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Some extensions for the several combinatorial identities

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Dedicated to Honor Professor Hari Mohan Srivastava on his 80th Birth Anniversary

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Abstract

In this paper, we give some extensions for Mortenson's identities in series with the Bell polynomial using the partial fraction decomposition. As applications, we obtain some combinatorial identities involving the harmonic numbers.

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1 Introduction

The higher-order harmonic numbers are defined by

$$H_0^{(r)} = 1 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \ldots.$$

When $n = 1$, they reduce to the classical harmonic numbers $H_n = H_n^{(1)}$.

We also define the generalized higher-order harmonic numbers $H_n^{(r)}(z)$ as

$$H_0^{(r)}(z) = 1 \quad \text{and} \quad H_n^{(r)}(z) = \sum_{\substack{k=1 \\ k \neq -z}}^{n} \frac{1}{(k+z)^r}. \tag{1}$$

When $z = 0$, they reduce to the higher-order harmonic numbers $H_n^{(r)}(0) = H_n^{(r)}$.

The standard Bell polynomials are presented in Comtet's book [5]. The modified Bell polynomials $L_n(x_1, x_2, \ldots)$ are defined by

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = 1 + \sum_{n=1}^{\infty} \mathbf{L}_n(x_1, x_2, \ldots) z^n. \tag{2}$$

This expansion gives

$$\mathbf{L}_n(x_1,x_2,\ldots)=\sum_{m_1+2m_2+3m_3+\cdots=n}\frac{1}{m_1!m_2!m_3!\cdots}\left(\frac{x_1}{1}\right)^{m_1}\left(\frac{x_2}{2}\right)^{m_2}\left(\frac{x_3}{3}\right)^{m_3}\cdots. \tag{3}$$

Mortenson [9, p. 990, Lemma 3.1] gave the following identities:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{r}{r+k}=\frac{(1-r)_n}{(1+r)_n},\quad n,r\in\mathbb{N}, \tag{4}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}(H_{m+k}-H_k)=0,\quad 1\le m\le n, \tag{5}$$

which are called Mortenson's identities, where $(z)_n$ are the Pochhammer symbols defined by $(z)_0=1$, $(z)_n=z(z+1)\cdots(z+n-1)$.

H.M. Srivastava, J. Choi, G. Dattoli, and A. Sofo et al. investigated some infinite combinatorial series identities involving the harmonic numbers and generalized harmonic numbers by applying the hypergeometric series, Vandermonde convolutions, and Riemann zeta and polygamma functions; for details, see [1, 6–8, 10–13]. W. Chu studied some finite combinatorial identities involving the harmonic numbers by applying the partial fraction decomposition [2–4].

In this paper, we give some extensions of Mortenson's identities using the partial fraction decomposition. We obtain some new or old combinatorial identities involving the harmonic numbers and generalized harmonic numbers and propose two open problems.

2 Extensions of Mortenson's identities

First, we give an extension of Mortenson's identity (4).

Theorem 1 *For $n\in\mathbb{N}$, $r\in\mathbb{N}_0$, and $x>0$, we have*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(\frac{x}{x+k}\right)^r\\&\quad=\left(\prod_{k=1}^{n}\frac{k-x}{k+x}\right)\sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{x^{r-1}}{m_1!m_2!m_3!\cdots}\left(\frac{U_1}{1}\right)^{m_1}\left(\frac{U_2}{2}\right)^{m_2}\left(\frac{U_3}{3}\right)^{m_3}\cdots,\end{aligned} \tag{6}$$

where $U_k=(-1)^{k-1}H_n^{(k)}(-x)+H_{n+1}^{(k)}(x-1)$.

Proof By means of the standard partial fraction decomposition we easily obtain

$$\begin{aligned}&\frac{(z+1)_n}{z(z-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r\\&\quad=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}\left(\frac{x}{x+k}\right)^r\frac{1}{z-k}+\frac{\lambda}{(z+x)^r}+\cdots+\frac{\mu}{z+x}.\end{aligned} \tag{7}$$

Multiplying both sides of (7) by z and then letting $z\to\infty$, we obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}\left(\frac{x}{x+k}\right)^r+\mu=0.$$

By (2), (3), and (7) we get

$$
\begin{aligned}
\mu &= \left[(z+x)^{-1}\right]\frac{(z+1)\cdots(z+n)}{z(z-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r \\
&= (-1)^{n+1}\left(\prod_{k=1}^{n}\frac{1}{x+k}\right)\left(\prod_{k=1}^{n}(k-x)\right)x^{r-1}\left[z^{r-1}\right]\exp\left(\sum_{k\geq 1}U_k\frac{z^k}{k}\right) \\
&= (-1)^{n+1}\left(\prod_{k=1}^{n}\frac{1}{x+k}\right)\left(\prod_{k=1}^{n}(k-x)\right)x^{r-1} \\
&\quad \times \sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{1}{m_1!m_2!m_3!\cdots}\left(\frac{U_1}{1}\right)^{m_1}\left(\frac{U_2}{2}\right)^{m_2}\left(\frac{U_3}{3}\right)^{m_3}\cdots.
\end{aligned}
$$

This completes the proof. □

We next give an extension of Mortenson's identity (5) by Theorem 1.

Theorem 2 *For $n, r, M \in \mathbb{N}$ and $x \geq -j$, $j = 1, 2, \ldots, M$, we have*

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(H_{M+k}^{(r)}(x+1)-H_k^{(r)}(x+1)\right) \\
&\quad = \sum_{j=1}^{M}\left(\prod_{k=1}^{n+1}\frac{1}{x+k+j}\right)\left(\prod_{k=0}^{n-1}(k-j-x)\right) \\
&\qquad \times \sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{1}{m_1!m_2!m_3!\cdots}\left(\frac{U_1}{1}\right)^{m_1}\left(\frac{U_2}{2}\right)^{m_2}\left(\frac{U_3}{3}\right)^{m_3}\cdots,
\end{aligned}
\tag{8}
$$

where $U_k = (-1)^{k-1}H_n^{(k)}(-x-j-1) + H_{n+1}^{(k)}(x+j)$.

Proof Letting $x \longmapsto x+1$ in (6) and then letting $x \longmapsto x+j+1$, we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(H_{M+k}^{(r)}(x+1)-H_k^{(r)}(x+1)\right) \\
&\quad = \sum_{j=1}^{M}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{1}{(x+k+j+1)^r} \\
&\quad = \sum_{j=1}^{M}\left(\prod_{k=1}^{n+1}\frac{1}{x+k+j}\right)\left(\prod_{k=0}^{n-1}(k-j-x)\right) \\
&\qquad \times \sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{1}{m_1!m_2!m_3!\cdots}\left(\frac{U_1}{1}\right)^{m_1}\left(\frac{U_2}{2}\right)^{m_2}\left(\frac{U_3}{3}\right)^{m_3}\cdots.
\end{aligned}
$$

The proof is complete. □

Taking $x = -1$ in (8), we easily obtain the following corollary.

Corollary 3 *Let $n, r, M \in \mathbb{N}$.*

When $1 \le M \le n$, we have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\left(H_{M+k}^{(r)} - H_k^{(r)}\right) = 0, \tag{9}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\left(H_{n+k}^{(r)} + H_{M+k}^{(r)} - 2H_k^{(r)}\right) = 0. \tag{10}$$

When $M > n$, we have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\left(H_{M+k}^{(r)} - H_k^{(r)}\right) \\ = \sum_{j=n+1}^{M}\left(\prod_{k=1}^{n+1}\frac{1}{k+j-1}\right)\left(\prod_{k=0}^{n-1}(k-j+1)\right) \\ \times \sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{1}{m_1!m_2!m_3!\cdots}\left(\frac{V_1}{1}\right)^{m_1}\left(\frac{V_2}{2}\right)^{m_2}\left(\frac{V_3}{3}\right)^{m_3}\cdots,\end{aligned} \tag{11}$$

where $V_k = (-1)^{k-1}H_n^{(k)}(-j) + H_{n+1}^{(k)}(j-1)$.

3 Some applications and two open problems

In this section, we deduce several combinatorial identities involving the harmonic numbers from Theorems 1 and 2. We also suggest two open problems on Mortenson's identities.

Setting $r = 0, 1$ in (6), we obtain the familiar formulas

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k} = (-1)^n,$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\frac{x}{x+k} = \prod_{k=1}^{n}\frac{k-x}{k+x},$$

respectively. Setting $r = 1$ in Corollary 3, we obtain the following combinatorial identities involving the harmonic numbers:

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}(H_{M+k} - H_k) = 0, \quad 1 \le M \le n,$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}(H_{n+k} + H_{M+k} - 2H_k) = 0, \quad 1 \le M \le n,$$

$$\sum_{k=0}^{n}(-1)^{n-k} \binom{n}{k}\binom{n+k}{k}(H_{M+k} - H_k) = \sum_{j=n+1}^{M}\frac{\binom{j-1}{n}}{j\binom{n+j}{j}}, \quad M > n.$$

Finally, we propose the following two open problems.

Open Problem 1 For $m > n$, how do we calculate the combinatorial sums

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+k}{k} \quad \text{and} \quad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+k}{k}\frac{x}{x+k}?$$

Open Problem 2 For $n \in \mathbb{N}$, $m, r, M \in \mathbb{N}_0$, $x > 0$, what are the combinatorial sums

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+k}{k}\left(\frac{x}{x+k}\right)^r \quad \text{and} \quad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+k}{k}\left(H_{M+k}^{(r)} - H_k^{(r)}\right)?$$

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Authors' contributions
There was an equal amount of contributions from two authors. The authors read and approved the final manuscript.

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