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Approximation by bivariate generalized Bernstein–Schurer operators and associated GBS operators

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Abstract

We construct the bivariate form of Bernstein–Schurer operators based on parameter α. We establish the Voronovskaja-type theorem and give an estimate of the order of approximation with the help of Peetre's K-functional of our newly defined operators. Moreover, we define the associated generalized Boolean sum (shortly, GBS) operators and estimate the rate of convergence by means of mixed modulus of smoothness. Finally, the order of approximation for Bögel differentiable function of our GBS operators is presented.

MSC: 41A10; 41A25; 41A36

Keywords: α-Bernstein operators; Bivariate α-Bernstein–Schurer operators; GBS operators; Bögel differentiable function; Modulus of continuity

1 Introduction

Schurer [41] presented the modification of the classical Bernstein operators with the help of nonnegative parameter and nowadays called Bernstein–Schurer operators, which are linear and positive. Suppose that $\mathbb{Z}_0^+$ and $C[a,b]$ are used to denote the space of nonnegative integers and continuous functions on $[a,b]$, respectively. Let us take $\eta \in \mathbb{Z}_0^+$. The well-known Bernstein–Schurer operators

$$M_{j,\eta} : C[0,1+\eta] \longrightarrow C[0,1]$$

are defined as

$$M_{j,\eta}(g;y) = \sum_{k=0}^{j+\eta} g\left(\frac{k}{j}\right) M_{j,\eta,k}(y) \tag{1.1}$$

for any $g \in [0,1+\eta]$, $j \in \mathbb{N}$, and $y \in [0,1]$, where

$$M_{j,\eta,k}(y) = \binom{j+\eta}{k} y^k (1-y)^{j+\eta-k}.$$

When $\eta = 0$ in (1.1), we obtain

$$M_{j,0}(g;y) = \sum_{k=0}^{j} g\left(\frac{k}{j}\right) M_{j,0,k}(y) = M_j(g;y), \quad \text{say}. \tag{1.2}$$

In this case, the operators $M_j(g;y)$ and $M_{j,0,k}(y)$, respectively, are called Bernstein operators and polynomials [11].

Most recently, the generalization of Bernstein operators was demonstrated by the authors Chen et al. [17] by taking the parameter $\alpha \in \mathbb{R}$. However, they showed that their operators are positive and linear for the choice of $0 \leq \alpha \leq 1$, so they considered this assumption in their work and then studied several approximation properties for their α-Bernstein operators. Thereafter, many researchers, keeping the idea of this meaningful parameter α into account, constructed several operators. For example, Mohiuddine et al. [26] introduced the family of α-Bernstein–Kantorovich operators and associated bivariate form and demonstrated the results regarding the rate of convergence via Peetre's K-functional together with modulus of continuity. In addition to this, the Stancu type α-Bernstein–Kantorovich, α-Baskakov and their Kantorovich form, and α-Baskakov–Durrmeyer operators were analyzed by Mohiuddine and Özger [32], Aral et al. [6, 20], and Mohiuddine et al. [31], respectively, and for other blending type operators, see [23, 27, 36]. Some other modifications of Bernstein operators have been studied in [2, 15, 16, 25, 33, 34, 37, 42]. Furthermore, Acar and Kajla [3] gave the bivariate α-Bernstein operators and associated generalized Boolean sum operators and then studied the degree of approximation of their operators. For more details on approximation by related operators and statistical approximation, we refer to [1, 4, 10, 14, 21, 28–30, 35, 40, 44, 45].

Motivated by the operators defined in (1.1) and α-Bernstein operators, very recently, Ozger et al. [38] defined the α-Bernstein–Schurer operators $M^{\alpha}_{j,\eta} : C[0,1+\eta] \longrightarrow C[0,1]$ by

$$M^{\alpha}_{j,\eta}(g;y) = \sum_{k=0}^{j+\eta} g_k M^{(\alpha)}_{j,\eta,k}(y) \tag{1.3}$$

and

$$g_k = g\left(\frac{k}{j}\right)$$

for any $g \in C[0,1+\eta]$, $y \in [0,1]$, $j \in \mathbb{N}$, and $0 \leq \alpha \leq 1$, where

$$M^{(\alpha)}_{1,\eta,0}(y) = 1 - y, \qquad M^{(\alpha)}_{1,\eta,1}(y) = y$$

and

$$\begin{aligned} M^{(\alpha)}_{j,\eta,k}(y) = \Bigg[& y(1-\alpha)\binom{j+\eta-2}{k} + (1-y)(1-\alpha)\binom{j+\eta-2}{k-2} \\ & + y\alpha(1-y)\binom{j+\eta}{k}\Bigg] y^{k-1}(1-y)^{j+\eta-k-1} \end{aligned}$$

for $j \geq 2$. Note that $M^{\alpha}_{j,\eta}(t-y;y) = \eta y/j$. When $\eta = 0$, the operators $M^{\alpha}_{j,\eta}(g;y)$ coincide with α-Bernstein operators. In addition to $\eta = 0$, take $\alpha = 1$, then $M^{\alpha}_{j,\eta}(g;y)$ reduces to $M_j(g;y)$. When only $\alpha = 1$, the operators $M^{\alpha}_{j,\eta}(g;y)$ reduce to the operators $M_{j,\eta}(g;y)$. In the same paper, authors investigated global approximation, local approximation, and Voronovskaja-type approximation results of the operators $M^{\alpha}_{j,\eta}(g;y)$. They also established shape preserving properties such as monotonicity and convexity.

2 Bivariate generalized Bernstein–Schurer operators

Here, we construct the bivariate form of α-Bernstein–Schurer operators and demonstrate their basic properties.

Throughout the manuscript, we suppose that $C(I^2)$ is the space of continuous function on $I^2(= I \times I) = [0,1+\eta] \times [0,1+\eta]$, where η in $\mathbb{Z}^+_0$. For any $h \in C(I^2)$, $(y_1,y_2) \in [0,1] \times [0,1]$, $s_1, s_2 \in \mathbb{N}$, and α_1, α_2 in $[0,1]$, we define

$$M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2) = \sum_{\ell_1=0}^{s_1+\eta}\sum_{\ell_2=0}^{s_2+\eta} h\left(\frac{\ell_1}{s_1}, \frac{\ell_2}{s_2}\right) M^{(\alpha_1,\alpha_2)}_{s_1+\eta,s_2+\eta,\ell_1,\ell_2}(y_1,y_2), \tag{2.1}$$

where the polynomials $M^{(\alpha_1,\alpha_2)}_{s_1+\eta,s_2+\eta,\ell_1,\ell_2}(y_1,y_2) = M^{(\alpha_1)}_{s_1+\eta,\ell_1}(y_1) M^{(\alpha_2)}_{s_2+\eta,\ell_2}(y_2)$ are considered by

$$\begin{aligned} M^{(\alpha_1,\alpha_2)}_{s_1+\eta,s_2+\eta,\ell_1,\ell_2}(y_1,y_2) = {}& \left[(1-\alpha_1)y_1\binom{s_1+\eta-2}{\ell_1} + (1-\alpha_1)(1-y_1)\binom{s_1+\eta-2}{\ell_1-2}\right. \\ & \left. + \alpha_1 y_1(1-y_1)\binom{s_1+\eta}{\ell_1}\right] y_1^{\ell_1-1}(1-y_1)^{s_1+\eta-(\ell_1+1)} \\ & \times \left[(1-\alpha_2)y_2\binom{s_2+\eta-2}{\ell_2} + (1-\alpha_2)(1-y_2)\binom{s_2+\eta-2}{\ell_2-2}\right. \\ & \left. + \alpha_2 y_2(1-y_2)\binom{s_2+\eta}{\ell_2}\right] y_2^{\ell_2-1}(1-y_2)^{s_2+\eta-(\ell_2+1)}. \end{aligned}$$

Lemma 2.1 *Suppose that $e_{jk}(y_1,y_2) = y_1^j y_2^k$ for $(j,k) = \mathbb{N}_0 \times \mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) with $j + k \leq 4$. Then*

$$\begin{aligned} & M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{00};y_1,y_2) = 1, \\ & M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{10};y_1,y_2) = \left(1+\frac{\eta}{s_1}\right)y_1, \\ & M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{01};y_1,y_2) = \left(1+\frac{\eta}{s_2}\right)y_2, \\ & M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{20};y_1,y_2) = y_1^2 + \frac{(s_1+\eta+2(1-\alpha_1))(y_1-y_1^2)}{s_1^2} + \frac{\eta(\eta+2s_1)y_1^2}{s_1^2}, \\ & M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{02};y_1,y_2) = y_2^2 + \frac{(s_2+\eta+2(1-\alpha_2))(y_2-y_2^2)}{s_2^2} + \frac{\eta(\eta+2s_2)y_2^2}{s_2^2}, \\ & M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{30};y_1,y_2) \\ & \quad = y_1^3 + \frac{s_1+\eta+6(1-\alpha_1)}{s_1^3}y_1 + \bigl(-6\alpha_1\eta - 6\alpha_1 s_1 + 3\eta^2 + 6\eta s_1 + 3s_1^2 \\ & \qquad + 18\alpha_1 + 3\eta + 3s_1 - 18\bigr)\frac{y_1^2}{s_1^3} + \bigl(\eta^3 + 3\eta^2 s_1 + 3\eta s_1^2 + 6\alpha_1\eta + 6\alpha_1 s_1 \end{aligned}$$

$$- 3\eta^2 - 6\eta s_1 - 3s_1^2 - 12\alpha_1 - 4\eta - 4s_1 + 12\big)\frac{y_1^3}{s_1^3},$$

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{03};y_1,y_2)$$
$$= y_2^3 + \frac{s_2 + \eta + 6(1-\alpha_2)}{s_2^3}y_2 + \big(-6\alpha_2\eta - 6\alpha_2 s_2 + 3\eta^2 + 6\eta s_2 + 3s_2^2$$
$$+ 18\alpha_2 + 3\eta + 3s_2 - 18\big)\frac{y_2^2}{s_2^3} + \big(\eta^3 + 3\eta^2 s_2 + 3\eta s_2^2 + 6\alpha_2\eta + 6\alpha_2 s_2$$
$$- 3\eta^2 - 6\eta s_2 - 3s_2^2 - 12\alpha_2 - 4\eta - 4s_2 + 12\big)\frac{y_2^3}{s_2^3},$$

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{40};y_1,y_2)$$
$$= y_1^4 + \frac{s_1 + \eta + 14(1-\alpha_1)}{s_1^4}y_1 + \big(-36\alpha_1\eta - 36\alpha_1 s_1 + 7\eta^2 + 14\eta s_1$$
$$+ 7s_1^2 + 86\alpha_1 + 29\eta + 29s_1 - 86\big)\frac{y_1^2}{s_1^4} + \big(-12\alpha_1\eta^2 - 24\alpha_1\eta s_1$$
$$- 12\alpha_1 s_1^2 + 6\eta^3 + 18\eta^2 s_1 + 18\eta s_1^2 + 6s_1^3 + 96\alpha_1\eta + 96\alpha_1 s_1 - 6\eta^2$$
$$- 12\eta s_1 - 6s_1^2 - 144\alpha_1 - 84\eta - 84s_1 + 144\big)\frac{y_1^3}{s_1^4} + \big(\eta^4 + 4\eta^3 s_1$$
$$+ 6\eta^2 s_1^2 + 4\eta s_1^3 + 12\alpha_1\eta^2 + 24\alpha_1\eta s_1 + 12\alpha_1 s_1^2 - 6\eta^3 - 18\eta^2 s_1$$
$$- 18\eta s_1^2 - 6s_1^3 - 60\alpha_1\eta - 60\alpha_1 s_1 - \eta^2 - 2\eta s_1 - s_1^2 + 72\alpha_1 + 54\eta$$
$$+ 54s_1 - 72\big)\frac{y_1^4}{s_1^4},$$

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{04};y_1,y_2)$$
$$= y_2^4 + \frac{s_2 + \eta + 14(1-\alpha_2)}{s_2^4}y_2 + \big(-36\alpha_2\eta - 36\alpha_2 s_2 + 7\eta^2 + 14\eta s_2$$
$$+ 7s_2^2 + 86\alpha_2 + 29\eta + 29s_2 - 86\big)\frac{y_2^2}{s_2^4} + \big(-12\alpha_2\eta^2 - 24\alpha_2\eta s_2$$
$$- 12\alpha_2 s_2^2 + 6\eta^3 + 18\eta^2 s_2 + 18\eta s_2^2 + 6s_2^3 + 96\alpha_2\eta + 96\alpha_2 s_2 - 6\eta^2$$
$$- 12\eta s_2 - 6s_2^2 - 144\alpha_2 - 84\eta - 84s_2 + 144\big)\frac{y_2^3}{s_2^4} + \big(\eta^4 + 4\eta^3 s_2$$
$$+ 6\eta^2 s_2^2 + 4\eta s_2^3 + 12\alpha_2\eta^2 + 24\alpha_2\eta s_2 + 12\alpha_2 s_2^2 - 6\eta^3 - 18\eta^2 s_2$$
$$- 18\eta s_2^2 - 6s_2^3 - 60\alpha_2\eta - 60\alpha_2 s_2 - \eta^2 - 2\eta s_2 - s_2^2 + 72\alpha_2 + 54\eta$$
$$+ 54s_2 - 72\big)\frac{y_2^4}{s_2^4}.$$

Proof We shall use Lemma 3 of [38] to prove Lemma 2.1. Clearly, from (2.1), we obtain

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{00};y_1,y_2) = \sum_{\ell_1=0}^{s_1+\eta}\sum_{\ell_2=0}^{s_2+\eta} M_{s_1+\eta,s_2+\eta,\ell_1,\ell_2}^{(\alpha_1,\alpha_2)}(y_1,y_2) = 1.$$

Next, we write

$$\begin{aligned}
M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{10};y_1,y_2) &= \sum_{\ell_1=0}^{s_1+\eta}\sum_{\ell_2=0}^{s_2+\eta}\frac{\ell_1}{s_1}M_{s_1+\eta,s_2+\eta,\ell_1,\ell_2}^{(\alpha_1,\alpha_2)}(y_1,y_2)\\
&= \sum_{\ell_1=0}^{s_1+\eta}\frac{\ell_1}{s_1}M_{s_1+\eta,\ell_1}^{(\alpha_1)}(y_1)\sum_{\ell_2=0}^{s_2+\eta}M_{s_2+\eta,\ell_2}^{(\alpha_2)}(y_2)\\
&= M_{s_1,\eta}^{\alpha}(e_1;y_1)M_{s_2,\eta}^{\alpha}(e_0;y_2)\\
&= \left(1+\frac{\eta}{s_1}\right)y_1.
\end{aligned}$$

Similarly, we get

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{01};y_1,y_2) = M_{s_1,\eta}^{\alpha}(e_0;y_1)M_{s_2,\eta}^{\alpha}(e_1;y_2) = \left(1+\frac{\eta}{s_2}\right)y_2.$$

Further,

$$\begin{aligned}
M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{20};y_1,y_2) &= \sum_{\ell_1=0}^{s_1+\eta}\left(\frac{\ell_1}{s_1}\right)^2 M_{s_1+\eta,\ell_1}^{(\alpha_1)}(y_1)\sum_{\ell_2=0}^{s_2+\eta}M_{s_2+\eta,\ell_2}^{(\alpha_2)}(y_2)\\
&= M_{s_1,\eta}^{\alpha}(e_2;y_1)M_{s_2,\eta}^{\alpha}(e_0;y_2)\\
&= y_1^2+\frac{(s_1+\eta+2(1-\alpha_1))(y_1-y_1^2)}{s_1^2}+\frac{\eta(\eta+2s_1)y_1^2}{s_1^2},
\end{aligned}$$

and similarly, we obtain

$$\begin{aligned}
M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{02};y_1,y_2) &= \sum_{\ell_1=0}^{s_1+\eta}M_{s_1+\eta,\ell_1}^{(\alpha_1)}(y_1)\sum_{\ell_2=0}^{s_2+\eta}\left(\frac{\ell_2}{s_2}\right)^2 M_{s_2+\eta,\ell_2}^{(\alpha_2)}(y_2)\\
&= M_{s_1,\eta}^{\alpha}(e_0;y_1)M_{s_2,\eta}^{\alpha}(e_2;y_2)\\
&= y_2^2+\frac{(s_2+\eta+2(1-\alpha_2))(y_2-y_2^2)}{s_2^2}+\frac{\eta(\eta+2s_2)y_2^2}{s_2^2}.
\end{aligned}$$

Similarly, we obtain the last two moments. □

Corollary 2.2 *The following identities hold*:

$$\begin{aligned}
&M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(t_1-y_1;y_1,y_2) = \frac{\eta y_1}{s_1},\\
&M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(t_2-y_2;y_1,y_2) = \frac{\eta y_2}{s_2},\\
&M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^2;y_1,y_2\big) = \frac{(s_1+\eta+2(1-\alpha_1))(y_1-y_1^2)+\eta^2y_1^2}{s_1^2},\\
&M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_2-y_2)^2;y_1,y_2\big) = \frac{(s_2+\eta+2(1-\alpha_2))(y_2-y_2^2)+\eta^2y_2^2}{s_2^2},\\
&M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^4;y_1,y_2\big)
\end{aligned}$$

$$\begin{aligned}
&= \big(s_1 + \eta + 14(1 - \alpha_1)\big)\frac{y_1}{s_1^4} + \big(-36\alpha_1\eta - 12\alpha_1 s_1 + 7\eta^2 \\
&\quad + 10\eta s_1 + 3s_1^2 + 86\alpha_1 + 29\eta + 5s_1 - 86\big)\frac{y_1^2}{s_1^4} + \big(-12\alpha_1\eta^2 \\
&\quad + 6\eta^3 + 6\eta^2 s_1 + 96\alpha_1\eta + 24\alpha_1 s_1 - 6\eta^2 - 24\eta s_1 - 6s_1^2 \\
&\quad - 144\alpha_1 - 84\eta - 12s_1 + 144\big)\frac{y_1^3}{s_1^4} + \big(\eta^4 + 12\alpha_1\eta^2 - 6\eta^3 \\
&\quad - 6\eta^2 s_1 - 60\alpha_1\eta - 12\alpha_1 s_1 - \eta^2 + 14\eta s_1 + 3s_1^2 + 72\alpha_1 \\
&\quad + 54\eta + 6s_1 - 72\big)\frac{y_1^4}{s_1^4},
\end{aligned}$$

$$\begin{aligned}
&M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_2 - y_2)^4; y_1, y_2\big) \\
&= \big(s_2 + \eta + 14(1 - \alpha_2)\big)\frac{y_2}{s_2^4} + \big(-36\alpha_2\eta - 12\alpha_2 s_2 + 7\eta^2 \\
&\quad + 10\eta s_2 + 3s_2^2 + 86\alpha_2 + 29\eta + 5s_2 - 86\big)\frac{y_2^2}{s_2^4} + \big(-12\alpha_2\eta^2 \\
&\quad + 6\eta^3 + 6\eta^2 s_2 + 96\alpha_2\eta + 24\alpha_2 s_2 - 6\eta^2 - 24\eta s_2 - 6s_2^2 \\
&\quad - 144\alpha_2 - 84\eta - 12s_2 + 144\big)\frac{y_2^3}{s_2^4} + \big(\eta^4 + 12\alpha_2\eta^2 - 6\eta^3 \\
&\quad - 6\eta^2 s_2 - 60\alpha_2\eta - 12\alpha_2 s_2 - \eta^2 + 14\eta s_2 + 3s_2^2 + 72\alpha_2 \\
&\quad + 54\eta + 6s_2 - 72\big)\frac{y_2^4}{s_2^4}.
\end{aligned}$$

3 Order of convergence and Voronovskaja-type results

In this section, we obtain the Voronovskaja-type result and the order of convergence with the help of Peetre's K-functional for our operators $M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h; y_1, y_2)$. For $g \in C(I^2)$, the norm of the bivariate function g is considered by

$$\|g\|_{C(I^2)} = \sup_{(y_1,y_2)\in\mathcal{I}^2} \big|g(y_1, y_2)\big|.$$

Theorem 3.1 *For any $h \in C(I^2)$, one has*

$$\lim_{s_1,s_2\to\infty} \big\|M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h) - h\big\|_{C(I_1^2)} = 0, \tag{3.1}$$

where $I_1^2 = [0,1] \times [0,1]$.

Proof We see that

$$\begin{aligned}
&\big\|M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{00}) - e_{00}\big\|_{C(I_1^2)} \to 0, \qquad \big\|M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{10}) - e_{10}\big\|_{C(I_1^2)} \to 0, \\
&\big\|M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{01}) - e_{01}\big\|_{C(I_1^2)} \to 0, \qquad \big\|M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(e_{20} + e_{02}) - (e_{20} + e_{02})\big\|_{C(I_1^2)} \to 0
\end{aligned}$$

as $s_1, s_2 \to \infty$. Thus (3.1) holds by Volkov's theorem [43]. □

We will use $C^2(I^2)$ to denote the space of all functions $h \in C(I^2)$ such that $\frac{\partial^j h}{\partial y_1^j}, \frac{\partial^j h}{\partial y_2^j}, \frac{\partial^2 h}{\partial y_1\,\partial y_2} \in C(I^2)$ ($j = 1,2$) and equipped with the norm

$$\|h\|_{C^2(I^2)} = \|h\|_{C(I^2)} + \sum_{j=1}^{2}\left(\left\|\frac{\partial^j h}{\partial y_1^j}\right\|_{C(I^2)} + \left\|\frac{\partial^j h}{\partial y_2^j}\right\|_{C(I^2)}\right) + \left\|\frac{\partial^2 h}{\partial y_1\,\partial y_2}\right\|_{C(I^2)} \quad \bigl(h \in C(I^2)\bigr).$$

Theorem 3.2 *Suppose that $h \in C^2(I^2)$. Then*

$$\begin{aligned}
&\lim_{s\to\infty} s\bigl(M^{\alpha_1,\alpha_2}_{s,s,\eta}(h;y_1,y_2) - h(y_1,y_2)\bigr)\\
&\quad = \eta\bigl(y_1 h_{y_1}(y_1,y_2) + y_2 h_{y_2}(y_1,y_2)\bigr)\\
&\qquad + \frac{y_1(1-y_1)}{2} h_{y_1,y_1}(y_1,y_2) + \frac{y_2(1-y_2)}{2} h_{y_2,y_2}(y_1,y_2)
\end{aligned}$$

uniformly on I_1^2.

Proof Suppose that $h \in C^2(I^2)$. Then Taylor's theorem gives

$$\begin{aligned}
h(t_1,t_2) &= h(y_1,y_2) + h_{y_1}(y_1,y_2)(t_1-y_1) + h_{y_2}(y_1,y_2)(t_2-y_2)\\
&\quad + \frac{1}{2}\bigl[h_{y_1y_1}(y_1,y_2)(t_1-y_1)^2 + 2h_{y_1y_2}(y_1,y_2)(t_1-y_1)(t_2-y_2)\\
&\quad + h_{y_2y_2}(y_1,y_2)(t_2-y_2)^2\bigr] + \rho(t_1,t_2,y_1,y_2)\sqrt{(t_1-y_1)^4 + (t_2-y_2)^4},
\end{aligned} \tag{3.2}$$

where $\rho(t_1,t_2,y_1,y_2) \in C(I^2)$ and

$$\rho(t_1,t_2,y_1,y_2) \to 0\bigl((t_1,t_2) \to (y_1,y_2)\bigr).$$

Since $M^{\alpha_1,\alpha_2}_{s,s,\eta}$ is linear so, by operating on (3.2), we obtain

$$\begin{aligned}
&M^{\alpha_1,\alpha_2}_{s,s,\eta}\bigl(h(t_1,t_2);y_1,y_2\bigr)\\
&\quad = h(y_1,y_2) + h_{y_1}(y_1,y_2)M^{\alpha_1,\alpha_2}_{s,s,\eta}(t_1-y_1;y_1,y_2)\\
&\qquad + h_{y_2}(y_1,y_2)M^{\alpha_1,\alpha_2}_{s,s,\eta}(t_2-y_2;y_1,y_2)\\
&\qquad + \frac{1}{2}\bigl[h_{y_1y_1}(y_1,y_2)M^{\alpha_1,\alpha_2}_{s,s,\eta}\bigl((t_1-y_1)^2;y_1,y_2\bigr)\\
&\qquad + 2h_{y_1y_2}(y_1,y_2)M^{\alpha_1,\alpha_2}_{s,s,\eta}\bigl((t_1-y_1)(t_2-y_2);y_1,y_2\bigr)\\
&\qquad + h_{y_2y_2}(y_1,y_2)M^{\alpha_1,\alpha_2}_{s,s,\eta}\bigl((t_2-y_2)^2;y_1,y_2)\bigr)\bigr]\\
&\qquad + M^{\alpha_1,\alpha_2}_{s,s,\eta}\bigl(\rho(t_1,t_2,y_1,y_2)\sqrt{(t_1-y_1)^4+(t_2-y_2)^4};y_1,y_2\bigr).
\end{aligned} \tag{3.3}$$

With a view of Corollary 2.2, we find that

$$\begin{aligned}
\lim_{s\to\infty} sM^{\alpha_1,\alpha_2}_{s,s,\eta}\bigl((t_1-y_1)(t_2-y_2);y_1,y_2\bigr) &= \lim_{s\to\infty} s\bigl[M^{\alpha_1}_{s,\eta}(t_1-y_1;y_1)\\
&\quad \times M^{\alpha_2}_{s,\eta}(t_2-y_2;y_2)\bigr]\\
&= 0
\end{aligned} \tag{3.4}$$

and also

$$\lim_{s\to\infty} sM_{s,s,\eta}^{\alpha_1,\alpha_2}(t_1-y_1;y_1,y_2)=\eta y_1, \qquad \lim_{s\to\infty} sM_{s,s,\eta}^{\alpha_1,\alpha_2}(t_2-y_2;y_1,y_2)=\eta y_2, \tag{3.5}$$

$$\begin{aligned}&\lim_{s\to\infty} sM_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_1-y_1)^2;y_1,y_2\bigr)=y_1(1-y_1),\\ &\lim_{s\to\infty} sM_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_2-y_2)^2;y_1,y_2\bigr)=y_2(1-y_2).\end{aligned} \tag{3.6}$$

It follows from (3.3)–(3.6) that

$$\begin{aligned}&\lim_{s\to\infty} s\bigl(M_{s,s,\eta}^{\alpha_1,\alpha_2}(h;y_1,y_2)-h(y_1,y_2)\bigr)\\ &\quad=\eta y_1h_{y_1}(y_1,y_2)+\eta y_2h_{y_2}(y_1,y_2)\\ &\qquad+\frac{1}{2}\bigl[y_1(1-y_1)h_{y_1y_1}(y_1,y_2)+y_2(1-y_2)h_{y_2y_2}(y_1,y_2)\bigr]\\ &\qquad+\lim_{s\to\infty} sM_{s,s,\eta}^{\alpha_1,\alpha_2}\Bigl(\rho(t_1,t_2,y_1,y_2)\sqrt{(t_1-y_1)^4+(t_2-y_2)^4};y_1,y_2\Bigr).\end{aligned} \tag{3.7}$$

By the Cauchy–Schwarz inequality, one gets

$$\begin{aligned}&sM_{s,s,\eta}^{\alpha_1,\alpha_2}\Bigl(\rho(t_1,t_2,y_1,y_2)\sqrt{(t_1-y_1)^4+(t_2-y_2)^4};y_1,y_2\Bigr)\\ &\quad\le\sqrt{M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl(\rho^2(t_1,t_2,y_1,y_2);y_1,y_2\bigr)}\sqrt{s^2M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_1-y_1)^4+(t_2-y_2)^4;y_1,y_2\bigr)}\\ &\quad=\sqrt{M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl(\rho^2(t_1,t_2,y_1,y_2);y_1,y_2\bigr)}\\ &\qquad\times\sqrt{s^2\bigl(M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_1-y_1)^4;y_1,y_2\bigr)+M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_2-y_2)^4;y_1,y_2\bigr)\bigr)}.\end{aligned} \tag{3.8}$$

Since $\rho(t_1,t_2,y_1,y_2)\in C(I^2)$ and $\lim_{(t_1,t_2)\to(y_1,y_2)}\rho(t_1,t_2,y_1,y_2)=0$, we have

$$\lim_{s\to\infty} M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl(\rho^2(t_1,t_2,y_1,y_2);y_1,y_2\bigr)=0$$

uniformly on I_1^2 by Theorem 3.1. Corollary 2.2 gives

$$s^2M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_1-y_1)^4;y_1,y_2\bigr)\longrightarrow 3y_1^2-6y_1^3+3y_1^4\quad(s\longrightarrow\infty)$$

and

$$s^2M_{s,s,\eta}^{\alpha_1,\alpha_2}\bigl((t_2-y_2)^4;y_1,y_2\bigr)\longrightarrow 3y_2^2-6y_2^3+3y_2^4\quad(s\longrightarrow\infty).$$

Employing the last three relations in Eq. (3.8) and then using Eq. (3.7), we obtain

$$\begin{aligned}&\lim_{s\to\infty} s\bigl(M_{s,s,\eta}^{\alpha_1,\alpha_2}(h;y_1,y_2)-h(y_1,y_2)\bigr)\\ &\quad=\eta y_1h_{y_1}(y_1,y_2)+\eta y_2h_{y_2}(y_1,y_2)\\ &\qquad+\frac{1}{2}\bigl[y_1(1-y_1)h_{y_1y_1}(y_1,y_2)+y_2(1-y_2)h_{y_2y_2}(y_1,y_2)\bigr]\end{aligned}$$

uniformly on I_1^2. □

For $h \in C(I^2)$ and $\delta > 0$, Peetre's K-functional is given by

$$K(h;\delta) = \inf_{h_1 \in C^2(I^2)} \left\{ \|h - h_1\|_{C(I^2)} + \delta \|h_1\|_{C(I^2)} \right\}$$

and the modulus of continuity of h is

$$\omega(h;\delta) = \sup\left\{ \left| h(u_1,u_2) - h(y_1,y_2) \right| : (u_1,u_2),(y_1,y_2) \in I_1^2, \sqrt{(u_1-y_1)^2 + (u_2-y_2)^2} \le \delta \right\}.$$

By Theorem 9 (see [18]), there is a constant $C > 0$ such that

$$K(h;\delta) \le C\omega_2(h;\sqrt{\delta}). \tag{3.9}$$

In the above relation, $\omega_2(h;\sqrt{\delta})$ is the second-order modulus of continuity of $h \in C(I^2)$ (for details, see [5]).

Theorem 3.3 *For any $h \in C(I^2)$, one has*

$$\left| M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h;y_1,y_2) - h(y_1,y_2) \right| \le 4K\left(h;\delta_{s_1,s_2}(y_1,y_2)\right) + \omega\left(h;\sqrt{\left(\frac{\eta y_1}{s_1}\right)^2 + \left(\frac{\eta y_2}{s_2}\right)^2}\right),$$

where

$$\delta_{s_1,s_2}(y_1,y_2)(=\delta) = \frac{1}{4}\left(\delta_{s_1}^2(y_1) + \delta_{s_2}^2(y_2) + \left(\frac{\eta y_1}{s_1}\right)^2 + \left(\frac{\eta y_2}{s_2}\right)^2\right) > 0$$

and

$$\delta_{s_1}^2(y_1) = \mathcal{M}_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\left((t_1-y_1)^2;y_1,y_2\right), \qquad \delta_{s_2}^2(y_2) = \mathcal{M}_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\left((t_2-y_2)^2;y_1,y_2\right).$$

Proof Assume that $h_1 \in C^2(I^2)$. Then, by Taylor's theorem, we have

$$\begin{aligned} h_1(t_1,t_2) - h_1(y_1,y_2) &= \frac{\partial h_1(y_1,y_2)}{\partial y_1}(t_1-y_1) + \int_{y_1}^{t_1}(t_1-u_1)\frac{\partial^2 h_1(u_1,y_2)}{\partial u_1^2}\,du_1 \\ &\quad + \frac{\partial h_1(y_1,y_2)}{\partial y_2}(t_2-y_2) + \int_{y_2}^{t_2}(t_2-u_2)\frac{\partial^2 h_1(y_1,u_2)}{\partial u_2^2}\,du_2. \end{aligned} \tag{3.10}$$

We are now defining the auxiliary operators by

$$\mathcal{M}_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h;y_1,y_2) = M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h;y_1,y_2) - h\left(\frac{(\eta+s_1)y_1}{s_1}, \frac{(\eta+s_2)y_2}{s_2}\right) + h(y_1,y_2).$$

Simple calculation together with Corollary 2.2 gives that

$$\mathcal{M}_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(t_1-y_1;y_1,y_2) = 0 \quad \text{and} \quad \mathcal{M}_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(t_2-y_2;y_1,y_2) = 0.$$

By operating $\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}$ in (3.10) and using the last relation, we obtain

$$\begin{aligned}
&\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h_1;y_1,y_2)-h_1(y_1,y_2)\\
&\quad=\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\left(\int_{y_1}^{t_1}(t_1-u_1)\frac{\partial^2 h_1(u_1,y_2)}{\partial u_1^2}\,du_1;y_1,y_2\right)\\
&\qquad+\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\left(\int_{y_2}^{t_2}(t_2-u_2)\frac{\partial^2 h_1(y_1,u_2)}{\partial u_2^2}\,du_2;y_1,y_2\right)\\
&\quad=M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\left(\int_{y_1}^{t_1}(t_1-u_1)\frac{\partial^2 h_1(u_1,y_2)}{\partial u_1^2}\,du_1;y_1,y_2\right)\\
&\qquad-\left(\int_{y_1}^{\frac{(\eta+s_1)y_1}{s_1}}\left(\frac{(\eta+s_1)y_1}{s_1}-u_1\right)\frac{\partial^2 h_1(u_1,y_2)}{\partial u_1^2}\,du_1;y_1,y_2\right)\\
&\qquad+M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\left(\int_{y_2}^{t_2}(t_2-u_2)\frac{\partial^2 h_1(y_1,u_2)}{\partial u_2^2}\,du_2;y_1,y_2\right)\\
&\qquad-\left(\int_{y_2}^{\frac{(\eta+s_2)y_2}{s_2}}\left(\frac{(\eta+s_2)y_2}{s_2}-u_2\right)\frac{\partial^2 h_1(y_1,u_2)}{\partial u_2^2}\,du_2;y_1,y_2\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\left|\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h_1;y_1,y_2)-h_1(y_1,y_2)\right|\\
&\quad\le\left(M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\left((t_1-y_1)^2;y_1,y_2\right)+\left(\frac{\eta y_1}{s_1}\right)^2\right)\|h_1\|_{C^2(I^2)}\\
&\qquad+\left(M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\left((t_2-y_2)^2;y_1,y_2\right)+\left(\frac{\eta y_2}{s_2}\right)^2\right)\|h_1\|_{C^2(I^2)}\\
&\quad=\left(\delta^2_{s_1}(y_1)+\delta^2_{s_2}(y_2)+\left(\frac{\eta y_1}{s_1}\right)^2+\left(\frac{\eta y_2}{s_2}\right)^2\right)\|h_1\|_{C^2(I^2)}.
\end{aligned}$$

We can see that

$$\left|\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)\right|\le\left|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)\right|+\left|h\left(\frac{(\eta+s_1)y_1}{s_1},\frac{(\eta+s_2)y_2}{s_2}\right)\right|+\left|h(y_1,y_2)\right|,$$

which yields $|\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)|\le 3\|h\|$ for any $h\in C(I^2)$. We therefore write

$$\begin{aligned}
&\left|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\right|\\
&\quad\le\left|\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h-h_1;y_1,y_2)\right|+\left|h(y_1,y_2)-h_1(y_1,y_2)\right|\\
&\qquad+\left|\mathcal{M}^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h_1;y_1,y_2)-h_1(y_1,y_2)\right|\\
&\qquad+\left|h\left(\frac{(\eta+s_1)y_1}{s_1},\frac{(\eta+s_2)y_2}{s_2}\right)-h(y_1,y_2)\right|\\
&\quad\le 4\|h-h_1\|+\left(\delta^2_{s_1}(y_1)+\delta^2_{s_2}(y_2)+\left(\frac{\eta y_1}{s_1}\right)^2+\left(\frac{\eta y_2}{s_2}\right)^2\right)\|h_1\|_{C^2(I^2)}\\
&\qquad+\omega\left(h;\sqrt{\left(\frac{\eta y_1}{s_1}\right)^2+\left(\frac{\eta y_2}{s_2}\right)^2}\right).
\end{aligned}$$

Now, taking $\inf_{h_1\in C^2(I^2)}$, we get

$$\left|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\right|\le 4K\bigl(h;\delta_{s_1,s_2}(y_1,y_2)\bigr)+\omega\left(h;\sqrt{\left(\frac{\eta y_1}{s_1}\right)^2+\left(\frac{\eta y_2}{s_2}\right)^2}\right),$$

which completes the proof. □

The following corollary follows from Theorem 3.3 and inequality (3.9).

Corollary 3.4 *Let $h\in C(I^2)$. Then*

$$\begin{aligned}&\left|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\right|\\&\quad\le C\omega_2\Bigl(h;\sqrt{\delta_{s_1,s_2}(y_1,y_2)}\Bigr)+\omega\left(h;\sqrt{\left(\frac{\eta y_1}{s_1}\right)^2+\left(\frac{\eta y_2}{s_2}\right)^2}\right).\end{aligned}$$

4 GBS operators of bivariate generalized Bernstein–Schurer type

For any compact real intervals X and Y, the function $h:Y_1\times Y_2\to\mathbb{R}$ is B-bounded (or Bögel bounded) on $Y_1\times Y_2$ if there exists $H>0$ such that

$$\left|\Delta_{(y_1,y_2)}h[u_1,u_2;y_1,y_2]\right|\le H\quad\bigl(\forall(u_1,u_2),(y_1,y_2)\in Y_1\times Y_2\bigr),$$

where $\Delta_{(y_1,y_2)}h[u_1,u_2;y_1,y_2]$ is the mixed difference of h defined by

$$\Delta_{(y_1,y_2)}h[u_1,u_2;y_1,y_2]=h(y_1,y_2)-h(y_1,u_2)-h(u_1,y_2)+h(u_1,u_2).\tag{4.1}$$

A function h is said to be B-continuous (or Bögel continuous) at a point (u_1,u_2) if

$$\lim_{(y_1,y_2)\to(u_1,u_2)}\Delta_{(y_1,y_2)}h[u_1,u_2;y_1,y_2]=0$$

for any $(u_1,u_2),(y_1,y_2)\in Y_1\times Y_2$ (see [13]).

Given a function $h\in C(I^2)$, for any $s_1,s_2\in\mathbb{N}$, $\eta\in\mathbb{Z}_0^+$, and $\alpha_1,\alpha_2\in[0,1]$, we define the generalized Boolean sum (or GBS) operators of the bivariate form of generalized Bernstein–Schurer operators (2.1) by

$$\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)=M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(h(t_1,y_2)+h(y_1,t_2)-h(t_1,t_2);y_1,y_2\bigr),\quad\bigl((y_1,y_2)\in I_1^2\bigr),$$

or, equivalently, we write

$$\begin{aligned}\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)={}&\sum_{\ell_1=0}^{s_1+\eta}\sum_{\ell_2=0}^{s_2+\eta}M^{(\alpha_1,\alpha_2)}_{s_1+\eta,s_2+\eta,\ell_1,\ell_2}(y_1,y_2)\\&\times\left(h\left(\frac{\ell_1}{s_1},y_2\right)+h\left(y_1,\frac{\ell_2}{s_2}\right)-h\left(\frac{\ell_1}{s_1},\frac{\ell_2}{s_2}\right)\right).\end{aligned}\tag{4.2}$$

Note that $\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)$ is well defined on $C_B(I^2)$ (the space of all B-continuous functions on I^2) into $C(I^2)$ and $h\in C_B(I^2)$ as well as linear and positive.

Recall that the mixed modulus of smoothness of $h \in C_B(I^2)$ is given by

$$\omega_{\text{mixed}}(h;\delta_1,\delta_2) = \sup\bigl\{\bigl|\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2]\bigr| : |y_1-t_1|<\delta_1, |y_2-t_2|<\delta_2; (y_1,y_2),(u_1,u_2)\in I^2\bigr\} \tag{4.3}$$

for any $\delta_1,\delta_2>0$ (see [7, 9]).

A function h is B-differentiable (or Bögel differentiable) at the point $(u_1,u_2)\in Y_1\times Y_2$ if

$$\lim_{(y_1,y_2)\to(u_1,u_2)} \frac{\Delta_{(y_1,y_2)}f[u_1,u_2;y_1,y_2]}{(y_1-u_1)(y_2-u_2)}$$

exists. The limit is called B-differentiable of h at (u_1,u_2) and denoted by $D_{y_1y_2}h(u_1,u_2) = D_B(h;u_1,u_2)$. By $D_b(Y_1\times Y_2)$, we denote the set of all B-differentiable functions.

For more details and related results, we refer to [8, 12, 19, 22, 24, 39].

The following theorem gives an estimate of the rate of convergence of $\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}$ to $h\in C_B(I^2)$.

Theorem 4.1 *For any $h\in C_B(I^2)$, the inequality*

$$\bigl|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\bigr| \le 4\,\omega_{\text{mixed}}\bigl(h;\sqrt{\beta_{s_1,\eta}},\sqrt{\beta_{s_2,\eta}}\bigr)$$

holds, where

$$\beta_{s_1,\eta} = \frac{3+\eta(1+\eta)}{s_1} \quad \textit{and} \quad \beta_{s_2,\eta} = \frac{3+\eta(1+\eta)}{s_2}.$$

Proof It follows from (4.3) and

$$\omega_{\text{mixed}}(h;\lambda_2\delta_1,\lambda_2\delta_2) \le (1+\lambda_1)(1+\lambda_2)\omega_{\text{mixed}}(h;\delta_1,\delta_2) \quad (\lambda_1,\lambda_1>0)$$

that

$$\begin{aligned}\bigl|\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2]\bigr| &\le \omega_{\text{mixed}}\bigl(h;|t_1-y_1|,|t_2-y_2|\bigr)\\ &\le \left(1+\frac{|t_1-y_1|}{\delta_1}\right)\left(1+\frac{|t_2-y_2|}{\delta_2}\right)\omega_{\text{mixed}}(h;\delta_1,\delta_2)\end{aligned} \tag{4.4}$$

for all $(y_1,y_2),(t_1,t_2)\in I^2$ and for any $\delta_1,\delta_2>0$. Rewrite (4.1) as

$$h(y_1,t_2)+h(t_1,y_2)-h(t_1,t_2) = h(y_1,y_2)-\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2].$$

Operating $M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}$ and using the definition of $\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}$, we obtain

$$\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2) = h(y_1,y_2)M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{00};y_1,y_2) - M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2];y_1,y_2\bigr),$$

which yields

$$\bigl|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\bigr| \le M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(\bigl|\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2]\bigr|;y_1,y_2\bigr).$$

Employing (4.4) and then using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\left|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\right| \\
&\quad\le \Big(M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(e_{00};y_1,y_2)+\delta_1^{-1}\sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\big((t_1-y_1)^2;y_1,y_2\big)} \\
&\qquad+\delta_2^{-1}\sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\big((t_2-y_2)^2;y_1,y_2\big)} \\
&\qquad+\delta_1^{-1}\delta_2^{-1}\sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\big((t_1-y_1)^2;y_1,y_2\big)M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\big((t_2-y_2)^2;y_1,y_2\big)}\Big)\omega_{\mathrm{mixed}}(h;\delta_1,\delta_2).
\end{aligned}$$

Using Corollary 2.2, we write

$$\begin{aligned}
M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\big((t_1-y_1)^2;y_1,y_2\big) &\le \frac{y_1(1-y_1)}{s_1}+\frac{y_1(1-y_1)(\eta+2)}{s_1^2}+\frac{\eta^2y_1^2}{s_1^2} \\
&\le \frac{3+\eta(1+\eta)}{s_1}=\beta_{s_1,\eta}
\end{aligned}$$

and

$$M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\big((t_2-y_2)^2;y_1,y_2\big)\le\frac{3+\eta(1+\eta)}{s_2}=\beta_{s_2,\eta}.$$

We therefore obtain

$$\begin{aligned}
\left|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\right| \le \Bigg(1&+\frac{1}{\delta_1}\sqrt{\frac{3+\eta(1+\eta)}{s_1}} \\
&+\frac{1}{\delta_1\delta_2}\sqrt{\frac{3+\eta(1+\eta)}{s_1}}\sqrt{\frac{3+\eta(1+\eta)}{s_2}} \\
&+\frac{1}{\delta_2}\sqrt{\frac{3+\eta(1+\eta)}{s_2}}\Bigg)\omega_{\mathrm{mixed}}(h;\delta_1,\delta_2),
\end{aligned}$$

which gives the assertion of Theorem 4.1 by choosing $\delta_1=\sqrt{\beta_{s_1,\eta}}$ and $\delta_2=\sqrt{\beta_{s_2,\eta}}$. □

Finally, we study the order of approximation for B-differentiable functions of our operators $\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}$.

Theorem 4.2 *Suppose that $h\in D_b(I^2)$ and D_Bh in $B(I^2)$ (the space of all bounded functions on I^2). Then*

$$\left|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2)-h(y_1,y_2)\right|\le\frac{N}{\sqrt{s_1s_2}}\left\{\omega_{\mathrm{mixed}}\left(D_Bh;\sqrt{\frac{1}{s_1}},\sqrt{\frac{1}{s_2}}\right)+\|D_Bh\|_\infty\right\},$$

where $N>0$ is a constant.

Proof Let $h\in D_b(I^2)$. Then, from (see [13], p. 62), we write

$$\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2]=(t_1-y_1)(t_2-y_2)D_Bh(\alpha,\gamma)$$

for $y_1 < \alpha < t_1$, $y_2 < \gamma < t_2$. Thus, we fairly have

$$D_B h(\alpha,\gamma) = \Delta_{(y_1,y_2)} D_B h(\alpha,\gamma) + D_B h(\alpha,y_2) + D_B h(y_1,\gamma) - D_B h(y_1,y_2).$$

Since $D_B h \in B(I^2)$, we have $|D_B h(y_1,y_2)| \leq \|D_B h\|_\infty$. In view of the last two equalities, we obtain

$$\begin{aligned}
&\bigl|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(\Delta_{(y_1,y_2)} h[t_1,t_2;y_1,y_2];y_1,y_2\bigr)\bigr| \\
&\quad = \bigl|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)(t_2-y_2)D_B h(\alpha,\gamma);y_1,y_2\bigr)\bigr| \\
&\quad \leq M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(|t_1-y_1||t_2-y_2|\bigl|\Delta_{y_1,y_2} D_B h(\alpha,\gamma)\bigr|;y_1,y_2\bigr) \\
&\qquad + M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(|t_1-y_1||t_2-y_2|\bigl(\bigl|D_B h(\alpha,y_2)\bigr| + \bigl|D_B h(y_1,\gamma)\bigr| \\
&\qquad + \bigl|D_B h(y_1,y_2)\bigr|\bigr);y_1,y_2\bigr) \\
&\quad \leq M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(|t_1-y_1||t_2-y_2|\omega_{\text{mixed}}\bigl(D_B h;|\alpha-y_1|,|\gamma-y_2|\bigr);y_1,y_2\bigr) \\
&\qquad + 3\|D_B h\|_\infty\, M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(|t_1-y_1||t_2-y_2|;y_1,y_2\bigr).
\end{aligned} \tag{4.5}$$

Also, we have

$$\begin{aligned}
\omega_{\text{mixed}}\bigl(D_B h;|\alpha-y_1|,|\gamma-y_2|\bigr) &\leq \omega_{\text{mixed}}\bigl(D_B h;|t_1-y_1|,|t_2-y_2|\bigr) \\
&\leq \biggl(1+\frac{|t_1-y_1|}{\delta_1}\biggr)\biggl(1+\frac{|t_2-y_2|}{\delta_2}\biggr)\omega_{\text{mixed}}(h;\delta_1,\delta_2).
\end{aligned} \tag{4.6}$$

We thus have from (4.5) and (4.6) together with the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\bigl|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2) - h(y_1,y_2)\bigr| \\
&\quad = \bigl|M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(\Delta_{(y_1,y_2)} h[t_1,t_2;y_1,y_2];y_1,y_2\bigr)\bigr| \\
&\quad \leq \bigl\{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(|t_1-y_1||t_2-y_2|;y_1,y_2\bigr) \\
&\qquad + \delta_1^{-1} M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)^2|t_2-y_2|;y_1,y_2\bigr) \\
&\qquad + \delta_2^{-1} M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl(|t_1-y_1|(t_2-y_2)^2;y_1,y_2\bigr) \\
&\qquad + \delta_1^{-1}\delta_2^{-1} M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)^2(t_2-y_2)^2;y_1,y_2\bigr)\bigr\} \\
&\qquad \times \omega_{\text{mixed}}(D_B h;\delta_1,\delta_2) \\
&\qquad + 3\|D_B h\|_\infty \sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)^2(t_2-y_2)^2;y_1,y_2\bigr)},
\end{aligned}$$

which gives

$$\begin{aligned}
&\bigl|\Phi^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}(h;y_1,y_2) - h(y_1,y_2)\bigr| \\
&\quad \leq \biggl\{\sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)^2(t_2-y_2)^2;y_1,y_2\bigr)} \\
&\qquad + \delta_1^{-1}\sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)^4(t_2-y_2)^2;y_1,y_2\bigr)} \\
&\qquad + \delta_2^{-1}\sqrt{M^{\alpha_1,\alpha_2}_{s_1,s_2,\eta}\bigl((t_1-y_1)^2(t_2-y_2)^4;y_1,y_2\bigr)}
\end{aligned}$$

$$+ \delta_1^{-1}\delta_2^{-1} M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^2(t_2-y_2)^2; y_1,y_2\big)\Big\}$$
$$\times\, \omega_{\text{mixed}}(D_B h; \delta_1,\delta_2)$$
$$+ 3\|D_B h\|_\infty \sqrt{M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^2(t_2-y_2)^2; y_1,y_2\big)}.$$

By straightforward calculation (from Corollary 2.2), we obtain

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^2; y_1,y_2\big) \le \frac{3+\eta(1+\eta)}{s_1} = \frac{N_1}{s_1}, \quad \text{say}$$
$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_2-y_2)^2; y_1,y_2\big) \le \frac{3+\eta(1+\eta)}{s_2} = \frac{N_1}{s_2}, \quad \text{say}$$
$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^4; y_1,y_2\big) \le \frac{N_2}{s_1^2} \quad \text{and} \quad M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_2-y_2)^4; y_1,y_2\big) \le \frac{N_2}{s_2^2}$$

for some constant $N_1, N_2 > 0$. Also

$$M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^{2m}(t_2-y_2)^{2n}; y_1,y_2\big) = M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_1-y_1)^{2m}; y_1,y_2\big) \times M_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}\big((t_2-y_2)^{2n}; y_1,y_2\big)$$

for $(t_1-y_1),(t_2-y_2) \in I^2$ and $m,n = 1,2$. From the above and by choosing $\delta_1 = \sqrt{\frac{1}{s_1}}$ and $\delta_2 = \sqrt{\frac{1}{s_2}}$, we have

$$\begin{aligned}
&\big|\Phi_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h; y_1,y_2) - h(y_1,y_2)\big| \\
&\quad\le \left\{N_1\sqrt{\frac{1}{s_1s_2}} + 2\sqrt{\frac{N_1N_2}{s_1s_2}} + N_1^2\sqrt{\frac{1}{s_1s_2}}\right\} \\
&\qquad\times \omega_{\text{mixed}}\left(D_B h; \sqrt{\frac{1}{s_1}}, \sqrt{\frac{1}{s_2}}\right) + 3\|D_B h\|_\infty N_1 \sqrt{\frac{1}{s_1s_2}} \\
&\quad= \frac{1}{\sqrt{s_1s_2}}\left\{\big(N_1 + 2\sqrt{N_1N_2} + N_1^2\big)\omega_{\text{mixed}}\left(D_B h; \sqrt{\frac{1}{s_1}}, \sqrt{\frac{1}{s_2}}\right)\right. \\
&\qquad\left. + 3N_1\|D_B h\|_\infty\right\},
\end{aligned}$$

which yields

$$\big|\Phi_{s_1,s_2,\eta}^{\alpha_1,\alpha_2}(h; y_1,y_2) - h(y_1,y_2)\big| \le \frac{N}{\sqrt{s_1s_2}}\left\{\omega_{\text{mixed}}\left(D_B h; \sqrt{\frac{1}{s_1}}, \sqrt{\frac{1}{s_2}}\right) + \|D_B h\|_\infty\right\},$$

where

$$N = \max\big\{N_1 + 2\sqrt{N_1N_2} + N_1^2, 3N_1\big\},$$

which completes the proof. □

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