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An oscillation criterion of linear delay differential equations

George E. Chatzarakis[1*], Božena Dorociaková[2] and Rudolf Olach[2]

*Correspondence:
geaxatz@otenet.gr;
gea.xatz@aspete.gr
[1]Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), Athens, Greece
Full list of author information is available at the end of the article

Abstract

In this paper, we present a new sufficient condition for the oscillation of all solutions of linear delay differential equations. The obtained result improves known conditions in the literature. We also give an example to illustrate the applicability and strength of the obtained condition over known ones.

MSC: Primary 34K06; secondary 34K11

Keywords: Oscillation; Delay differential equation; Oscillatory solutions; Nonoscillatory solutions; Sufficient conditions

1 Introduction

This paper is devoted to studying the oscillation of the first-order delay differential equation of the form

$$x'(t) + p(t)x\big(t - r(t)\big) = 0, \quad t \geq T_0, \tag{1}$$

where $T_0 \in \mathbb{R}_+, p, r \in C([T_0, \infty), (0, \infty))$, and $0 < r(t) < t$, and $\lim_{t\to\infty}(t - r(t)) = \infty$.

The problem of the oscillatory properties of the solutions of delay differential equations has been recently investigated by many authors. See, for example [1–11] and the references therein. We mention some results for the purpose of this paper.

Chatzarakis and Li [5] studied the oscillation of delay differential equations with nonmonotone arguments. The results reported in this paper (regarding the oscillation of first-order delay differential equations) have numerous applications (e.g., comparison principles) in the study of oscillation and asymptotic behavior of higher-order differential equations; see, for instance, [1, 6, 10, 11] for more detail.

In 1972, Ladas, Lakshmikantham, and Papadakis [9] proved that if

$$\limsup_{t\to\infty} \int_{t-r(t)}^{t} p(s)\, ds > 1, \tag{2}$$

then all solutions of (1) are oscillatory.

Ladas [8] in 1979, and Koplatadze and Chanturiya [7] in 1982 improved (2) to

$$\liminf_{t\to\infty}\int_{t-r(t)}^{t} p(s)\,ds > \frac{1}{e}. \tag{3}$$

Concerning the constant $\frac{1}{e}$ in (3), it is to be pointed out that if the inequality

$$\int_{t-r(t)}^{t} p(s)\,ds \le \frac{1}{e}$$

eventually holds, then, according to a result in [4], (1) has a nonoscillatory solution.

In the recent paper [3] the authors established the following oscillation criterion for (1) when $r(t) = \tau, \tau > 0$.

Theorem 1.1 ([3]) *Let $p : [T_0,\infty) \to \mathbb{R}_+$ be a nonnegative, bounded, and uniformly continuous function such that*

$$\liminf_{t\to\infty}\int_{t-\tau}^{t} p(s)\,ds > 0.$$

Moreover, suppose that the function

$$A(t) = \int_{t-\tau}^{t} p(s)\,ds, \quad t \ge T_0 + \tau,$$

is slowly varying at infinity. Then

$$\limsup_{t\to\infty}\int_{t-\tau}^{t} p(s)\,ds > \frac{1}{e}$$

implies that all solutions of (1) *are oscillatory.*

Our aim is establishing a new condition for the oscillation of all solutions of (1), including the cases where conditions (2)–(3) and Theorem 1.1 cannot be applied. We also give an example illustrating the applicability and strength of the obtained condition over the known ones.

2 Main result

The proof of our main result is essentially based on the following lemmas.

Lemma 2.1 *Let x be an eventually positive solution of* (1). *Then for sufficiently large $t_0 > T_0$,*

$$\ln\frac{x(t-\tau)}{x(t)} = \int_{t-\tau}^{t} p(s)\frac{x(s-r(s))}{x(s)}\,ds, \quad t \ge t_0 + \tau.$$

Proof Let x be an eventually positive solution of (1). Then $x(t - r(t)) > 0$ for $t \ge t_0 + \tau$, where $t_0 > T_0$ is sufficiently large. From (1), for $t \ge t_0 + \tau$, we obtain

$$\frac{x'(t)}{x(t)} + p(t)\frac{x(t-r(t))}{x(t)} = 0,$$

or

$$\int_{t-\tau}^{t} \frac{x'(s)}{x(s)}\,ds + \int_{t-\tau}^{t} p(s)\frac{x(s-r(s))}{x(s)}\,ds = 0,$$

that is,

$$\ln\frac{x(t-\tau)}{x(t)} = \int_{t-\tau}^{t} p(s)\frac{x(s-r(s))}{x(s)}\,ds, \quad t \ge t_0+\tau.$$

The proof of the lemma is complete. □

Lemma 2.2 *Let x be an eventually positive solution of* (1). *Then*

$$\begin{aligned}\ln\frac{x(t-\tau)}{x(t)} &= p(t)\int_{t-\tau}^{t} \frac{x(s-r(s))}{x(s)}\,ds \\ &\quad + \bigl[p(t)-p(t-\tau)\bigr]\int_{t_0}^{t-\tau} \frac{x(s-r(s))}{x(s)}\,ds \\ &\quad - \int_{t-\tau}^{t} p'(s)\int_{t_0}^{s} \frac{x(u-r(u))}{x(u)}\,du\,ds, \quad t \ge t_0+\tau. \end{aligned} \tag{4}$$

Proof It is obvious that

$$\begin{aligned}&\int_{t-\tau}^{t} p(s)\frac{x(s-r(s))}{x(s)}\,ds \\ &\quad = p(t)\int_{t-\tau}^{t} \frac{x(s-r(s))}{x(s)}\,ds \\ &\qquad + \bigl[p(t)-p(t-\tau)\bigr]\int_{t_0}^{t-\tau} \frac{x(s-r(s))}{x(s)}\,ds - \int_{t-\tau}^{t} p'(s)\int_{t_0}^{s} \frac{x(u-r(u))}{x(u)}\,du\,ds,\end{aligned}$$

or

$$\begin{aligned}\ln\frac{x(t-\tau)}{x(t)} &= p(t)\int_{t-\tau}^{t} \frac{x(s-r(s))}{x(s)}\,ds \\ &\quad + \bigl[p(t)-p(t-\tau)\bigr]\int_{t_0}^{t-\tau} \frac{x(s-r(s))}{x(s)}\,ds \\ &\quad - \int_{t-\tau}^{t} p'(s)\int_{t_0}^{s} \frac{x(u-r(u))}{x(u)}\,du\,ds, \quad t \ge t_0+\tau. \end{aligned}$$

The proof of the lemma is complete. □

Now we focus on the function

$$R(t) = -\int_{t-\tau}^{t} p'(s)\int_{t_0}^{s} \frac{x(u-r(u))}{x(u)}\,du\,ds, \quad t \ge t_0+\tau.$$

Lemma 2.3 *Let x be an eventually positive solution of* (1). *Assume that*:

(H_1) *the function $p \in C^1([T_0,\infty),(0,\infty))$*;

(H_2) $p((2n+1)\tau) - p(2n\tau) = 0, n \in \mathbb{N}$;

(H_3) *there exists* $T_n \in (2n\tau, (2n+1)\tau)$ *such that* $p'(t) > 0$ *for* $t \in (T_n - \tau, T_n)$ *and* $p'(t) < 0$ *for* $t \in (T_n, (2n+1)\tau]$, $n \in \mathbb{N}$;

(H_4) $\inf\{-\int_{T_n}^{(2n+1)\tau}(t - T_n)p'(t)\,dt, n \in \mathbb{N}\} > 0$.

Then

$$\inf\bigl\{R\bigl((2n+1)\tau\bigr), n \in \mathbb{N}\bigr\} > 0.$$

Proof We easily see that

$$\begin{aligned}
&R\bigl((2n+1)\tau\bigr) \\
&\quad = -\int_{2n\tau}^{(2n+1)\tau} p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt \\
&\quad = -\int_{2n\tau}^{T_n} p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt - \int_{T_n}^{(2n+1)\tau} p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt \\
&\quad \geq -\int_{2n\tau}^{T_n} p'(t)\,dt\int_{t_0}^{T_n}\frac{x(s-r(s))}{x(s)}\,ds - \int_{T_n}^{(2n+1)\tau} p'(t)\,dt\int_{t_0}^{T_n}\frac{x(s-r(s))}{x(s)}\,ds \\
&\qquad - \int_{T_n}^{(2n+1)\tau} p'(t)\int_{T_n}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt \\
&\quad = \left(-\int_{2n\tau}^{T_n} p'(t)\,dt - \int_{T_n}^{(2n+1)\tau} p'(t)\,dt\right)\int_{t_0}^{T_n}\frac{x(s-r(s))}{x(s)}\,ds \\
&\qquad - \int_{T_n}^{(2n+1)\tau} p'(t)\int_{T_n}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt \\
&\quad = \bigl[p(2n\tau) - p\bigl((2n+1)\tau\bigr)\bigr]\int_{t_0}^{T_n}\frac{x(s-r(s))}{x(s)}\,ds - \int_{T_n}^{(2n+1)\tau} p'(t)\int_{T_n}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt \\
&\quad = -\int_{T_n}^{(2n+1)\tau} p'(t)\int_{T_n}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt.
\end{aligned}$$

Since $x(t)$ is decreasing, $x(t - r(t)) \geq x(t)$, $t \geq t_0 + \tau$. Thus

$$\begin{aligned}
R\bigl((2n+1)\tau\bigr) &\geq -\int_{T_n}^{(2n+1)\tau} p'(t)\int_{T_n}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt \\
&\geq -\int_{T_n}^{(2n+1)\tau}(t - T_n)p'(t)\,dt, \quad n \in \mathbb{N}.
\end{aligned}$$

In view of (H_4), we get $\inf\{R((2n+1)\tau), n \in \mathbb{N}\} > 0$.

The proof of the lemma is complete. □

Theorem 2.1 *Suppose that* (H_1)–(H_4) *hold,* $r(t) \geq \tau$, $p(t)$ *is periodic with period* 2τ, *and*

$$p'(t-\tau) - p'(t) > 0, \quad t \in \bigl(T_n, (2n+1)\tau\bigr), n \in \mathbb{N}, \tag{5}$$

$$\liminf_{t\to\infty}\int_{t-r(t)}^{t} p(s)\,ds > 0. \tag{6}$$

Then all solutions of (1) *are oscillatory.*

Proof Assume that (1) has a positive solution x. The derivative of the function $R(t)$ is

$$\begin{aligned} R'(t) &= -p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds + p'(t-\tau)\int_{t_0}^{t-\tau}\frac{x(s-r(s))}{x(s)}\,ds \\ &= -p'(t)\int_{t-\tau}^{t}\frac{x(s-r(s))}{x(s)}\,ds - p'(t)\int_{t_0}^{t-\tau}\frac{x(s-r(s))}{x(s)}\,ds \\ &\quad + p'(t-\tau)\int_{t_0}^{t-\tau}\frac{x(s-r(s))}{x(s)}\,ds \\ &= -p'(t)\int_{t-\tau}^{t}\frac{x(s-r(s))}{x(s)}\,ds \\ &\quad + \bigl[p'(t-\tau)-p'(t)\bigr]\int_{t_0}^{t-\tau}\frac{x(s-r(s))}{x(s)}\,ds, \quad t \ge t_0+\tau. \end{aligned}$$

Condition (5) implies that $R'(t) > 0$ for $t \in (T_n, (2n+1)\tau)$. Thus the function $R(t)$ is increasing on $(T_n, (2n+1)\tau)$, $n \in \mathbb{N}$. Since $R(T_n) < 0$, $n \in \mathbb{N}$, by Lemma 2.3 there exist $t_n \in (T_n, (2n+1)\tau)$ such that $R(t_n) = 0$, $n \in \mathbb{N}$. Condition (H_4) implies that $\inf\{(2n+1)\tau - T_n, n \in \mathbb{N}\} > 0$. Put

$$H(t) = p(t) - p(t-\tau), \quad t \in \bigl(T_n, (2n+1)\tau\bigr], n \in \mathbb{N}.$$

According to (5) and (H_2), we have

$$H'(t) = p'(t) - p'(t-\tau) < 0, \quad t \in \bigl(T_n, (2n+1)\tau\bigr),$$

and $H((2n+1)\tau) = 0$, $n \in \mathbb{N}$. Then

$$H(t) = p(t) - p(t-\tau) > 0 \quad \text{for } t \in \bigl(T_n, (2n+1)\tau\bigr), n \in \mathbb{N}. \tag{7}$$

Now assume that

$$t_n \le b_n = (2n+1)\tau - \varepsilon, \quad n \in \mathbb{N},$$

where $0 < \varepsilon < \inf\{(2n+1)\tau - T_n, n \in \mathbb{N}\}$. In view of (4), we get

$$\begin{aligned} &\ln\frac{x(b_n-\tau)}{x(b_n)} \\ &\quad = p(b_n)\int_{b_n-\tau}^{b_n}\frac{x(s-\tau)}{x(s)}\,ds \\ &\qquad + \bigl[p(b_n)-p(b_n-\tau)\bigr]\int_{t_0}^{b_n-\tau}\frac{x(s-\tau)}{x(s)}\,ds + R(b_n), \quad b_n \ge t_0+\tau, n \in \mathbb{N}. \end{aligned}$$

Condition (6) implies that $x(t-r(t))/x(t)$ is bounded [7]. Since $x(t-r(t))/x(t) \ge x(t-\tau)/x(t)$, it is obvious that there exists a constant $K > 0$ such that $x(t-\tau)/x(t) \le K$, $t \ge T \ge t_0+\tau$, where T is sufficiently large. Thus

$$\ln K \ge p(b_n)\int_{b_n-\tau}^{b_n}\frac{x(s-r(s))}{x(s)}\,ds$$

$$+\left[p(b_n)-p(b_n-\tau)\right]\int_{t_0}^{b_n-\tau}\frac{x(s-r(s))}{x(s)}\,ds+R(b_n),\quad b_n\geq T. \tag{8}$$

Otherwise, for sufficiently large $b_n \geq T$, by (7) and the periodicity of $p(t)$, we get

$$\left[p(b_n)-p(b_n-\tau)\right]\int_{t_0}^{b_n-\tau}\frac{x(s-r(s))}{x(s)}\,ds>\ln K,$$

which contradicts (8).

Now assume that there exists a sequence $\{t_n\}$ such that

$$t_n\to(2n+1)\tau\quad\text{as } n\to\infty,\qquad R(t_n)=0,\quad t_n\in\left(T_n,(2n+1)\tau\right), n\in\mathbb{N}.$$

Then

$$\begin{aligned}
&R\left((2n+1)\tau\right)\\
&\quad=-\int_{2n\tau}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\quad=-\int_{2n\tau}^{T_n}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt-\int_{T_n}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\quad=-\int_{t_n-\tau}^{T_n}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt+\int_{t_n-\tau}^{2n\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\qquad-\int_{T_n}^{t_n}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt-\int_{t_n}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\quad=-\int_{t_n-\tau}^{t_n}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt+\int_{t_n-\tau}^{2n\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\qquad-\int_{t_n}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\quad=R(t_n)+\int_{t_n-\tau}^{2n\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt-\int_{t_n}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\quad=\int_{t_n-\tau}^{2n\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\\
&\qquad-\int_{t_n}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt,\quad n\in\mathbb{N}.
\end{aligned}$$

Since $t_n\to(2n+1)\tau\quad\text{as } n\to\infty$, clearly,

$$\int_{t_n-\tau}^{2n\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\to 0$$

and

$$-\int_{t_n}^{(2n+1)\tau}p'(t)\int_{t_0}^{t}\frac{x(s-r(s))}{x(s)}\,ds\,dt\to 0.$$

Thus

$$R\bigl((2n+1)\tau\bigr) \to 0 \quad \text{as } t_n \to (2n+1)\tau \text{ and } n \to \infty.$$

This contradicts $\inf\{R((2n+1)\tau), n \in \mathbb{N}\} > 0$.

The proof of the theorem is complete. □

Example Consider the delay differential equation

$$x'(t) + \left(\frac{a}{\pi e} + \delta \sin at\right) x\left(t - \frac{\pi}{a}\right) = 0, \qquad t \geq 0, \tag{9}$$

where $a > 0, \delta \in (0, \frac{a}{\pi e})$.

Equation (9) is a particular case of (1) when $r(t) = \tau = \frac{\pi}{a}, T_0 = 0$, and

$$p(t) = \frac{a}{\pi e} + \delta \sin at.$$

It is easy to see that (H_1) is satisfied. For condition (H_2), we have

$$\begin{aligned} &p\bigl((2n+1)\tau\bigr) - p(2n\tau) \\ &\quad = \frac{a}{\pi e} + \delta \sin a(2n+1)\frac{\pi}{a} - \frac{a}{\pi e} - \delta \sin a2n\frac{\pi}{a} \\ &\quad = \delta\bigl[\sin(2n+1)\pi - \sin 2n\pi\bigr] = 0, \qquad n \in \mathbb{N}. \end{aligned}$$

In condition (H_3), $T_n = (2n+0.5)\frac{\pi}{a}$, and

$$\begin{aligned} &p'(t) = a\delta \cos at > 0 \quad \text{for } t \in \left((2n-0.5)\frac{\pi}{a}, (2n+0.5)\frac{\pi}{a}\right), \\ &p'(t) < 0 \quad \text{for } t \in \left((2n+0.5)\frac{\pi}{a}, (2n+1)\frac{\pi}{a}\right], n \in \mathbb{N}. \end{aligned}$$

For condition (H_4), we get

$$\begin{aligned} &-a\delta \int_{(2n+0.5)\frac{\pi}{a}}^{(2n+1)\frac{\pi}{a}} \left(t - (2n+0.5)\frac{\pi}{a}\right) \cos at\, dt \\ &\quad = a\delta(2n+0.5)\frac{\pi}{a} \int_{(2n+0.5)\frac{\pi}{a}}^{(2n+1)\frac{\pi}{a}} \cos at\, dt - a\delta \int_{(2n+0.5)\frac{\pi}{a}}^{(2n+1)\frac{\pi}{a}} t \cos at\, dt \\ &\quad = \delta(2n+0.5)\frac{\pi}{a}\bigl[\sin(2n+1)\pi - \sin(2n+0.5)\pi\bigr] - a\delta\left[\frac{1}{a^2}\cos(2n+1)\pi\right. \\ &\qquad \left. + \frac{1}{a}(2n+1)\frac{\pi}{a}\sin(2n+1)\pi - \frac{1}{a^2}\cos(2n+0.5)\pi - \frac{1}{a}(2n+0.5)\frac{\pi}{a}\sin(2n+0.5)\pi\right] \\ &\quad = -\delta(2n+0.5)\frac{\pi}{a} - a\delta\left[-\frac{1}{a^2} - \frac{1}{a}(2n+0.5)\frac{\pi}{a}\right] \\ &\quad = -\delta(2n+0.5)\frac{\pi}{a} - \delta\left[-\frac{1}{a} - (2n+0.5)\frac{\pi}{a}\right] = \frac{\delta}{a}. \end{aligned}$$

Thus

$$\begin{aligned}
&\inf\left\{-\int_{T_n}^{(2n+1)\tau}(t-T_n)p'(t)\,dt, n\in\mathbb{N}\right\} \\
&\quad=\inf\left\{-a\delta\int_{(2n+0.5)\frac{\pi}{a}}^{(2n+1)\frac{\pi}{a}}\left(t-(2n+0.5)\frac{\pi}{a}\right)\cos at\,dt, n\in\mathbb{N}\right\}=\frac{\delta}{a}>0.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
p'(t-\tau)-p'(t)=p'\left(t-\frac{\pi}{a}\right)-p'(t)&=a\delta\cos a\left(t-\frac{\pi}{a}\right)-a\delta\cos at \\
&=a\delta\bigl[\cos(at-\pi)-\cos at\bigr]=a\delta(-\cos at-\cos at) \\
&=-2a\delta\cos at>0\quad\text{for } t\in\left((2n+0.5)\frac{\pi}{a},(2n+1)\frac{\pi}{a}\right),\quad n\in\mathbb{N},
\end{aligned}$$

that is, condition (5) is satisfied. Also,

$$\begin{aligned}
&\int_{t-\frac{\pi}{a}}^{t}\left(\frac{a}{\pi e}+\delta\sin as\right)ds \\
&\quad=\frac{a}{\pi e}t-\frac{\delta}{a}\cos at-\frac{a}{\pi e}\left(t-\frac{\pi}{a}\right)+\frac{\delta}{a}\cos a\left(t-\frac{\pi}{a}\right) \\
&\quad=-\frac{\delta}{a}\cos at+\frac{1}{e}+\frac{\delta}{a}\cos(at-\pi)=\frac{1}{e}-\frac{\delta}{a}\cos at-\frac{\delta}{a}\cos at \\
&\quad=\frac{1}{e}-\frac{2\delta}{a}\cos at.
\end{aligned}$$

Therefore

$$\begin{aligned}
\liminf_{t\to\infty}\int_{t-\frac{\pi}{a}}^{t}p(s)\,ds&=\liminf_{t\to\infty}\int_{t-\frac{\pi}{a}}^{t}\left(\frac{a}{\pi e}+\delta\sin as\right)ds \\
&=\frac{1}{e}-\frac{2\delta}{a}>0,\quad\delta\in\left(0,\frac{a}{\pi e}\right),
\end{aligned}$$

so that all conditions of Theorem 2.1 are satisfied, which means that all solutions of (9) are oscillatory.

Observe, however, that

$$\limsup_{t\to\infty}\int_{t-\frac{\pi}{a}}^{t}p(s)\,ds=\frac{1}{e}+\frac{2\delta}{a}<\frac{\pi+2}{\pi e}<1,\quad\delta\in\left(0,\frac{a}{\pi e}\right)$$

and

$$\liminf_{t\to\infty}\int_{t-\frac{\pi}{a}}^{t}p(s)\,ds=\frac{1}{e}-\frac{2\delta}{a}<\frac{1}{e},\quad\delta\in\left(0,\frac{a}{\pi e}\right),$$

which means that conditions (2) and (3) are not satisfied.

Moreover, the function $f(t)$ is not slowly varying at infinity. Indeed,

$$f(t)=\int_{t-\frac{\pi}{a}}^{t}p(s)\,ds=\frac{1}{e}-\frac{2\delta}{a}\cos at,\quad\delta\in\left(0,\frac{a}{\pi e}\right),$$

and

$$f(t+s)-f(t) = -\frac{2\delta}{a}\cos a(t+s) + \frac{2\delta}{a}\cos at$$
$$= \frac{2\delta}{a}\bigl[\cos at - \cos a(t+s)\bigr], \quad s \in \mathbb{R}.$$

For $s = \pi/a$, we get

$$f\left(t+\frac{\pi}{a}\right) - f(t) = \frac{2\delta}{a}\left[\cos at - \cos a\left(t+\frac{\pi}{a}\right)\right]$$
$$= \frac{2\delta}{a}[\cos at + \cos at] = \frac{4\delta}{a}\cos at \not\to 0 \quad \text{as } t\to\infty, \delta \in \left(0, \frac{a}{\pi e}\right).$$

Thus Theorem 1.1 cannot be applied. Recall (see, e.g., [3, 12]) that a function $f:[t_0,\infty)\to\mathbb{R}$ is slowly varying at infinity if for every $s\in\mathbb{R}$,

$$f(t+s)-f(t)\to 0 \quad \text{as } t\to\infty.$$

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Authors' contributions
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Author details
[1]Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), Athens, Greece. [2]Department of Applied Mathematics, Faculty of Mechanical Engineering, University of Žilina, Žilina, Slovak Republic.

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