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Entire solutions for several second-order partial differential-difference equations of Fermat type with two complex variables

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Abstract

This paper is concerned with description of the existence and the forms of entire solutions of several second-order partial differential-difference equations with more general forms of Fermat type. By utilizing the Nevanlinna theory of meromorphic functions in several complex variables we obtain some results on the forms of entire solutions for these equations, which are some extensions and generalizations of the previous theorems given by Xu and Cao (Mediterr. J. Math. 15:1–14, 2018; Mediterr. J. Math. 17:1–4, 2020) and Liu et al. (J. Math. Anal. Appl. 359:384–393, 2009; Electron. J. Differ. Equ. 2013:59–110, 2013; Arch. Math. 99:147–155, 2012). Moreover, by some examples we show the existence of transcendental entire solutions with finite order of such equations.

MSC: 30D35; 35M30; 32W50; 39A45

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1 Introduction

The main purpose of this paper is investigation of the existence and the forms of transcendental entire solutions with finite order of second-order differential difference equations

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1+c_1,z_2+c_2)^2 = e^{g(z_1,z_2)}$$

and

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + \left[f(z_1+c_1,z_2+c_2) - f(z_1,z_2)\right]^2 = e^{g(z_1,z_2)},$$

where $g(z_1,z_2)$ is a polynomial in $\mathbb{C}^2$. In general, for the Fermat-type functional equation

$$f^m + g^n = 1, \tag{1.1}$$

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Gross [6] discussed the existence of solutions of equation (1.1) and showed that the entire solutions are $f = \cos a(z)$, $g = \sin a(z)$ for $m = n = 2$, where $a(z)$ is an entire function. Montel [7] proved that there are no nonconstant entire solutions for equation (1.1) for $m = n > 2$.

Recently, Han and Lü [8] gave a description of meromorphic solutions for the functional equation (1.1) when $g(z) = f'(z)$, $m = n$, and 1 is replaced by $e^{\alpha z+\beta}$, where $\alpha, \beta \in \mathbb{C}$, and obtained the following results.

Thereom A (See [8, Theorem 1.1]) *The meromorphic solutions f of the differential equation*

$$f^n(z) + \bigl(f'\bigr)^n(z) = e^{\alpha z+\beta} \tag{1.2}$$

must be entire functions, and the following statements hold:

(A) *For $n = 1$, the general solutions of (1.2) are $f(z) = \frac{e^{\alpha z+\beta}}{\alpha+1} + ae^{-z}$ for $\alpha \neq -1$ and $f(z) = ze^{-z+\beta} + ae^{-z}$.*

(B) *For $n = 2$, either $\alpha = 0$ and the general solutions of (1.2) are $f(z) = e^{\frac{\beta}{2}} \sin(z + b)$, or $f(z) = de^{\frac{\alpha z+\beta}{2}}$.*

(C) *For $n \geq 3$, the general solutions of (1.2) are $f(z) = de^{\frac{\alpha z+\beta}{n}}$.*

Here $\alpha, \beta, a, b, d \in \mathbb{C}$ with $d^n(1 + (\frac{\alpha}{n})^n) = 1$ for $n \geq 1$.

They also proved that all the trivial meromorphic solutions of $f^n(z) + f^n(z + c) = e^{\alpha z+\beta}$ are the functions $f(z) = de^{\frac{\alpha z+\beta}{n}}$ with $d^n(1 + e^{\alpha c}) = 1$ for $n \geq 1$ (see [8, p. 99]).

An equation is called differential-difference equation (DDE) if the equation includes derivatives and shifts or differences of f (see [9]). In many previous papers [10–15], Naftalevich [11, 12] in 1995 discussed the meromorphic solutions of complex differential-difference equations with one complex variable by using the operator theory and iteration method, but recently, many researchers have begun to discuss this kind of equations by using the difference analogues of Nevanlinna theory (see [16–19]). In particular, Liu et al. [3–5] investigated the existence of entire solutions with finite order of the Fermat-type differential-difference equations

$$f'(z)^2 + f(z + c)^2 = 1, \tag{1.3}$$

$$f'(z)^2 + \bigl[f(z + c) - f(z)\bigr]^2 = 1. \tag{1.4}$$

They proved that the transcendental entire solutions with finite order of equation (1.3) must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant, $c = 2k\pi$ or $c = (2k + 1)\pi$ with integer k, and the transcendental entire solutions with finite order of equation (1.4) must satisfy $f(z) = 12 \sin(2z + Bi)$, where $c = (2k + 1)\pi$ with integer k, and B is a constant. In 2019, Liu and Gao [20] further studied the entire solutions of second-order differential and difference equation with single complex variable and obtained the following:

Thereom B (See [20, Theorem 2.1]) *Let f be a transcendental entire solution with finite order of the complex differential-difference equation*

$$f''(z)^2 + f(z + c)^2 = Q(z).$$

Then $Q(z) = c_1c_2$ *is a constant, and* $f(z)$ *satisfies*

$$f(z) = \frac{c_1e^{az+b} + c_2e^{-az-b}}{2a^2},$$

with $a, b \in \mathbb{C}$ *such that* $a^4 = 1$ *and* $c = \frac{\log(-ia^2)+2k\pi i}{a}$, $k \in \mathbb{Z}$.

Now let us recall some previous results on Fermat-type partial differential equations with several complex variables (including [21–25]). Khavinson [22] in 1995 pointed out that any entire solution of the partial differential equation $(\frac{\partial f}{\partial z_1})^2 + (\frac{\partial f}{\partial z_2})^2 = 1$ in $\mathbb{C}^2$ is necessarily linear. This partial differential equation in real variable case occurs in the study of characteristic surfaces and wave propagation theory, and it is the two-dimensional eiconal equation, one of the main equations of geometric optics (see [26, 27]). In 2005, Li [28] discussed the partial differential equation of Fermat-type

$$\left(\frac{\partial u}{\partial z_1}\right)^2 + \left(\frac{\partial u}{\partial z_2}\right)^2 = e^g, \tag{1.5}$$

where g is a polynomial or an entire function in $\mathbb{C}^2$, and obtained some results on the forms of entire solution of equation (1.5).

Thereom C ([28, Theorem 2.1]) *Let* g *be a polynomial in* $\mathbb{C}^2$. *Then* u *is an entire solution of the partial differential equation* (1.5) *if and only if*

(i) $u = f(c_1z_1 + c_2z_2)$; *or*

(ii) $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$,

where f *is an entire function in* $\mathbb{C}$ *satisfying* $f'(c_1z_1 + c_2z_2) = \pm e^{\frac{1}{2}g(z)}$, c_1 *and* c_2 *are two constants satisfying* $c_1^2 + c_2^2 = 1$, *and* ϕ_1 *and* ϕ_2 *are entire functions in* $\mathbb{C}$ *satisfying* $\phi_1'(z_1 + iz_2)\phi_2'(z_1 - iz_2) = \frac{1}{4}e^{g(z)}$.

Very recently, Xu and Cao [1, 2, 29] investigated the existence of solutions for some Fermat-type partial differential-difference equations with several variables by using the difference logarithmic derivative lemma of several complex variables and obtained the following theorem (see [29–31]).

Thereom D (See [1, Theorem 1.2]) *Let* $c = (c_1, c_2)$ *be a constant in* $\mathbb{C}^2$. *Then any transcendental entire solution with finite order of the partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, *where* $A \in \mathbb{C}$ *is a constant satisfying* $Ae^{iAc_1} = 1$, *and* $B \in \mathbb{C}$ *is a constant*; *in the particular case* $c_1 = 0$, *we have* $f(z_1, z_2) = \sin(z_1 + B)$.

Theorems B, C, and D suggest the following questions as open problems.

Question 1.1 What will happen when the right side of those equations, 1, is replaced by a function e^g in Theorem D, where g is a polynomial in $\mathbb{C}^2$?

Question 1.2 What will happen when $\frac{\partial f(z_1,z_2)}{\partial z_1}$ is replaced by $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}$ or $\frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}$ in Theorem D?

2 Results and some examples

In view of the above questions, this paper is concerned with description of entire solutions for several second-order partial differential-difference equations of Fermat type of more general form. The main tools used in this paper are the Nevanlinna theory and difference Nevanlinna theory with several complex variables. Our principal results generalize the previous theorems given by Xu and Cao [1] and Liu, Cao, and Cao [5]. Throughout this paper, for convenience, we assume that $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2)$, $w = (w_1, w_2)$. We now state the main results of this paper.

Theorem 2.1 *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $c_2 \neq 0$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1,z_2)} \tag{2.1}$$

admits a transcendental entire solution $f(z_1,z_2)$ of finite order, then $g(z_1,z_2)$ must be a linear function of the form $g(z_1,z_2) = A_1z_1 + A_2z_2 + B$, where $A_1, A_2, B \in \mathbb{C}$. Further, $f(z_1,z_2)$ must satisfy one of the following cases:

(i)

$$f(z_1,z_2) = \frac{4(\xi^2+1)}{A_1^2\xi}e^{\frac{1}{2}g(z_1,z_2)},$$

with $\xi\,(\neq 0), A_1, A_2, B \in \mathbb{C}$ satisfying

$$\frac{\xi^2-1}{4(\xi^2+1)i}A_1^2 = e^{\frac{1}{2}(A_1c_1+A_2c_2)};$$

(ii)

$$f(z_1,z_2) = \frac{A_{21}^2 e^{L_1(z)+B_1} + A_{11}^2 e^{L_2(z)+B_2}}{2A_{11}^2A_{21}^2},$$

where $L_1(z) = A_{11}z_1 + A_{12}z_2 + B_1$, $L_2(z) = A_{21}z_1 + A_{22}z_2 + B_2$, $A_{j1}, A_{j2}, B_j \in \mathbb{C}$ $(j = 1,2)$ satisfy

$$L_1(z) \neq L_2(z), \qquad g(z) = L_1(z) + L_2(z) + B_1 + B_2,$$

and

$$-iA_{11}^2e^{-L_1(c)} = iA_{21}^2e^{-L_2(c)} = 1.$$

The following examples show that the forms of solutions are precise to some extent.

Example 2.1 Let $A_1 = 2$, $A_2 = 1$, $B = 0$, and

$$f(z_1,z_2) = \frac{\sqrt{2}}{2}e^{z_1+\frac{1}{2}z_2}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.1) with $g(z) = 2z_1 + z_2$, $c_1 = \pi i$, and $c_2 = 2\pi i$.

Example 2.2 Let $L_1(z) = iz_1 + \frac{1}{2}z_2$, $L_2(z) = -iz_1 - \frac{5}{2}z_2$, $B_1 = B_2 = 0$, and

$$f(z_1,z_2) = -\frac{e^{iz_1+\frac{1}{2}z_2} + e^{-iz_1-\frac{5}{2}z_2}}{2}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.1) with $g(z) = -\frac{3}{2}z_2$, $c_1 = \pi$, and $c_2 = -\pi i$.

From Theorem 2.1 we easily get the following:

Corollary 2.1 *Let $c = (c_1,c_2) \in \mathbb{C}^2$, $c_2 \neq 0$, and let $g(z_1,z_2)$ be not a linear function of the form $L(z) = A_1z_1 + A_2z_2 + B$, where $A_1,A_2,B \in \mathbb{C}$. Then the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1+c_1,z_2+c_2)^2 = e^{g(z_1,z_2)} \tag{2.2}$$

admits no transcendental entire solution of finite order.

The following example shows that the condition $c_2 \neq 0$ in Corollary 2.1 cannot be removed.

Example 2.3 Let $f(z_1,z_2) = \frac{\sqrt{2}}{2}e^{z_1+z_2-2\pi z_2^2}$. Then $f(z_1,z_2)$ is a transcendental entire solution of finite order of equation (2.2) with $c = (c_1,c_2) = (2\pi i,0)$ and $g(z_1,z_2) = 2z_1 + 2z_2 - 4\pi z_2^2$.

Remark 2.1 In addition, in view of Theorem 2.1, we can obtain the conclusions of Theorem 1.2 in [1] if $\alpha = 1$, $\beta = 0$, and $g(z) = 2k\pi i$, $k \in \mathbb{Z}$, in equation (2.1).

For the difference counterpart of Theorem 2.1, we have the following:

Theorem 2.2 *Let $c = (c_1,c_2) \in \mathbb{C}^2$, $c_2 \neq 0$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + \left[f(z_1+c_1,z_2+c_2) - f(z_1,z_2)\right]^2 = e^{g(z_1,z_2)} \tag{2.3}$$

admits a transcendental entire solution $f(z_1,z_2)$ of finite order, then $g(z_1,z_2)$ must be a linear function of the form $g(z_1,z_2) = A_1z_1 + A_2z_2 + B$ with $A_1,A_2,B \in \mathbb{C}$. Further, $f(z_1,z_2)$ must satisfy one of the following cases:

(i)

$$f(z_1,z_2) = \frac{4(\xi^2+1)}{A_1^2\xi}e^{\frac{1}{2}g(z_1,z_2)} + z_1G_1(z_2) + G_2(z_2),$$

where $G_1(z_2)$ is a finite-order entire period function in z_2 with period c_2, $\xi(\neq 0)$, $A_1,A_2,B \in \mathbb{C}$ satisfying

$$G_2(z_2+c_2) = G_2(z_2) - c_1G_1(z_2), \qquad \frac{\xi^2-1}{2i(\xi^2+1)}A_1^2 + 1 = e^{\frac{1}{2}(A_1c_1+A_2c_2)} = e^{\frac{1}{2}g(c_1,c_2)};$$

(ii)

$$f(z_1,z_2)=\frac{A_{21}^2e^{L_1(z)+B_1}+A_{11}^2e^{L_2(z)+B_2}}{2A_{11}^2A_{21}^2}+z_1G_1(z_2)+G_2(z_2),$$

where $G_1(z_2)$ is a finite-order entire period function in z_2 with period c_2, $L_1(z)=A_{11}z_1+A_{12}z_2+B_1$, $L_2(z)=A_{21}z_1+A_{22}z_2+B_2$, $A_{ij},B_i\in\mathbb{C}$ satisfy

$$G_2(z_2+c_2)=G_2(z_2)-c_1G_1(z_2),\qquad L_1(z)\not\equiv L_2(z),$$
$$g(z)=L_1(z)+L_2(z)+B_1+B_2,$$

and

$$\left(1-iA_{11}^2\right)e^{-(A_{11}c_1+A_{12}c_2)}=1,\qquad \left(1+iA_{21}^2\right)e^{-(A_{21}c_1+A_{22}c_2)}=1.$$

The following examples explain the existence of transcendental finite-order entire solutions of (2.3).

Example 2.4 Let $A_1=2$, $A_2=-1$, $G_1(z_2)=e^{z_2}$, $G_2(z_2)=e^{2z_2}-z_2e^{z_2}$, $B=0$, and

$$f(z_1,z_2)=\frac{\sqrt{5}}{5}e^{z_1-\frac{1}{2}z_2}+(z_1-z_2)e^{z_2}+e^{2z_2}.$$

Then $\rho(f)=1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.3) with $g(z)=2z_1-z_2$, $c_1=2\pi i$, and $c_2=2\pi i$.

Example 2.5 Let $L_1(z)=z_1+z_2$, $L_2(z)=z_1-z_2$, $G_1(z)=e^{\frac{4\pi i}{\log(-i)}z_2}$, $G_2(z_2)=-\frac{\log(-i)}{\log 2}e^{\frac{4\pi i z_2}{\log(-i)}}$, $B_1=B_2=0$, and

$$f(z_1,z_2)=\frac{e^{z_1+z_2}+e^{z_1-z_2}}{2}+\left(z_1-\frac{\log(-i)}{\log 2}z_2\right)e^{\frac{4\pi i}{\log(-i)}z_2}.$$

Then $f(z_1,z_2)$ is a transcendental finite-order entire solution of equation (2.3) with $g(z)=2z_1$, $c_1=\frac{1}{2}\log 2$, and $c_2=\frac{1}{2}\log(-i)$.

In view of Theorem 2.2, we obtain the following:

Corollary 2.2 *Let $c=(c_1,c_2)\in\mathbb{C}^2$, $c_2\neq 0$, and let $g(z_1,z_2)$ be not a linear function of the form $L(z)=A_1z_1+A_2z_2+B$ with $A_1,A_2,B\in\mathbb{C}$. Then the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2+\left[f(z_1+c_1,z_2+c_2)-f(z_1,z_2)\right]^2=e^{g(z_1,z_2)}\tag{2.4}$$

has no transcendental entire solution of finite order.

The following example shows that the condition $c_2\neq 0$ in Corollary 2.2 cannot be removed.

Example 2.6 Let $f(z_1,z_2) = e^{z_1+z_2-4\pi i z_2^3}$. Then $f(z_1,z_2)$ is a transcendental finite-order entire solution of equation (2.4) with $c = (c_1,c_2) = (2\pi i, 0)$ and $g(z_1,z_2) = 2z_1 + 2z_2 - 8\pi i z_2^3$.

When $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}$ is replaced by $\frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}$ in Theorems 2.1 and 2.2, we have the following:

Theorem 2.3 *Let $c = (c_1,c_2) \in \mathbb{C}^2$, $c_1 \neq 0$, $c_2 \neq 0$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1+c_1, z_2+c_2)^2 = e^{g(z_1,z_2)} \tag{2.5}$$

admits a transcendental entire solution of finite order, then $g(z_1,z_2)$ must be a linear function of the form $g(z_1,z_2) = A_1z_1 + A_2z_2 + B$ with $A_1, A_2, B \in \mathbb{C}$. Further, $f(z_1,z_2)$ must satisfy one of the following cases:

(i)

$$f(z_1,z_2) = \frac{4(\xi^2+1)}{A_1A_2\xi} e^{\frac{1}{2}g(z_1,z_2)}$$

with $\xi(\neq 0)$, $A_1, A_2, B \in \mathbb{C}$ satisfying

$$\frac{\xi^2-1}{4(\xi^2+1)i} A_1A_2 = e^{\frac{1}{2}(A_1c_1+A_2c_2)};$$

(ii)

$$f(z_1,z_2) = \frac{A_{21}A_{22}e^{L_1(z)+B_1} + A_{11}A_{12}e^{L_2(z)+B_2}}{2A_{11}A_{12}A_{21}A_{22}},$$

where $L_1(z) = A_{11}z_1 + A_{12}z_2 + B_1$, $L_2(z) = A_{21}z_1 + A_{22}z_2 + B_2$, $A_{j1}, A_{j2}, B_j \in \mathbb{C}$ $(j = 1,2)$ satisfy

$$L_1(z) \neq L_2(z), \qquad g(z) = L_1(z) + L_2(z) + B_1 + B_2,$$

and

$$-iA_{11}A_{12}e^{-L_1(c)} = iA_{21}A_{22}e^{-L_2(c)} = 1.$$

Example 2.7 Let $A_1 = 2$, $A_2 = 2$, $B = 0$, and

$$f(z_1,z_2) = \frac{\sqrt{2}}{2} e^{z_1+z_2}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.5) with $g(z) = 2z_1 + 2z_2$, $c_1 = \pi i$, and $c_2 = \pi i$.

Example 2.8 Let $L_1(z) = z_1 + z_2$, $L_2(z) = z_1 - z_2$, $B_1 = B_2 = 0$, and

$$f(z_1,z_2) = \frac{e^{z_1+z_2} - e^{z_1-z_2}}{2}.$$

Then $\rho(f)=1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.5) with $g(z)=2z_1$, $c_1=\frac{\pi}{2}i$, and $c_2=\pi i$.

From Theorem 2.3 we get the following:

Corollary 2.3 *Let $c=(c_1,c_2)\in\mathbb{C}^2$, $c_1\neq 0$, $c_2\neq 0$, and let $g(z_1,z_2)$ be not a linear function of the form $L(z)=A_1z_1+A_2z_2+B$ with $A_1,A_2,B\in\mathbb{C}$. Then the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1\partial z_2}\right)^2+f(z_1+c_1,z_2+c_2)^2=e^{g(z_1,z_2)} \tag{2.6}$$

admits no transcendental entire solution of finite order.

The following example shows that the condition $c_1\neq 0$, $c_2\neq 0$ in Corollary 2.3 cannot be removed.

Example 2.9 Let $f(z_1,z_2)=e^{z_2+z_2^3}$. Then $f(z_1,z_2)$ is a transcendental finite-order entire solution of equation (2.6) with $c=(c_1,c_2)=(2\pi i,0)$ and $g(z_1,z_2)=2z_2+2z_2^3$.

Theorem 2.4 *Let $c=(c_1,c_2)\neq(0,0)\in\mathbb{C}^2$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1\partial z_2}\right)^2+\left[f(z_1+c_1,z_2+c_2)-f(z_1,z_2)\right]^2=e^{g(z_1,z_2)} \tag{2.7}$$

admits a transcendental entire solution of finite order, then $g(z_1,z_2)$ must be a linear function of the form $g(z_1,z_2)=A_1z_1+A_2z_2+B$ with $A_1,A_2,B\in\mathbb{C}$. Further, $f(z_1,z_2)$ must satisfy one of the following cases:

(i)

$$f(z_1,z_2)=\frac{4(\xi^2+1)}{A_1A_2\xi}e^{\frac{1}{2}g(z_1,z_2)}+G_3(z_1)+D_1z_1+G_4(z_2)+D_2z_2,$$

where $G_3(z_1)$ and $G_4(z_2)$ are finite-order entire periodic functions in z_1 and z_2 with periods c_1 and c_2, respectively, and $\xi(\neq 0),A_1,A_2,B,D_1,D_2\in\mathbb{C}$ satisfy

$$\frac{\xi^2-1}{4i(\xi^2+1)}A_1A_2+1=e^{\frac{1}{2}(A_1c_1+A_2c_2)}=e^{\frac{1}{2}g(c_1,c_2)},\qquad D_1c_1+D_2c_2=0;$$

(ii)

$$f(z_1,z_2)=\frac{A_{21}A_{22}e^{L_1(z)+B_1}+A_{11}A_{12}e^{L_2(z)+B_2}}{2A_{11}A_{12}A_{21}A_{22}}+G_3(z_1)+D_1z_1+G_4(z_2)+D_2z_2,$$

where $G_3(z_1)$ and $G_4(z_2)$ are finite-order entire periodic functions in z_1 and z_2 with periods c_1 and c_2, respectively, $L_1(z)=A_{11}z_1+A_{12}z_2+B_1$, $L_2(z)=A_{21}z_1+A_{22}z_2+B_2$, $A_{j1},A_{j2},B_j\in\mathbb{C}$ $(j=1,2)$ satisfy

$$L_1(z)\neq L_2(z),\qquad g(z)=L_1(z)+L_2(z)+B_1+B_2,\qquad D_1c_1+D_2c_2=0,$$

and

$$(1 - iA_{11}A_{12})e^{-(A_{11}c_1+A_{12}c_2)} = 1, \qquad (1 + iA_{21}A_{22})e^{-(A_{21}c_1+A_{22}c_2)} = 1.$$

Example 2.10 Let $A_1 = 2$, $A_2 = 2$, $B = 0$, $G_3(z_1) = e^{2z_1}$, $G_4(z_2) = e^{4z_2}$, and

$$f(z_1,z_2) = \frac{\sqrt{5}}{5}e^{2z_1+2z_2} + e^{2z_1} + z_1 + e^{4z_2} + 2z_2.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.7) with $g(z) = 2z_1 + 2z_2$, $c_1 = \pi i$, and $c_2 = -\frac{\pi}{2}i$.

Example 2.11 Let $L_1(z) = z_1 + z_2$, $L_2(z) = z_1 - 2z_2$, $B_1 = B_2 = 0$, $G_3(z) = e^{\frac{6\pi i}{\log[-2(2+i)]}z_1}$, $G_4(z) = e^{\frac{6\pi i}{\log(1-i)-\log(1-2i)}z_2}$, and

$$f(z_1,z_2) = \frac{e^{z_1+z_2}}{2} - \frac{e^{z_1-2z_2}}{4} + e^{\frac{6\pi i}{\log[-2(2+i)]}z_1} - \frac{\log\frac{1-i}{1-2i}}{\log(-2-2i)}z_1 + e^{\frac{6\pi i}{\log(1-i)-\log(1-2i)}z_2}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (2.7) with $g(z) = 2z_1 - z_2$, $c_1 = \frac{\log[-2(2+i)]}{3}$, and $c_2 = \frac{\log(1-i)-\log(1-2i)}{3}$.

In view of Theorem 2.4, we obtain the following:

Corollary 2.4 *Let $c = (c_1,c_2) \neq (0,0) \in \mathbb{C}^2$, and let $g(z_1,z_2)$ be not a linear function of the form $L(z) = A_1z_1 + A_2z_2 + B$ with $A_1, A_2, B \in \mathbb{C}$. Then the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + \left[f(z_1+c_1,z_2+c_2) - f(z_1,z_2)\right]^2 = e^{g(z_1,z_2)} \tag{2.8}$$

admits no transcendental entire solution of finite order.

In view of Theorems 2.1 and 2.3, we also get the following:

Corollary 2.5 *Let f be a finite-order transcendental entire solution of the partial differential equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1,z_2)^2 = 1, \qquad \left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1,z_2)^2 = 1.$$

Then $f(z_1,z_2)$ must be of the form

$$f(z_1,z_2) = \frac{e^{L(z)+B} - e^{-L(z)-B}}{2i} = \sin\bigl(-i\bigl(L(z)+B\bigr)\bigr),$$

where $L(z) = A_1z_1 + A_2z_2$ with $A_1, A_2, B \in \mathbb{C}$ satisfying $A_1^4 = 1$ and $A_1^2A_2^2 = 1$.

3 Some lemmas

The following lemmas play the key role in proving our results.

Lemma 3.1 ([32, 33]) *For an entire function F on $\mathbb{C}^n$ with $F(0) \neq 0$, put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the particular case $n = 1$, f_F is the canonical Weierstrass product.*

Remark 3.1 Here $\rho(n_F)$ is the order of the counting function of zeros of F.

Lemma 3.2 ([34]) *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases*:

(a) *the internal function h is a polynomial, and the external function g is of finite order*;

(b) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 3.3 ([35, Theorem 1.106]) *Suppose that $a_0(z), a_1(z), \ldots, a_n(z)$ $(n \geq 1)$ are meromorphic functions on $\mathbb{C}^m$ and $g_0(z), g_1(z), \ldots, g_n(z)$ are entire functions on $\mathbb{C}^m$ such that $g_j(z) - g_k(z)$ are not constants for $0 \leq j < k \leq n$. If*

$$\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$$

and

$$\| T(r, a_j) = o\bigl(T(r)\bigr), \quad j = 0, 1, \ldots, n,$$

where $T(r) = \min_{0 \leq j < k \leq n} T(r, e^{g_j - g_k})$, then $a_j(z) \equiv 0$ $(j = 0, 1, 2, \ldots, n)$.

Lemma 3.4 ([35, Lemma 3.1]) *Let $f_j(\not\equiv 0)$, $j = 1, 2, 3$, be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, $f_1 + f_2 + f_3 = 1$, and*

$$\sum_{j=1}^{3} \left\{ N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + O\bigl(\log^+ T(r, f_1)\bigr)$$

for all r outside possibly a set of finite logarithmic measure, where $\lambda < 1$ is a positive number. Then either $f_2 = 1$ or $f_3 = 1$.

Remark 3.2 Here $N_2(r, \frac{1}{f})$ is the counting function of zeros of f in $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.

4 The proof of Theorem 2.1

Proof Let $f(z_1, z_2)$ be a transcendental finite-order entire solution of equation (2.1). We first rewrite (2.1) in the form

$$\left(\frac{\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}}{e^{\frac{g(z_1,z_2)}{2}}}\right)^2 + \left(\frac{f(z_1 + c_1, z_2 + c_2)}{e^{\frac{g(z_1,z_2)}{2}}}\right)^2 = 1$$

or

$$\left(\frac{\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}}{e^{\frac{g(z_1,z_2)}{2}}}+i\frac{f(z_1+c_1,z_2+c_2)}{e^{\frac{g(z_1,z_2)}{2}}}\right)\left(\frac{\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}}{e^{\frac{g(z_1,z_2)}{2}}}-i\frac{f(z_1+c_1,z_2+c_2)}{e^{\frac{g(z_1,z_2)}{2}}}\right)=1. \tag{4.1}$$

Since f is a finite-order transcendental entire function and g is a polynomial, by Lemmas 3.1 and 3.2 there exists a polynomial $p(z)$ such that

$$\begin{cases}\frac{\frac{\partial^2 f(z)}{\partial z_1^2}}{e^{\frac{g(z)}{2}}}+i\frac{f(z+c)}{e^{\frac{g(z)}{2}}}=e^{p(z)},\\ \frac{\frac{\partial^2 f(z)}{\partial z_1^2}}{e^{\frac{g(z)}{2}}}-i\frac{f(z+c)}{e^{\frac{g(z)}{2}}}=e^{-p(z)}.\end{cases} \tag{4.2}$$

Denote

$$\gamma_1(z)=\frac{g(z)}{2}+p(z),\qquad \gamma_2(z)=\frac{g(z)}{2}-p(z). \tag{4.3}$$

By combining with (4.2) it follows that

$$\frac{\partial^2 f(z)}{\partial z_1^2}=\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2}, \tag{4.4}$$

$$f(z+c)=\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i}. \tag{4.5}$$

This leads to

$$-iQ_1(z)e^{\gamma_1(z)-\gamma_1(z+c)}+iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)}-e^{\gamma_2(z+c)-\gamma_1(z+c)}\equiv 1, \tag{4.6}$$

where

$$Q_1(z)=\frac{\partial^2\gamma_1}{\partial z_1^2}+\left(\frac{\partial\gamma_1}{\partial z_1}\right)^2,\qquad Q_2(z)=\frac{\partial^2\gamma_2}{\partial z_1^2}+\left(\frac{\partial\gamma_2}{\partial z_1}\right)^2.$$

We consider two cases.

Case 1. If $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, then $\gamma_2(z+c)-\gamma_1(z+c)$ is a constant. Set $\gamma_2(z+c)-\gamma_1(z+c)=\kappa$, $\kappa\in\mathbb{C}$. In view of (4.3), $p(z)$ is a constant. Let $\xi=e^{p(z)}$. Then equations (4.4)–(4.5) can be represented as

$$\frac{\partial^2 f(z)}{\partial z_1^2}=K_1e^{\frac{g(z)}{2}},\qquad f(z+c)=K_2e^{\frac{g(z)}{2}}, \tag{4.7}$$

where $K_1=\frac{\xi+\xi^{-1}}{2}$, $K_2=\frac{\xi-\xi^{-1}}{2i}$, and $K_1^2+K_2^2=1$.

This leads to

$$\frac{K_2}{2K_1}\left(\frac{\partial^2 g}{\partial z_1^2}+\frac{1}{2}\left(\frac{\partial g}{\partial z_1}\right)^2\right)=e^{\frac{g(z+c)-g(z)}{2}}. \tag{4.8}$$

Since $g(z)$ is a polynomial, (4.8) implies that $g(z+c)-g(z)$ is a constant in $\mathbb{C}$. Otherwise, we obtain a contradiction from the fact that the left-hand side of this equation is not transcendental but the right-hand side is transcendental. Thus it follows that $g(z) = L(z) + H(s) + B$, where $L(z) = A_1z_1 + A_2z_2$, $A_1 \neq 0$, and $H(s)$ is a polynomial in s in $\mathbb{C}$, $s = c_2z_1 - c_1z_2$.

We will prove that $H(s) \equiv 0$. If $\deg_s H = n$, then equation (4.8) implies

$$4A_1c_2\frac{dH}{ds} + c_2^2\frac{d^2H}{ds^2} + 2c_2^2\left(\frac{dH}{ds}\right)^2 \equiv \zeta_0,$$

that is,

$$4A_1c_2\frac{dH}{ds} + c_2^2\frac{d^2H}{ds^2} \equiv \zeta_0 - 2c_2^2\left(\frac{dH}{ds}\right)^2,$$

where $\zeta_0 \in \mathbb{C}$. By comparing the degree of s in both sides of the above equation we have $2(n-1) = n-1$, that is, $n = 1$. Thus the form of $L(z) + H(s) + B$ is still the linear form of $A_1z_1 + A_2z_2 + B$, which means that $H(s) \equiv 0$. Hence it follows that $g(z) = L(z) + B = A_1z_1 + A_2z_2 + B$. By combining with (4.6)–(4.8) we conclude that

$$\begin{aligned} &f(z_1,z_2) = K_2e^{\frac{g(z-c)}{2}} = K_2e^{\frac{1}{2}[A_1z_1+A_2z_2+B-(A_1c_1+A_2c_2)]}, \\ &\frac{\xi^2-1}{4i(\xi^2+1)}A_1^2 = e^{\frac{1}{2}(A_1c_1+A_2c_2)}, \end{aligned}$$

which implies that

$$f(z_1,z_2) = \frac{2K_1}{A_1^2}e^{\frac{1}{2}[A_1z_1+A_2z_2+B]} = \frac{\xi^2+1}{A_1^2\xi}e^{\frac{1}{2}g(z_1,z_2)}. \tag{4.9}$$

This completes the proof of Theorem 2.1(i).

Case 2. $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant. Obviously, $Q_1(z) \equiv 0$ and $Q_2(z) \equiv 0$ cannot hold at the same time. Otherwise, it would follows from (4.6) that $e^{\gamma_2(z+c)-\gamma_1(z+c)} = -1$, a contradiction. If $Q_1(z) \equiv 0$ and $Q_2(z) \not\equiv 0$, then from (4.6) this yields that

$$iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} - e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \tag{4.10}$$

Thus we conclude that $e^{\gamma_2(z)-\gamma_1(z+c)}$ is a nonconstant because $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant. Moreover, it follows that $e^{\gamma_2(z+c)-\gamma_2(z)}$ is not a constant. Otherwise, $\gamma_2(z+c) = \gamma_2(z) + \zeta$, where $\zeta \in \mathbb{C}$. Then from (4.10) we have $[iQ_2(z)e^{-\zeta} - 1]e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1$, which is a contradiction with the nonconstant $e^{\gamma_2(z+c)-\gamma_1(z+c)}$. Thus (4.10) can be written in the form

$$iQ_2(z)e^{\gamma_2(z)} - e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0. \tag{4.11}$$

By applying Lemma 3.3 for (4.11) we easily get a contradiction. If $Q_2(z) \equiv 0$ and $Q_1(z) \not\equiv 0$, by using the same argument as before, we can get a contradiction. Hence we have that $Q_1(z) \not\equiv 0$ and $Q_2(z) \not\equiv 0$.

Since $\gamma_1(z)$, $\gamma_2(z)$ are polynomials and $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant, by applying Lemma 3.4 to (4.6) it follows that

$$-iQ_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1 \quad \text{or} \quad iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1. \tag{4.12}$$

Subcase 2.1. Suppose that $-iQ_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$. Then it follows from (4.6) that $iQ_2(z)e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1$. This means that $\gamma_1(z) - \gamma_1(z+c) = \zeta_1$ and $\gamma_2(z) - \gamma_2(z+c) = \zeta_2$, where $\zeta_1, \zeta_2 \in \mathbb{C}$. Hence we have that $\gamma_1(z) = L_1(z) + H_1(s) + B_1$ and $\gamma_2(z) = L_2(z) + H_2(s) + B_2$, where $L_j(z) = A_{j1}z_1 + A_{j2}z_2$, $H_j(s)$, $j = 1, 2$, are polynomials in $s = c_2z_1 - c_1z_2$, $A_{j1}, A_{j2}, B_j \in \mathbb{C}$, $j = 1, 2$. In view of the definitions of Q_1, Q_2, similarly to the argument in Case 1, we can conclude that $H_1(s) = H_2(s) \equiv 0$. In addition, it follows that $L_1(z) \not\equiv L_2(z)$. Otherwise, $\gamma_2(z+c) - \gamma_1(z+c)$ is a constant, which implies that $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, a contradiction. Substituting these into (4.12), we have

$$iA_{11}^2 e^{-L_1(c)} = iA_{11}^2 e^{-(A_{11}c_1+A_{12}c_2)} = -1, \qquad iA_{21}^2 e^{-L_2(c)} = iA_{21}^2 e^{-(A_{21}c_1+A_{22}c_2)} = 1.$$

By combining with (4.5) we have

$$f(z) = \frac{e^{L_1(z)+B_1-L_1(c)} - e^{L_2(z)+B_2-L_2(c)}}{2i} = \frac{A_{21}^2 e^{L_1(z)+B_1} + A_{11}^2 e^{L_2(z)+B_2}}{2A_{11}^2 A_{21}^2}.$$

From the definitions of $\gamma_1(z)$ and $\gamma_2(z)$ we can see that

$$g(z) = \gamma_1(z) + \gamma_2(z) = L(z) + B,$$

where $L(z) = L_1(z) + L_2(z)$, $B = B_1 + B_2$.

Subcase 2.2. Suppose that $iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. Then it follows from (4.6) that $-iQ_1(z)e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1$. This means that $\gamma_2(z) - \gamma_1(z+c) = \zeta_1$ and $\gamma_1(z) - \gamma_2(z+c) = \zeta_2$, where $\zeta_1, \zeta_2 \in \mathbb{C}$. Thus it follows that $\gamma_1(z+2c) - \gamma_1(z) = -\zeta_1 - \zeta_2$ and $\gamma_2(z+c) - \gamma_2(z) = -\zeta_1 - \zeta_2$. We can obtain that $\gamma_1(z) = L(z) + H(s) + B_1$ and $\gamma_2(z) = L(z) + H(s) + B_2$, where $L(z) = a_1z_1 + a_2z_2$, and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$, $a_1, a_2, B_1, B_2 \in \mathbb{C}$. This yields that $\gamma_2(z+c) - \gamma_1(z+c) = B_2 - B_1$, which implies that $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, a contradiction.

This completes the proof of Theorem 2.1. □

5 The proof of Theorem 2.2

Proof Let $f(z_1, z_2)$ be a finite-order transcendental entire solution of equation (2.3). We first rewrite (2.3) in the form

$$\left(\frac{\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}}{e^{\frac{g(z_1,z_2)}{2}}}\right)^2 + \left(\frac{f(z_1+c_1, z_2+c_2) - f(z_1,z_2)}{e^{\frac{g(z_1,z_2)}{2}}}\right)^2 = 1$$

or

$$\left(\frac{\frac{\partial^2 f(z)}{\partial z_1^2}}{e^{\frac{g(z)}{2}}} + i\frac{f(z+c) - f(z)}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\frac{\partial^2 f(z)}{\partial z_1^2}}{e^{\frac{g(z)}{2}}} - i\frac{f(z+c) - f(z)}{e^{\frac{g(z)}{2}}}\right) = 1. \tag{5.1}$$

Since f is a finite-order transcendental entire function and g is a polynomial, by Lemmas 3.1 and 3.2 there exists a polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases} \frac{\frac{\partial^2 f(z)}{\partial z_1^2}}{e^{\frac{g(z)}{2}}} + i\frac{f(z+c)-f(z)}{e^{\frac{g(z)}{2}}} = e^{p(z)}, \\ \frac{\frac{\partial^2 f(z)}{\partial z_1^2}}{e^{\frac{g(z)}{2}}} - i\frac{f(z+c)-f(z)}{e^{\frac{g(z)}{2}}} = e^{-p(z)}. \end{cases} \tag{5.2}$$

Denote

$$\gamma_1(z) = \frac{g(z)}{2} + p(z), \qquad \gamma_2(z) = \frac{g(z)}{2} - p(z). \tag{5.3}$$

By combining with (5.2) it follows that

$$\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} = \frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2}, \tag{5.4}$$

$$f(z+c) - f(z) = \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2i}. \tag{5.5}$$

This leads to

$$Q_3(z)e^{\gamma_1(z)-\gamma_1(z+c)} + Q_4(z)e^{\gamma_2(z)-\gamma_1(z+c)} - e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1, \tag{5.6}$$

where

$$Q_3(z) = 1 - i\left(\frac{\partial^2\gamma_1}{\partial z_1^2} + \left(\frac{\partial\gamma_1}{\partial z_1}\right)^2\right), \qquad Q_4(z) = 1 + i\left(\frac{\partial^2\gamma_2}{\partial z_1^2} + \left(\frac{\partial\gamma_2}{\partial z_1}\right)^2\right).$$

We consider two cases.

Case 1. If $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, then $\gamma_2(z+c) - \gamma_1(z+c)$ is a constant. Set $\gamma_2(z+c) - \gamma_1(z+c) = \kappa$, $\kappa \in \mathbb{C}$. In view of (5.3), this yields that $p(z)$ is a constant. Let $\xi = e^{p(z)}$. Then equations (5.4)–(5.5) can be represented as

$$\frac{\partial^2 f(z)}{\partial z_1^2} = K_1 e^{\frac{g(z)}{2}}, \qquad f(z+c) - f(z) = K_2 e^{\frac{g(z)}{2}}, \tag{5.7}$$

where $K_1 = \frac{\xi+\xi^{-1}}{2}$, $K_2 = \frac{\xi-\xi^{-1}}{2i}$, and $K_1^2 + K_2^2 = 1$.

This leads to

$$\frac{K_2}{2K_1}\left(\frac{\partial^2 g}{\partial z_1^2} + \frac{1}{2}\left(\frac{\partial g}{\partial z_1}\right)^2\right) + 1 = e^{\frac{g(z+c)-g(z)}{2}}. \tag{5.8}$$

Since $g(z)$ is a polynomial, (5.8) implies $g(z+c) - g(z)$, and thus $e^{\frac{g(z+c)-g(z)}{2}}$ must be a constant. Denote $g(z+c) - g(z) = \zeta$, where ζ is a constant in $\mathbb{C}$. By using the same argument as in Case 1 of Theorem 2.1, we obtain that $g(z) = L(z) + B$, where $L(z) = A_1z_1 + A_2z_2$, $B \in \mathbb{C}$.

By combining with (5.8) it follows that

$$\frac{K_2}{4K_1}A_1^2 + 1 = e^{\frac{1}{2}(A_1c_1+A_2c_2)}. \tag{5.9}$$

Solving the first equation in (5.7), we have

$$\begin{aligned} f(z_1,z_2) &= \frac{4K_1}{A_1^2}e^{\frac{1}{2}g(z)} + z_1G_1(z_2) + G_2(z_2) \\ &= \frac{4(\xi^2+1)}{A_1^2\xi}e^{\frac{1}{2}(L(z)+B)} + z_1G_1(z_2) + G_2(z_2). \end{aligned} \tag{5.10}$$

Substituting (5.10) into the second equation in (5.7) and combining with (5.9), we get that $G_1(z_2+c_2) = G_1(z_2)$ and $G_2(z_2+c_2) - G_2(z_2) = c_1G_1(z_2)$, which means that $G_1(z_2)$ is a finite-order entire period functions in z_2 with period c_2.

Case 2. $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant. Obviously, $Q_3(z)\equiv 0$ and $Q_4(z)\equiv 0$ cannot hold at the same time. Otherwise, it would follow from (5.6) that $e^{\gamma_2(z+c)-\gamma_1(z+c)} = -1$, a contradiction. If $Q_3(z)\equiv 0$ and $Q_4(z)\not\equiv 0$, then from (5.6) it follows that

$$Q_3(z)e^{\gamma_2(z)-\gamma_1(z+c)} - e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \tag{5.11}$$

Thus we conclude that $e^{\gamma_2(z)-\gamma_1(z+c)}$ is not a constant because $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant. Moreover, it follows that $e^{\gamma_2(z+c)-\gamma_2(z)}$ is not a constant. Otherwise, $\gamma_2(z+c) = \gamma_2(z)+\zeta$, where $\zeta\in\mathbb{C}$. Then from (5.11) we have $[Q_4(z)e^{-\zeta}-1]e^{\gamma_2(z+c)-\gamma_1(z+c)}\equiv 1$, which is a contradiction with the nonconstant $e^{\gamma_2(z+c)-\gamma_1(z+c)}$. Thus (5.11) can be written in the form

$$Q_4(z)e^{\gamma_2(z)} - e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0. \tag{5.12}$$

By applying Lemma 3.3 to (5.12) we easily get a contradiction. If $Q_4(z)\equiv 0$ and $Q_3(z)\not\equiv 0$, by using the same argument as before we can get a contradiction. Hence we have that $Q_3(z)\not\equiv 0$ and $Q_4(z)\not\equiv 0$.

Since $\gamma_1(z)$, $\gamma_2(z)$ are polynomials and $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a nonconstant, by applying Lemma 3.4 to (5.6) it follows that

$$Q_3(z)e^{\gamma_1(z)-\gamma_1(z+c)}\equiv 1 \quad\text{or}\quad Q_4(z)e^{\gamma_2(z)-\gamma_1(z+c)}\equiv 1. \tag{5.13}$$

Subcase 2.1. Suppose that $Q_3(z)e^{\gamma_1(z)-\gamma_1(z+c)}\equiv 1$. Then it follows from (5.6) that $Q_4(z)e^{\gamma_2(z)-\gamma_2(z+c)}\equiv 1$. This means that $\gamma_1(z)-\gamma_1(z+c)=\zeta_1$, $\gamma_2(z)-\gamma_2(z+c)=\zeta_2$, where $\zeta_1,\zeta_2\in\mathbb{C}$. Hence we have that $\gamma_1(z)=L_1(z)+H_1(s)+B_1$ and $\gamma_2(z)=L_2(z)+H_2(s)+B_2$, where $L_j(z)=A_{j1}z_1+A_{j2}z_2$, $H_j(s_1)$, $j=1,2$, are polynomials in $s_1=c_2z_1-c_1z_2$, $A_{j1},A_{j2},B_j\in\mathbb{C}$, $j=1,2$. Similarly to the argument in Case 1 of Theorem 2.2, we have $H_1(s)=H_2(s)\equiv 0$. Thus it follows that $\gamma_1(z)=L_1(z)+B_1$ and $\gamma_2(z)=L_2(z)+B_2$. Obviously, $L_1(z)\not\equiv L_2(z)$. Otherwise, $\gamma_2(z+c)-\gamma_1(z+c)$ is a constant, which implies that $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, a contradiction. Substituting these into (5.6), we have

$$\left(1-iA_{11}^2\right)e^{-(A_{11}c_1+A_{12}c_2)}=1, \qquad \left(1+iA_{21}^2\right)e^{-(A_{21}c_1+A_{22}c_2)}=1. \tag{5.14}$$

By solving the equation

$$\frac{\partial^2 f(z)}{\partial z_1^2} = \frac{e^{L_1(z)+B_1}+e^{L_2(z)+B_2}}{2} \tag{5.15}$$

we have

$$f(z_1,z_2)=\frac{A_{21}^2e^{L_1(z)+B_1}+A_{11}^2e^{L_2(z)+B_2}}{2A_{11}^2A_{21}^2}+z_1G_1(z_2)+G_2(z_2). \tag{5.16}$$

Substituting (5.16) into (5.5) and combining with (5.14), we have $G_1(z_2+c_2)=G_1(z_2)$ and $G_2(z_2+c_2)-G_2(z_2)=c_1G_1(z_2)$, which means that $G_1(z_2)$ is a finite-order entire periodic function in z_2 with period c_2.

From the definitions of $\gamma_1(z)$ and $\gamma_2(z)$ we can see that

$$g(z)=\gamma_1(z)+\gamma_2(z)=L(z)+B,$$

where $L(z)=L_1(z)+L_2(z)$, $B=B_1+B_2$.

Subcase 2.2. Suppose that $Q_4(z)e^{\gamma_2(z)-\gamma_1(z+c)}\equiv 1$. Similarly to the argument in Subcase 2.2 in Theorem 2.1, we can get a contradiction.

Therefore this completes the proof of Theorem 2.2. □

6 Proofs of Theorems 2.3 and 2.4

6.1 Proof of Theorem 2.4

Suppose that $f(z_1,z_2)$ is a finite-order transcendental entire solution of equation (2.7). We first rewrite (2.7) in the form

$$\left(\frac{\frac{\partial^2 f(z)}{\partial z_1\partial z_2}}{e^{\frac{g(z)}{2}}}+i\frac{f(z+c)-f(z)}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\frac{\partial^2 f(z)}{\partial z_1\partial z_2}}{e^{\frac{g(z)}{2}}}-i\frac{f(z+c)-f(z)}{e^{\frac{g(z)}{2}}}\right)=1. \tag{6.1}$$

Since f is a finite-order transcendental entire function and g is a polynomial, by Lemmas 3.1 and 3.2 there exists a polynomial $p(z)$ such that

$$\begin{cases}\frac{\frac{\partial^2 f(z)}{\partial z_1\partial z_2}}{e^{\frac{g(z)}{2}}}+i\frac{f(z+c)-f(z)}{e^{\frac{g(z)}{2}}}=e^{p(z)},\\ \frac{\frac{\partial^2 f(z)}{\partial z_1\partial z_2}}{e^{\frac{g(z)}{2}}}-i\frac{f(z+c)-f(z)}{e^{\frac{g(z)}{2}}}=e^{-p(z)}.\end{cases} \tag{6.2}$$

Denote

$$\gamma_1(z)=\frac{g(z)}{2}+p(z),\qquad \gamma_2(z)=\frac{g(z)}{2}-p(z). \tag{6.3}$$

By combining with (6.2) it follows that

$$\frac{\partial^2 f(z_1,z_2)}{\partial z_1\partial z_2}=\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2}, \tag{6.4}$$

$$f(z+c)-f(z)=\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i}. \tag{6.5}$$

This leads to

$$Q_5(z)e^{\gamma_1(z)-\gamma_1(z+c)}+Q_6(z)e^{\gamma_2(z)-\gamma_1(z+c)}-e^{\gamma_2(z+c)-\gamma_1(z+c)}\equiv 1, \tag{6.6}$$

where

$$Q_5(z) = 1 - i\left(\frac{\partial^2\gamma_1}{\partial z_1\partial z_2} + \frac{\partial\gamma_1}{\partial z_1}\frac{\partial\gamma_1}{\partial z_2}\right), \qquad Q_4(z) = 1 + i\left(\frac{\partial^2\gamma_2}{\partial z_1\partial z_2} + \frac{\partial\gamma_2}{\partial z_1}\frac{\partial\gamma_2}{\partial z_2}\right).$$

We consider two cases.

Case 1. If $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, then $\gamma_2(z+c)-\gamma_1(z+c)$ is a constant. Set $\gamma_2(z+c)-\gamma_1(z+c)=\kappa$, $\kappa\in\mathbb{C}$. In view of (6.3), this yields that $p(z)$ is a constant. Let $\xi = e^{p(z)}$. Then equations (6.4)–(6.5) become

$$\frac{\partial^2 f(z)}{\partial z_1\partial z_2} = K_1 e^{\frac{g(z)}{2}}, \qquad f(z+c)-f(z) = K_2 e^{\frac{g(z)}{2}}, \tag{6.7}$$

where $K_1 = \frac{\xi+\xi^{-1}}{2}$, $K_2 = \frac{\xi-\xi^{-1}}{2i}$, and $K_1^2+K_2^2=1$.

This leads to

$$\frac{K_2}{2K_1}\left(\frac{\partial^2 g}{\partial z_1\partial z_2} + \frac{1}{2}\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2}\right) + 1 = e^{\frac{g(z+c)-g(z)}{2}}. \tag{6.8}$$

Since $g(z)$ is a polynomial, (6.8) implies $g(z+c)-g(z)$, and thus $e^{\frac{g(z+c)-g(z)}{2}}$ must be a constant. Denote $g(z+c)-g(z)=\zeta$, where ζ is a constant in $\mathbb{C}$. Thus it follows that $g(z)=L(z)+H(s)+B$, where $L(z)=A_1z_1+A_2z_2$, and $H(s)$ is a polynomial in s in $\mathbb{C}$, $s=c_2z_1-c_1z_2$. Substituting this into (6.8), we deduce that

$$\frac{K_2}{2K_1}\left(-\frac{1}{4}c_1c_2H'' - \frac{1}{2}c_1c_2\left(H'\right)^2 + \frac{1}{2}(A_2c_2-A_1c_1)H' + \frac{1}{2}A_1A_2\right) + 1 = e^{\frac{g(z+c)-g(z)}{2}}. \tag{6.9}$$

Since $g(z)$ is a polynomial, then (6.9) implies that $g(z+c)-g(z)$ is a constant in $\mathbb{C}$. Otherwise, we would obtain a contradiction from the fact that the left-hand side of the above equation is not transcendental but the right-hand side is transcendental. Hence it follows that

$$\frac{K_2}{2K_1}\left(-\frac{1}{4}c_1c_2H'' - \frac{1}{2}c_1c_2\left(H'\right)^2 + \frac{1}{2}(A_2c_2-A_1c_1)H' + \frac{1}{2}A_1A_2\right) + 1 = \zeta_0, \tag{6.10}$$

where $\zeta_0\in\mathbb{C}$. If $c_1=0$, $c_2\neq 0$, that is, $\frac{K_2}{2K_1}(\frac{1}{2}A_2c_2H' + \frac{1}{2}A_1A_2)+1=\zeta_0$. Thus, either $A_2=0$, or H' is a constant. If $A_2=0$, then $\zeta_0=1$, that is, $e^{\frac{g(z+c)-g(z)}{2}}$ is a constant. By combining with $c_1=0$ this means that $g(z)$ is a constant. Set $e^{\frac{g}{2}}=\theta$. In view of the first equation of (6.5), we have

$$f(z_1,z_2) = K_1\theta z_1z_2 + \mu(z_1), \tag{6.11}$$

where $\mu(z_1)$ is a finite-order transcendental entire function. Substituting this into the second equation of (6.5), we have

$$K_1\theta(z_1+c_1)(z_2+c_2) + \mu(z_1+c_1) - K_1\theta z_1z_2 - \mu(z_1) = K_2\theta.$$

Combining with $c_1=0$, this yields that $K_1\theta c_2z_1 = K_2\theta$, which is impossible. Hence H' is a constant, that is, $H(s)=c2z_1$.

If $c_2 = 0$, $c_1 \neq 0$, similarly to the above argument, we can obtain that $H(s) = -c_1 z_2$.

Let $c_1 \neq 0$ and $c_2 \neq 0$. If $A_2c_2 - A_1c_1 = 0$, noting that the left-hand side of (6.10) is a constant, we have $\deg_s H \leq 1$, that is, $H(s) = c_2z_1 - c_1z_2 + \tau$, where $\tau \in \mathbb{C}$. If $A_2c_2 - A_1c_1 \neq 0$, we easily obtain that $\deg_s H \leq 1$, that is, $H(s) = c_2z_1 - c_1z_2 + +\tau$, where $\tau \in \mathbb{C}$. Thus the form of $L(z) + H(s) + B$ is still the linear form of $A_1z_1 + A_2z_2 + B$, which means that $H(s) \equiv 0$. Hence we obtain that $g(z) = L(z) + B$, where $L(z) = A_1z_1 + A_2z_2$, $B \in \mathbb{C}$.

By combining with (6.8) it follows that

$$\frac{K_2}{4K_1}A_1A_2 + 1 = e^{\frac{1}{2}(A_1c_1+A_2c_2)}. \tag{6.12}$$

Solving the first equation in (6.7), we have

$$\begin{aligned} f(z_1,z_2) &= \frac{4K_1}{A_1A_2}e^{\frac{1}{2}g(z)} + \phi(z_1) + \varphi(z_2) \\ &= \frac{4(\xi^2+1)}{A_1A_2\xi}e^{\frac{1}{2}(L(z)+B)} + \phi(z_1) + \varphi(z_2). \end{aligned} \tag{6.13}$$

Substituting (6.13) into the second equation in (6.7) and combining with (6.12), we get that

$$\phi(z_1 + c_1) - \phi(z_1) = -\bigl[\varphi(z_2 + c_2) - \varphi(z_2)\bigr],$$

which yields that $\phi(z_1) = G_3(z_1) + D_1z_1$ and $\varphi(z_2) = G_4(z_2) + D_2z_2$, where $D_1c_1 + D_2c_2 = 0$ and $G_3(z_1)$, $G_4(z_2)$ are finite-order entire period functions in z_1, z_2 with periods c_1, c_2, respectively.

Case 2. $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant. Obviously, $Q_5(z) \equiv 0$ and $Q_6(z) \equiv 0$ cannot hold at the same time. Otherwise, it would follow from (6.6) that $e^{\gamma_2(z+c)-\gamma_1(z+c)} = -1$, a contradiction. If $Q_5(z) \equiv 0$ and $Q_6(z) \not\equiv 0$, then from (6.6) we get that

$$Q_5(z)e^{\gamma_2(z)-\gamma_1(z+c)} - e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \tag{6.14}$$

Thus we conclude that $e^{\gamma_2(z)-\gamma_1(z+c)}$ is not a constant because $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant. Moreover, it follows that $e^{\gamma_2(z+c)-\gamma_2(z)}$ is not a constant. Otherwise, $\gamma_2(z + c) = \gamma_2(z) + \zeta$, where $\zeta \in \mathbb{C}$. Then from (6.14) we have $[Q_6(z)e^{-\zeta} - 1]e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1$, which is a contradiction with the nonconstant $e^{\gamma_2(z+c)-\gamma_1(z+c)}$. Thus (6.14) can be written in the form

$$Q_6(z)e^{\gamma_2(z)} - e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0. \tag{6.15}$$

By applying Lemma 3.3 for (6.15) we easily get a contradiction. If $Q_6(z) \equiv 0$ and $Q_5(z) \not\equiv 0$, by using the same argument as before we can get a contradiction. Hence we have that $Q_5(z) \not\equiv 0$ and $Q_6(z) \not\equiv 0$.

Since $\gamma_1(z)$, $\gamma_2(z)$ are polynomials and $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is not a constant, by applying Lemma 3.4 to (6.6) it follows that

$$Q_5(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1 \quad \text{or} \quad Q_6(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1. \tag{6.16}$$

Subcase 2.1. Suppose that $Q_5(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$. Then it follows from (6.6) that $Q_6(z)e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1$. This means that $\gamma_1(z) - \gamma_1(z+c) = \zeta_1$, $\gamma_2(z) - \gamma_2(z+c) = \zeta_2$, where $\zeta_1, \zeta_2 \in \mathbb{C}$. Hence we have that $\gamma_1(z) = L_1(z) + H_1(s) + B_1$ and $\gamma_2(z) = L_2(z) + H_2(s) + B_2$, where $L_j(z) = A_{j1}z_1 + A_{j2}z_2$, $H_j(s)$, $j = 1,2$, are polynomials in $s = c_2z_1 - c_1z_2$, $A_{j1}, A_{j2}, B_j \in \mathbb{C}$, $j = 1,2$. Similarly to the argument in Case 1, we have $H_1(s) = H_2(s) \equiv 0$. Thus it follows that $\gamma_1(z) = L_1(z) + B_1$ and $\gamma_2(z) = L_2(z) + B_2$. Obviously, $L_1(z) \not\equiv L_2(z)$. Otherwise, $\gamma_2(z+c) - \gamma_1(z+c)$ would be a constant, which implies that $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, a contradiction. Substituting these into (6.6), we have

$$(1 - iA_{11}A_{12})e^{-(A_{11}c_1+A_{12}c_2)} = 1, \qquad (1 + iA_{21}A_{22})e^{-(A_{21}c_1+A_{22}c_2)} = 1. \tag{6.17}$$

By solving the equation

$$\frac{\partial^2 f(z)}{\partial z_1 \partial z_2} = \frac{e^{L_1(z)+B_1} + e^{L_2(z)+B_2}}{2} \tag{6.18}$$

we have

$$f(z_1,z_2) = \frac{e^{L_1(z)+B_1}}{2A_{11}A_{12}} + \frac{e^{L_2(z)+B_2}}{2A_{21}A_{22}} + \phi(z_1) + \varphi(z_2). \tag{6.19}$$

Substituting (6.19) into (6.5) and combining with (6.17), we get that

$$\phi(z_1 + c_1) - \phi(z_1) = -\big[\varphi(z_2 + c_2) - \varphi(z_2)\big],$$

which yields that $\phi(z_1) = G_3(z_1) + D_1z_1$ and $\varphi(z_2) = G_4(z_2) + D_2z_2$, where $D_1c_1 + D_2c_2 = 0$ and $G_3(z_1)$, $G_4(z_2)$ are finite-order entire periodic functions in z_1, z_2 with period c_1, c_2, respectively.

From the definitions of $\gamma_1(z)$ and $\gamma_2(z)$ we can see that

$$g(z) = \gamma_1(z) + \gamma_2(z) = L(z) + B,$$

where $L(z) = L_1(z) + L_2(z)$, $B = B_1 + B_2$.

Subcase 2.2. Suppose that $Q_6(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. Similarly to the argument in Subcase 2.2 in Theorem 2.1, we can get a contradiction.

This completes the proof of Theorem 2.4.

6.2 Proof of Theorem 2.3

Similar to the argument in the proof of Theorem 2.1, we can easily prove the statements of Theorem 2.3.

7 Remarks

In view of the arguments in the proofs of Theorems 2.1 and 2.3, we easily get the following theorems.

Theorem 7.1 *Let $c = (c_1,c_2) \in \mathbb{C}^2$ with $c_1 \neq 0$, $c_2 \neq 0$, and $c_1 + c_2 \neq 0$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1,z_2)} \tag{7.1}$$

admits a transcendental entire solution $f(z_1,z_2)$ of finite order, then $g(z)$ must be a polynomial function of the form $g(z) = L(z) + B$, where $L(z)$ is a linear function of the form $L(z) = A_1z_1 + A_2z_2 + B$, $A_1, A_2, B \in \mathbb{C}$. Further, $f(z_1,z_2)$ must satisfy one of the following cases:

(i)

$$f(z_1,z_2) = \frac{4(\xi^2+1)}{A_1(A_1+A_2)\xi} e^{\frac{1}{2}g(z_1,z_2)}$$

with $\xi(\neq 0), A_1, A_2, B \in \mathbb{C}$ satisfying

$$\frac{1}{4}\frac{\xi^2-1}{(\xi^2+1)i}A_1(A_1+A_2) = e^{\frac{1}{2}(A_1c_1+A_2c_2)};$$

(ii)

$$f(z_1,z_2) = \frac{e^{L_1(z)+B_1}}{2A_{11}(A_{11}+A_{12})} + \frac{e^{L_2(z)+B_2}}{2A_{21}(A_{21}+A_{12})},$$

where $L_1(z) = A_{11}z_1 + A_{12}z_2 + B_1$, $L_2(z) = A_{21}z_1 + A_{22}z_2 + B_2$, $A_{j1}, A_{j2}, B_j \in \mathbb{C}$ $(j = 1,2)$ satisfy

$$L_1(z) \neq L_2(z), \qquad g(z) = L_1(z) + L_2(z) + B_1 + B_2,$$

and

$$-iA_{11}(A_{11}+A_{12})e^{-L_1(c)} = iA_{21}(A_{21}+A_{22})e^{-L_2(c)} = 1.$$

We give some examples showing the existence of finite-order transcendental entire solutions of equation (7.1).

Example 7.1 Let $A_1 = 1$, $A_2 = 1$, $B = 0$, and

$$f(z_1,z_2) = \frac{2\sqrt{5}}{5}e^{\frac{1}{2}(z_1+z_2)}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (7.1) with $g(z) = z_1 + z_2$, $c_1 = 2\pi i$, and $c_2 = 2\pi i$.

Example 7.2 Let $L_1(z) = iz_1 + (1-i)z_2$, $L_2(z) = z_1 + (i-1)z_2$, $B_1 = B_2 = 0$, and

$$f(z_1,z_2) = \frac{e^{iz_1+(1-i)z_2}}{2i} + \frac{e^{z_1+(i-1)z_2}}{2i}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (7.1) with $g(z) = (1+i)z_1$, $c_1 = \frac{3\pi}{2}(1+i)$, and $c_2 = \frac{\pi}{2}(1+2i)$.

Theorem 7.2 *Let $c = (c_1,c_2) \in \mathbb{C}^2$ with $c_1 \neq 0$, $c_2 \neq 0$, and $c_1 + c_2 \neq 0$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^2 + f(z_1+c_1,z_2+c_2)^2 = e^{g(z_1,z_2)} \tag{7.2}$$

admits a transcendental entire solution $f(z_1,z_2)$ of finite order, then $g(z)$ must be a polynomial function of the form $g(z) = L(z) + B$, where $L(z)$ is a linear function of the form $L(z) = A_1z_1 + A_2z_2 + B$, $A_1, A_2, B \in \mathbb{C}$. Further, $f(z_1,z_2)$ must satisfy one of the following cases:

(i)

$$f(z_1,z_2) = \frac{4(\xi^2+1)}{(A_1^2+A_2^2)\xi}e^{\frac{1}{2}g(z_1,z_2)}$$

with $\xi(\neq 0), A_1, A_2, B \in \mathbb{C}$ *satisfying*

$$\frac{1}{4}\frac{\xi^2-1}{(\xi^2+1)i}\big(A_1^2+A_2^2\big) = e^{\frac{1}{2}(A_1c_1+A_2c_2)};$$

(ii)

$$f(z_1,z_2) = \frac{e^{L_1(z)+B_1}}{2(A_{11}^2+A_{12}^2)} + \frac{e^{L_2(z)+B_2}}{2(A_{21}^2+A_{12}^2)},$$

where $L_1(z) = A_{11}z_1 + A_{12}z_2 + B_1$, $L_2(z) = A_{21}z_1 + A_{22}z_2 + B_2$, $A_{j1}, A_{j2}, B_j \in \mathbb{C}$ $(j = 1, 2)$ *satisfy*

$$L_1(z) \neq L_2(z), \qquad g(z) = L_1(z) + L_2(z) + B_1 + B_2,$$

and

$$-i\big(A_{11}^2+A_{12}^2\big)e^{-L_1(c)} = i\big(A_{21}^2+A_{22}^2\big)e^{-L_2(c)} = 1.$$

Some examples explain the existence of finiteorder – entire solutions of equation (7.2).

Example 7.3 Let $A_1 = 1$, $A_2 = 1$, $B = 0$, and

$$f(z_1,z_2) = \frac{2\sqrt{5}}{5}e^{\frac{1}{2}(z_1+z_2)}.$$

Then $\rho(f) = 1$, and $f(z_1,z_2)$ is a transcendental entire solution of equation (7.2) with $g(z) = z_1 + z_2$, $c_1 = \pi i$, and $c_2 = \pi i$.

Example 7.4 Let $L_1(z) = iz_1 + \sqrt{2}z_2$, $L_2(z) = \sqrt{2}iz_1 + z_2$, $B_1 = B_2 = 0$, and

$$f(z_1,z_2) = \frac{e^{iz_1+\sqrt{2}z_2}}{2} - \frac{e^{iz_1+\sqrt{2}z_2}}{2}.$$

Then $\rho(f) = 1$ and $f(z_1,z_2)$ is a transcendental entire solution of equation (7.2) with $g(z) = (\sqrt{2}+1)iz_1 + (\sqrt{2}+1)z_2$, $c_1 = \frac{(\sqrt{2}-3)\pi}{2}$, and $c_2 = \frac{(3\sqrt{2}-1)}{2}\pi i$.

In view of Theorems 7.1 and 7.2, we easily get the following.

Corollary 7.1 *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $c_1 \neq 0$, $c_2 \neq 0$, $\alpha, \beta, \gamma \in \mathbb{C}$, and let $g(z_1, z_2)$ be not a linear function of the form $L(z) = A_1z_1 + A_2z_2 + B$, where $A_1, A_2, B \in \mathbb{C}$. If $\alpha c_1^2 - \beta c_1c_2 + \gamma c_2^2 \neq 0$, then the partial differential-difference equation*

$$\left(\alpha \frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \beta \frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2} + \gamma \frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1,z_2)}$$

has no finite-order transcendental entire solution.

Corollary 7.2 *The finite-order transcendental entire solution $f(z_1, z_2)$ of the partial differential equations*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1,z_2)^2 = 1,$$
$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^2 + f(z_1,z_2)^2 = 1$$

must be of the form

$$f(z_1,z_2) = \frac{e^{L(z)+B} - e^{-L(z)-B}}{2i} = \sin\bigl(-i\bigl(L(z) + B\bigr)\bigr),$$

where $L(z) = A_1z_1 + A_2z_2$, $A_1, A_2, B \in \mathbb{C}$ satisfy $A_1^2(A_1 + A_2)^2 = 1$ and $(A_1^2 + A_2^2)^2 = 1$.

Corresponding to Theorems 7.1 and 7.2, we can obtain some results on the existence of solutions of the difference-type equations (7.1) and (7.2).

Theorem 7.3 *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $c_2 \neq 0$, $c_1 \neq c_2$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + \bigl[f(z_1 + c_1, z_2 + c_2) - f(z_1,z_2)\bigr]^2 = e^{g(z_1,z_2)} \tag{7.3}$$

admits a transcendental entire solution of finite order, then $g(z_1, z_2)$ must be a linear function of the form $g(z_1, z_2) = A_1z_1 + A_2z_2 + B$, where $A_1, A_2, B \in \mathbb{C}$.

Theorem 7.4 *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$ such that $c_1 \neq \pm ic_2$. If the partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^2 + \bigl[f(z_1 + c_1, z_2 + c_2) - f(z_1,z_2)\bigr]^2 = e^{g(z_1,z_2)} \tag{7.4}$$

admits a transcendental entire solution of finite order, then $g(z_1, z_2)$ must be a linear function of the form $g(z_1, z_2) = A_1z_1 + A_2z_2 + B$, where $A_1, A_2, B \in \mathbb{C}$.

Remark 7.1 Although we give the conditions for the existence of finite-order transcendental entire solutions of equations (7.3) and (7.4) in Theorems 7.3 and 7.4, in view of Theorems 2.2 and 2.4, there naturally arises an open question: *How to describe the forms of finite-order transcendental entire solutions of equations* (7.3) *and* (7.4)?

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