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On Chandrasekhar functional integral inclusion and Chandrasekhar quadratic integral equation via a nonlinear Urysohn–Stieltjes functional integral inclusion

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Abstract

We investigate the existence of solutions for a nonlinear integral inclusion of Urysohn–Stieltjes type. As applications, we give a Chandrasekhar quadratic integral equation and a nonlinear Chandrasekhar integral inclusion.

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1 Introduction

The integral equations of Urysohn–Stieltjes (U-S) type have been studied by some authors; see, for example, [3, 5, 11–15], and [16–22], and reference therein.

The quadratic Chandrasekhar integral equation

$$x(t) = a(t) + x(t)\int_0^1 \frac{t}{t+s} b_1(s)x(s)\,ds, \quad t \in I = [0,1]$$

has been studied in some papers; see, for example, [1, 4, 7–10], and [24] and references therein.

Our aim is to study the existence of solutions $x \in C[0,1]$ of the U-S nonlinear functional integral inclusion

$$x(t) - a(t) \in \int_0^1 F\left(t, s, x(s), \int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\right) d_s g_1(t,s), \quad t \in I = [0,1]. \quad (1.1)$$

As applications, we will prove the existence of solutions $x \in C[0,1]$ of the nonlinear Chandrasekhar functional integral inclusion

$$x(t) - a(t) \in \int_0^1 \frac{t}{t+s} F\left(b_1(s)x(s), \int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta)\, d\theta\right) ds, \quad t \in I = [0,1],$$

and the Chandrasekhar quadratic integral equation

$$x(t) = a(t) + \int_0^1 \frac{t}{t+s} b_1(s)x(s) \cdot \left(\int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta)\, d\theta\right) ds, \quad t \in I = [0,1].$$

The paper is organized as follows. In Sect. 2, we establish the existence and uniqueness results for single-valued nonlinear U-S equations. We also prove the continuous dependence of the unique solution on the g_i $(i = 1,2)$. As an application, we discuss some particular cases by presenting the existence of solutions of nonlinear Chandrasekhar quadratic functional integral equations. In Sect. 3, we add conditions to our problem in order to obtain a new existence result with an application. Our results are generalized in Sect. 4, where we discuss the existence of solutions for set-valued equation (1.1) with continuous dependence on the set S_F and demonstrate a particular case of inclusion by presenting the existence of solutions for set-valued Chandrasekhar nonlinear functional integral equations.

2 Single-valued problem

Here we consider the nonlinear single-valued functional integral equation of U-S type

$$x(t) = a(t) + \int_0^1 f\left(t, s, x(s), \int_0^1 h\big(s,\theta,x(\theta)\big)\, d_\theta g_2(s,\theta)\right) d_s g_1(t,s), \quad t \in [0,1]. \tag{2.1}$$

2.1 Existence of solutions I

Consider the U-S functional integral equation (2.1) under the following assumptions:

(i) $a : [0,1] \to [0,1]$ is a continuous function, with $a = \sup_{t\in[0,1]} |a(t)|$.

(ii) a) $f : [0,1] \times [0,1] \times R \times R \to R$ is a continuous function, and there exist two continuous functions $m_1, k_1 : [0,1] \times [0,1] \to R$ such that

$$\big|f(t,s,x,y)\big| \le m_1(t,s) + k_1(t,s)\big(|x| + |y|\big).$$

b) $h : [0,1] \times [0,1] \times R \to R$ is a continuous function, and there exist two continuous functions $m_2, k_2 : [0,1] \times [0,1] \to R$ such that

$$\big|h(t,s,x)\big| \le m_2(t,s) + k_2(t,s)|x|.$$

c) $k = \sup\{k_i(t,s) : t,s \in [0,1]\}$, and $m = \sup\{m_i(t,s) : t,s \in [0,1], i = 1,2\}$.

(iii) $g_i : [0,1] \times R \to R$, $i = 1,2$, are continuous functions with

$$\mu = \max\big\{\sup\big|g_i(t,1)\big| + \sup\big|g_i(t,0)\big|, \text{on } [0,1]\big\}.$$

(iv) For all $t_1, t_2 \in I$, $t_1 < t_2$, the functions $s \to g_i(t_2,s) - g_i(t_1,s)$ are nondecreasing on $[0,1]$.

(v) $g_i(0,s) = 0$ for $s \in [0,1]$.
(vi) $k\mu + k^2\mu^2 < 1$.

Let E be a Banach space with the norm $\|\cdot\|_E$, and let $I = [0,1]$. Denote by $C = C(I,E)$ the space of all continuous functions on I taking values in the space E. This space becomes a Banach space with supnorm

$$\|x\|_C = \sup_{t\in I}\bigl\|x(t)\bigr\|_E.$$

Remark 2.1 (see [11]) Note that the function $s \to g(t,s)$ is nondecreasing on the interval $[0,1]$. Indeed, for $s_1, s_2 \in [0,1]$ with $s_1 < s_2$, from assumptions (iv) and (v) we obtain

$$g(t,s_2) - g(t,s_1) = \bigl[g(t,s_2) - g(0,s_2)\bigr] - \bigl[g(t,s_1) - g(0,s_1)\bigr] \geq 0.$$

Lemma 2.2 ([11]) *Assume that a function g satisfies assumption* (*v*). *Then for arbitrary $s_1, s_2 \in I$ with $s_1 < s_2$, the function $t \to g(t,s_2) - g(t,s_1)$ is nondecreasing on I.*

Indeed, take $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. Then by assumption (vi) we get

$$\bigl[g(t_2,s_2)-g(t_2,s_1)\bigr]-\bigl[g(t_1,s_2)-g(t_1,s_1)\bigr] = \bigl[g(t_2,s_2)-g(t_1,s_2)\bigr]-\bigl[g(t_2,s_1)-g(t_1,s_1)\bigr] \geq 0.$$

For the existence of at least one solution of the U-S nonlinear functional integral equation (2.1), we have the following theorem.

Theorem 2.3 *Let the assumptions* (*i*)–(*vi*) *be satisfied. Then the functional integral equation* (2.1) *has at least one solution $x \in C[0,1]$.*

Proof Define the operator A by

$$Ax(t) = a(t) + \int_0^1 f\left(t,s,x(s),\int_0^1 h\bigl(s,\theta,x(\theta)\bigr)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s), \quad t \in I, \tag{2.2}$$

and define let the set

$$Q_r = \bigl\{x \in R : |x| \leq r\bigr\} \subseteq C[0,1],$$

where

$$r = \frac{a + m\mu + km\mu^2}{1 - [k\mu + k^2\mu^2]}.$$

It is clear that Q_r is a nonempty, bounded, closed, and convex set.

Let $x \in Q_r$. Then

$$\begin{aligned}\bigl|Ax(t)\bigr| &= \left|a(t) + \int_0^1 f\left(t,s,x(s),\int_0^1 h\bigl(s,\theta,x(\theta)\bigr)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\right| \\ &\leq \bigl|a(t)\bigr| + \int_0^1 \left|f\left(t,s,x(s),\int_0^1 h\bigl(s,\theta,x(\theta)\bigr)\,d_\theta g_2(s,\theta)\right)\right|d_s g_1(t,s)\end{aligned}$$

$$\begin{aligned}
&\le a + \int_0^1 \Big(m_1(t,s) + k_1(t,s)\Big(\big|x(t)\big| + \int_0^1 \big|h\big(s,\theta,x(\theta)\big)\big| \, d_\theta g_2(s,\theta)\Big)\Big) \, d_s g_1(t,s) \\
&\le a + \int_0^1 \Big(m_1(t,s) + k_1(t,s)\Big(\big|x(t)\big| \\
&\quad + \int_0^1 \big(m_2(s,\theta) + k_2(s,\theta)\big|x(\theta)\big| \, d_\theta g_2(s,\theta)\big)\Big) \, d_s g_1(t,s)\Big) \\
&\le a + \int_0^1 \big(m_1(t,s) + k_1(t,s)\big(\big|x(t)\big| + (m+kr)\mu\big) \, d_s g_1(t,s) \\
&\le a + \big(m + k\big(r + (m+kr)\mu\big)\big)\mu \le r.
\end{aligned}$$

This proves that the operator $A : Q_r \to Q_r$ and the class $\{Ax\}$ is uniformly bounded on Q_r.

Then, for $x \in Q_r$ and $y(s) = \int_0^1 h(s,\theta,x(\theta)) \, d_\theta g_2(s,\theta)$, define the set

$$\begin{aligned}
\theta(\delta) = \sup\big\{ &\big|f(t_2,s,x,y) - f(t_1,s,x,y)\big| : t_1,t_2,s \in [0,1], t_1 < t_2, \\
&|t_2 - t_1| < \delta, |x| \le r, |y| \le r \big\}.
\end{aligned} \tag{2.3}$$

Then from the uniform continuity of the function $f : [0,1] \times [0,1] \times Q_r \times Q_r \to R$ and assumption (ii) we deduce that $\theta(\delta) \to 0$ as $\delta \to 0$, independently of $x \in Q_r$.

Now let $t_2, t_1 \in [0,1]$, $|t_2 - t_1| < \delta$. Then we have

$$\begin{aligned}
&\big|Ax(t_2) - Ax(t_1)\big| \\
&\quad = \bigg|a(t_2) + \int_0^1 f\Big(t_2,s,x(s), \int_0^1 h\big(s,\theta,x(\theta)\big) \, d_\theta g_2(s,\theta)\Big) \, d_s g_1(t_2,s) \\
&\qquad - a(t_1) - \int_0^1 f\Big(t_1,s,x(s), \int_0^1 h\big(s,\theta,x(\theta)\big) \, d_\theta g_2(s,\theta)\Big) \, d_s g_1(t_1,s)\bigg| \\
&\quad \le \big|a(t_2) - a(t_1)\big| + \bigg|\int_0^1 f\Big(t_2,s,x(s), \int_0^1 h\big(s,\theta,x(\theta)\big) \, d_\theta g_2(s,\theta)\Big) \, d_s g_1(t_2,s) \\
&\qquad - \int_0^1 f\Big(t_1,s,x(s), \int_0^1 h\big(s,\theta,x(\theta)\big) \, d_\theta g_2(s,\theta)\Big) \, d_s g_1(t_1,s)\bigg| \\
&\quad \le \big|a(t_2) - a(t_1)\big| \\
&\qquad + \bigg|\int_0^1 f\big(t_2,s,x(s),y(s)\big) \, d_s g_1(t_2,s) - \int_0^1 f\big(t_1,s,x(s),y(s)\big) \, d_s g_1(t_2,s) \\
&\qquad + \int_0^1 f\big(t_1,s,x(s),y(s)\big) \, d_s g_1(t_2,s) - \int_0^1 f\big(t_1,s,x(s),y(s)\big) \, d_s g_1(t_1,s)\bigg| \\
&\quad \le \big|a(t_2) - a(t_1)\big| + \int_0^1 \big|(f\big(t_2,s,x(s),y(s)\big) - f\big(t_1,s,x(s),y(s)\big)\big| \, d_s g_1(t_2,s) \\
&\qquad + \int_0^1 \big|f\big(t_1,s,x(s),y(s)\big)\big| \, d_s\big[g_1(t_2,s) - g_1(t_1,s)\big] \\
&\quad \le \big|a(t_2) - a(t_1)\big| + \int_0^1 \theta(\delta) \, d_s g_1(t_2,s) \\
&\qquad + \int_0^1 \big(m_1(t,s) + k_1(t,s)\big(|x| + |y|\big)\big) \, d_s\big[g_1(t_2,s) - g_1(t_1,s)\big].
\end{aligned}$$

This inequality means that the class of functions $\{Ax\}$ is equicontinuous.

Therefore by the Arzelà–Ascoli theorem [25] A is compact.

Let $\{x_n\} \subset Q_r$, $x_n \to x$. Then

$$\begin{aligned}
&Ax_n(t)\\
&\quad= a(t) + \int_0^1 f\left(t,s,x_n(s),\int_0^1 h\big(s,\theta,x_n(\theta)\big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s),\\
&\lim_{n\to\infty} Ax_n(t)\\
&\quad= \lim_{n\to\infty}\left(a(t) + \int_0^1 f\left(t,s,x_n(s),\int_0^1 h\big(s,\theta,x_n(\theta)\big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\right),
\end{aligned}$$

and from assumption (ii) (see [23]) we get

$$\begin{aligned}
&\lim_{n\to\infty} Ax_n(t)\\
&\quad= a(t) + \int_0^1 \lim_{n\to\infty} f\left(t,s,x_n(s),\int_0^1 h\big(s,\theta,x_n(\theta)\big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\\
&\quad= a(t) + \int_0^1 f\left(t,s,\lim_{n\to\infty} x_n(s),\int_0^1 h\Big(s,\theta,\lim_{n\to\infty} x_n(\theta)\Big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\\
&\quad= a(t) + \int_0^1 f\left(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\\
&\quad= Ax(t).
\end{aligned}$$

This proves that $Ax_n(t) \to Ax(t)$ and A is continuous.

Now (see [23]) A has at least one fixed point $x \in Q_r$, and (2.1) has at least one solution $x \in Q_r \subset C[0,1]$. □

2.2 Uniqueness of the solution

To prove the existence of a unique solution of U-S functional integral equation (2.1), let us replace condition (ii) by

(ii)* a) the function $f : I \times I \times R \times R \to R$ is continuous and satisfies the Lipschitz condition

$$\big|f(t,s,x_1,y_1) - f(t,s,x_2,y_2)\big| \le k_1\big(|x_1 - x_2| + |y_1 - y_2|\big).$$

b) $h : I \times I \times R \to R$ is continuous and satisfies the Lipschitz condition

$$\big|h(t,s,x) - h(t,s,y)\big| \le k_2|x - y|.$$

By condition (ii)* we have

$$\big|f\big(t,s,x(s),y(s)\big)\big| - \big|f(t,s,0,0)\big| \le \big|f\big(t,s,x(s),y(s)\big) - f(t,s,0,0)\big| \le k_1\big(|x| + |y|\big).$$

Then

$$\big|f\big(t,s,x(s),y(s)\big)\big| \le k_1\big(|x| + |y|\big) + \big|f_1(t,s,0,0)\big|,$$

and

$$\left|f\big(t,s,x(s),y(s)\big)\right| \le k_1\big(|x|+|y|\big)+m_1,$$

where $m_1 = \sup_{t\times s\in I\times I} |f(t,s,0,0)|$, and

$$\left|h\big(t,s,x(s)\big)\right| - \left|h(t,s,0)\right| \le \left|h\big(t,s,x(s)\big)-h(t,s,0)\right| \le k_2|x|.$$

Then

$$\left|h\big(t,s,x(s)\big)\right| \le k_2|x| + \left|f_2(t,s,0)\right|,$$

and

$$\left|h\big(t,s,x(s)\big)\right| \le k_2|x| + m_2,$$

where $m_2 = \sup_{t\times s\in I\times I} |h(t,s,0)|$, $m = \max\{m_1,m_2\}$, and $k = \max\{k_1,k_2\}$.

Theorem 2.4 *Let conditions* (*i*), (*ii*)*, (*iii*), *and* (*iv*)–(*v*) *be satisfied with* $\mu k + k^2\mu^2 \le 1$. *Then the functional integral equation* (2.1) *has unique solution* $x \in C[0,1]$.

Proof Let x_1, x_2 be solutions of the integral equation (2.1). Then

$$\begin{aligned}
&\left|x_1(t)-x_2(t)\right| \\
&\quad= \left|a(t) + \int_0^1 f\left(t,s,x_1(s),\int_0^1 h\big(s,\theta,x_1(\theta)\big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\right. \\
&\qquad\left. - a(t) + \int_0^1 f\left(t,s,x_2(s),\int_0^1 h\big(s,\theta,x_2(\theta)\big)\,d_\theta g_2(s,\theta)\right)d_s g_1(t,s)\right| \\
&\quad\le \int_0^1 \left| f\left(t,s,x_1(s),\int_0^1 h\big(s,\theta,x_1(\theta)\big)\,d_\theta g_2(s,\theta)\right)\right. \\
&\qquad\left. - f\left(t,s,x_2(s),\int_0^1 h\big(s,\theta,x_2(\theta)\big)\,d_\theta g_2(s,\theta)\right)\right| d_s g_1(t,s) \\
&\quad\le \int_0^1 k_1\left(\left|x_1(s)-x_2(s)\right| + \int_0^1 \left|\big(h\big(s,\theta,x_1(\theta)\big)-h\big(s,\theta,x_2(\theta)\big)\big)\right| d_\theta g_2(s,\theta)\right) d_s g_1(t,s) \\
&\quad\le \int_0^1 k_1\left(\left|x_1(s)-x_2(s)\right| + \int_0^1 k_2\big(\left|x_1(\theta)-x_2(\theta)\right|\big)\,d_\theta g_2(s,\theta)\right) d_s g_1(t,s) \\
&\quad\le \int_0^1 k_1\big(\left|x_1(s)-x_2(s)\right| + k_2\|x_1-x_2\|\mu\big)\,d_s g_1(t,s) \\
&\quad\le k\|x_1-x_2\|\mu + k^2\|x_1-x_2\|\mu^2.
\end{aligned}$$

Hence we have

$$\|x_1-x_2\| \le \big(\mu k + k^2\mu^2\big)\|x_1-x_2\|$$

and

$$\big(1 - (\mu + k^2\mu^2)\big)\|x_1 - x_2\| \leq 0,$$

which implies

$$x_1(t) = x_2(t). \qquad \square$$

2.2.1 Continuous dependence of solution on functions $g_i(t,s)$

Here we show that the solution of U-S functional integral equation (2.1) continuously depends on the functions g_i.

Definition 2.5 The solutions of functional integral equation (2.1) continuously depends on the functions $g_i(t,s)$, $i = 1,2$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\big|g_i(t,s) - g_i^*(t,s)\big| \leq \delta \quad \Rightarrow \quad \big\|x - x^*\big\| \leq \epsilon.$$

Theorem 2.6 *Let the assumptions of Theorem 2.4 be satisfied. Then the solution of* (2.1) *depends continuously on functions $g_i(t,s)$, $i = 1,2$.*

Proof Let $\delta > 0$ be such that $|g_i(t,s) - g_i^*(t,s)| \leq \delta$ for all $t \geq 0$. Then

$$\begin{aligned}
&\big|x(t) - x^*(t)\big| \\
&\quad = \bigg|a(t) + \int_0^1 f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) \\
&\qquad - a(t) + \int_0^1 f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\bigg)\,d_s g_1^*(t,s)\bigg| \\
&\quad \leq \bigg|\int_0^1 f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) \\
&\qquad - \int_0^1 f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) \\
&\qquad + \int_0^1 f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s)) \\
&\qquad - \int_0^1 f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\bigg)\,d_s g_1^*(t,s)\bigg| \\
&\quad \leq \int_0^1 \bigg|f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad -f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\bigg|\,d_s g_1(t,s) \\
&\qquad +\bigg|\int_0^1 f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad -f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\bigg)\,d_s g_1^*(t,s)\bigg| \\
&\quad \leq \int_0^1 k_1\bigg(\big|x(s) - x^*(s)\big|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \big|h\big(s,\theta,x(\theta)\big)-h\big(s,\theta,x^*(\theta)\big)\big|\,d_\theta g_2(s,\theta)\Big)\,d_s g_1(t,s)\\
&+\Big|\int_0^1 f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\Big)\,d_s g_1(t,s))\\
&-\int_0^1 f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big)\,d_s g_1(t,s))\\
&+\int_0^1 f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big)\,d_s g_1(t,s))\\
&-\int_0^1 f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big)\,d_s g_1^*(t,s))\Big|\\
\le{}&\int_0^1 k\Big(\big|x(s)-x^*(s)\big|+\int_0^1 k\big|x(\theta)-x^*(\theta)\big|\,d_\theta g_2(s,\theta)\Big)\,d_s g_1(t,s)\\
&+\int_0^1\Big|f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\Big)\\
&-f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big)\Big|\,d_s g_1(t,s))\\
&+\int_0^1\Big|f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big)\Big|\big[d_s g_1(t,s)-d_s g_1^*(t,s)\big]\\
\le{}&\int_0^1 k\Big(\big|x(s)-x^*(s)\big|+\int_0^1 k\big|x(\theta)-x^*(\theta)\big|\,d_\theta g_2(s,\theta)\Big)\,d_s g_1(t,s)\\
&+\int_0^1 k\Big(\Big|\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)-\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big|\Big)\,d_s g_1(t,s))\\
&+\int_0^1\Big|f\Big(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2^*(s,\theta)\Big)\Big|\big[d_s g_1(t,s)-d_s g_1^*(t,s)\big]\\
\le{}&\int_0^1 k\Big(\big|x(s)-x^*(s)\big|+\int_0^1 k\big|x(\theta)-x^*(\theta)\big|\,d_\theta g_2(s,\theta)\Big)\,d_s g_1(t,s)\\
&+\int_0^1 k\Big(\int_0^1\big|h\big(s,\theta,x^*(\theta)\big)\big|\big[d_\theta g_2(s,\theta)-d_\theta g_2^*(s,\theta)\big]\Big)\,d_s g_1(t,s))\\
&+\int_0^1\Big[m+k\Big(\big|x^*(s)\big|+\int_0^1\big|h\big(s,\theta,x^*(\theta)\big)\big|\,d_\theta g_2^*(s,\theta)\Big)\Big]\big[d_s g_1(t,s)-d_s g_1^*(t,s)\big]\\
\le{}&\int_0^1 k\Big(\big|x(s)-x^*(s)\big|+\int_0^1 k\big|x(\theta)-x^*(\theta)\big|\,d_\theta g_2(s,\theta)\Big)\,d_s g_1(t,s)\\
&+\int_0^1 k\Big(\int_0^1\big[m+k\big|x^*(\theta)\big|\big]\big[d_\theta g_2(s,\theta)-d_\theta g_2^*(s,\theta)\big]\Big)\,d_s g_1(t,s)\\
&+\int_0^1\Big[m+k\Big(\big|x^*(s)\big|+\int_0^1\big[m+k\big|x^*(\theta)\big|\big]\,d_\theta g_2^*(s,\theta)\Big)\Big]\big[d_s g_1(t,s)-d_s g_1^*(t,s)\big]\\
\le{}&k\mu\big\|x-x^*\big\|+k^2\mu^2\big\|x-x^*\big\|+k[m+kr]\mu\big[g_2(s,1)-g_2^*(s,1)\big]\\
&+\big[m+k[r+m+kr]\big]\mu\big[g_1(t,1)-g_1^*(t,1)\big].
\end{aligned}
$$

Taking the supremum over $t\in I$, we get

$$\big\|x-x^*\big\|\le k\mu\big\|x-x^*\big\|+k^2\mu^2\big\|x-x^*\big\|+[km+kr]\mu\delta+\big[m+k[r+kr+m]\big]\mu\delta.$$

Then

$$\|x - x^*\| \leq \frac{(2km + 2kr + k^2r + m)\mu\delta}{1 - (k\mu + k^2\mu^2)} = \epsilon.$$

Now we get that the solution of (2.1) continuously depends on the functions g_i, $i = 1, 2$. □

3 Existence of solutions II

Now we replace assumptions (ii) a), (vi) by

(ii*) a*) $f : [0,1] \times [0,1] \times R \times R \to R$ is a function, and there exist two continuous functions $m_1, k_1 : [0,1] \times [0,1] \to R$ such that

$$\big|f(t,s,x,y)\big| \leq m_1(t,s) + k_1(t,s)|x| \cdot |y|.$$

(vi*) There exists a positive root l of the algebraic equation

$$\mu^2k^2l^2 + \big(k\mu^2m - 1\big)l + (a + m\mu) = 0.$$

Theorem 3.1 *Let the assumptions of Theorem 2.3 be satisfied with (ii) a) and (vi) replaced by (ii*) a*) and (vi*), respectively. Then equation (2.1) has at least one solution $x \in C[0,1]$.*

Proof Define the operator A^* by

$$A^*x(t) = a(t) + \int_0^1 f\left(t, s, x(s), \int_0^1 h\big(s, \theta, x(\theta)\big)\, d_\theta g_2(s,\theta)\right) d_s g_1(t,s), \quad t \in [0,1],$$

and define the set

$$Q_l = \big\{x \in R : |x| \leq l\big\} \subseteq C\big([0,1]\big),$$

where l is a positive root of the algebraic equation

$$\mu^2k^2l^2 + \big(k\mu^2m - 1\big)l + (a + m\mu) = 0.$$

It is clear that Q_l is a nonempty, bounded, closed, and convex set.

Now let $x \in Q_l$. Then

$$\begin{aligned}
&\big|A^*x(t)\big| \\
&\quad = \left|a(t) + \int_0^1 f\left(t, s, x(s), \int_0^1 h\big(s, \theta, x(\theta)\big)\, d_\theta g_2(s,\theta)\right) d_s g_1(t,s)\right| \\
&\quad \leq a + \int_0^1 \left|f\left(t, s, x(s), \int_0^1 h\big(s, \theta, x(\theta)\big)\, d_\theta g_2(s,\theta)\right)\right| d_s g_1(t,s) \\
&\quad \leq a + \int_0^1 \left(m_1(t,s) + k_1(t,s)\left(\big|x(t)\big| \cdot \left|\int_0^1 h\big(s, \theta, x(\theta)\big)\, d_\theta g_2(s,\theta)\right)\right)\right| d_s g_1(t,s) \\
&\quad \leq a + \int_0^1 (m_1(t,s) + k_1(t,s)(\big|x(t)\big| \cdot \int_0^1 \big(m_2(s,\theta) + k_2(s,\theta)\big|x(\theta)\big|\, d_\theta g_2(s,\theta)\big)\, d_s g_1(t,s)
\end{aligned}$$

$$
\begin{aligned}
&\leq a + \int_0^1 (m_1(t,s) + k_1(t,s)(|x(t)| \cdot (m + kl)\mu)\, d_s g_1(t,s) \\
&\leq a + (m + k(l \cdot (m + kl)\mu))\mu \leq l.
\end{aligned}
$$

This proves that $A^* : Q_l \to Q_l$ and the class $\{A^* x\}$ is uniformly bounded on Q_l.

Now for $x \in Q_r$ and $y(s) = \int_0^1 h(s,\theta,x(\theta))\, d_\theta g_2(s,\theta)$, define the set

$$
\begin{aligned}
\theta(\delta) = \sup\{ &|f(t_2,s,x,y) - f(t_1,s,x,y)| : t_1,t_2,s \in [0,1], t_1 < t_2, \\
&|t_2 - t_1| < \delta, |x| \leq l, |y| \leq l\}.
\end{aligned}
$$

Then from the uniform continuity of the function $f : [0,1] \times [0,1] \times Q_l \times Q_l \to R$ and assumption (ii*) we deduce that $\theta(\delta) \to 0$ as $\delta \to 0$, independently of $x \in Q_l$.

Now let $t_2, t_1 \in [0,1]$ be such that $|t_2 - t_1| < \delta$. Then we have

$$
\begin{aligned}
&|A^* x(t_2) - A^* x(t_1)| \\
&\quad = \left| a(t_2) + \int_0^1 f\left(t_2,s,x(s), \int_0^1 h(s,\theta,x(\theta))\, d_\theta g_2(s,\theta)\right) d_s g_1(t_2,s) \right. \\
&\qquad \left. - a(t_1) - \int_0^1 f\left(t_1,s,x(s), \int_0^1 h(s,\theta,x(\theta))\, d_\theta g_2(s,\theta)\right) d_s g_1(t_1,s) \right| \\
&\quad \leq |a(t_2) - a(t_1)| + \left| \int_0^1 f\left(t_2,s,x(s), \int_0^1 h(s,\theta,x(\theta))\, d_\theta g_2(s,\theta)\right) d_s g_1(t_2,s) \right. \\
&\qquad \left. - \int_0^1 f\left(t_1,s,x(s), \int_0^1 h(s,\theta,x(\theta))\, d_\theta g_2(s,\theta)\right) d_s g_1(t_1,s) \right| \\
&\quad \leq |a(t_2) - a(t_1)| \\
&\qquad + \left| \int_0^1 f(t_2,s,x(s),y(s))\, d_s g_1(t_2,s) - \int_0^1 f(t_1,s,x(s),y(s))\, d_s g_1(t_2,s) \right. \\
&\qquad \left. + \int_0^1 f(t_1,s,x(s),y(s))\, d_s g_1(t_2,s) - \int_0^1 f(t_1,s,x(s),y(s))\, d_s g_1(t_1,s) \right| \\
&\quad \leq |a(t_2) - a(t_1)| + \int_0^1 |(f(t_2,s,x(s),y(s)) - f(t_1,s,x(s),y(s)))|\, d_s g_1(t_2,s) \\
&\qquad + \int_0^1 |f(t_1,s,x(s),y(s))|\, d_s[g_1(t_2,s) - g_1(t_1,s)] \\
&\quad \leq |a(t_2) - a(t_1)| \\
&\qquad + \int_0^1 \theta(\delta)\, d_s g_1(t_2,s) + \int_0^1 (m_1(t,s) + k_1(t,s)(|x| \cdot |y|))\, d_s[g_1(t_2,s) - g_1(t_1,s)].
\end{aligned}
$$

This inequality means that the class of functions $\{A^* x\}$ is equicontinuous. Therefore A^* is compact by the Arzelà–Ascoli theorem [25].

Let $\{x_n\} \subset Q_l$, $x_n \to x$. Then

$$
A^* x_n(t) = a(t) + \int_0^1 f\left(t,s,x_n(s), \int_0^1 h(s,\theta,x_n(\theta))\, d_\theta g_2(s,\theta)\right) d_s g_1(t,s),
$$

$$
\begin{aligned}
\lim_{n\to\infty} A^* x_n(t) = \lim_{n\to\infty} \Bigg(& a(t) \\
& + \int_0^1 f\bigg(t,s,x_n(s),\int_0^1 h\big(s,\theta,x_n(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s)\Bigg),
\end{aligned}
$$

and by assumption (ii*) (see [23]) we get

$$
\begin{aligned}
& \lim_{n\to\infty} A^* x_n(t) \\
& \quad = a(t) + \int_0^1 \lim_{n\to\infty} f\bigg(t,s,x_n(s),\int_0^1 h\big(s,\theta,x_n(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) \\
& \quad = a(t) + \int_0^1 f\bigg(t,s,\lim_{n\to\infty} x_n(s),\int_0^1 h\Big(s,\theta,\lim_{n\to\infty} x_n(\theta)\Big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) \\
& \quad = a(t) + \int_0^1 f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) = A^* x(t).
\end{aligned}
$$

This proves that $A^* x_n(t) \to A^* x(t)$ and A^* is continuous. So (see [23]) A^* has at least one fixed point $x \in Q_r$, and (2.1) has at least one solution $x \in Q_l \subset C([0,1])$. □

3.1 Application

Let in equation (2.1), $h(t,s,x(s)) = b_2(t)x(s)$,

$$
g_1(t,s) = \begin{cases} t \ln \frac{t+s}{t} & \text{for } t \in (0,1], s \in I, \\ 0 & \text{for } t = 0, s \in I, \end{cases}
$$

and

$$
g_2(s,\theta) = \begin{cases} s \ln \frac{s+\theta}{s} & \text{for } s \in (0,1], \theta \in I, \\ 0 & \text{for } s = 0, \theta \in I. \end{cases}
$$

Then g_1, g_2 satisfy our assumptions (iii)–(v), and we obtain the nonlinear Chandrasekhar functional integral equation

$$
x(t) = a(t) + \int_0^1 \frac{t}{t+s} f\bigg(t,s,x(s),\int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta)\,d\theta\bigg)\,ds. \tag{3.1}
$$

Let, in equation (3.1), $f(t,s,x(s),y(s)) = b_1(s)x(s)\cdot y(s)$, where

$$
y(s) = \int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta)\,d\theta.
$$

Then we obtain the Chandrasekhar quadratic functional integral equation of the form

$$
x(t) = a(t) + \int_0^1 \frac{t}{t+s} b_1(s)x(s)\cdot\bigg(\int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta)\,d\theta\bigg)\,ds. \tag{3.2}
$$

Now, under the assumptions of Theorem 3.1, the Chandrasekhar quadratic functional integral equation (3.2) has at least one solution $x \in C[0,1]$.

3.2 Example

Consider the following Chandrasekhar quadratic functional integral equation:

$$x(t) = \frac{e^{-t}}{9+e^{t}} + \int_0^1 \frac{t}{t+s} \cdot \frac{2\cos(s)x(s)}{7e^{2s}(1+\cos^2(s))} \cdot \left(\int_0^1 \frac{s}{s+\theta} \cdot \frac{\sin(s)}{4(1+\sin^2(s))} x(\theta)\, d\theta \right) ds. \quad (3.3)$$

First, note that equation (3.3) is a particular case of equation (3.2) if we put

$$\begin{aligned}
a(t) &= \frac{e^{-t}}{9+e^{t}}, \\
h\bigl(t,s,x(s)\bigr) &= \frac{\sin(t)}{4(1+\sin^2(t))} x(s), \\
f\bigl(t,s,x(s),y(s)\bigr) &= \frac{2\cos(s)x(s)}{7e^{2s}(1+\cos^2(s))} \cdot y(s), \\
y(s) &= \int_0^1 \frac{s}{s+\theta} \frac{\sin(s)}{4(1+\sin^2(s))} x(\theta)\, d\theta,
\end{aligned}$$

$b_1(s) = \frac{2\cos(s)}{7e^{2s}(1+\cos^2(s))}$, $b_2(s) = \frac{\sin(s)}{4(1+\sin^2(s))}$, with $k_1 = \frac{2}{7}$ and $k_2 = \frac{1}{4}$.

Thus conditions (i), (ii*) and (iii) are satisfied with $a = \frac{1}{10}$, $k = \frac{1}{4}$, and $m = 0$. By all facts established above, we deduce that condition (vi*) of the form

$$\mu^2 k^2 l^2 + \bigl(k\mu^2 m - 1\bigr) l + (a + m\mu) = 0$$

has a positive solution l. For example, if $l \approx 0.1$ or $l \approx 33$, then assumption (vi*) will be satisfied if we choose one of this values.

As all the conditions of Theorem 3.1 are satisfied, equation (3.3) has at least one solution $x \in C[0,1]$.

4 Set-valued problem

Consider the U-S nonlinear functional integral inclusion (1.1),

$$x(t) \in a(t) + \int_0^1 F\left(t, s, x(s), \int_0^1 h\bigl(s,\theta,x(\theta)\bigr)\, d_\theta g_2(s,\theta) \right) d_s g_1(t,s), \quad t \in I,$$

under the following assumptions:

(i) $a : [0,1] \to [0,1]$ is a continuous function.

(ii)*** (a) $F : [0,1] \times [0,1] \times R \times R \to P(R)$, is a Lipschitzian set-valued map with a nonempty compact convex subset of 2^R, with a Lipschitz constant $k_1 > 0$:

$$\bigl\| F(t,s,x_1,y_1) - F(t,s,x_2,y_2) \bigr\| \le k_1 \bigl(|x_1 - x_2| + |y_1 - y_2| \bigr).$$

Remark. From this assumption and Theorem 1 from [2, Sect. 9, Chap. 1] on the existence of Lipschitzian selection we deduce that the set of Lipschitz selections of F is not empty and there exists $f \in F$ such that

$$\bigl| f(t,s,x_1,y_1) - f(t,s,x_2,y_2) \bigr| \le k_1 \bigl(|x_1 - x_2| + |y_1 - y_2| \bigr).$$

(b) $h : [0,1] \times [0,1] \times R \to R$ is a continuous function such that

$$\left|h(t,s,x)\right| \le m_2(t,s) + k_2(t,s)|x|.$$

(c) $k = \sup_{(t,s)\in[0,1]\times[0,1]} k_i(t,s)$ and $m = \sup_{(t,s)\in[0,1]\times[0,1]} m_i(t,s)$.

(iii) $g_i : [0,1] \times R \to R$, $i = 1,2$, are continuous with

$$\mu = \max\left\{\sup\left|g_i\left(t,\varphi(t)\right)\right| + \sup\left|g_i(t,0)\right| \text{ on } [0,1]\right\}.$$

(iv) For all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, the functions $s \to g_i(t_2,s) - g_i(t_1,s)$ are nondecreasing on $[0,1]$.

(v) $g_i(0,s) = 0$ for any $s \in [0,1]$.

(vi) $k\mu + k^2\mu^2 < 1$.

4.1 Existence of solution

Theorem 4.1 *Let assumptions* (*i*)–(*ii*)***, *and* (*iv*)–(*vi*) *be satisfied. Then* (1.1) *has at least one solution* $x \in C[0,1]$.

Proof By assumption (ii)***-(a) it is clear that the set of Lipschitz selection of F is nonempty. So, the solution of the single-valued (2.1) where $f \in S_F$ is a solution to (1.1).

Note that the Lipschitz selection $f : [0,1] \times [0,1] \times R \times R \to R$ satisfies

$$\left|f(t,s,x_1,y_1) - f(t,s,x_2,y_2)\right| \le k_1\left(|x_1 - x_2| + |y_1 - y_2|\right).$$

From this condition with $m_1 = \sup_{(t,s)\in I\times I} |f(t,s,0,0)|$ we have

$$\left|f\left(t,s,x(s),y(s)\right)\right| - \left|f(t,s,0,0)\right| \le \left|f\left(t,s,x(s),y(s)\right) - f(t,s,0,0)\right| \le k_1\left(|x| + |y|\right).$$

Then

$$\left|f\left(t,s,x(s),y(s)\right)\right| \le k_1\left(|x| + |y|\right) + \left|f(t,s,0,0)\right|,$$

and

$$\left|f\left(t,s,x(s),y(s)\right)\right| \le k_1\left(|x| + |y|\right) + m_1,$$

that is, assumption (ii) of Theorem 2.3 is satisfied. So, all conditions of Theorem 2.3 hold.

Note that if $x \in C(I,R)$ is a solution of (2.1), then x is a solution to (1.1). □

4.1.1 Continuous dependence on the set of selection S_F

Here we study the continuous dependence on the set S_F of all selections of the set-valued function F.

Definition 4.2 The solution of (1.1) continuously depends on the set S_F if for all $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\left|f(t,s,x,y) - f^*(t,s,x,y)\right| < \delta, \quad f, f^* \in S_F, t \in [0,1],$$

then $\|x - x^*\| < \epsilon$.

Now we have the following theorem.

Theorem 4.3 *Let the assumptions of Theorem 4.1 be satisfied with*

$$\big|h(t,s,x)-h(t,s,y)\big| \le k_2|x-y|.$$

Then the solution of (1.1) continuously depends on the set S_F of all Lipschitzian selections of F.

Proof For two solutions $x(t)$ and $x^*(t)$ of (1.1) corresponding to two selections $f,f^* \in S_F$, we have

$$\begin{aligned}
&\big|x(t)-x^*(t)\big| \\
&\quad= \bigg|a(t)+\int_0^1 f\bigg(t,s,x(s),\int_0^1 f\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s) \\
&\qquad - a(t)+\int_0^1 f^*\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg))\,d_s g_1(t,s)\bigg| \\
&\quad\le \int_0^1 \bigg|f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad - f^*\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg))\bigg|\,d_s g_1(t,s) \\
&\quad\le \int_0^1 \bigg|f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad - f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\bigg|\,d_s g_1(t,s) \\
&\qquad + \int_0^1 \bigg|f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad - f^*\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\bigg|\,d_s g_1(t,s) \\
&\quad\le \int_0^1 \bigg|f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad - f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\bigg|\,d_s g_1(t,s) + \delta\int_0^1 d_s g_1(t,s) \\
&\quad\le \int_0^1 \bigg|f\bigg(t,s,x(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad - f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\bigg|\,d_s g_1(t,s) \\
&\qquad + \int_0^1 \bigg|f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x(\theta)\big)\,d_\theta g_2(s,\theta)\bigg) \\
&\qquad - f\bigg(t,s,x^*(s),\int_0^1 h\big(s,\theta,x^*(\theta)\big)\,d_\theta g_2(s,\theta)\bigg)\bigg|\,d_s g_1(t,s) + \delta\int_0^1 d_s g_1(t,s) \\
&\quad\le \int_0^1 k_1\bigg(\big|x(s)-x^*(s)\big| + \int_0^1 \big|h\big(s,\theta,x(\theta)\big)-h\big(s,\theta,x^*(\theta)\big)\big|\,d_\theta g_2(s,\theta)\bigg)\,d_s g_1(t,s)
\end{aligned}$$

$$
\begin{aligned}
&+ \delta \int_0^1 d_s g_1(t,s) \\
\le& \int_0^1 k_1 \left(\left|x(s) - x^*(s)\right| + \int_0^1 k_2 \left|x(\theta) - x^*(\theta)\right| d_\theta g_2(s,\theta) \right) d_s g_1(t,s) \\
&+ \delta \int_0^1 d_s g_1(t,s).
\end{aligned}
$$

Now, taking the supremum over $t \in I$, we get

$$\left\|x - x^*\right\| \le k\mu \left\|x - x^*\right\| + k^2\mu^2 \left\|x - x^*\right\| + \delta\mu.$$

Hence

$$\left\|x - x^*\right\| \le \frac{\delta\mu}{1 - (k\mu + k^2\mu^2)} = \epsilon.$$

Thus from last inequality we get

$$\left\|x - x^*\right\| \le \epsilon.$$

This proves the continuous dependence of the solution on the set S_F. □

4.2 Set-valued Chandrasekhar nonlinear quadratic functional integral inclusion

Now, as an application of the nonlinear set-valued functional integral equations of U-S type (1.1), we have the following. Let the functions g_i be defined by

$$g_1(t,s) = \begin{cases} t \ln \frac{t+s}{t} & \text{for } t \in (0,1], s \in I, \\ 0 & \text{for } t = 0, s \in I, \end{cases}$$

and

$$g_2(s,\theta) = \begin{cases} s \ln \frac{s+\theta}{s} & \text{for } s \in (0,1], \theta \in I, \\ 0 & \text{for } s = 0, \theta \in I. \end{cases}$$

Let, in (1.1), $h(t,s,x(s)) = b_2(s)x(s)$ and $F(t,s,x(s),y(s)) = F(b_1(s)x(s),y(s))$, where

$$y(s) = \int_0^s \frac{s}{s+\theta} b_2(s)x(\theta)\, d\theta.$$

Further, since the functions g_i satisfy assumptions (iii)–(v) (see [6]), we obtain the nonlinear Chandrasekhar functional integral inclusion

$$x(t) \in a(t) + \int_0^1 \frac{t}{t+s} F\left(b_1(s)x(s), \int_0^1 \frac{s}{s+\theta} b_2(s)x(\theta)\, d\theta \right) ds, \quad t \in [0,1]. \tag{4.1}$$

Now we can state the following existence result for (4.1).

Theorem 4.4 *Under the assumptions of Theorem 4.1, inclusion (4.1) has at least one continuous solution $x \in C[0,1]$.*

4.3 Example

Consider the following nonlinear Chandrasekhar functional integral inclusion:

$$x(t) \in te^{-4t} + \int_0^1 \frac{t}{t+s} \frac{\sqrt{\pi} e^{-2t} x(s)}{\pi + e^t} \int_0^1 \frac{s}{s+\theta} \frac{\sqrt{s}}{e^{s+1}} x(\theta)\, d\theta\, ds, \quad t \in [0,1]. \tag{4.2}$$

Note that this inclusion is a particular case of inclusion (4.1) if we choose $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^+}$ in (4.2) as follows:

$$F\big(b_1(s)x(s), y(s)\big) = \left[0, \frac{s}{s^2+1} x(s) \int_0^1 \frac{s}{s+\theta} \frac{\sqrt{s}}{e^{s+1}} x(\theta)\, d\theta\, ds\right].$$

Further, note that now the terms involved in (4.1) have the form

$$a(t) = te^{-4t}, \qquad y(s) = \int_0^s \frac{s}{s+\theta} \frac{1}{s^2+1} x(\theta)\, d\theta, \qquad h\big(t,s,x(s)\big) = \frac{\sqrt{s}}{e^{s+1}} x(\theta),$$

with $b_1(s) = \frac{1}{s^2+1}$ and $b_2(s) = \frac{\sqrt{s}}{e^{s+1}}$.

Let $f : [0,1] \times R \to R$ be a continuous map. Note that if $f \in S_F$, then we have

$$\big|f\big(b_1(s)x_1(s), y_1(s)\big) - f\big(b_1(s)x_2(s), y_2(s)\big)\big| \le \frac{\sqrt{\pi}}{e^2(\pi+1)} |x_1 - x_2|$$

and

$$\big|h\big(t,s,x_1(t)\big) - h\big(t,s,x_2(t)\big)\big| \le \frac{1}{e^2} |x_1 - x_2|.$$

Thus conditions (i) and (ii)* are satisfied with $a = e$, $k_1 = \frac{\sqrt{\pi}}{e^2(\pi+1)}$, and $k_2 = \frac{1}{e^2}$.

Moreover, we have

$$k\mu + k^2\mu^2 \approx 0.102607 < 1.$$

This shows that assumption (vii) is satisfied. So, as all the conditions of Theorem 4.4 are satisfied, inclusion (4.2) has at least one solution $x \in C[0,1]$.

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