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A new generalization of Mittag-Leffler function via q-calculus

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Abstract
The present paper deals with a new different generalization of the Mittag-Leffler function through q-calculus. We then investigate its remarkable properties like convergence, recurrence relation, integral representation, q-derivative formula, q-Laplace transformation, and image formula under q-derivative operator. In addition to this, we consider some specific cases to give the utilization of our main results.

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1 Introduction

The Swedish mathematician Gösta Mittag-Leffler discovered a special function in 1903 (see [12, 13]) defined as

$$E_{\eta}(u)=\sum_{m=0}^{\infty}\frac{u^m}{\Gamma(\eta m+1)m!},\quad \bigl(\eta,u\in\mathbb{C};\Re(\eta)>0\bigr), \tag{1.1}$$

where $\Gamma(\cdot)$ is a classical gamma function [17]. The special function defined in (1.1) is called the Mittag-Leffler function.

For the very first time, in 1905, Wiman [21] firstly proposed the generalization of the Mittag-Leffler $E_{\eta,\kappa}(u)$ as follows:

$$E_{\eta,\kappa}(u)=\sum_{m=0}^{\infty}\frac{u^m}{\Gamma(\eta m+\kappa)m!},\quad \bigl(\eta,\kappa\in\mathbb{C};\Re(\eta)>0,\Re(\kappa)>0\bigr). \tag{1.2}$$

Subsequently, the generalized form of series (1.1) and (1.2) was studied by Prabhakar [16] in 1971:

$$E^{\sigma}_{\eta,\kappa}(u)=\sum_{m=0}^{\infty}\frac{u^m(\sigma)_m}{\Gamma(\eta m+\kappa)m!},\quad \bigl(\eta,\kappa,\sigma\in\mathbb{C};\Re(\eta)>0,\Re(\kappa)>0,\Re(\sigma)>0\bigr), \tag{1.3}$$

where $(\sigma)_m=\frac{\Gamma(\sigma+m)}{\Gamma(\sigma)}$ denotes the Pochhammer symbol [17].

The Mittag-Leffler function plays a vital role in the solution of fractional order differential and integral equations. It has recently become a subject of rich interest in the field of fractional calculus and its applications. Nowadays some mathematicians consider the classical Mittag-Leffler function as the *queen function* in fractional calculus. An enormous amount of research in the theory of Mittag-Leffler functions has been published in the literature. For a detailed account of the various generalizations, properties, and applications of the Mittag-Leffler function, readers may refer to the literature (see [3, 8–10, 14, 15, 18, 20]).

The q-calculus is the q-extension of the ordinary calculus. The theory of q-calculus operators has been recently applied in the areas of ordinary fractional calculus, optimal control problem, in finding solutions of the q-difference and q-integral equations, and q-transform analysis.

In 2009, Mansoor [11] proposed a new form of q-analogue of the Mittag-Leffler function given as

$$e_{\eta,\kappa}(u;q)=\sum_{m=0}^{\infty}\frac{u^m}{\Gamma_q(\eta m+\kappa)},\quad \bigl(|u|<(1-q)^{-\eta}\bigr), \tag{1.4}$$

where $\eta>0$, $\kappa\in\mathbb{C}$.

For other analogues of the Mittag-Leffler functions on the quantum time scale by means of the linear Caputo q-fractional initial value problems and of better imitation to the theory of time scales, we refer the reader to Definition 10 and Remark 11 in [1]. For the Kilbas–Saigo q-analogue of the Mittag-Leffler function, we refer to [2].

Recently, Sharma and Jain [19] introduced the following q-analogue of the generalized Mittag-Leffler function:

$$E^{\sigma}_{\eta,\kappa}(u;q)=\sum_{m=0}^{\infty}\frac{(q^{\sigma};q)_m}{(q;q)_m}\frac{u^m}{\Gamma_q(\eta m+\kappa)}, \tag{1.5}$$

$$\bigl(\eta,\kappa,\sigma\in\mathbb{C};\Re(\eta)>0,\Re(\kappa)>0,\Re(\sigma)>0,|q|<1\bigr).$$

2 Prelude

In the theory of q-series (see [6]), for complex λ and $0<q<1$, the q-shifted factorial is defined as follows:

$$(\lambda;q)_m=\begin{cases}1; & m=0,\\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{m-1}); & m\in\mathbb{N},\end{cases} \tag{2.1}$$

which is equivalent to

$$(\lambda;q)_m=\frac{(\lambda;q)_\infty}{(\lambda q^m;q)_\infty} \tag{2.2}$$

and its extension naturally is

$$(\lambda;q)_\eta=\frac{(\lambda;q)_\infty}{(\lambda q^\eta;q)_\infty},\quad \eta\in\mathbb{C}, \tag{2.3}$$

where the principal value of q^η is taken.

For $s, t \in \mathbb{R}$, the q-analogue of the exponent $(s-t)^m$ is

$$(s-t)^{(m)} = \begin{cases} 1; & m = 0, \\ \prod_{i=0}^{m-1}(s - tq^i); & m \neq 0 \end{cases} \tag{2.4}$$

and connected by the following relationship:

$$(s-t)^{(m)} = s^m (t/s;q)_m \quad (s \neq 0).$$

Obviously, its expansion for $\tau \in \mathbb{R}$ is as follows:

$$(s-t)^{(m)} = s^m \frac{(t/s;q)_\infty}{(q^m t/s;q)_\infty}, \qquad (s;q)_\tau = \frac{(s;q)_\infty}{(sq^\tau;q)_\infty}. \tag{2.5}$$

Note that

$$(s-t)^{(\tau)} = s^\tau (t/s;q)_\tau.$$

The q-analogue of binomial coefficient is defined for $s, t > 0$ as

$$\binom{s}{t}_q = \frac{[s]_q!}{[t]_q![s-t]_q!} = \frac{(q;q)_s}{(q;q)_t (q;q)_{s-t}} = \binom{s}{s-t}_q. \tag{2.6}$$

The definition can be generalized in the following way. For arbitrary complex τ, we have

$$\binom{\tau}{m}_q = \frac{(q^{-\tau};q)_m}{(q;q)_m}(-1)^m q^{\tau m - \binom{m}{2}} = \frac{\Gamma_q(\tau+1)}{\Gamma_q(m+1)\Gamma_q(\tau - m + 1)}, \tag{2.7}$$

where $\Gamma_q(u)$ is the q-gamma function.

The q-gamma and q-beta functions [6] are defined by

$$\Gamma_q(u) = \frac{(q;q)_\infty}{(q^u;q)_\infty}(1-q)^{1-u} \tag{2.8}$$

for $u \in \mathbb{R} \setminus \{0, -1, -2, -3, \ldots\}$; $|q| < 1$.

Clearly,

$$\Gamma_q(u+1) = [u]_q \Gamma_q(u) \tag{2.9}$$

and

$$B_q(\eta,\kappa) = \frac{\Gamma_q(\eta)\Gamma_q(\kappa)}{\Gamma_q(\eta+\kappa)} = \int_0^1 u^{\eta-1} \frac{(qu;q)_\infty}{(q^\kappa u;q)_\infty} \, d_q u = \int_0^1 u^{\eta-1}(uq;q)_{\kappa-1} \, d_q u, \tag{2.10}$$

$$\bigl(\Re(\eta), \Re(\kappa) > 0\bigr).$$

Also, the q-difference operator and q-integration of a function $f(u)$ defined on a subset of $\mathbb{C}$ are given by [6] respectively:

$$D_q f(u) = \frac{f(u) - f(uq)}{u(1-q)} \quad (u \neq 0, q \neq 1), (D_q f)(0) = \lim_{u \to 0} (D_q f)(u) \tag{2.11}$$

and

$$\int_0^u f(t)\,d(t;q) = u(1-q)\sum_{m=0}^{\infty} q^m f\bigl(uq^m\bigr). \tag{2.12}$$

3 Generalized q-Mittag-Leffler function and its properties

In this section, we generalize definition (1.5) by introducing the following relation for $(q^c,q)_m$:

$$\frac{(q^c;q)_m}{(q^\sigma;q)_m} = \frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}. \tag{3.1}$$

Now, we define the generalization of Mittag-Leffler function (1.5) using the above relation as follows:

$$E_{\eta,\kappa}^{(\sigma;c)}(u;q) = \sum_{m=0}^{\infty} \frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)} \frac{(q^c;q)_m}{(q;q)_m} \frac{u^m}{\Gamma_q(\eta m+\kappa)} \tag{3.2}$$
$$\bigl(\Re(c) > \Re(\sigma) > 0, |q| < 1\bigr),$$

where $B_q(\cdot)$ is the q-analogue of beta function.

We enumerate the relations as particular cases of q-analogue of the generalized Mittag-Leffler function with other special functions as given below.

(i) On setting $c = 1$ in (3.2), we obtain

$$E_{\eta,\kappa}^{(\sigma;1)}(u;q) = \sum_{m=0}^{\infty} \frac{(q^\sigma;q)_m}{(q;q)_m} \frac{u^m}{\Gamma_q(\eta m+\kappa)} = E_{\eta,\kappa}^{\sigma}(u;q), \tag{3.3}$$

which is given by equation (1.5).

(ii) Again, on setting $\sigma = 1$ in (3.2), we obtain

$$E_{\eta,\kappa}^{(1;c)}(u;q) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma_q(\eta m+\kappa)} = e_{\eta,\kappa}(u;q), \tag{3.4}$$

the function $e_{\eta,\kappa}(u;q)$ can be termed as q-analogue of the Mittag-Leffler function defined in (1.4).

(iii) On setting $\eta = \kappa = \sigma = 1$, in (3.2), we obtain

$$E_{1,1}^{(1;c)}(u;q) = \sum_{m=0}^{\infty} \frac{(q^c;q)_m}{(q;q)_m} u^m = \frac{(q^c u;q)_\infty}{(q;q)_\infty} = {}_1\phi_0\bigl(q^c;-;q,u\bigr), \tag{3.5}$$

where the function ${}_1\phi_0(q^c;-;q,u) = (1-u)^{-c}$ can be termed as q-binomial function.

(iv) On setting $c = c+\sigma$, in (3.2), we obtain q-analogue of the Mittag-Leffler function $E_{\eta,\kappa}^{\sigma}(u;q)$ defined in (1.5).

4 Convergence of $E_{\eta,\kappa}^{(\sigma;c)}(u;q)$

Theorem 4.1 *The q-analogue of the generalized Mittag-Leffler function defined by the summation formula* (3.2) *converges absolutely for* $|u| < (1-q)^{-\eta}$ *provided that* $0 < q < 1$, $\eta > 0$, $\Re(c) > \Re(\sigma)$, $c,\sigma \in \mathbb{C}$.

Proof Writing the summation formula (3.2) as $E_{\eta,\kappa}^{(\sigma;c)}(u;q)=\sum_{m=0}^{\infty}s_m$ and by applying the ratio formula, we find

$$\begin{aligned}
\lim_{m\to\infty}\left|\frac{s_{m+1}}{s_m}\right| &= \left|\frac{B_q(\sigma+m+1,c-\sigma)}{B_q(\sigma+m,c-\sigma)}\right|\left|\frac{(q^c,q)_{m+1}}{(q^c,q)_m}\right|\left|\frac{(q,q)_m}{(q,q)_{m+1}}\right|\left|\frac{\Gamma(\eta m+\kappa)}{\Gamma(\eta m+\eta+\kappa)}u\right| \\
&= \lim_{m\to\infty}\left|\frac{(q^{c+m},q)_\infty}{(q^{c+m+1},q)_\infty}\frac{(q^{\sigma+m},q)_\infty}{(q^{\sigma+m+1},q)_\infty}\frac{(q^{\eta m+\kappa},q)_\infty}{(q^{\eta m+\kappa+\eta},q)_\infty}\frac{(q^{m+1},q)_\infty}{(q^m,q)_\infty}(1-q)^{-\eta}u\right| \\
&= \lim_{m\to\infty}\left|\left(1-q^{c+m}\right)\left(1-q^{\sigma+m}\right)\left(1-q^{\eta m+\kappa}\right)^{\eta}\frac{(1-q)^{-\eta}}{(1-q^m)}u\right| \\
&= \left|(1-q)^{-\eta}\right||u| \quad \text{for } 0<|q|<1.
\end{aligned} \tag{4.1}$$

□

5 Recurrence relations

Theorem 5.1 *If* $\eta,\kappa,\sigma\in\mathbb{C}$, $\Re(\eta)>0$, $\Re(\kappa)>0$, $\Re(\sigma)>0$, *and* $\sigma\neq c$, *then*

$$E_{\eta,\kappa}^{(\sigma;c)}(u;q)=E_{\eta,\kappa}^{(\sigma+1;c+1)}(u;q)-uq^cE_{\eta,\eta+\kappa}^{(\sigma+1;c+1)}(u;q).$$

Proof Using definition (3.2), we obtain

$$\begin{aligned}
E_{\eta,\kappa}^{(\sigma;c)}(u;q) &= \sum_{m=0}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(q^c;q)_m}{(q;q)_m}\frac{u^m}{\Gamma_q(\eta m+\kappa)}, \\
&= \frac{1}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(1-q^c)(q^{c+1};q)_{m-1}}{(q;q)_m}\frac{u^m}{\Gamma_q(\eta m+\kappa)}.
\end{aligned}$$

Since $(1-q^c)=(1-q^{c+m})-q^c(1-q^m)$, the above equation reduces to

$$\begin{aligned}
E_{\eta,\kappa}^{(\sigma;c)}(u;q) &= \frac{1}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(1-q^{c+m})(q^{c+1};q)_{m-1}}{(q;q)_m}\frac{u^m}{\Gamma_q(\eta m+\kappa)} \\
&\quad -q^c\sum_{m=1}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(1-q^m)(q^{c+1};q)_{m-1}}{(q;q)_m}\frac{u^m}{\Gamma_q(\eta m+\kappa)}.
\end{aligned}$$

On replacing m with $m+1$ in the second summation, it becomes

$$\begin{aligned}
E_{\eta,\kappa}^{(\sigma;c)}(u;q) &= \frac{1}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(q^{c+1};q)_m}{(q;q)_m}\frac{u^m}{\Gamma_q(\eta m+\kappa)} \\
&\quad -q^c\sum_{m=1}^{\infty}\frac{B_q(\sigma+m+1,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(q^{c+1};q)_m}{(q;q)_m}\frac{u^{m+1}}{\Gamma_q(\eta m+\eta+\kappa)},
\end{aligned}$$

which leads to the required result (5.1). □

6 Some elementary properties of the generalized q-Mittag-Leffler function

We begin with the following theorem, which shows the integral representation of the generalized q-Mittag-Leffler function.

Theorem 6.1 (Integral representation) *For the generalized q-Mittag-Leffler function, we have*

$$E_{\eta,\kappa}^{(\sigma;c)}(u;q) = \frac{1}{B_q(\sigma,c-\sigma)} \int_0^1 t^{\sigma-1} \frac{(tq;q)_\infty}{(tq^{c-\sigma};q)_\infty} E_{\eta,\kappa}^{(c)}(tu;q)\, d_q t, \tag{6.1}$$

provided that $\eta,\kappa,\sigma \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\kappa) > 0$, $\Re(\sigma) > 0$, *and* $\sigma \neq c$.

Proof By the definition of q-analogue of beta function, we can rewrite equation (3.2) as follows:

$$\begin{aligned} E_{\eta,\kappa}^{(\sigma;c)}(u;q) &= \sum_{m=0}^{\infty} \left\{ \int_0^1 t^{\sigma+m-1} \frac{(tq;q)_\infty}{(tq^{c-\sigma};q)_\infty}\, d_q t \right\} \frac{1}{B_q(\sigma,c-\sigma)} \\ &\quad \times \frac{(q^c;q)_m}{\Gamma_q(\eta m+\kappa)} \frac{u^m}{(q;q)_m} \\ &= \frac{1}{B_q(\sigma,c-\sigma)} \sum_{m=0}^{\infty} \left\{ \int_0^1 t^{\sigma-1} \frac{(tq;q)_\infty}{(tq^{c-\sigma};q)_\infty}\, d_q t \left(\frac{(q^c;q)_m}{(q;q)_m} \frac{tu^m}{\Gamma_q(\eta m+\kappa)} \right) \right\}, \end{aligned}$$

which leads to the required result (6.1). □

Theorem 6.2 *For* $\eta,\kappa,\sigma \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\kappa) > 0$, $\Re(\sigma) > 0$, $c \neq \sigma$, *then for any* $m \in \mathbb{N}$, *we have*

$$D_q^m \left[u^{\kappa-1} E_{\eta,\kappa}^{(\sigma;c)}(\lambda u^\eta;q) \right] = u^{\kappa-m-1} E_{\eta,\kappa-m}^{(\sigma;c)}(\lambda u^\eta;q). \tag{6.2}$$

Proof By considering the function

$$f(u) = u^{\kappa-1} E_{\eta,\kappa}^{(\sigma;c)}(\lambda u^\eta;q).$$

In view of (2.11) and using definition (3.2), we obtain

$$\begin{aligned} D_q\left[u^{\kappa-1} E_{\eta,\kappa}^{(\sigma;c)}(\lambda u^\eta) \right] = \sum_{m=0}^{\infty} & \frac{B_q(\sigma+m+1,c-\sigma)}{B_q(\sigma,c-\sigma)} \frac{(q^c;q)_m}{(q;q)_m} \\ & \times \frac{\lambda^m (1-q^{\eta m+\kappa-1})}{1-q} \frac{u^{\eta m+\kappa-2}}{\Gamma_q(\eta m+\kappa)}. \end{aligned}$$

Since, according to the functional equation (2.9), the right-hand side of the above expression can be written as

$$\sum_{m=0}^{\infty} \frac{B_q(\sigma+m+1,c-\sigma)}{B_q(\sigma,c-\sigma)} \frac{(q^c;q)_m}{(q;q)_m} \frac{\lambda^m u^{\eta m+\kappa-2}}{\Gamma_q(\eta m+\kappa-1)} = u^{\kappa-2} E_{\eta,\kappa-1}^{(\sigma;c)}(\lambda u^\eta;q).$$

Conclusively, we obtain

$$D_q\left[u^{\kappa-1} E_{\eta,\kappa}^{(\sigma;c)}(\lambda u^\eta;q) \right] = u^{\kappa-2} E_{\eta,\kappa-1}^{(\sigma;c)}(\lambda u^\eta;q).$$

Iterating the above result $m-1$ times, we obtain the required result (6.2). □

Theorem 6.3 *Let* $\xi,\zeta,\sigma,\kappa\in\mathbb{C}$; $\Re(\xi),\Re(\kappa),\Re(\sigma)>0$; $\zeta\neq 0,-1,-2,\dots$, *then*

$$\begin{aligned}&\int_0^1 u^{\xi-1}(1-qu)_{(\zeta-1)}E_{\eta,\kappa}^{(\sigma;c)}\bigl(xu^{\rho};q\bigr)\,d_qu\\&\quad=\sum_{m=0}^{\infty}\frac{B_q(\sigma+m,c-\sigma)(q^c;q)_m}{B_q(\sigma,c-\sigma)(q;q)_m}\frac{x^m\Gamma_q(\xi+\rho m)\Gamma_q(\xi)}{\Gamma_q(\eta m+\kappa)\Gamma_q(\xi+\zeta+\rho m)}.\end{aligned}\tag{6.3}$$

In particular,

$$\int_0^1 u^{\xi-1}(1-qu)_{(\zeta-1)}E_{\eta,\kappa}^{(\sigma;c)}\bigl(xu^{\rho};q\bigr)\,d_qu=\Gamma_q(\zeta)E_{\eta,\kappa+\zeta}^{(\sigma;c)}(x;q).\tag{6.4}$$

Proof By using definition (3.2), the left-hand side of equation (6.3) can be written as

$$\int_0^1 u^{\xi-1}(1-qu)_{(\zeta-1)}\sum_{m=0}^{\infty}\frac{B_q(\sigma+m,c-\sigma)(q^c;q)_m}{B_q(\sigma,c-\sigma)(q;q)_m}\frac{u^{\rho m}x^m}{\Gamma_q(\eta m+\kappa)}\,d_qu.$$

Interchanging the order of summation and integration and in view of equation (2.10), we obtain the required result (6.3).

In equation (6.3) replacing $\eta=\rho$, $\xi=\kappa$, then in view of equation (3.2), we can clearly obtain (6.4). □

Theorem 6.4 (q-Laplace transform) *The q-analogue of the generalized Laplace transform is defined as follows*:

$$\begin{aligned}{}_qL_s\bigl[E_{\eta,\kappa}^{(\sigma;c)}\bigl(xu^{\rho};q\bigr)\bigr]&=\frac{1}{s}\sum_{m=0}^{\infty}\frac{B_q(\sigma+m,c-\sigma)(q^c;q)_m}{B_q(\sigma,c-\sigma)(q;q)_m}\frac{\Gamma_q(1+\rho m)}{\Gamma_q(\eta m+\kappa)}\\&\quad\times\left(\frac{(1-q)^{\rho}x}{s^{\rho}}\right)^m\end{aligned}\tag{6.5}$$

provided that $\kappa,\sigma,s\in\mathbb{C}$; $\Re(\beta),\Re(\kappa),\Re(s)>0$.

Proof The q-Laplace transform of a suitable function is given by means of the following q-integral [7]:

$${}_qL_s\bigl\{f(u)\bigr\}=\frac{1}{(1-q)}\int_0^{s^{-1}}E_q^{qsu}f(u)\,d_qu.\tag{6.6}$$

The q-extension of the exponential function [6] is given by

$$E_q^u={}_0\phi_0(-,-;q,-u)=\sum_{m=0}^{\infty}\frac{q^{\binom{m}{2}}u^m}{(q;q)_m}=(-u;q)_{\infty}\tag{6.7}$$

and

$$e_q^u={}_1\phi_0(0,-;q,-u)=\sum_{m=0}^{\infty}\frac{u^m}{(q;q)_m}=\frac{1}{(u;q)_{\infty}},\quad |u|<1.\tag{6.8}$$

By using the above q-exponential series and the q-integral equation (2.12), we can write equation (6.6) as

$$ {}_qL_s\{f(u)\} = \frac{(q;q)_\infty}{s}\sum_{j=0}^{\infty}\frac{q^j f(s^{-1}q^j)}{(q;q)_j}. \tag{6.9}$$

Using definition (3.2) and the definition of q-Laplace transform, we obtain

$$\begin{aligned} {}_qL_s\bigl[E_{\eta,\kappa}^{(\sigma;c)}(xu^\rho;q)\bigr] &= \frac{(q;q)_\infty}{s}\sum_{j=0}^{\infty}\frac{q^j}{(q;q)_j} \\ &\quad\times\sum_{m=0}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(q^\sigma;q)_m}{(q;q)_m}\frac{[u(s^{-1}q^j)^\sigma]^m}{\Gamma_q(\eta m+\kappa)}. \end{aligned}$$

On interchanging the order of summation and writing the j series as ${}_1\phi_0$, which can be summed up as $\frac{1}{(q^{1+\rho m};q)_\infty}$, and after some simplifications, we obtain the required result (6.5). □

7 Kober-type fractional q-calculus operators

Agarwal [4] established Kober-type fractional q-integral operator in the following manner:

$$\bigl(I_q^{\nu,\mu}f\bigr)(u) = \frac{u^{-\nu-\mu}}{\Gamma_q(u)}\int_0^u (u-tq)_{\mu-1}t^\nu f(t)\,d_qt, \tag{7.1}$$

where $\Re(\mu) > 0$. Also, Garg *et al.* [5] introduced Kober fractional q-derivative operator given by

$$\bigl(D_q^{\nu,\mu}f\bigr)(u) = \prod_{i=0}^{m}\bigl([\nu+j]_q + uq^{\nu+j}D_q\bigr)\bigl(I_q^{\nu+\mu,m-\mu}f\bigr)(u), \tag{7.2}$$

where $m = [\Re(\mu)] + 1$, $m \in \mathbb{N}$.

The image formulas of the power function u^m under the above operators [5] are given as follows:

$$I_q^{\nu,\mu}\bigl\{u^m\bigr\} = \frac{\Gamma_q(\nu+m+1)}{\Gamma_q(\nu+\mu+m+1)}u^m, \tag{7.3}$$

$$D_q^{\nu,\mu}\bigl\{u^m\bigr\} = \frac{\Gamma_q(\nu+\mu+m+1)}{\Gamma_q(\nu+m+1)}u^m. \tag{7.4}$$

Theorem 7.1 *The following assumption holds true*:

$$\begin{aligned} I_q^{\nu,\mu}\bigl\{E_{\eta,\kappa}^{(\sigma;c)}(u;q)\bigr\} &= \sum_{m=0}^{\infty}\frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)}\frac{(q^c;q)_m}{(q;q)_m} \\ &\quad\times\frac{\Gamma_q(\nu+m+1)}{\Gamma_q(\nu+\mu+m+1)}\frac{u^m}{\Gamma_q(\eta m+\kappa)}, \end{aligned} \tag{7.5}$$

particularly,

$$I_q^{\nu,\mu}E_{\eta,\kappa}^{(\nu+\mu;1)}(u;q) = \frac{\Gamma_q(\nu+1)}{\Gamma_q(\nu+\mu+1)}E_{\eta,\kappa}^{(\nu+1;1)}(u;q), \tag{7.6}$$

provided that if $\eta, c > 0$, $\kappa,\sigma,u \in \mathbb{C}$; $\Re(\kappa),\Re(\sigma) > 0$.

Proof The proof of (7.5) can easily be obtained by making use of definition (3.2) and result (7.3).

Now, on setting $\sigma = \nu + \mu$ in definition (3.2), we obtain result (7.6). □

Theorem 7.2 *The following assumption holds true*:

$$D_q^{\nu,\mu}\left\{E_{\eta,\kappa}^{(\sigma;c)}(u;q)\right\} = \sum_{m=0}^{\infty} \frac{B_q(\sigma+m,c-\sigma)}{B_q(\sigma,c-\sigma)} \frac{(q^c;q)_m}{(q;q)_m} \times \frac{\Gamma_q(\nu+\mu+m+1)}{\Gamma_q(\nu+m+1)} \frac{u^m}{\Gamma_q(\eta m+\kappa)}, \tag{7.7}$$

particularly,

$$D_q^{\nu,\mu} E_{\eta,\kappa}^{(\nu+1;1)}(u;q) = \frac{\Gamma_q(\nu+\mu+1)}{\Gamma_q(\nu+1)} E_{\eta,\kappa}^{\nu+\mu}(u;q) \tag{7.8}$$

provided that if $\eta, c > 0$, $\kappa, \sigma, u \in \mathbb{C}$; $\Re(\kappa), \Re(\sigma) > 0$.

Proof The proof of (7.7) can easily be obtained by making use of definition (3.2) and result (7.4). Similarly, on setting $\sigma = \nu + 1$ in definition (3.2), we obtain result (7.8). □

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