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Hilbert-type inequalities for time scale nabla calculus

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Abstract
This paper deals with the derivation of some new dynamic Hilbert-type inequalities in time scale nabla calculus. In proving the results, the basic idea is to use some algebraic inequalities, Hölder's inequality, and Jensen's time scale inequality. This generalization allows us not only to unify all the related results that exist in the literature on an arbitrary time scale, but also to obtain new outcomes that are analytical to the results of the delta time scale calculation.

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1 Introduction

In recent years, Hilbert's dual-series inequality and its integral form [1, pp. 253–254] have been granted significant attention by many scholars (for example, see [2–10]). In particular, B. G. Pachpatte [11] established a new inequality close to that of Hilbert as follows. Let $k,r \geq 1$, $A_s = \sum_{m=1}^{s} a_m \geq 0$ and $B_\vartheta = \sum_{n=1}^{\vartheta} b_n \geq 0$. Then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{A_s^k B_\vartheta^r}{s+\vartheta} \leq C(k,r,p,q)\left(\sum_{s=1}^{p}(p-s+1)\left(A_s^{k-1}a_s\right)^2\right)^{\frac{1}{2}} \times\left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)\left(B_\vartheta^{r-1}b_\vartheta\right)^2\right)^{\frac{1}{2}}, \tag{1}$$

where

$$C(k,r,p,q) = \frac{1}{2}kr\sqrt{pq}.$$

In the same article [11], Pachpatte demonstrated the integral version of (1) as follows. Let $k,r \geq 1$, $\Pi(s) = \int_0^s \omega_1(\xi)\,d\xi \geq 0$ and $\Omega(\vartheta) = \int_0^\vartheta \omega_2(\nu)\,d\nu \geq 0$, for $s,\xi \in (0,x)$ and $\vartheta,\nu \in (0,y)$.

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Then

$$\int_0^x \int_0^y \frac{\Pi^k(s)\Omega^r(\vartheta)}{s+\vartheta}\,ds\,d\vartheta \le C^*(k,r,x,y)\left(\int_0^x (x-s)\bigl(\Pi^{k-1}(s)\omega_1(s)\bigr)^2\,ds\right)^{\frac{1}{2}} \times \left(\int_0^y (y-\vartheta)\bigl(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\bigr)^2\,d\vartheta\right)^{\frac{1}{2}}, \tag{2}$$

where

$$C^*(k,r,x,y) = \frac{1}{2}kr\sqrt{xy}.$$

In [12], Young-Ho Kim gave some generalizations of (1) and (2) by introducing a parameter $\gamma > 0$ as follows. Let $k,r \ge 1$, $A_s = \sum_{m=1}^{s} a_m \ge 0$ and $B_\vartheta = \sum_{n=1}^{\vartheta} b_n \ge 0$. Then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q} \frac{A_s^k B_\vartheta^r}{(s^\gamma+\vartheta^\gamma)^{\frac{1}{\gamma}}} \le D(k,r,\gamma,p,q)\left(\sum_{s=1}^{p}(p-s+1)\bigl(A_s^{k-1}a_s\bigr)^2\right)^{\frac{1}{2}} \times \left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)\bigl(B_\vartheta^{r-1}b_\vartheta\bigr)^2\right)^{\frac{1}{2}}, \tag{3}$$

where

$$D(k,r,\gamma,p,q) = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} kr\sqrt{pq}.$$

The integral version of (3) is established in the next consequence. Let $k,r \ge 1$, $\gamma > 0$, $\Pi(s) = \int_0^s \omega_1(\xi)\,d\xi \ge 0$, and $\Omega(\vartheta) = \int_0^\vartheta \omega_2(\nu)\,d\nu \ge 0$, for $s,\xi \in (0,x)$ and ϑ, $\nu \in (0,y)$. Then

$$\int_0^x \int_0^y \frac{\Pi^k(s)\Omega^r(\vartheta)}{(s^\gamma+\vartheta^\gamma)^{\frac{1}{\gamma}}}\,ds\,d\vartheta \le D^*(k,r,\gamma,x,y)\left(\int_0^x (x-s)\bigl(\Pi^{k-1}(s)\omega_1(s)\bigr)^2\,ds\right)^{\frac{1}{2}} \times \left(\int_0^y (y-\vartheta)\bigl(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\bigr)^2\,d\vartheta\right)^{\frac{1}{2}}, \tag{4}$$

where

$$D^*(k,r,\gamma,x,y) = \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} kr\sqrt{xy}.$$

Another refinement of inequalities (1) and (2) has been made by Yang [13] as follows. Let $k,r \ge 1$ and $\lambda,\mu > 1$ be constants such that $1/\lambda + 1/\mu = 1$, $A_s = \sum_{m=1}^{s} a_m \ge 0$, and $B_\vartheta = \sum_{n=1}^{\vartheta} b_n \ge 0$. Then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q} \frac{A_s^k B_\vartheta^r}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \le E(k,r,\lambda,\mu,p,q)\left(\sum_{s=1}^{p}(p-s+1)\bigl(A_s^{k-1}a_s\bigr)^\lambda\right)^{\frac{1}{\lambda}} \times \left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)\bigl(B_\vartheta^{r-1}b_\vartheta\bigr)^\mu\right)^{\frac{1}{\mu}}, \tag{5}$$

where

$$E(k,r,\lambda,\mu,p,q)=\frac{kr}{\lambda+\mu}p^{\frac{\lambda-1}{\lambda}}q^{\frac{\mu-1}{\mu}}.$$

The integral version of (5) is established in the next consequence. Let $k,r\geq 1$ and $\lambda,\mu>1$ be constants such that $1/\lambda+1/\mu=1$, $\Pi(s)=\int_0^s\omega_1(\xi)\,d\xi\geq 0$, and $\Omega(\vartheta)=\int_0^{\vartheta}\omega_2(\nu)\,d\nu\geq 0$, for $s,\xi\in(0,x)$ and $\vartheta,\nu\in(0,y)$. Then

$$\begin{aligned}&\int_0^x\int_0^y\frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\,ds\,d\vartheta\\&\quad\leq E^*(k,r,\lambda,\mu,x,y)\left(\int_0^x(x-s)\left(\Pi^{k-1}(s)\omega_1(s)\right)^{\lambda}ds\right)^{\frac{1}{\lambda}}\\&\qquad\times\left(\int_0^y(y-\vartheta)\left(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\right)^{\mu}d\vartheta\right)^{\frac{1}{\mu}},\end{aligned}\tag{6}$$

where

$$E^*(k,r,\lambda,\mu,x,y)=\frac{kr}{\lambda+\mu}x^{\frac{\lambda-1}{\lambda}}y^{\frac{\mu-1}{\mu}}.$$

After construction of time scale calculus, dynamic inequalities have become the focus of interest, and classical inequalities have been established for any time scale $\mathbb{T}$. We can refer two surveys [14, 15] and a monograph [16] for exhibition of these results.

In [17] the researchers concluded some generalizations of inequalities (1) and (2) for time scale delta calculus. Specifically, they proved that if $s,\vartheta,\vartheta_0\in\mathbb{T}$, $\omega_1(s)\in C_{rd}([\vartheta_0,x)_{\mathbb{T}},\mathbb{R}^+)$, $\omega_2(\vartheta)\in C_{rd}([\vartheta_0,y)_{\mathbb{T}},\mathbb{R}^+)$, $k,r\geq 1$ and $\lambda,\mu>1$ are constants such that $1/\lambda+1/\mu=1$, then for $s\in[\vartheta_0,x)_{\mathbb{T}}$ and $\vartheta\in[\vartheta_0,y)_{\mathbb{T}}$, one has

$$\begin{aligned}&\int_{\vartheta_0}^x\int_{\vartheta_0}^y\frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\frac{1}{\lambda}}+\lambda(\vartheta-\vartheta_0)^{\frac{1}{\mu}}}\Delta s\Delta\vartheta\\&\quad\leq G(k,r,\lambda,\mu,x,y)\left(\int_{\vartheta_0}^x\left(\sigma(x)-s\right)\left(\Pi^{k-1}\left(\sigma(s)\right)\omega_1(s)\right)^{\lambda}\Delta s\right)^{\frac{1}{\lambda}}\\&\qquad\times\left(\int_{\vartheta_0}^y\left(\sigma(y)-\vartheta\right)\left(\Omega^{r-1}\left(\sigma(\vartheta)\right)\omega_2(\vartheta)\right)^{\mu}\Delta\vartheta\right)^{\frac{1}{\mu}},\end{aligned}\tag{7}$$

where $\Pi(s)=\int_{\vartheta_0}^s\omega_1(\xi)\Delta\xi$, $\Omega(\vartheta)=\int_{\vartheta_0}^{\vartheta}\omega_2(\xi)\Delta\xi$, and

$$G(k,r,\lambda,\mu,x,y)=\frac{kr}{\lambda\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.\tag{8}$$

Another refinement of (7) for time scale delta calculus has been made by Rezk et al. [18] as follows. Let $s,\vartheta,\vartheta_0\in\mathbb{T}$, $\omega_1(s)\in C_{rd}([\vartheta_0,x)_{\mathbb{T}},\mathbb{R}^+)$, $\omega_2(\vartheta)\in C_{rd}([\vartheta_0,y)_{\mathbb{T}},\mathbb{R}^+)$, $k,r\geq 1$ and $\lambda,\mu>1$ be constants such that $1/\lambda+1/\mu=1$, then for $s\in[\vartheta_0,\rho)_{\mathbb{T}}$ and $\vartheta\in[\vartheta_0,\tau)_{\mathbb{T}}$, one has

$$\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\Delta s\Delta\vartheta$$
$$\leq G^*(k,r,\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\big(\sigma(x)-s\big)\big(\Pi^{k-1}\big(\sigma(s)\big)\omega_1(s)\big)^{\lambda}\Delta s\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_{\vartheta_0}^{y}\big(\sigma(y)-\vartheta\big)\big(\Omega^{r-1}\big(\sigma(\vartheta)\big)\omega_2(\vartheta)\big)^{\mu}\Delta\vartheta\right)^{\frac{1}{\mu}}, \tag{9}$$

where $\Pi(s)=\int_{\vartheta_0}^{s}\omega_1(\xi)\Delta\xi$, $\Omega(\vartheta)=\int_{\vartheta_0}^{\vartheta}\omega_2(\xi)\Delta\xi$, and

$$G^*(k,r,\lambda,\mu,x,y)=\frac{kr}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}. \tag{10}$$

For developing of Hilbert's inequalities for time scale delta calculus, we refer the reader to the articles [19–29]. Although there are many results for time scale calculus in the sense of delta derivative, there is not much done for the nabla derivative. Therefore the major contribution of this article is to extend Hilbert-type inequalities for the nabla time scale calculus and to unify them for an arbitrary time scale. The main theorems are inspired from the paper [18] which presents the corresponding results for time scale delta calculus. By obtaining their nabla versions, we can show the generalizations of these inequalities for different types of time scales $\mathbb{T}$, such as real numbers and integers.

The structure of this paper can be listed as follows. Section 2 presents the fundamental concepts of the time scale calculus in terms of delta and nabla derivatives. Section 3 is devoted to main results, which are to generalize inequalities (5) and (6) for the nabla time scale calculus and so, to obtain nabla calculus versions of (9) and several inequalities of Hilbert's type in [18].

2 Preliminaries

In this section, the fundamental theories of the time scale delta and nabla calculi will be presented. Time scale calculus whose detailed information can be found in [30, 31] has been invented in order to unify continuous and discrete analysis.

A nonempty closed subset of $\mathbb{R}$ is named a time scale and is denoted by $\mathbb{T}$. For $\vartheta\in\mathbb{T}$, if $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$, then the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ are defined as $\sigma(\vartheta)=\inf(\vartheta,\infty)_{\mathbb{T}}$ and $\rho(\vartheta)=\sup(-\infty,\vartheta)_{\mathbb{T}}$, respectively. From the above two concepts, it can be mentioned that a point $\vartheta\in\mathbb{T}$ with $\inf\mathbb{T}<\vartheta<\sup\mathbb{T}$ is named right-scattered if $\sigma(\vartheta)>\vartheta$, right-dense if $\sigma(\vartheta)=\vartheta$, left-scattered if $\rho(\vartheta)<\vartheta$ and left-dense if $\rho(\vartheta)=\vartheta$.

The Δ-derivative of $\psi:\mathbb{T}\to\mathbb{R}$ at $\vartheta\in\mathbb{T}^k=\mathbb{T}/(\rho(\sup\mathbb{T}),\sup\mathbb{T}]$ denoted by $\psi^{\Delta}(\vartheta)$ is the number enjoying the property that for all $\varepsilon>0$ there is a neighborhood U of $\vartheta\in\mathbb{T}^k$ such that

$$\big|\psi\big(\sigma(\vartheta)\big)-\psi(s)-\psi^{\Delta}(\vartheta)\big(\sigma(\vartheta)-s\big)\big|\leq\varepsilon\big|\sigma(\vartheta)-s\big|,\quad\text{for all } s\in U.$$

The ∇-derivative of $\psi:\mathbb{T}\to\mathbb{R}$ at $\vartheta\in\mathbb{T}_k=\mathbb{T}/[\inf\mathbb{T},\sigma(\inf\mathbb{T}))$ denoted by $\psi^{\nabla}(\xi)$ is the number enjoying the property that for all $\varepsilon>0$ there is a neighborhood V of $\vartheta\in\mathbb{T}_k$ such that

$$\left|\psi(\vartheta) - \psi\big(\rho(s)\big) - \psi^{\nabla}(\vartheta)\big(\vartheta - \rho(s)\big)\right| \leq \varepsilon\left|s - \rho(\vartheta)\right|, \quad \text{for all } s \in V.$$

A function $\psi : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at each right-dense point in $\mathbb{T}$ and $\lim_{s\to\vartheta^-} \psi(s)$ exists as a finite number for all left-dense points in $\mathbb{T}$. The set $C_{rd}(\mathbb{T},\mathbb{R})$ represents the class of real, rd-continuous functions defined on $\mathbb{T}$. If $\psi \in C_{rd}(\mathbb{T},\mathbb{R})$, then there exists a function $\Psi(\vartheta)$ such that $\Psi^{\Delta}(\vartheta) = \psi(\vartheta)$ and the delta integral of ψ is defined by

$$\int_{x_0}^{x} \psi(\vartheta)\Delta\vartheta = \Psi(x) - \Psi(x_0).$$

A function $\psi : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at each left-dense point in $\mathbb{T}$ and $\lim_{s\to\vartheta^+} \psi(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T},\mathbb{R})$ represents the class of real, ld-continuous functions defined on $\mathbb{T}$. If $\psi \in C_{ld}(\mathbb{T},\mathbb{R})$, then there exists a function $\Psi(\vartheta)$ such that $\Psi^{\nabla}(\vartheta) = \psi(\vartheta)$ and the nabla integral of ψ is defined by

$$\int_{x_0}^{x} \psi(\vartheta)\nabla\vartheta = \Psi(x) - \Psi(x_0).$$

In the following, we display some basic lemmas and algebraic inequalities that play a key role in proving the major findings of this paper.

Lemma 2.1 (Nabla Hölder's Inequality [32]) *Let* $x_0, x \in \mathbb{T}$. *For* $\xi, \psi \in C_{ld}([x_0,x]_{\mathbb{T}},\mathbb{R})$, *we have*

$$\int_{x_0}^{x} \xi(\vartheta)\psi(\vartheta)\nabla\vartheta \leq \left(\int_{x_0}^{x} \xi^{\lambda}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\lambda}}\left(\int_{x_0}^{x} \psi^{\mu}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\mu}}, \tag{11}$$

where $\lambda, \mu > 1$ *with* $1/\lambda + 1/\mu = 1$.

Lemma 2.2 (Nabla Jensen's Inequality [33, Theorem 3.4]) *Let* x_0, $x \in \mathbb{T}$ *and* $m, n \in \mathbb{R}$. *Assume that* $\xi \in C_{ld}([x_0,x]_{\mathbb{T}},(m,n))$ *and* $\psi \in C_{ld}([x_0,x]_{\mathbb{T}},\mathbb{R})$ *are nonnegative with* $\int_{x_0}^{x} \xi(\eta)\Delta\eta > 0$. *If* $\Theta \in C((m,n),\mathbb{R})$ *is a convex function, then*

$$\Theta\left(\frac{\int_{x_0}^{x} \xi(\eta)\psi(\eta)\nabla\eta}{\int_{x_0}^{x} \xi(\eta)\nabla\eta}\right) \leq \frac{\int_{x_0}^{x} \xi(\eta)\Theta(\psi(\eta))\nabla\eta}{\int_{x_0}^{x} \xi(\eta)\nabla\eta}. \tag{12}$$

Lemma 2.3 (The power rule for nabla derivative [33, Lemma 3.1]) *Let* x_0, $x \in \mathbb{T}$, $\psi \in C_{ld}([x_0,x]_{\mathbb{T}},\mathbb{R})$ *be a nonnegative function, and* $\gamma \geq 1$ *a real constant. Then*

$$\left(\int_{x_0}^{x} \psi(\xi)\nabla\xi\right)^{\gamma} \leq \gamma\int_{x_0}^{x} \psi(\nu)\left(\int_{a}^{\nu} \psi(\xi)\nabla\xi\right)^{\gamma-1}\nabla\nu. \tag{13}$$

Lemma 2.4 (Young's inequality [34]) *Let $\delta > 0$, $\Lambda_q > 0$ and $\sum_{q=1}^{n} \Lambda_q = \Upsilon_n$. Then*

$$\left\{\prod_{q=1}^{n} s_q^{\Lambda_q}\right\}^{\frac{1}{\Upsilon_n}} \leq \left\{\frac{1}{\Upsilon_n}\sum_{q=1}^{n} \Lambda_q s_q^{\delta}\right\}^{\frac{1}{\delta}}. \tag{14}$$

Lemma 2.5 ([33, Lemma 3.2]) *Let $s, \vartheta, \vartheta_0 \in \mathbb{T}$ with $s, \vartheta \geq \vartheta_0$ and $\psi \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$. Then*

$$\int_{\vartheta_0}^{s}\left(\int_{\vartheta_0}^{\vartheta} \psi(\xi)\nabla\xi\right)\nabla\vartheta = \int_{\vartheta_0}^{s}\left(\int_{\rho(\xi)}^{s} \psi(\xi)\nabla s\right)\nabla\xi = \int_{\vartheta_0}^{s}\big(s-\rho(\xi)\big)\psi(\xi)\nabla\xi. \tag{15}$$

3 Key results

In this section, we focus on obtaining the corresponding outcomes for the nabla time scale calculation in [18]. We must assume that all functions found in the theorem statements are nonnegative, ld-continuous, ∇-differentiable, and locally nabla integrable.

Theorem 3.1 *Let $s, \vartheta, \vartheta_0 \in \mathbb{T}$ and $\omega_1 \in C_{ld}([\vartheta_0, x]_{\mathbb{T}}, \mathbb{R}^+)$, $\omega_2 \in C_{ld}([\vartheta_0, y]_{\mathbb{T}}, \mathbb{R}^+)$. Define*

$$\Pi(s) = \int_{\vartheta_0}^{s} \omega_1(\xi)\nabla\xi \quad \textit{and} \quad \Omega(\vartheta) = \int_{\vartheta_0}^{\vartheta} \omega_2(\xi)\nabla\xi. \tag{16}$$

Then for $s \in [\vartheta_0, x]_{\mathbb{T}}$ and $\vartheta \in [\vartheta_0, y]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\quad \leq H(k,r,\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\big(x-\rho(s)\big)\big(\Pi^{k-1}(s)\omega_1(s)\big)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\qquad \times \left(\int_{\vartheta_0}^{y}\big(y-\rho(\vartheta)\big)\big(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\big)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned} \tag{17}$$

where

$$H(k,r,\lambda,\mu,x,y) = \frac{kr}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

Proof By using (13), we obtain

$$\Pi^k(s) \leq k\int_{\vartheta_0}^{s} \Pi^{k-1}(\xi)\omega_1(\xi)\nabla\xi \tag{18}$$

and

$$\Omega^r(\vartheta) \leq r\int_{\vartheta_0}^{\vartheta} \Omega^{r-1}(\xi)\omega_2(\xi)\nabla\xi. \tag{19}$$

Then, we have

$$\Pi^k(s)\Omega^r(\vartheta) \leq kr\left(\int_{\vartheta_0}^{s} \Pi^{k-1}(\xi)\omega_1(\xi)\nabla\xi\right)\left(\int_{\vartheta_0}^{\vartheta} \Omega^{r-1}(\xi)\omega_2(\xi)\nabla\xi\right). \tag{20}$$

Applying (11) to $\int_{\vartheta_0}^{s}\Pi^{k-1}(\xi)\omega_1(\xi)\nabla\xi$ with indices λ and $\lambda/(\lambda-1)$, we find that

$$\int_{\vartheta_0}^{s}\Pi^{k-1}(\xi)\omega_1(\xi)\nabla\xi \le (s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{s}\left(\Pi^{k-1}(\xi)\omega_1(\xi)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}, \tag{21}$$

while doing the same to the integral $\int_{\vartheta_0}^{\vartheta}\Omega^{r-1}(\xi)\omega_2(\xi)\nabla\xi$ with indices μ and $\mu/(\mu-1)$, we find that

$$\int_{\vartheta_0}^{\vartheta}\Omega^{r-1}(\xi)\omega_2(\xi)\nabla\xi \le (\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{\vartheta}\left(\Omega^{r-1}(\xi)\omega_2(\xi)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}. \tag{22}$$

From (20), (21), and (22), we get

$$\begin{aligned}\Pi^{k}(s)\Omega^{r}(\vartheta) \le{}& kr(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{s}\left(\Pi^{k-1}(\xi)\omega_1(\xi)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\\ &\times\left(\int_{\vartheta_0}^{\vartheta}\left(\Omega^{r-1}(\xi)\omega_2(\xi)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.\end{aligned} \tag{23}$$

Using inequality (14), we note

$$\left(s_1^{\Lambda_1}s_2^{\Lambda_2}\right)^{\frac{\delta}{\Lambda_1+\Lambda_2}} \le \frac{1}{\Lambda_1+\Lambda_2}\left(\Lambda_1 s_1^{\delta}+\Lambda_2 s_2^{\delta}\right). \tag{24}$$

Now, by setting $s_1=(s-\vartheta_0)^{\lambda-1}$, $s_2=(\vartheta-\vartheta_0)^{\mu-1}$, $\Lambda_1=1/\lambda$, $\Lambda_1=1/\mu$, and $\delta=\Lambda_1+\Lambda_2$ in (24), we get

$$(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}} \le \frac{\lambda\mu}{\lambda+\mu}\left(\frac{(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}}{\lambda}+\frac{(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}{\mu}\right). \tag{25}$$

Substituting (25) into (23) yields

$$\begin{aligned}\Pi^{k}(s)\Omega^{r}(\vartheta) \le{}& \frac{kr\lambda\mu}{\lambda+\mu}\left(\frac{(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}}{\lambda}+\frac{(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}{\mu}\right)\\ &\times\left(\int_{\vartheta_0}^{s}\left(\Pi^{k-1}(\xi)\omega_1(\xi)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\left(\int_{\vartheta_0}^{\vartheta}\left(\Omega^{r-1}(\xi)\omega_2(\xi)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.\end{aligned} \tag{26}$$

Dividing both sides of (26) by $\mu(s-\vartheta_0)^{[(\lambda-1)(\lambda+\mu)]/\lambda\mu}+\lambda(\vartheta-\vartheta_0)^{[(\mu-1)(\lambda+\mu)]/\lambda\mu}$, we obtain

$$\begin{aligned}&\frac{\Pi^{k}(s)\Omega^{r}(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\\ &\quad\le \frac{kr}{\lambda+\mu}\left(\int_{\vartheta_0}^{s}\left(\Pi^{k-1}(\xi)\omega_1(\xi)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\\ &\qquad\times\left(\int_{\vartheta_0}^{\vartheta}\left(\Omega^{r-1}(\xi)\omega_2(\xi)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.\end{aligned} \tag{27}$$

Integrating both sides of (27) and using (11) again, we get

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\le \frac{kr}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}} \\
&\quad\times\left(\int_{\vartheta_0}^{x}\left(\int_{\vartheta_0}^{s}\left(\Pi^{k-1}(\xi)\omega_1(\xi)\right)^{\lambda}\nabla\xi\right)\nabla s\right)^{\frac{1}{\lambda}} \\
&\quad\times\left(\int_{\vartheta_0}^{y}\left(\int_{\vartheta_0}^{\vartheta}\left(\Omega^{r-1}(\xi)\omega_2(\xi)\right)^{\mu}\nabla\xi\right)\nabla\vartheta\right)^{\frac{1}{\mu}}.
\end{aligned} \tag{28}$$

Applying Lemma 2.5 on (28), we conclude that

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\le \frac{kr}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{x}\left(x-\rho(s)\right)\left(\Pi^{k-1}(s)\omega_1(s)\right)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\quad\times\left(\int_{\vartheta_0}^{y}\left(y-\rho(\vartheta)\right)\left(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\right)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}} \\
&= H(k,r,\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\left(x-\rho(s)\right)\left(\Pi^{k-1}(s)\omega_1(s)\right)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\quad\times\left(\int_{\vartheta_0}^{y}\left(y-\rho(\vartheta)\right)\left(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\right)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}$$

that is, (17) is true. □

Remark 3.2 By setting $1/\lambda + 1/\mu = 1$ in (24), we obtain

$$\left(s_1^{\Lambda_1}s_2^{\Lambda_2}\right) \le \frac{1}{\Lambda_1+\Lambda_2}\left(\Lambda_1 s_1^{\Lambda_1+\Lambda_2}+\Lambda_2 s_2^{\Lambda_1+\Lambda_2}\right). \tag{29}$$

Hence, by applying (29) on the right-hand side of (17) in Theorem 3.1, we get

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\le \frac{\lambda\mu kr}{(\lambda+\mu)^2}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}} \\
&\quad\times\left\{\frac{1}{\lambda}\left(\int_{\vartheta_0}^{x}\left(x-\rho(s)\right)\left(\Pi^{k-1}(s)\omega_1(s)\right)^{\lambda}\nabla s\right)^{\frac{\lambda+\mu}{\lambda\mu}}\right. \\
&\quad\left.+\frac{1}{\mu}\left(\int_{\vartheta_0}^{y}\left(y-\rho(\vartheta)\right)\left(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\right)^{\mu}\nabla\vartheta\right)^{\frac{\lambda+\mu}{\lambda\mu}}\right\}.
\end{aligned}$$

Corollary 3.1 *If we take* $1/\lambda + 1/\mu = 1$ *in* (17), *then*

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{\Pi^k(s)\Omega^r(\vartheta)}{\mu(s-\vartheta_0)^{\lambda-1}+\lambda(\vartheta-\vartheta_0)^{\mu-1}}\nabla s\nabla\vartheta \\
&\quad\le H^*(k,r,\lambda,\mu,x,y)\biggl(\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(\Pi^{k-1}(s)\omega_1(s)\bigr)^{\lambda}\nabla s\biggr)^{\frac{1}{\lambda}} \\
&\qquad\times\biggl(\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\bigr)^{\mu}\nabla\vartheta\biggr)^{\frac{1}{\mu}},
\end{aligned}\tag{30}$$

where

$$H^*(k,r,\lambda,\mu,x,y)=\frac{kr}{\lambda\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

Remark 3.3 As a particular case of Corollary 3.1, if $\lambda=\mu=2$, then we have

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{\Pi^k(s)\Omega^r(\vartheta)}{s+\vartheta-2\vartheta_0}\nabla s\nabla\vartheta \\
&\quad\le \frac{1}{2}kr\biggl((x-\vartheta_0)\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(\Pi^{k-1}(s)\omega_1(s)\bigr)^{2}\nabla s\biggr)^{\frac{1}{2}} \\
&\qquad\times\biggl((y-\vartheta_0)\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(\Omega^{r-1}(\vartheta)\omega_2(\vartheta)\bigr)^{2}\nabla\vartheta\biggr)^{\frac{1}{2}},
\end{aligned}\tag{31}$$

which is [33, Theorem 3.3].

Remark 3.4 Clearly, for $\mathbb{T}=\mathbb{Z}$ or $\mathbb{T}=\mathbb{R}$, and $\vartheta_0=0$, together with $\rho(u)=u-1$ or $\rho(u)=u$, (17) reduces to (5) or (6), respectively.

Remark 3.5 In Theorem 3.1, if we take $k=r=1$, then we have

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{\Pi(s)\Omega(\vartheta)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\quad\le H^{**}(\lambda,\mu,x,y)\biggl(\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(\omega_1(s)\bigr)^{\lambda}\nabla s\biggr)^{\frac{1}{\lambda}} \\
&\qquad\times\biggl(\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(\omega_2(\vartheta)\bigr)^{\mu}\nabla\vartheta\biggr)^{\frac{1}{\mu}},
\end{aligned}\tag{32}$$

where

$$H^{**}(\lambda,\mu,x,y)=\frac{1}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

For $\lambda=\mu=2$, this is Anderson's result [33, Remark 4].

In what follows, we give a further generalization of (32) obtained in Remark 3.5. Before giving our results, we presume that there are Φ and Ψ which are real-valued, nonnegative, convex and submultiplicative functions defined on $[0,\infty)$. A function ψ is submultiplicative if $\psi(s\vartheta) \le \psi(s)\psi(\vartheta)$ for $s,\vartheta \ge 0$.

Theorem 3.6 *Let $s,\vartheta,\vartheta_0 \in \mathbb{T}$ and $\Pi(s)$, $\Omega(\vartheta)$ be as in Theorem 3.1 and let $k(\xi)$, $l(\xi)$ be two positive functions defined for $\xi \in [\vartheta_0,x]_{\mathbb{T}}$ and $\xi \in [\vartheta_0,y]_{\mathbb{T}}$. Suppose that*

$$K(s) = \int_{\vartheta_0}^{s} k(\xi)\nabla\xi \quad \text{and} \quad L(\vartheta) = \int_{\vartheta_0}^{\vartheta} l(\xi)\nabla\xi. \tag{33}$$

Then for $s \in [\vartheta_0,x]_{\mathbb{T}}$ and $\vartheta \in [\vartheta_0,y]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\quad \le M(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\left(k(s)\Phi\left(\frac{\omega_1(s)}{k(s)}\right)\right)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\qquad \times \left(\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\left(l(\vartheta)\Psi\left(\frac{\omega_2(\vartheta)}{l(\vartheta)}\right)\right)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
M(\lambda,\mu,x,y) = {}& \frac{1}{\lambda+\mu}\left(\int_{\vartheta_0}^{x}\left(\frac{\Phi(K(s))}{K(s)}\right)^{\frac{\lambda}{\lambda-1}}\nabla s\right)^{\frac{\lambda-1}{\lambda}} \\
&\times \left(\int_{\vartheta_0}^{y}\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right)^{\frac{\mu}{\mu-1}}\nabla\vartheta\right)^{\frac{\mu-1}{\mu}}.
\end{aligned}$$

Proof Using Jensen's inequality (12) and the properties of Φ, we obtain

$$\begin{aligned}
\Phi\bigl(\Pi(s)\bigr) &= \Phi\left(\frac{K(s)\int_{\vartheta_0}^{s} k(\xi)\frac{\omega_1(\xi)}{k(\xi)}\nabla\xi}{\int_{\vartheta_0}^{s} k(\xi)\nabla\xi}\right) \\
&\le \Phi\bigl(K(s)\bigr)\Phi\left(\frac{\int_{\vartheta_0}^{s} k(\xi)\frac{\omega_1(\xi)}{k(\xi)}\nabla\xi}{\int_{\vartheta_0}^{s} k(\xi)\nabla\xi}\right) \\
&\le \frac{\Phi(K(s))}{K(s)}\int_{\vartheta_0}^{s} k(\xi)\Phi\left(\frac{\omega_1(\xi)}{k(\xi)}\right)\nabla\xi.
\end{aligned} \tag{35}$$

Further, by (11), we find that

$$\Phi\bigl(\Pi(s)\bigr) \le \frac{\Phi(K(s))}{K(s)}(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{s}\left(k(\xi)\Phi\left(\frac{\omega_1(\xi)}{k(\xi)}\right)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}. \tag{36}$$

Analogously,

$$\Psi\bigl(\Omega(\vartheta)\bigr) \le \frac{\Psi(L(\vartheta))}{L(\vartheta)}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{\vartheta}\left(l(\xi)\Psi\left(\frac{\omega_2(\xi)}{l(\xi)}\right)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}. \tag{37}$$

By multiplying (36) and (37), we get

$$\begin{aligned}
&\Phi\bigl(\Pi(s)\bigr)\Psi\bigl(\Omega(\vartheta)\bigr)\\
&\quad\leq (s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\frac{\Phi(K(s))}{K(s)}\left(\int_{\vartheta_0}^{s}\left(k(\xi)\Phi\left(\frac{\omega_1(\xi)}{k(\xi)}\right)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\right)\\
&\qquad\times\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left(\int_{\vartheta_0}^{\vartheta}\left(l(\xi)\Psi\left(\frac{\omega_2(\xi)}{l(\xi)}\right)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}\right). \qquad (38)
\end{aligned}$$

Applying (24) on the term $(s-\vartheta_0)^{(\lambda-1)/\lambda}\times(\vartheta-\vartheta_0)^{(\mu-1)/\mu}$ gives

$$\begin{aligned}
&\Phi\bigl(\Pi(s)\bigr)\Psi\bigl(\Omega(\vartheta)\bigr)\\
&\quad\leq \frac{\lambda\mu}{\lambda+\mu}\left(\frac{(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}}{\lambda}+\frac{(t-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}{\mu}\right)\\
&\qquad\times\left(\frac{\Phi(K(s))}{K(s)}\left(\int_{\vartheta_0}^{s}\left(k(\xi)\Phi\left(\frac{\omega_1(\xi)}{k(\xi)}\right)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\right)\\
&\qquad\times\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left(\int_{\vartheta_0}^{\vartheta}\left(l(\xi)\Psi\left(\frac{\omega_2(\xi)}{l(\xi)}\right)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}\right). \qquad (39)
\end{aligned}$$

From (39), we observe that

$$\begin{aligned}
&\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\\
&\quad\leq \frac{1}{\lambda+\mu}\left(\frac{\Phi(K(s))}{K(s)}\left(\int_{\vartheta_0}^{s}\left(k(\xi)\Phi\left(\frac{\omega_1(\xi)}{k(\xi)}\right)\right)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\right)\\
&\qquad\times\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left(\int_{\vartheta_0}^{\vartheta}\left(l(\xi)\Psi\left(\frac{\omega_2(\xi)}{l(\xi)}\right)\right)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}\right). \qquad (40)
\end{aligned}$$

Integrating both sides of (40) and using (11) again with indices λ, $\lambda/(\lambda-1)$ and μ, $\mu/(\mu-1)$, we find that

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta\\
&\quad\leq \frac{1}{\lambda+\mu}\left(\int_{\vartheta_0}^{x}\left(\frac{\Phi(K(s))}{K(s)}\right)^{\frac{\lambda}{\lambda-1}}\nabla s\right)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{s}\left(k(\xi)\Phi\left(\frac{\omega_1(\xi)}{k(\xi)}\right)\right)^{\lambda}\nabla\xi\nabla s\right)^{\frac{1}{\lambda}}\\
&\qquad\times\left(\int_{\vartheta_0}^{y}\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right)^{\frac{\mu}{\mu-1}}\nabla\vartheta\right)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{y}\int_{\vartheta_0}^{\vartheta}\left(l(\xi)\Psi\left(\frac{\omega_2(\xi)}{l(\xi)}\right)\right)^{\mu}\nabla\xi\nabla\vartheta\right)^{\frac{1}{\mu}}. \qquad (41)
\end{aligned}$$

Applying Lemma 2.5 to (41), we get

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta\\
&\quad\leq \frac{1}{\lambda+\mu}\left(\int_{\vartheta_0}^{x}\left(\frac{\Phi(K(s))}{K(s)}\right)^{\frac{\lambda}{\lambda-1}}\nabla s\right)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{y}\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right)^{\frac{\mu}{\mu-1}}\nabla\vartheta\right)^{\frac{\mu-1}{\mu}}
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_{\vartheta_0}^{x}(x-\rho(s))\left(k(s)\Phi\left(\frac{\omega_1(s)}{k(s)}\right)\right)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\times\left(\int_{\vartheta_0}^{y}(y-\rho(\vartheta))\left(l(\vartheta)\Psi\left(\frac{\omega_2(\vartheta)}{l(\vartheta)}\right)\right)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}} \\
={}& M(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}(x-\rho(s))\left(k(s)\Phi\left(\frac{\omega_1(s)}{k(s)}\right)\right)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\times\left(\int_{\vartheta_0}^{y}(y-\rho(\vartheta))\left(l(\vartheta)\Psi\left(\frac{\omega_2(\vartheta)}{l(\vartheta)}\right)\right)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}
$$

which is (34). □

Corollary 3.2 *If we take* $1/\lambda+1/\mu=1$ *in* (34), *then we get*

$$
\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\lambda-1}+\lambda(\vartheta-\vartheta_0)^{\mu-1}}\nabla s\nabla\vartheta \\
&\quad\le M^{*}(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}(x-\rho(s))\left(k(s)\Phi\left(\frac{\omega_1(s)}{k(s)}\right)\right)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\quad\quad\times\left(\int_{\vartheta_0}^{y}(y-\rho(\vartheta))\left(l(\vartheta)\Psi\left(\frac{\omega_2(\vartheta)}{l(\vartheta)}\right)\right)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}
\tag{42}
$$

where

$$
\begin{aligned}
M^{*}(\lambda,\mu,x,y)={}&\frac{1}{\lambda\mu}\left(\int_{\vartheta_0}^{x}\left(\frac{\Phi(K(s))}{K(s)}\right)^{\frac{\lambda}{\lambda-1}}\nabla s\right)^{\frac{\lambda-1}{\lambda}} \\
&\times\left(\int_{\vartheta_0}^{y}\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right)^{\frac{\mu}{\mu-1}}\nabla\vartheta\right)^{\frac{\mu-1}{\mu}}.
\end{aligned}
$$

Remark 3.7 As a particular case of Corollary 3.2, if $\lambda=\mu=2$, then we get

$$
\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{s+\vartheta-2\vartheta_0}\nabla s\nabla\vartheta \\
&\quad\le M^{**}(x,y)\left(\int_{\vartheta_0}^{x}(x-\rho(s))\left(k(s)\Phi\left(\frac{\omega_1(s)}{k(s)}\right)\right)^{2}\nabla s\right)^{\frac{1}{2}} \\
&\quad\quad\times\left(\int_{\vartheta_0}^{y}(y-\rho(\vartheta))\left(l(\vartheta)\Psi\left(\frac{\omega_2(\vartheta)}{l(\vartheta)}\right)\right)^{2}\nabla\vartheta\right)^{\frac{1}{2}},
\end{aligned}
\tag{43}
$$

where

$$
M^{**}(x,y)=\frac{1}{2}\left(\int_{\vartheta_0}^{x}\left(\frac{\Phi(K(s))}{K(s)}\right)^{2}\nabla s\right)^{\frac{1}{2}}\left(\int_{\vartheta_0}^{y}\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right)^{2}\nabla\vartheta\right)^{\frac{1}{2}},
$$

which is [33, Theorem 3.5].

Remark 3.8 As a particular case of Theorem 3.6 if $\mathbb{T}=\mathbb{Z}$, $\vartheta_0=0$, then $\rho(u)=u-1$ and (34) reduces to

$$\begin{aligned}\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}&\frac{\Phi(\Pi_s)\Psi(\Omega_\vartheta)}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\\&\leq M_0(\lambda,\mu,p,q)\left(\sum_{s=1}^{p}(p-s+1)\left(k_s\Phi\left(\frac{\omega_s}{k_s}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\\&\quad\times\left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)\left(l_\vartheta\Phi\left(\frac{\omega_\vartheta}{l_\vartheta}\right)\right)^{\mu}\right)^{\frac{1}{\mu}},\end{aligned}\tag{44}$$

where

$$M_0(\lambda,\mu,p,q)=\frac{1}{\lambda+\mu}\left(\sum_{s=1}^{p}\left(\frac{\Phi(K_s)}{K_s}\right)^{\frac{\lambda}{\lambda-1}}\right)^{\frac{\lambda-1}{\lambda}}\left(\sum_{\vartheta=1}^{q}\left(\frac{\Psi(L_\vartheta)}{L_\vartheta}\right)^{\frac{\mu}{\mu-1}}\right)^{\frac{\mu-1}{\mu}},$$

which is [13, Theorem 2.2].

Remark 3.9 As a particular case of Theorem 3.6 if $\mathbb{T}=\mathbb{R}$, $t_0=0$, then $\rho(u)=u$ and (34) reduces to

$$\begin{aligned}\int_0^x\int_0^y&\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\,ds\,dt\\&\leq M_0^*(\lambda,\mu,x,y)\left(\int_0^x(x-s)\left(k(s)\Phi\left(\frac{\omega_1(s)}{k(s)}\right)\right)^{\lambda}ds\right)^{\frac{1}{\lambda}}\\&\quad\times\left(\int_0^y(y-\vartheta)\left(l(\vartheta)\Psi\left(\frac{\omega_2(\vartheta)}{l(\vartheta)}\right)\right)^{\mu}d\vartheta\right)^{\frac{1}{\mu}}.\end{aligned}\tag{45}$$

where

$$\begin{aligned}M_0^*(\lambda,\mu,x,y)&=\frac{1}{\lambda+\mu}\left(\int_0^x\left(\frac{\Phi(K(s))}{K(s)}\right)^{\frac{\lambda}{\lambda-1}}ds\right)^{\frac{\lambda-1}{\lambda}}\\&\quad\times\left(\int_0^y\left(\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right)^{\frac{\mu}{\mu-1}}d\vartheta\right)^{\frac{\mu-1}{\mu}},\end{aligned}$$

which is [13, Theorem 3.2].

Our next outcome deals with a further generalization of the inequality in (34).

Theorem 3.10 *Let $s,\vartheta,\vartheta_0\in\mathbb{T}$, and ω_1,ω_2 be as in Theorem 3.1. Define*

$$\Pi(s)=\frac{1}{s-\vartheta_0}\int_{\vartheta_0}^{s}\omega_1(\xi)\nabla\xi\quad\text{and}\quad\Omega(\vartheta)=\frac{1}{\vartheta-\vartheta_0}\int_{\vartheta_0}^{\vartheta}\omega_2(\xi)\nabla\xi.\tag{46}$$

Then for $s \in [\vartheta_0, x]_{\mathbb{T}}$ *and* $\vartheta \in [\vartheta_0, y]_{\mathbb{T}}$, *we have*

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))(s-\vartheta_0)(\vartheta-\vartheta_0)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\
&\quad\le N(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(\Phi\bigl(\omega_1(s)\bigr)\bigr)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\qquad\times\left(\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}\tag{47}$$

where

$$N(\lambda,\mu,x,y) = \frac{1}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

Proof Based on the assumptions and the inequality of Jensen (12), we can see that

$$\begin{aligned}
\Phi\bigl(\Pi(s)\bigr) &= \Phi\left(\frac{1}{s-\vartheta_0}\int_{\vartheta_0}^{s}\omega_1(\xi)\nabla\xi\right) \\
&\le \frac{1}{s-\vartheta_0}\int_{\vartheta_0}^{s}\Phi\bigl(\omega_1(\xi)\bigr)\nabla\xi.
\end{aligned}\tag{48}$$

By applying (11) to (48) with indices λ, $\lambda/(\lambda-1)$, we have

$$\Phi\bigl(\Pi(s)\bigr) \le \frac{1}{s-\vartheta_0}(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{s}\bigl(\Phi\bigl(\omega_1(\xi)\bigr)\bigr)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}.\tag{49}$$

This implies that

$$\Phi\bigl(\Pi(s)\bigr)(s-\vartheta_0) \le (s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{s}\bigl(\Phi\bigl(\omega_1(\xi)\bigr)\bigr)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}.\tag{50}$$

Analogously,

$$\Psi\bigl(\Omega(\vartheta)\bigr)(\vartheta-\vartheta_0) \le (\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{t}\bigl(\Psi\bigl(\omega_2(\xi)\bigr)\bigr)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.\tag{51}$$

From (50) and (51), we get

$$\begin{aligned}
&\Phi\bigl(\Pi(s)\bigr)\Psi\bigl(\Omega(\vartheta)\bigr)(s-\vartheta_0)(\vartheta-\vartheta_0) \\
&\quad\le (s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{s}\bigl(\Phi\bigl(\omega_1(\xi)\bigr)\bigr)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}} \\
&\qquad\times\left(\int_{\vartheta_0}^{t}\bigl(\Psi\bigl(\omega_2(\xi)\bigr)\bigr)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.
\end{aligned}\tag{52}$$

Applying (24) to the term $(s-\vartheta_0)^{(\lambda-1)/\lambda} \times (\vartheta-\vartheta_0)^{(\mu-1)/\mu}$ gives

$$\begin{aligned}
&\Phi\big(\Pi(s)\big)\Psi\big(\Omega(\vartheta)\big)(s-\vartheta_0)(\vartheta-\vartheta_0)\\
&\quad\le \frac{\lambda\mu}{\lambda+\mu}\left(\frac{(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}}{\lambda}+\frac{(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}{\mu}\right)\\
&\qquad\times\left(\int_{\vartheta_0}^{s}\big(\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\left(\int_{\vartheta_0}^{t}\big(\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.
\end{aligned}\tag{53}$$

From (53), we have

$$\begin{aligned}
&\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))(s-\vartheta_0)(\vartheta-\vartheta_0)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\\
&\quad\le\frac{1}{\lambda+\mu}\left(\int_{\vartheta_0}^{s}\big(\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\left(\int_{\vartheta_0}^{t}\big(\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}.
\end{aligned}\tag{54}$$

Integrating both sides of (54) and using (11) again with indices λ, $\lambda/(\lambda-1)$ and μ, $\mu/(\mu-1)$, we get

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))(s-\vartheta_0)(\vartheta-\vartheta_0)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta\\
&\quad\le\frac{1}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{x}\left(\int_{\vartheta_0}^{s}\big(\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)\nabla s\right)^{\frac{1}{\lambda}}\\
&\qquad\times\left(\int_{\vartheta_0}^{y}\left(\int_{\vartheta_0}^{\vartheta}\big(\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)\nabla\vartheta\right)^{\frac{1}{\mu}}.
\end{aligned}\tag{55}$$

Applying Lemma 2.5 to (55), we find that

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))(s-\vartheta_0)(\vartheta-\vartheta_0)}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta\\
&\quad\le\frac{1}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}\\
&\qquad\times\left(\int_{\vartheta_0}^{x}\big(x-\rho(s)\big)\big(\Phi\big(\omega_1(s)\big)\big)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}}\\
&\qquad\times\left(\int_{\vartheta_0}^{y}\big(y-\rho(\vartheta)\big)\big(\Psi\big(\omega_2(\vartheta)\big)\big)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}}\\
&\quad=N(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\big(x-\rho(s)\big)\big(\Phi\big(\omega_1(s)\big)\big)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}}\\
&\qquad\times\left(\int_{\vartheta_0}^{y}\big(y-\rho(\vartheta)\big)\big(\Psi\big(\omega_2(\vartheta)\big)\big)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}$$

which is (47). □

Corollary 3.3 *If we take* $1/\lambda + 1/\mu = 1$ *in* (47), *then we get*

$$\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))(s-\vartheta_0)(\vartheta-\vartheta_0)}{\mu(s-\vartheta_0)^{\lambda-1}+\lambda(\vartheta-\vartheta_0)^{\mu-1}}\nabla s\nabla\vartheta$$
$$\leq N^*(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(\Phi\bigl(\omega_1(s)\bigr)\bigr)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}}, \tag{56}$$

where

$$N^*(\lambda,\mu,x,y)=\frac{1}{\lambda\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

Remark 3.11 As a particular case of Corollary 3.3, if $\lambda = \mu = 2$, then we get

$$\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))(s-\vartheta_0)(\vartheta-\vartheta_0)}{s+\vartheta-2\vartheta_0}\nabla s\nabla\vartheta$$
$$\leq\frac{1}{2}\left((x-\vartheta_0)\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(\Phi\bigl(\omega_1(s)\bigr)\bigr)^{2}\nabla s\right)^{\frac{1}{2}}$$
$$\times\left((y-\vartheta_0)\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{2}\nabla\vartheta\right)^{\frac{1}{2}}, \tag{57}$$

which is [33, Theorem 3.6].

Remark 3.12 As a particular case of Theorem 3.10, if $\mathbb{T}=\mathbb{Z}$, $\vartheta_0 = 0$, then $\rho(u) = u - 1$ and (47) reduces to

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{\Phi(\Pi_s)\Psi(\Omega_\vartheta)s\vartheta}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}$$
$$\leq N_0(\lambda,\mu,p,q)\left(\sum_{s=1}^{p}(p-s+1)\bigl(\Phi(\omega_s)\bigr)^{\lambda}\right)^{\frac{1}{\lambda}}\left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)\bigl(\Phi(\omega_\vartheta)\bigr)^{\mu}\right)^{\frac{1}{\mu}}, \tag{58}$$

where

$$N_0(\lambda,\mu,p,q)=\frac{1}{\lambda+\mu}p^{\frac{\lambda-1}{\lambda}}q^{\frac{\mu-1}{\mu}},$$

which is [13, Theorem 2.3].

Remark 3.13 As a particular state of Theorem 3.10, if $\mathbb{T}=\mathbb{R}$, $t_0 = 0$, then $\rho(u) = u$ and (47) reduces to

$$\int_{0}^{x}\int_{0}^{y}\frac{\Phi(\Pi(s))\Psi(\Omega(\vartheta))s\vartheta}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\,ds\,d\vartheta$$
$$\leq N_0^*(\lambda,\mu,x,y)\left(\int_{0}^{x}(x-s)\bigl(\Phi\bigl(\omega_1(s)\bigr)\bigr)^{\lambda}\,ds\right)^{\frac{1}{\lambda}}$$
$$\times\left(\int_{0}^{y}(y-\vartheta)\bigl(\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{\mu}\,d\vartheta\right)^{\frac{1}{\mu}}, \tag{59}$$

where

$$N_0^*(\lambda,\mu,x,y)=\frac{1}{\lambda+\mu}x^{\frac{\lambda-1}{\lambda}}y^{\frac{\mu-1}{\mu}},$$

which is [13, Theorem 3.3].

Theorem 3.14 *Let $s,\vartheta,\vartheta_0\in\mathbb{T}$ and $\omega_1,\omega_2,k,l,H,L$ be as in Theorem 3.6. Define*

$$\Pi(s)=\frac{1}{K(s)}\int_{\vartheta_0}^{s}k(\xi)\omega_1(\xi)\nabla\xi \quad \textit{and} \quad \Omega(\vartheta)=\frac{1}{L(\vartheta)}\int_{\vartheta_0}^{\vartheta}l(\xi)\omega_2(\xi)\nabla\xi. \tag{60}$$

Then for $s\in[\vartheta_0,y]_{\mathbb{T}}$ and $\vartheta\in[\vartheta_0,x]_{\mathbb{T}}$, we get

$$\begin{aligned}&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta\\&\quad\le W(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\big(x-\rho(s)\big)\big(k(s)\Phi\big(\omega_1(s)\big)\big)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}}\\&\qquad\times\left(\int_{\vartheta_0}^{y}\big(y-\rho(\vartheta)\big)\big(l(\vartheta)\Psi\big(\omega_2(\vartheta)\big)\big)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},\end{aligned} \tag{61}$$

where

$$W(\lambda,\mu,x,y)=\frac{1}{\lambda+\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

Proof Based on the assumptions and the inequality of Jensen (12), we find that

$$\begin{aligned}\Phi\big(\Pi(s)\big)&=\Phi\left(\frac{1}{K(s)}\int_{\vartheta_0}^{s}k(\xi)\omega_1(\xi)\nabla\xi\right)\\&\le\frac{1}{K(s)}\int_{\vartheta_0}^{s}k(\xi)\Phi\big(\omega_1(\xi)\big)\nabla\xi.\end{aligned} \tag{62}$$

By applying (11) to (62) with indices λ, $\lambda/(\lambda-1)$, we have

$$\Phi\big(\Pi(s)\big)\le\frac{1}{K(s)}(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{s}\big(k(\xi)\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}. \tag{63}$$

From (63), we get

$$\Phi\big(\Pi(s)\big)K(s)\le(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}\left(\int_{\vartheta_0}^{s}\big(k(\xi)\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}. \tag{64}$$

Similarly, we also obtain

$$\Psi(\Omega(\vartheta)L(\vartheta)\le(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{\vartheta}\big(l(\xi)\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}. \tag{65}$$

From (64) and (65), we find that

$$\begin{aligned} &K(s)L(\vartheta)\Phi\big(\Pi(s)\big)\Psi\big(\Omega(\vartheta)\big) \\ &\quad\le (s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{s}\big(k(\xi)\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}} \\ &\qquad\times\left(\int_{\vartheta_0}^{\vartheta}\big(l(\xi)\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}. \end{aligned} \tag{66}$$

Applying (24) to the term $(s-\vartheta_0)^{(\lambda-1)/\lambda}\times(\vartheta-\vartheta_0)^{(\mu-1)/\mu}$ gives

$$\begin{aligned} &K(s)L(\vartheta)\Phi\big(\Pi(s)\big)\Psi\big(\Omega(\vartheta)\big) \\ &\quad\le \frac{\lambda\mu}{\lambda+\mu}\left(\frac{(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}}{\lambda}+\frac{(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}{\mu}\right) \\ &\qquad\times\left(\int_{\vartheta_0}^{s}\big(k(\xi)\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\left(\int_{\vartheta_0}^{\vartheta}\big(l(\xi)\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}. \end{aligned} \tag{67}$$

This implies that

$$\begin{aligned} &\frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \\ &\quad\le \frac{1}{\lambda+\mu}\left(\int_{\vartheta_0}^{s}\big(k(\xi)\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)^{\frac{1}{\lambda}}\left(\int_{\vartheta_0}^{\vartheta}\big(l(\xi)\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)^{\frac{1}{\mu}}. \end{aligned} \tag{68}$$

Integrating both sides of (68) and using (11) again with indices λ, $\lambda/(\lambda-1)$ and μ, $\mu/(\mu-1)$, we get

$$\begin{aligned} &\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\ &\quad\le \frac{1}{\lambda+\mu}(s-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(\vartheta-\vartheta_0)^{\frac{\mu-1}{\mu}}\left(\int_{\vartheta_0}^{x}\left(\int_{\vartheta_0}^{s}\big(k(\xi)\Phi\big(\omega_1(\xi)\big)\big)^{\lambda}\nabla\xi\right)\nabla s\right)^{\frac{1}{\lambda}} \\ &\qquad\times\left(\int_{\vartheta_0}^{y}\left(\int_{\vartheta_0}^{\vartheta}\big(l(\xi)\Psi\big(\omega_2(\xi)\big)\big)^{\mu}\nabla\xi\right)\nabla\vartheta\right)^{\frac{1}{\mu}}. \end{aligned} \tag{69}$$

Applying Lemma 2.5 to (69), we find that

$$\begin{aligned} &\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y}\frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu(s-\vartheta_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-\vartheta_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla\vartheta \\ &\quad\le W(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\big(x-\rho(s)\big)\big(k(s)\Phi\big(\omega_1(s)\big)\big)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\ &\qquad\times\left(\int_{\vartheta_0}^{y}\big(y-\rho(\vartheta)\big)\big(l(\vartheta)\Psi\big(\omega_2(\vartheta)\big)\big)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}}, \end{aligned}$$

which is (61). □

Corollary 3.4 *If we take* $1/\lambda + 1/\mu = 1$ *in* (61), *then*

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta)}{\mu(s-\vartheta_0)^{\lambda-1}+\lambda(\vartheta-\vartheta_0)^{\mu-1}}\nabla s\nabla\vartheta \\
&\quad\le W^{*}(\lambda,\mu,x,y)\left(\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(k(s)\Phi\bigl(\omega_1(s)\bigr)\bigr)^{\lambda}\nabla s\right)^{\frac{1}{\lambda}} \\
&\qquad\times\left(\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(l(\vartheta)\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{\mu}\nabla\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}\tag{70}$$

where

$$W^{*}(\lambda,\mu,x,y)=\frac{1}{\lambda\mu}(x-\vartheta_0)^{\frac{\lambda-1}{\lambda}}(y-\vartheta_0)^{\frac{\mu-1}{\mu}}.$$

Remark 3.15 As a particular case of Corollary 3.4, if $\lambda=\mu=2$, then we get

$$\begin{aligned}
&\int_{\vartheta_0}^{x}\int_{\vartheta_0}^{y} \frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta)}{s+\vartheta-2\vartheta_0}\nabla s\nabla\vartheta \\
&\quad\le \frac{1}{2}\left((x-\vartheta_0)\int_{\vartheta_0}^{x}\bigl(x-\rho(s)\bigr)\bigl(k(s)\Phi\bigl(\omega_1(s)\bigr)\bigr)^{2}\nabla s\right)^{\frac{1}{2}} \\
&\qquad\times\left((y-\vartheta_0)\int_{\vartheta_0}^{y}\bigl(y-\rho(\vartheta)\bigr)\bigl(l(\vartheta)\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{2}\nabla\vartheta\right)^{\frac{1}{2}},
\end{aligned}\tag{71}$$

which is [33, Theorem 3.7].

Remark 3.16 As a particular case of Theorem 3.14, if $\mathbb{T}=\mathbb{Z}$, $\vartheta_0=0$, then $\rho(u)=u-1$ and (61) reduces to

$$\begin{aligned}
\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{K_sL_\vartheta\Phi(\Pi_s)\Psi(\Omega_\vartheta)}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} &\le W_0(\lambda,\mu,p,q)\left(\sum_{s=1}^{p}(p-s+1)\bigl(k_s\Phi(\omega_s)\bigr)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\quad\times\left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)\bigl(l_\vartheta\Phi(\omega_\vartheta)\bigr)^{\mu}\right)^{\frac{1}{\mu}},
\end{aligned}\tag{72}$$

where

$$W_0(\lambda,\mu,p,q)=\frac{1}{\lambda+\mu}p^{\frac{\lambda-1}{\lambda}}q^{\frac{\mu-1}{\mu}},$$

which is [13, Theorem 2.4].

Remark 3.17 As a particular case of Theorem 3.14, if $\mathbb{T}=\mathbb{R}$, $t_0=0$, then $\rho(u)=u$ and (61) reduces to

$$\begin{aligned}
&\int_{0}^{x}\int_{0}^{y} \frac{K(s)L(\vartheta)\Phi(\Pi(s))\Psi(\Omega(\vartheta))}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\,ds\,d\vartheta \\
&\quad\le W_0^{*}(\lambda,\mu,x,y)\left(\int_{0}^{x}(x-s)\bigl(k(s)\Phi\bigl(\omega_1(s)\bigr)\bigr)^{\lambda}\,ds\right)^{\frac{1}{\lambda}} \\
&\qquad\times\left(\int_{0}^{y}(y-\vartheta)\bigl(l(\vartheta)\Psi\bigl(\omega_2(\vartheta)\bigr)\bigr)^{\mu}\,d\vartheta\right)^{\frac{1}{\mu}},
\end{aligned}\tag{73}$$

where

$$W_0^*(\lambda,\mu,x,y)=\frac{1}{\lambda+\mu}x^{\frac{\lambda-1}{\lambda}}y^{\frac{\mu-1}{\mu}},$$

which is [13, Theorem 3.4].

Remark 3.18 Clearly, Theorems 3.1, 3.6, 3.10, and 3.14 present the corresponding results of Theorems 6, 9, 12, and 15 in [18], respectively, for time scale delta calculus. Likewise, Corollaries 3.1, 3.2, 3.3, and 3.4 display the corresponding results of Theorems 3.1, 3.2, 3.3, and 3.4 in [17], respectively, for delta time scale calculus.

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Authors' contributions
All authors conceived the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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