau)^{\sigma^* + 1} \Upsilon_2(\tau) \, d\tau \\ &\quad + \frac{\nabla_4 z + \nabla_2 z^2}{\alpha_1^* \Gamma(\sigma^* + \theta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* + \theta^* + 1} \Upsilon_2(\tau) \, d\tau \\ &\quad + \frac{\nabla_6 z + \nabla_3 z^2}{\alpha_1^* \Gamma(\sigma^* - \delta^* + 2)} \int_0^1 (1 - \tau)^{\sigma^* - \delta^* + 1} \Upsilon_2(\tau) \, d\tau \end{aligned}$$

for all $z \in \mathbb{J}$. By the same argument used in Theorem 4.1, we get

$$\begin{aligned} \|h_1^* - h_2^*\| &= \sup_{z \in \mathbb{J}} |h_1^*(z) - h_2^*(z)| + \sup_{z \in \mathbb{J}} |{}^C\mathfrak{D}_0^1 h_1^*(z) - {}^C\mathfrak{D}_0^1 h_2^*(z)| \\ &\leq (\Omega_1^* + \Omega_2^*) \lambda^* \psi(\|\nu - \hat{\nu}\|) \\ &= \psi(\|\nu - \hat{\nu}\|). \end{aligned}$$

Hence, $\mathcal{H}_d(\mathfrak{N}(\nu), \mathfrak{N}(\pi)) \leq \psi(\|\nu - \pi\|)$ for all $\nu, \pi \in \mathcal{M}$. By using hypothesis (xv), one can easily find that \mathfrak{N} has the (AEP)-property. By Theorem 2.15, there exists $\nu^* \in \mathcal{M}$ such that $\mathfrak{N}(\nu^*) = \{\nu^*\}$. This implies that ν^* satisfies the given problem (2) and the proof is completed. \square

5 Examples

Here, we provide two simulative examples to review our findings by a numerical view.

Example 5.1 Consider the boundary value problem in the multi-term form as follows:

$$\begin{cases} {}^C\mathfrak{D}_0^{0.2} [2 {}^C\mathfrak{D}_0^2] \nu(z) = \frac{z^3 |\sin(\nu(z))| + z^5 |{}^C\mathfrak{D}_0^1 \nu(z)|}{8 + 16z} & (z \in \mathbb{J} := [0, 1]), \\ \nu(0) = 0, \\ 2 {}^C\mathfrak{D}_0^1 \nu(0) + 3 {}^C\mathfrak{D}_0^{1.5} \nu(1) = 0, \\ 4 {}^C\mathfrak{D}_0^2 \nu(0) + 5 {}^{\mathcal{R}\mathcal{L}}\mathfrak{J}_0^2 \nu(1) = 0, \end{cases} \tag{14}$$

where the numerical data $\sigma^* = 0.2, \delta^* = 1.5, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5, \alpha_1^* = 2$, and $\theta^* = 2$ are chosen. Here, ${}^C\mathfrak{D}_0^{(\cdot)}$ and ${}^{\mathcal{R}\mathcal{L}}\mathfrak{J}_0^{(\cdot)}$ denote derivative and integral operators of the Caputo and Riemann–Liouville types. Consider the continuous function $\Phi^* : \mathbb{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$\Phi^*(z, s(z), r(z)) = \frac{z^3 |\sin(s(z))| + z^5 |{}^C\mathfrak{D}_0^1 r(z)|}{8 + 16z}.$$

For $s_1, s_2, r_1, r_2 \in \mathbb{R}$, we have

$$\begin{aligned} &|\Phi^*(z, s_1(z), r_1(z)) - \Phi^*(z, s_2(z), r_2(z))| \\ &\leq \frac{z}{8 + 16z} [|\sin(s_1(z)) - \sin(s_2(z))| + |{}^C\mathcal{D}_0^1 r_1(z) - {}^C\mathcal{D}_0^1 r_2(z)|] \\ &\leq \frac{z}{8 + 16z} [|s_1(z) - s_2(z)| + |{}^C\mathcal{D}_0^1 r_1(z) - {}^C\mathcal{D}_0^1 r_2(z)|]. \end{aligned}$$

Put $\xi(z) = \frac{z}{8+16z}$ for all $z \in \mathbb{J}$, then $\|\xi\| = \frac{1}{8} = 0.125$. Also, consider the continuous and non-decreasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $\psi(v) = v$ for $v \in \mathbb{R}^+$. Then

$$|\Phi^*(z, v(z), {}^C\mathcal{D}_0^1(v(z)))| \leq \frac{z}{8 + 16z} \psi(|v(z)| + |{}^C\mathcal{D}_0^1 v(z)|).$$

It is clear that $\eta^* : \mathbb{J} \rightarrow \mathbb{R}^+$ defined by $\eta^* = \frac{z}{8+16z}$ is a continuous function with $\|\eta^*\| = 0.125$. On the other hand, we get $\nabla_1 = -1.5, \Delta_1 = 5.5956, \nabla_2 = -0.8935, \nabla_3 = -0.2233, \nabla_4 = 1.5950, \nabla_5 = 0.7559, \nabla_6 = -0.7441, \Omega_1^* = 0.7492, \Omega_2^* = 1.1119, \Omega_3^* = 0.0678, \Omega_4^* = 0.0822$. Since $K^* = 0.1500 < 1$, by using Theorem 3.3, boundary value problem (14) has at least one solution.

Example 5.2 Consider the inclusion BVP in the multi-term form as follows:

$$\begin{cases} {}^C\mathcal{D}_0^{0.7} [3 {}^C\mathcal{D}_0^2 v(z) \in [0, \frac{z|\arctan z| + z|\arcsin(v(z))| + \frac{z^3}{2} |\sin({}^C\mathcal{D}_0^1 v(z))|}{3(20+60z)}]] & (z \in \mathbb{J} := [0, 1]), \\ v(0) = 0, \\ 11 {}^C\mathcal{D}_0^1 v(0) + 12 {}^C\mathcal{D}_0^{1.7} v(1) = 0, \\ 13 {}^C\mathcal{D}_0^2 v(0) + 14 {}^{\mathcal{R}\mathcal{L}}\mathcal{J}_0^4 v(1) = 0, \end{cases} \tag{15}$$

where the numerical data $\sigma^* = 0.7, \delta^* = 1.7, \alpha_1^* = 3, \theta^* = 4$ and $a_1 = 11, a_2 = 12, a_3 = 13, a_4 = 14$ are chosen. In view of the above data, we obtain $\nabla_1 = -1.090, \Delta_1 = 25.7554, \nabla_2 = -0.5435, \nabla_3 = 0.0049, \nabla_4 = 1.0155, \nabla_5 = -0.00119, \nabla_6 = -1.1019, \Omega_1^* = 0.4485, \Omega_2^* = 0.5798, \Omega_3^* = 0.0182, \Omega_4^* = 0.0182$. Here, ${}^C\mathcal{D}_0^{(\cdot)}$ and ${}^{\mathcal{R}\mathcal{L}}\mathcal{J}_0^{(\cdot)}$ denote derivative and integral operators of the Caputo and Riemann–Liouville types. Let $\mathfrak{H} : \mathbb{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a multifunction defined by

$$\mathfrak{H}(z, s(z), r(z)) = \left[0, \frac{z|\arctan z| + z^3|\arcsin(s(z))| + \frac{z^3}{2} |\sin({}^C\mathcal{D}_0^1 r(z))|}{3(20 + 60z)} \right].$$

For $s_1, s_2, r_1, r_2 \in \mathbb{R}$, we have

$$\begin{aligned} &\mathcal{H}_a(\mathfrak{H}(z, s_1(z), r_1(z)), \mathfrak{H}(z, s_2(z), r_2(z))) \\ &\leq \frac{z}{20} \frac{1}{3} (|\arcsin(s_1(z)) - \arcsin(s_2(z))| + |\sin({}^C\mathcal{D}_0^1 r_1(z)) - \sin({}^C\mathcal{D}_0^1 r_2(z))|) \\ &\leq \frac{z}{20} \frac{1}{3} (|s_1(z) - s_2(z)| + |{}^C\mathcal{D}_0^1 r_1(z) - {}^C\mathcal{D}_0^1 r_2(z)|) \\ &\leq \frac{z}{20} \psi(|s_1(z) - s_2(z)| + |{}^C\mathcal{D}_0^1 r_1(z) - {}^C\mathcal{D}_0^1 r_2(z)|) \\ &\leq \omega(z) \psi(|s_1(z) - s_2(z)| + |{}^C\mathcal{D}_0^1 r_1(z) - {}^C\mathcal{D}_0^1 r_2(z)|) \left(\frac{1}{\Omega_1^* + \Omega_2^*} \right), \end{aligned}$$

where $\Omega_1^* + \Omega_2^* = 1.0283$ and $\frac{1}{\Omega_1^* + \Omega_2^*} = 0.97247$. Choose the nonnegative function $\omega \in C(\mathbb{J}, [0, \infty))$ defined by $\omega(z) = \frac{z}{20}$ for all $z \in \mathbb{J}$. Then $\|\omega\| = \frac{1}{20} = 0.05$. Also, we consider the nonnegative and nondecreasing u.s.c map $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defined by $\psi(z) = \frac{z}{3}$ for almost all $z > 0$. Note that $\lim_{z \rightarrow \infty} \inf(z - \psi(z)) > 0$ holds with $\psi(z) < z$ ($\forall z > 0$). At last, consider $\mathfrak{N} : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M})$ by

$$\mathfrak{N}(v) = \{p \in \mathcal{M} : \text{there exists } \Upsilon \in \mathfrak{S}_{\mathfrak{N},v} \text{ s.t. } p(z) = \pi(z) \text{ for a.e. } z \in \mathbb{J}\},$$

where we have

$$\begin{aligned} \pi(z) = & \frac{1}{3\Gamma(2.7)} \int_0^z (z - \tau)^{1.7} \Upsilon(\tau) d\tau + \frac{1.0155z - 0.5435z^2}{3\Gamma(6.7)} \int_0^1 (1 - \tau)^{5.7} \Upsilon(\tau) d\tau \\ & + \frac{-1.1019z + 0.0049z^2}{3\Gamma(1)} \int_0^1 \Upsilon(\tau) d\tau. \end{aligned}$$

Since operator \mathfrak{N} has the (AEP)-property, by using Theorem 4.2, there exists at least one solution for the given BVP (15).

6 Conclusion

In the current research, we discussed some new conditions ensuring the existence of solution for given two multi-term BVPs in two different equation and inclusion versions. Indeed, we defined some operators based on the equivalent integral equation which belonged to a special category of α -admissible and α - ψ -contractions. Also, we investigated the (AEP)-property for such operators. At last, in two separate examples, the numerical simulation of given BVPs was done. As a future and next project, one can apply these techniques for the generalized BVPs having multi-strip, multi-point, multi-order integral conditions simultaneously.

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