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New type of source extension for a two-dimensional special lattice equation and determinant solutions

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Abstract

We present a new type of two-dimensional special lattice equations with self-consistent sources using the source generation procedure. Then we obtain the Grammy-type and Casorati-type determinant solutions of the coupled system. Further, we present the one-soliton and two-soliton solutions.

Keywords: Two-dimensional special lattice equation; Self-consistent sources; Determinant solution; Soliton solution

1 Introduction

Soliton equations with self-consistent sources (SESCSs) are integrable coupled generalizations of the soliton equations. These coupled systems are usually relevant to problems in certain areas of physics, such as hydrodynamics, solid-state physics, and plasma physics [1–4]. This branch has attracted considerable attention in recent years [5–11]. In [10] the authors proposed a method, termed the source generation procedure (SGP), to construct SESCSs, which has been applied to study different kinds of SESCSs [7, 8, 12–14]. Furthermore, new types of SESCSs have also been studied, including the AKP-type and BKP-type equations [1, 11, 15–17]. In this study, we consider the 2 + 1-dimensional KP equation

$$-4u_t + u_{xxx} + 6uu_x + 3\int^x u_{yy}\,dx = 0 \tag{1}$$

as an example. Through the dependent-variable transformation

$$u = 2(\ln\tau)_{xx},$$

the KP Eq. (1) can be represented in the bilinear form as

$$\left(D_x^4 - 4D_xD_t + 3D_y^2\right)\tau\cdot\tau = 0, \tag{2}$$

where D is Hirota's bilinear operator [18] given by

$$D_t^l D_y^m a \cdot b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m a(t,y)b(t',y')\Big|_{t'=t,y'=y}.$$

The application of the SGP is closely related to the bilinear form of the soliton equation. For example, applying the SGP to the operator D_t in (2) produces the first type of KPESCS [2, 5, 10], whereas a new type of KPESCS [1, 11] is obtained by applying the SGP method to the operator D_y in (2). In terms of the bilinear form, the operator D_t is of the first order, and D_y is of the second order, which results in different types of SESCSs. Therefore it is natural to determine if there are any other SESCSs of this new type, especially in differential–difference equations.

However, the following lattice equation was proposed by Blaszak and Szum [19] as an application of the "central extension procedure and operand formalism":

$$\frac{\partial u_n}{\partial t} = u_n H^{-1} p_{n-1}, \tag{3}$$

$$\frac{\partial v_n}{\partial t} = u_{n+1} - u_n + (E+1)^{-1}\frac{\partial p_n}{\partial y}, \tag{4}$$

$$\frac{\partial p_n}{\partial t} = v_{n+1} - v_n - p_n H^{-1} p_n, \tag{5}$$

where E is the shift operator, that is, $Eu_n = u_{n+1}$, and $H = (E+1)/(E-1)$. By setting $w_n = (E+1)^{-1}p_n$ Eqs. (3)–(5) can be rewritten as [20]

$$\frac{\partial u_n}{\partial t} = u_n(w_n - w_{n-1}), \tag{6}$$

$$\frac{\partial v_n}{\partial t} = u_{n+1} - u_n + \frac{\partial w_n}{\partial y}, \tag{7}$$

$$\frac{\partial w_{n+1}}{\partial t} + \frac{\partial w_n}{\partial t} = v_{n+1} - v_n - w_{n+1}^2 + w_n^2. \tag{8}$$

Through the dependent-variable transformations

$$u_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad v_n = \frac{D_t^2 \tau_n \cdot \tau_{n+1}}{\tau_n \tau_{n+1}}, \qquad w_n = \left(\ln \frac{\tau_{n+1}}{\tau_n}\right)_t,$$

and by introducing the auxiliary variable z, Eqs. (6)–(8) can be transformed into the bilinear forms as

$$\left(D_z e^{\frac{1}{2}D_n} - D_t^2 e^{\frac{1}{2}D_n}\right)\tau_n \cdot \tau_n = 0, \tag{9}$$

$$\left(D_t D_z - D_t D_y - 4\sinh^2\left(\frac{1}{2}D_n\right)\right)\tau_n \cdot \tau_n = 0, \tag{10}$$

where the difference operator $e^{\delta D_n}$ is defined as [18]

$$e^{\delta D_n} a \cdot b = a(n+\delta)b(n-\delta).$$

We can see that D_t in Eq. (9) is a second-order operator. It is important that D_t acts on different functions τ_{n+1} and τ_n, which is different from that for operator D_y in the bilinear KP Eq. (2). The purpose of this study is to construct and solve a new type of special lattice ESCS. The remainder of this paper is organized as follows. In Sect. 2, we propose a new type of special lattice ESCS using the SGP, and obtain its Grammian determinant solution. In Sect. 3, we derive the Casorati determinant solution. In Sect. 4, we describe the one-soliton and two-soliton solutions of the coupled system. Finally, we present conclusions in Sect. 5.

2 New type of special lattice ESCS and Grammian determinant solution

The Grammian determinant solutions of bilinear Eqs. (9)–(10) have the following forms [21]:

$$\tau_n = \det\left| c_{ij} + \int^t \varphi_i(n)\psi_j(-n)\,dt \right|_{1\le i,j\le N}, \tag{11}$$

where each c_{ij} is an arbitrary constant, and the functions $\varphi_i(n)$ and $\psi_j(-n)$ satisfy the following differential equations:

$$\frac{\partial \varphi_i(n)}{\partial t} = \varphi_i(n+1), \qquad \frac{\partial \varphi_i(n)}{\partial y} = \varphi_i(n+2) + \varphi_i(n-1), \tag{12}$$

$$\frac{\partial \varphi_i(n)}{\partial z} = \varphi_i(n+2), \qquad \frac{\partial \psi_j(-n)}{\partial y} = -\psi_j(-n+2) - \psi_j(-n-1), \tag{13}$$

$$\frac{\partial \psi_i(-n)}{\partial t} = -\psi_i(-n+1), \qquad \frac{\partial \psi_j(-n)}{\partial z} = -\psi_j(-n+2). \tag{14}$$

Now we construct the special lattice ESCS by applying the SGP. First, the Grammian determinant function (11) is changed into the following form:

$$f_n = \det\left| C_{ij}(t) + \int^t \varphi_i(n)\psi_j(-n)\,dt \right|_{1\le i,j\le N}, \tag{15}$$

wherein $C_{ij}(t)$ are functions satisfying

$$C_{ij}(t) = \begin{cases} C_i(t), & i=j, 1\le i\le M\le N, M\in Z^+, \\ c_{ij} & \text{otherwise}, 1\le i,j\le N. \end{cases}$$

Here each $C_i(t)$ is a differentiable function with respect to t, c_{ij} are arbitrary constants, and the functions $\varphi_i(n)$ and $\psi_i(-n)$ satisfy the dispersion relations (12)–(14). For calculation, the function f_n can be rewritten in the Pfaffian form as follows:

$$f_n = \left(1,2,\dots,N,N^*,\dots,2^*,1^*\right)_n \equiv (\circ)_n, \tag{16}$$

where the Pfaffian elements are defined as

$$\left(i,j^*\right)_n = C_{ij}(t) + \int^t \varphi_i(n)\psi_j(-n)\,dt, \qquad (i,j)_n = \left(i^*,j^*\right)_n = 0, \quad 1\le i,j\le N.$$

We introduce other new functions expressed as

$$\begin{aligned} g_{i,n} &= \sqrt{\dot{C}_i(t)}\left(d_{-1}^*, 1, 2, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n \\ &\triangleq \sqrt{\dot{C}_i(t)}\left(d_{-1}^*, \star_1\right)_n, \quad i = 1, 2, \ldots, M, \end{aligned} \tag{17}$$

$$\begin{aligned} h_{i,n} &= \sqrt{\dot{C}_i(t)}\left(d_{-1}, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, 2^*, 1^*\right)_n \\ &\triangleq \sqrt{\dot{C}_i(t)}(d_{-1}, \star_2)_n, \quad i = 1, 2, \ldots, M, \end{aligned} \tag{18}$$

where the dot and hat symbols above a variable respectively denote the derivative with respect to the variable t and the deletion of the letter under it. Here the above Pfaffian entries refer to

$$\begin{aligned} &\left(d_m^*, i\right)_n = \varphi_i(n+m), \qquad \left(d_m, j^*\right)_n = \psi_j(-n+m), \\ &\left(d_m, d_l^*\right)_n = (d_m, i)_n = \left(d_l^*, i^*\right)_n = 0, \quad m, l \in \mathrm{Z}. \end{aligned}$$

In this condition, we introduce another set of auxiliary functions in the following expression:

$$k_{i,n} = \dot{C}_i(t)\left(1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n, \tag{19}$$

$$\begin{aligned} P_{i,n} &= \frac{\ddot{C}_i(t)}{2\sqrt{\dot{C}_i(t)}}\left(d_{-1}^*, 1, 2, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n \\ &\quad + \sqrt{\dot{C}_i(t)}\left[\sum_{1 \le j \le N}^{j \ne i}\left(d_{-1}^*, 1, \ldots, \hat{j}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, \hat{j}^*, \ldots, 1^*\right)_n\right], \end{aligned} \tag{20}$$

$$\begin{aligned} Q_{i,n} &= \frac{\ddot{C}_i(t)}{2\sqrt{\dot{C}_i(t)}}\left(d_{-1}, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, 2^*, 1^*\right)_n \\ &\quad + \sqrt{\dot{C}_i(t)}\left[\sum_{1 \le j \le N}^{j \ne i}\left(d_{-1}, 1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, N, N^*, \ldots, \hat{j}^*, \ldots, 1^*\right)_n\right], \end{aligned} \tag{21}$$

where $i = 1, 2, \ldots, M$. Thus the new functions in Eqs. (16)–(18) and auxiliary functions in Eqs. (19)–(21) satisfy the following bilinear expressions:

$$\begin{aligned} \left(D_z - D_t^2\right)e^{\frac{D_n}{2}} f_n \cdot f_n &= -\sum_{i=1}^{M} D_t\left(e^{\frac{D_n}{2}} + e^{-\frac{D_n}{2}}\right)k_{i,n} \cdot f_n + \sum_{i=1}^{M} D_t e^{\frac{D_n}{2}} g_{i,n} \cdot h_{i,n} \\ &\quad + \sum_{i=1}^{M} e^{\frac{D_n}{2}}(g_{i,n} \cdot Q_{i,n} + P_{i,n} \cdot h_{i,n}), \end{aligned} \tag{22}$$

$$\left(D_t D_z - D_t D_y - 4\sinh^2\left(\frac{1}{2}D_n\right)\right)f_n \cdot f_n = -2\sum_{i=1}^{M} g_{i,n} \cdot h_{i,n}, \tag{23}$$

$$\left(e^{\frac{1}{2}D_n} - e^{-\frac{1}{2}D_n}\right)k_{i,n} \cdot f_n = -e^{\frac{1}{2}D_n} g_{i,n} \cdot h_{i,n}, \quad i = 1, 2, \ldots, M, \tag{24}$$

$$\left(D_t + e^{-D_n}\right)f_n \cdot g_{i,n} = \left(\sum_{j=1}^{M} k_{j,n}\right)g_{i,n} - f_n P_{i,n}, \quad i = 1, 2, \ldots, M, \tag{25}$$

$$\left(D_t + e^{-D_n}\right)h_{i,n} \cdot f_n = -h_{i,n}\left(\sum_{j=1}^{M} k_{j,n}\right) + f_n Q_{i,n}, \quad i = 1,2,\ldots,M, \tag{26}$$

$$(D_z - D_y)e^{-\frac{1}{2}D_n} f_n \cdot g_{i,n} = e^{\frac{1}{2}D_n} f_n \cdot g_{i,n}, \quad i = 1,2,\ldots,M, \tag{27}$$

$$(D_z - D_y)e^{-\frac{1}{2}D_n} h_{i,n} \cdot f_n = e^{\frac{1}{2}D_n} h_{i,n} \cdot f_n, \quad i = 1,2,\ldots,M. \tag{28}$$

In the following section, we consider Eqs. (22), (23), and (25) as examples for verification. The key to the proof is in the derivatives of functions f_n, f_{n+1}, $g_{i,n}$, and $h_{i,n}$. According to Eqs. (16)–(21), we have the following differential formulas concerning f_n and f_{n+1}:

$$f_{n+1} = f_n + \left(d_{-1}, d_0^*, \circ\right)_n, \qquad f_{n-1} = f_n - \left(d_0, d_{-1}^*, \circ\right)_n, \tag{29}$$

$$\frac{\partial f_n}{\partial z} = \left(d_0, d_1^*, \circ\right)_n + \left(d_1, d_0^*, \circ\right)_n, \tag{30}$$

$$\frac{\partial f_n}{\partial y} = \left(d_0, d_1^*, \circ\right)_n + \left(d_1, d_0^*, \circ\right)_n - \left(d_{-1}, d_{-1}^*, \circ\right)_n, \tag{31}$$

$$\frac{\partial f_n}{\partial t} = \sum_{i=1}^{M} k_{i,n} + \left(d_0, d_0^*, \circ\right)_n, \tag{32}$$

$$\begin{aligned}\frac{\partial^2 f_n}{\partial t^2} = {}& \sum_{i=1}^{M} \frac{\partial k_{i,n}}{\partial t} + \left(d_0, d_1^*, \circ\right)_n - \left(d_1, d_0^*, \circ\right)_n \\ & + \sum_{i=1}^{M} \dot{C}_i(t)\left(d_0, d_0^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n,\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial^2 f_n}{\partial t \partial y} = {}& \sum_{i=1}^{M} \dot{C}_i(t)\left[\left(d_0, d_1^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n\right. \\ & + \left(d_1, d_0^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n \\ & \left. - \left(d_{-1}, d_{-1}^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n\right] \\ & + \left(d_0, d_2^*, \circ\right)_n + \left(d_0, d_{-1}^*, \circ\right)_n - \left(d_2, d_0^*, \circ\right)_n - \left(d_{-1}, d_0^*, \circ\right)_n \\ & - \left(d_{-1}, d_{-1}^*, d_0, d_0^*, \circ\right)_n,\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\partial^2 f_n}{\partial t \partial z} = {}& \sum_{i=1}^{M} \dot{C}_i(t)\left[\left(d_0, d_1^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n\right. \\ & \left. + \left(d_1, d_0^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n\right. \\ & + \left(d_0, d_2^*, \circ\right)_n - \left(d_2, d_0^*, \circ\right)_n,\end{aligned} \tag{35}$$

$$\frac{\partial f_{n+1}}{\partial t} = \sum_{i=1}^{M} k_{i,n+1} + \left(d_{-1}, d_1^*, \circ\right)_n, \tag{36}$$

$$\begin{aligned}\frac{\partial^2 f_{n+1}}{\partial t^2} = {}& \sum_{i=1}^{M} \frac{\partial k_{i,n+1}}{\partial t} + \left(d_{-1}, d_1^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right)_n \\ & + \left(d_{-1}, d_2^*, \circ\right)_n - \left(d_0, d_1^*, \circ\right)_n + \left(d_0, d_0^*, d_{-1}, d_1^*, \circ\right)_n.\end{aligned} \tag{37}$$

In addition, the derivatives of $g_{i,n}$ and $h_{i,n}$ are given as follows:

$$g_{i,n+1} = \sqrt{\dot{C}_i(t)}\left(d_0^*,\star_1\right)_n, \tag{38}$$

$$\frac{\partial g_{i,n+1}}{\partial t} = \sqrt{\dot{C}_i(t)}\left(d_1^*,\star_1\right)_n + P_{i,n+1}, \tag{39}$$

$$\frac{\partial g_{i,n+1}}{\partial z} = \sqrt{\dot{C}_i(t)}\left[\left(d_2^*,\star_1\right)_n + \left(d_0^*,d_0,d_1^*,\star_1\right)_n\right], \tag{40}$$

$$\frac{\partial g_{i,n+1}}{\partial y} = \sqrt{\dot{C}_i(t)}\left[\left(d_2^*,\star_1\right)_n + \left(d_{-1}^*,\star_1\right)_n + \left(d_0^*,d_0,d_1^*,\star_1\right)_n - \left(d_0^*,d_{-1},d_{-1}^*,\star_1\right)_n\right], \tag{41}$$

$$h_{i,n-1} = \sqrt{\dot{C}_i(t)}(d_0,\star_2)_n, \tag{42}$$

$$\frac{\partial h_{i,n}}{\partial t} = \sqrt{\dot{C}_i(t)}\left[-(d_0,\star_2)_n + \left(d_{-1},d_0,d_0^*,\star_2\right)_n\right] + Q_{i,n}, \tag{43}$$

$$\frac{\partial h_{i,n-1}}{\partial z} = \sqrt{\dot{C}_i(t)}\left[-(d_2,\star_2)_n + \left(d_0,d_1,d_0^*,\star_2\right)_n\right], \tag{44}$$

$$\frac{\partial h_{i,n-1}}{\partial y} = \sqrt{\dot{C}_i(t)}\left[\left(d_0,d_1,d_0^*,\star_2\right)_n - (d_2,\star_2)_n - (d_{-1},\star_2)_n - \left(d_0,d_{-1},d_{-1}^*,\star_2\right)_n\right], \tag{45}$$

where $i = 1,2,\dots,M$. Substituting expressions (29)–(33), (34)–(35), and (39) into Eq. (22), we obtain the Jacobi identities of the determinants

$$\begin{aligned}
&2\left[\left(d_{-1},d_1^*,\circ\right)_n\left(d_0,d_0^*,\circ\right)_n - \left(d_{-1},d_0^*,\circ\right)_n\left(d_0,d_1^*,\circ\right)_n + \left(d_{-1},d_0^*,d_0,d_1^*,\circ\right)_n(\circ)_n\right] \\
&\quad + \sum_{i=1}^{M}\dot{C}_i(t)\left[\left(d_{-1},d_1^*,1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n(\circ)_n\right. \\
&\quad - \left(1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n\left(d_{-1},d_1^*,\circ\right)_n + \left.\left(d_1^*,\star_1\right)_n(d_{-1},\star_2)_n\right] \\
&\quad + \sum_{i=1}^{M}\dot{C}_i(t)\left[\left(d_0,d_0^*,1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n(\circ)_n\right. \\
&\quad - \left(1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n\left(d_0,d_0^*,\circ\right)_n + \left.\left(d_0^*,\star_1\right)_n(d_0,\star_2)_n\right] \\
&\quad + \sum_{i=1}^{M}\dot{C}_i(t)\left[\left(d_{-1},d_0^*,1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n\left(d_0,d_0^*,\circ\right)_n\right. \\
&\quad - \left(d_0,d_0^*,1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n\left(d_{-1},d_0^*,\circ\right)_n \\
&\quad + \left.\left(d_0^*,\star_1\right)_n\left(d_{-1},d_0,d_0^*,\star_2\right)_n\right] \equiv 0,
\end{aligned}$$

thereby indicating that Eq. (22) holds. Similarly, substitution of expressions (17)–(18) and (29)–(35) into Eq. (23) yields the following determinant identities:

$$\begin{aligned}
&2\left[\left(d_{-1},d_0^*,\circ\right)_n\left(d_0,d_{-1}^*,\circ\right)_n - \left(d_0,d_0^*,\circ\right)_n\left(d_{-1},d_{-1}^*,\circ\right)_n + \left(d_{-1},d_{-1}^*,d_0,d_0^*,\circ\right)_n(\circ)_n\right] \\
&\quad + 2\sum_{i=1}^{M}\dot{C}_i(t)\left[\left(d_{-1},d_{-1}^*,1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n(\circ)_n\right. \\
&\quad - \left(1,\dots,\hat{i},\dots,N,N^*,\dots,\hat{i}^*,\dots,1^*\right)_n\left(d_{-1},d_{-1}^*,\circ\right)_n \\
&\quad + \left.\left(d_{-1}^*,\star_1\right)_n(d_{-1},\star_2)_n\right] \equiv 0.
\end{aligned}$$

Finally, we substitute Eqs. (16)–(17), (20), (32), and (38)–(39) into Eq. (25) to derive the following determinant identity:

$$\sum_{i=1}^{M}\dot{C}_i(t)\big[\big(d_0^*,\star_1\big)_n\big(d_{-1},d_{-1}^*,\circ\big)_n-\big(d_{-1}^*,\star_1\big)_n\big(d_{-1},d_0^*,\circ\big)_n$$
$$+\big(d_0^*,d_{-1},d_{-1}^*,\star_1\big)_n(\circ)_n\big]\equiv 0.$$

The above results indicate that Eqs. (23) and (25) are true. In the same manner, we can prove the other bilinear equations in (22)–(28). Therefore the bilinear Eqs. (22)–(28) constitute the bilinear forms of the two-dimensional special lattice ESCS, and the functions in Eqs. (16)–(21) are the Grammian determinant solutions of the coupled system.

By the dependent-variable transformations

$$u_n=\frac{f_{n+1}f_{n-1}}{f_n^2},\qquad w_n=\left(\ln\frac{f_{n+1}}{f_n}\right)_t,\qquad v_n=\left(\ln\frac{f_{n+1}}{f_n}\right)_z,\tag{46}$$

$$\Phi_{i,n}=\frac{g_{i,n}}{f},\qquad \Psi_{i,n}=\frac{h_{i,n}}{f},\tag{47}$$

and auxiliary transformations

$$\lambda_{i,n}=\frac{k_{i,n}}{f},\qquad \phi_{i,n}=\frac{P_{i,n}}{f},\qquad \theta_{i,n}=\frac{Q_{i,n}}{f},\tag{48}$$

the bilinear system (22)–(28) can be transformed into the following differential–difference system:

$$\begin{aligned}&\frac{\partial w_{n+1}}{\partial t}+\frac{\partial w_n}{\partial t}-v_{n+1}+v_n+w_{n+1}^2-w_n^2+\sum_{i=1}^{M}(\Phi_{i,n+2}\Psi_{i,n+1}+\Phi_{i,n+1}\Psi_{i,n})_t\\&\quad=\sum_{i=1}^{M}\big(\triangle^2u_n\Phi_{i,n+1}\Psi_{i,n-1}-2\triangle w_n\Phi_{i,n+1}\Psi_{i,n}\big)-\triangle\left(\sum_{i=1}^{M}\Phi_{i,n+1}\Psi_{i,n}\right)^2,\end{aligned}\tag{49}$$

$$\frac{\partial v_n}{\partial t}-\frac{\partial w_n}{\partial y}-u_{n+1}+u_n=-\sum_{i=1}^{M}\triangle(\Phi_{i,n}\Psi_{i,n}),\tag{50}$$

$$\frac{\partial\Phi_{i,n}}{\partial y}-\frac{\partial\Phi_{i,n}}{\partial z}=v_{n-1}\Phi_{i,n}+\Phi_{i,n-1}-\Phi_{i,n}\int^t\frac{\partial w_n}{\partial y}\,\mathrm{d}t,\quad i=1,2,\ldots,M,\tag{51}$$

$$\frac{\partial\Psi_{i,n}}{\partial y}-\frac{\partial\Psi_{i,n}}{\partial z}=-v_n\Phi_{i,n}-\Psi_{i,n+1}+\Psi_{i,n}\int^t\frac{\partial w_n}{\partial y}\,\mathrm{d}t,\quad i=1,2,\ldots,M,\tag{52}$$

where the operator $\triangle$ is defined by $\triangle u_n=u_{n+1}-u_n$.

3 Casoratian determinant solution to the bilinear SESCS (22)–(28)

The authors in [21] derived the Casoratian determinant solution for the bilinear two-dimensional special lattice equation. Herein we present the Casoratian determinant solution to the bilinear special lattice ESCS in Eqs. (22)–(28), which can be expressed in the following Pfaffian form:

$$\begin{aligned} f_n &= \det\left|\phi_i(n+j-1)\right|_{1\le i,j\le N} \\ &\triangleq (d_0, d_1, \ldots, d_{N-1}, N, \ldots, 2, 1)_n, \end{aligned} \tag{53}$$

$$g_{i,n} = \sqrt{\dot{C}_i(t)}(d_{-1}, d_0, \ldots, d_{N-1}, N, \ldots, 2, 1, b_i)_n, \tag{54}$$

$$h_{i,n} = \sqrt{\dot{C}_i(t)}(d_1, \ldots, d_{N-1}, N, \ldots, \hat{\imath}, \ldots, 2, 1)_n. \tag{55}$$

Here the functions $\phi_i(n)$ are given by the expression

$$\phi_i(n) = \varphi_{i1}(n) + (-1)^{i-1}\varphi_{i2}(n), \quad 1 \le i \le N,$$

where $\varphi_{i1}(n)$ and $\varphi_{i2}(n)$ satisfy the dispersion relations of Eqs. (12)–(13), and the functions $C_i(t)$ have the form

$$C_i(t) = \begin{cases} C_i(t), & 1 \le i \le M \le N, \\ \text{constant}, & \text{otherwise}. \end{cases}$$

At the same time the Pfaffian entries are defined by

$$\begin{aligned} &(d_m, i)_n = \phi_i(n+m), \qquad (d_m, d_l)_n = (i,j)_n = 0, \quad m \in \mathbf{Z}, \\ &(d_m, b_i)_n = \varphi_{i2}(n+m), \qquad (b_i, b_j)_n = (b_i, j)_n = 0. \end{aligned}$$

Moreover, the other solutions k_i, $P_{i,n}$, and $Q_{i,n}$ have the forms

$$k_{i,n} = \dot{C}_i(t)(d_0, d_1, \ldots, d_{N-1}, N, \ldots, \hat{\imath}, \ldots, 1, b_i)_n, \tag{56}$$

$$\begin{aligned} P_{i,n} &= \frac{\ddot{C}_i(t)}{2\sqrt{\dot{C}_i(t)}}(d_{-1}, d_0, \ldots, d_{N-1}, N, \ldots, 2, 1, b_i)_n \\ &\quad + \sqrt{\dot{C}_i(t)}\left[\sum_{1\le j\le N}^{j\ne i}(d_{-1}, d_0, \ldots, d_{N-1}, N, \ldots, \hat{\jmath}, \ldots, 1, b_j, b_i)_n\right], \end{aligned} \tag{57}$$

$$\begin{aligned} Q_{i,n} &= \frac{\ddot{C}_i(t)}{2\sqrt{\dot{C}_i(t)}}(d_1, \ldots, d_{N-1}, N, \ldots, \hat{\imath}, \ldots, 1)_n \\ &\quad + \sqrt{\dot{C}_i(t)}\left[\sum_{1\le j\le N}^{j\ne i}(d_1, \ldots, d_{N-1}, N, \ldots, \hat{\jmath}, \ldots, \hat{\imath}, \ldots, 1, b_j)_n\right], \end{aligned} \tag{58}$$

where $i = 1, 2, \ldots, M$.

Herein we only provide the proof of Eqs. (22), (24), and (25). We use the following notation for the functions f_n, $g_{i,n}$, and $h_{i,n}$:

$$f_n = (d_0, d_1, \ldots, d_{N-1}, \bullet)_n, \tag{59}$$

$$g_{i,n} = \sqrt{\dot{C}_i(t)}(d_{-1}, d_0, \ldots, d_{N-1}, \bullet, b_i)_n, \tag{60}$$

$$h_{i,n} = \sqrt{\dot{C}_i(t)}(d_1, \ldots, d_{N-1}, \star)_n. \tag{61}$$

Then we have the following formulas:

$$\frac{\partial f_n}{\partial t} = \sum_{i=1}^{M} k_{i,n} + (d_0, \ldots, d_{N-2}, d_N, \bullet)_n, \tag{62}$$

$$\begin{aligned} \frac{\partial^2 f_n}{\partial t^2} = {} & \sum_{i=1}^{M} \frac{\partial k_{i,n}}{\partial t} + \sum_{i=1}^{M} \dot{C}_i(t)(d_0, \ldots, d_{N-2}, d_N, \star, b_i)_n \\ & + (d_0, \ldots, d_{N-3}, d_{N-1}, d_N, \bullet)_n + (d_0, \ldots, d_{N-2}, d_{N+1}, \bullet)_n, \end{aligned} \tag{63}$$

$$\begin{aligned} \frac{\partial^2 f_n}{\partial t \partial z} = {} & \sum_{i=1}^{M} \dot{C}_i(t)\big[(d_0, \ldots, d_{N-2}, d_{N+1}, \star, b_i)_n \\ & - (d_0, \ldots, d_{N-3}, d_{N-1}, d_N, \star, b_i)_n\big] + (d_0, \ldots, d_{N-2}, d_{N+2}, d_N, \bullet)_n \\ & - (d_0, \ldots, d_{N-4}, d_{N-2}, d_{N-1}, d_N, \bullet)_n. \end{aligned} \tag{64}$$

Further, we obtain the formulas for functions $k_{i,n}$, $g_{i,n}$, and $h_{i,n}$ by

$$k_{i,n+1} = \dot{C}_i(t)(d_0, d_1, \ldots, d_{N-1}, \star, b_i)_n, \tag{65}$$

$$g_{i,n+1} = \sqrt{\dot{C}_i(t)}(d_0, d_1, \ldots, d_N, \bullet, b_i)_n, \tag{66}$$

$$\frac{\partial g_{i,n+1}}{\partial t} = P_{i,n+1} + \sqrt{\dot{C}_i(t)}(d_0, \ldots, d_{N-1}, d_{N+1}, \bullet, b_i)_n, \tag{67}$$

$$\frac{\partial h_{i,n}}{\partial t} = Q_{i,n} + \sqrt{\dot{C}_i(t)}(d_1, \ldots, d_{N-2}, d_N, \star)_n. \tag{68}$$

Now substituting Eqs. (62)–(68) into Eq. (22) yields

$$\begin{aligned} & \dot{C}_i(t)\big[(d_1, d_2, \ldots, d_N, \star, b_i)_n(d_0, d_1, \ldots, d_{N-2}, d_N, \bullet)_n \\ & \quad - (d_0, d_1, \ldots, d_{N-2}, d_N, \star, b_i)_n(d_1, d_2, \ldots, d_N, \bullet)_n \\ & \quad + (d_0, d_1, \ldots, d_N, \bullet, b_i)_n(d_1, \ldots, d_{N-2}, d_N, \star)_n\big] \\ & \quad + \dot{C}_i(t)\big[(d_0, d_1, \ldots, d_{N-1}, \star, b_i)_n(d_1, d_2, \ldots, d_{N-1}, d_{N+1}, \bullet)_n \\ & \quad - (d_1, d_2, \ldots, d_{N-1}, d_{N+1}, \star, b_i)_n(d_0, d_1, \ldots, d_{N-1}, \bullet)_n \\ & \quad - (d_1, d_2, \ldots, d_{N-1}, d_{N+1}, \star)_n(d_0, d_1, \ldots, d_{N-1}, \bullet, b_i)_n\big] = 0, \end{aligned}$$

which is the sum of the Plücker identities of the determinants. Then substituting Eqs. (60)–(61) and (65)–(66) into Eq. (24) yields

$$\begin{aligned} & (d_1, d_2, \ldots, d_N, \star, b_i)_n(d_0, d_1, \ldots, d_{N-1}, \bullet)_n \\ & \quad - (d_0, d_1, \ldots, d_{N-1}, \star, b_i)_n(d_1, d_2, \ldots, d_N, \bullet)_n \\ & \quad + (d_0, d_1, \ldots, d_N, \bullet, b_i)_n(d_1, \ldots, d_{N-2}, d_N, \star)_n = 0, \end{aligned}$$

which are again the Plücker identities of the determinants. Finally, substituting f_n, $k_{i,n}$, $g_{i,n}$, and $P_{i,n}$ into Eq. (25) gives the following determinant identity:

$$\begin{aligned}&(d_{-1},d_0,\dots,d_{N-1},\bullet,b_i)_n(d_0,d_1,\dots,d_{N-2},d_N,\bullet)_n\\&\quad-(d_{-1},d_0,\dots,d_{N-2},d_N,\bullet,b_i)_n(d_0,d_1,\dots,d_{N-1},\bullet)_n\\&\quad+(d_0,d_1,\dots,d_N,\bullet,b_i)_n(d_{-1},d_0,\dots,d_{N-2},\bullet)_n=0.\end{aligned}$$

These expressions show that Eqs. (22), (24), and (25) are valid. The other Eqs. (22)–(28) can be proved similarly. Therefore functions in (53)–(58) constitute the Casoratian determinant solutions of the bilinear ESCS in Eqs. (22)–(28).

4 One- and two-soliton solutions of the SESCS in (49)–(52)

Starting from the Grammian determinant solutions of Eqs. (15)–(18) and the transformations of Eqs. (46)–(47), we can obtain explicit solutions of the two-dimensional special lattice ESCS Eqs. (49)–(52). In this section, we take $M=1$, and the coupled system is read as

$$\begin{aligned}&\frac{\partial w_{n+1}}{\partial t}+\frac{\partial w_n}{\partial t}-v_{n+1}+v_n+w_{n+1}^2-w_n^2+(\Phi_{n+2}\Psi_{n+1}+\Phi_{n+1}\Psi_n)_t\\&\quad=\triangle^2(u_n\Phi_{n+1}\Psi_{n-1})-2\triangle(w_n\Phi_{n+1}\Psi_n)-\triangle(\Phi_{n+1}\Psi_n)^2,\end{aligned}\tag{69}$$

$$\frac{\partial v_n}{\partial t}-\frac{\partial w_n}{\partial y}-u_{n+1}+u_n=-\triangle(\Phi_n\Psi_n),\tag{70}$$

$$\frac{\partial \Phi_n}{\partial y}-\frac{\partial \Phi_n}{\partial z}=v_{n-1}\Phi_n+\Phi_{n-1}-\Phi_n\int^t\frac{\partial w_n}{\partial y}\,\mathrm{d}t,\tag{71}$$

$$\frac{\partial \Psi_n}{\partial y}-\frac{\partial \Psi_n}{\partial z}=-v_n\Phi_n-\Psi_{n+1}+\Psi_n\int^t\frac{\partial w_n}{\partial y}\,\mathrm{d}t,\tag{72}$$

which is the special lattice equation with one self-consistent source. Now we derive the one-soliton and two-soliton solutions of this system.

Example 1 We choose the parameter $N=1$ in Eqs. (15)–(18). Then the functions $\varphi(n)$, $\psi(-n)$, and f_n can be expressed in the following form:

$$\begin{aligned}&\varphi(n)=p^ne^{\xi},\qquad \xi=\left(p^2+p^{-1}\right)y+pt+p^2z,\\&\psi(-n)=q^{-n}e^{\eta},\qquad \eta=-\left(q^2+q^{-1}\right)y-qt-q^2z,\\&f_n=C_1(t)+\frac{1}{p-q}\left(\frac{p}{q}\right)^ne^{\xi+\eta},\qquad C_1(t)=e^{2a(t)}/(p-q),\end{aligned}\tag{73}$$

where p and q are arbitrary constants, and $a(t)$ is a differentiable function of t. According to expressions (46)–(47), the explicit forms u_n, w_n, Φ_n, and Ψ_n are as follows:

$$u_n=\frac{[1+(\frac{p}{q})^{n+1}e^{\xi+\eta-2a(t)}]\cdot[1+(\frac{p}{q})^{n-1}e^{\xi+\eta-2a(t)}]}{[1+(\frac{p}{q})^ne^{\xi+\eta-2a(t)}]^2},\tag{74}$$

$$w_n=\frac{(p-q-2\dot{a}(t))(\frac{p-q}{q})(\frac{p}{q})^ne^{\xi+\eta-2a(t)}}{[1+(\frac{p}{q})^{n+1}e^{\xi+\eta-2a(t)}]\cdot[1+(\frac{p}{q})^ne^{\xi+\eta-2a(t)}]},\tag{75}$$

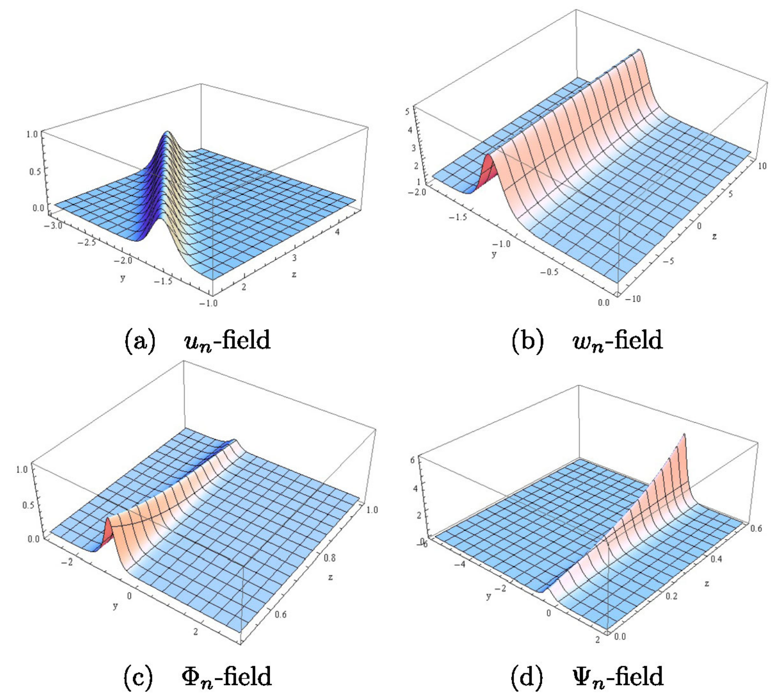

Figure 1 One soliton of u_n, w_n, Φ_n, and Ψ_n

$$\Phi_n = \frac{\sqrt{2(p-q)\dot{a}(t)}p^{n-1}e^{\xi-a(t)}}{1+(\frac{p}{q})^n e^{\xi+\eta-2a(t)}}, \tag{76}$$

$$\Psi_n = \frac{\sqrt{2(p-q)\dot{a}(t)}q^{-n-1}e^{\eta-a(t)}}{1+(\frac{p}{q})^n e^{\xi+\eta-2a(t)}}. \tag{77}$$

If we set $a(t) = t$, $p = 3$, $q = 1/5$, $t = 1$, and $n = 2$, then the profiles of these solutions are as shown in Fig. 1.

Example 2 We set $N = 2$ in expressions (15)–(18), and the functions $\varphi_i(n)$ and $\psi_i(-n)$ possess the following structures:

$$\varphi_i(n) = p_i^n e^{(p_i^2+p_i^{-1})y+p_i t+p_i^2 z} \triangleq p_i^n e^{\xi_i}, \quad i = 1, 2,$$

$$\psi_i(-n) = q_i^{-n} e^{-(q_i^2+q_i^{-1})y-q_i t-q_i^2 z} \triangleq q_i^{-n} e^{\eta_i}, \quad i = 1, 2,$$

where p_i and q_i are real constants. In this case the function f_n is a second-order determinant, and $C_1(t)$ included in f_n is chosen as $e^{2\beta(t)}/(p_1 - q_1)$. Through computations we obtain the following form:

$$f_n = \frac{e^{2\beta(t)}}{(p_1-q_1)(p_2-q_2)} \cdot \tilde{f}_n, \tag{78}$$

wherein the function $\tilde{f}_n$ is defined by

$$\tilde{f}_n = 1 + \left(\frac{p_1}{q_1}\right)^n e^{\xi_1+\eta_1-2\beta(t)} + \left(\frac{p_2}{q_2}\right)^n e^{\xi_2+\eta_2} + A\left(\frac{p_1 p_2}{q_1 q_2}\right)^n e^{\xi_1+\eta_1+\xi_2+\eta_2-2\beta(t)}, \tag{79}$$

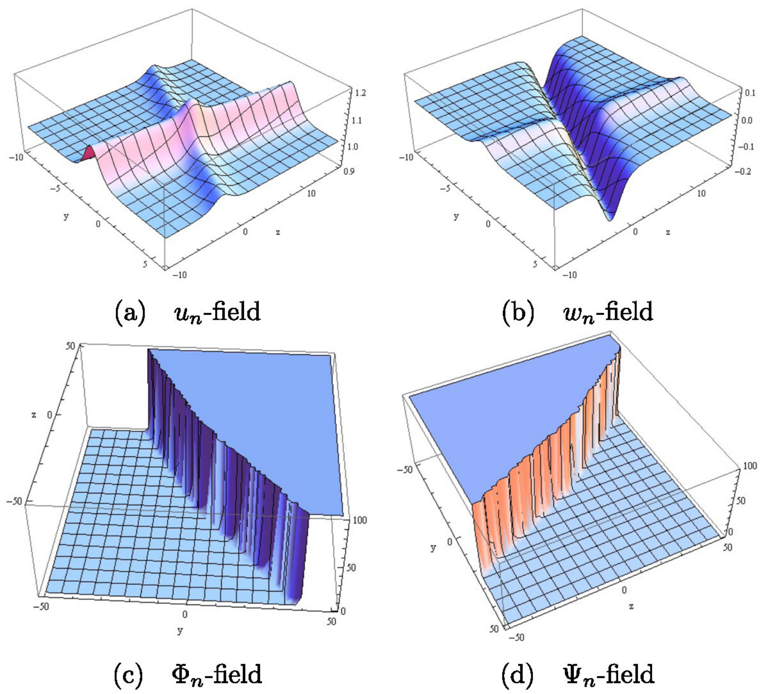

Figure 2 Interactions of two solitons in (**a**) and (**b**) with the sources Φ_n and Ψ_n as in (**c**) and (**d**)

and $A = \frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}$. Subsequently, we obtain the expressions for the solutions u_n and w_n as

$$u_n = \frac{\tilde{f}_{n+1}\tilde{f}_{n-1}}{\tilde{f}_n^2}, \qquad w_n = \left(\ln\frac{\tilde{f}_{n+1}}{\tilde{f}_n}\right)_t. \tag{80}$$

Contrarily, the functions g_n and h_n in Eqs. (16)–(17) have the Pfaffian forms

$$g_n = \sqrt{\dot{C}_1(t)}\left(d_{-1}^*, 1, 2, 2^*\right)_n \quad \text{and} \quad h_n = \sqrt{\dot{C}_1(t)}\left(d_{-1}, 2, 2^*, 1^*\right)_n \tag{81}$$

and can be rewritten as

$$g_n = \frac{\sqrt{\frac{2\dot{\beta}(t)}{p_1-q_1}}}{p_2-q_2}\left[p_1^{n-1}e^{\xi_1+\beta(t)} + \frac{(p_1-p_2)}{(p_1-q_2)}\left(\frac{p_1p_2}{q_2}\right)^{n-1}e^{\xi_1+\xi_2+\eta_2+\beta(t)}\right], \tag{82}$$

$$h_n = \frac{\sqrt{\frac{2\dot{\beta}(t)}{p_1-q_1}}}{p_2-q_2}\left[q_1^{-n-1}e^{\eta_1+\beta(t)} + \frac{(q_1-q_2)}{(q_1-p_2)}\left(\frac{p_2}{q_1q_2}\right)^{n+1}e^{\xi_2+\eta_1+\eta_2+\beta(t)}\right]. \tag{83}$$

Finally, the solutions Φ_n and Ψ_n have the following forms:

$$\Phi_n = \gamma(t)\cdot\frac{p_1^{n-1}e^{\xi_1-\beta(t)} + \frac{(p_1-p_2)}{(p_1-q_2)}(\frac{p_1p_2}{q_2})^{n-1}e^{\xi_1+\xi_2+\eta_2-\beta(t)}}{\tilde{f}_n}, \tag{84}$$

$$\Psi_n = \gamma(t)\cdot\frac{q_1^{-n-1}e^{\eta_1-\beta(t)} + \frac{(q_1-q_2)}{(q_1-p_2)}(\frac{p_2}{q_1q_2})^{n+1}e^{\xi_2+\eta_1+\eta_2-\beta(t)}}{\tilde{f}_n}, \tag{85}$$

where $\gamma(t) = \sqrt{2(p_1 - q_1)\dot{\beta}(t)}$. If we choose $\beta(t) = t$, $p_1 = 1.5$, $p_2 = 0.25$, $q_1 = 1$, $q_2 = 0.5$, $t = 1$, and $n = -2$, then the profiles of the above solutions are as shown in Fig. 2.

5 Discussion and conclusion

In this study, we applied SGP to the bilinear form of the two-dimensional special lattice equation and presented a new type of special lattice ESCS given by Eqs. (49)–(52). Additionally, we obtained the Grammian and Casoratian determinant solutions to the coupled system. According to the Grammian determinant solution, we considered the special lattice with one self-consistent source as an example to examine its one-soliton and two-soliton solutions. For further study of the integrability of the coupled system, we can examine the commutativity of the SGP and bilinear Bäcklund transformation, which will enable deriving the bilinear Bäcklund transformation for the coupled system.

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Availability of data and materials
The datasets used or analyzed during the current study are available from the corresponding author upon reasonable request.

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The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. All authors read and approved the final manuscript.

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