Chu et al. *Advances in Difference Equations* (2020) 2020:594
https://doi.org/10.1186/s13662-020-03059-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Generalizations of Hermite–Hadamard like inequalities involving χ_κ-Hilfer fractional integrals

Yu-Ming Chu[1], Muhammad Uzair Awan[2*], Sadia Talib[2], Muhammad Aslam Noor[3] and Khalida Inayat Noor[3]

*Correspondence:
awan.uzair@gmail.com
[2]Department of Mathematics, Government College University, Faisalabad, Pakistan
Full list of author information is available at the end of the article

Abstract

The main objective of this paper is to obtain a new κ-fractional analogue of Hermite–Hadamard's inequality using the class of s-convex functions and χ_κ-Hilfer fractional integrals. In order to obtain other main results of the paper we derive two new fractional integral identities using the definitions of χ_κ-Hilfer fractional integrals. For the validity of these identities we also take some particular examples. Using these identities we then obtain some more new variants of Hermite–Hadamard's inequality using s-convex functions.

MSC: 26D10; 26D15; 26A51; 26A33

Keywords: Convex; s-convex; Fractional; Hilfer; Hermite–Hadamard inequality

1 Introduction

Theory of convexity has played significant role in the development of theory of inequalities. Many famous results known in theory of inequalities are direct consequences of the applications of convex functions. In this regard Hermite–Hadamard's inequality which can be viewed as necessary and sufficient condition for a function to be convex is one of the most studied result. In recent years it has been observed that a number of new generalizations of classical Hermite–Hadamard's inequality have been obtained in the literature. Dragomir and Pearce [3] has written a very informative monograph on some recent developments and applications of Hermite–Hadamard's inequality.

Sarikaya et al. [16] utilized the concepts of fractional calculus and obtained fractional analogues of Hermite–Hadamard's inequality. This particular article has opened a new venue of research and consequently several new fractional analogues of Hermite–Hadamard's inequality have been obtained using different approaches. Since the birth of fractional calculus this subject has received special attention by the mathematicians and resultantly the classical concepts of fractional calculus have been extended and generalized in different directions according to the need of problem. This motivated inequalities experts and as a result they used new generalized concepts of fractional calculus in obtaining novel generalized fractional analogues of classical inequalities.

The main motivation of this paper is to derive some new fractional analogues of Hermite–Hadamard's inequality using χ_κ-Hilfer fractional integrals via s-convex functions of Breckner type. In order to obtain the main results of the paper we first derive new fractional integral identities. To check the validity of these new identities we take some particular examples. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

Before we proceed, let us recall some previously known concepts and results which will be used during the study of this paper.

Riemann–Liouville fractional integrals are defined as follows.

Definition 1.1 ([8]) Let $\Xi \in L_1[a,b]$. Then Riemann–Liouville integrals $J^{\alpha}_{a^+}\Xi$ and $J^{\alpha}_{b^-}\Xi$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+}\Xi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\lambda)^{\alpha-1}\Xi(\lambda)\,\mathrm{d}\lambda, \quad x > a,$$

and

$$J^{\alpha}_{b^-}\Xi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\lambda-x)^{\alpha-1}\Xi(\lambda)\,\mathrm{d}\lambda, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}\,\mathrm{d}x,$$

is the well-known Gamma function.

Mubeen and Habibullah [10] were the first to define the notion of κ-fractional integrals. Sarikaya et al. [15] introduced the κ-analogue of Riemann–Liouville fractional integrals and discussed some of its basic properties. They defined this concept in the following way: To be more precise let Ξ be piecewise continuous on $I^* = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then, for $\lambda > 0$, we consider κ-Riemann–Liouville fractional integral of Ξ of order α

$${}_\kappa J^{\alpha}_{a}\Xi(x) = \frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_a^x (x-\lambda)^{\frac{\alpha}{\kappa}-1}\Xi(\lambda)\,\mathrm{d}\lambda, \quad x > a, \kappa > 0.$$

If $\kappa \to 1$, then κ-Riemann–Liouville fractional integrals reduces to classical Riemann–Liouville fractional integral.

Another important generalization of Riemann–Liouville fractional integrals the generalized R–L integrals with respect to another function χ (in the Hilfer sense [5]).

Definition 1.2 ([8]) Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $\chi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\chi'(x)$ on (a,b). The left and right-sided χ-fractional integrals in the Hilfer sense of a function Ξ with respect to another function χ on $[a,b]$ are defined as

$$I^{\alpha;\chi}_{a^+}\Xi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \chi'(\lambda)\big(\chi(x)-\chi(\lambda)\big)^{\alpha-1}\Xi(\lambda)\,\mathrm{d}\lambda,$$

$$I_{b^-}^{\alpha;\chi}\,\Xi(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\chi'(\lambda)\big(\chi(\lambda)-\chi(x)\big)^{\alpha-1}\,\Xi(\lambda)\,\mathrm{d}\lambda,$$

respectively; $\Gamma(\cdot)$ is the gamma function.

Liu et al. [9] and Zhao et al. [20] obtained some interesting results pertaining to Hermite–Hadamard's inequality involving χ_κ-Riemann–Liouville fractional integrals.

Recently Awan et al. [1] introduced the notion of χ_κ-Hilfer fractional integrals and obtained some new variants of Hermite–Hadamard's inequality.

Definition 1.3 ([1]) Let (a,b) $(-\infty\le a<b\le\infty)$ be a finite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $\chi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\chi'(x)$ on (a,b). The left- and right-sided χ_κ-fractional integrals in the Hilfer sense of a function Ξ with respect to another function χ_κ on $[a,b]$ and $\kappa>0$ are defined as

$$
\begin{aligned}
{}_\kappa I_{a^+}^{\alpha;\chi}\,\Xi(x)&=\frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_a^x\chi'(\lambda)\big(\chi(x)-\chi(\lambda)\big)^{\frac{\alpha}{\kappa}-1}\,\Xi(\lambda)\,\mathrm{d}\lambda,\\
{}_\kappa I_{b^-}^{\alpha;\chi}\,\Xi(x)&=\frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_x^b\chi'(\lambda)\big(\chi(\lambda)-\chi(x)\big)^{\frac{\alpha}{\kappa}-1}\,\Xi(\lambda)\,\mathrm{d}\lambda,
\end{aligned}
$$

respectively;

$$\Gamma_\kappa(x)=\int_0^\infty\lambda^{x-1}e^{-\frac{\lambda^\kappa}{\kappa}}\,\mathrm{d}\lambda,\quad \Re(x)>0,$$

is the κ-analogue of the gamma function.

By taking $\kappa\to1$, Definition 1.3 reduces to Definition 1.2. This shows that χ_κ-fractional integrals in the Hilfer sense is the significant generalization of χ-fractional integrals in the Hilfer sense.

The κ-analogues of beta function and incomplete beta function are, respectively, defined as

$$\mathrm{B}_\kappa(x,y)=\frac{1}{\kappa}\int_0^1\lambda^{\frac{x}{\kappa}-1}(1-\lambda)^{\frac{x}{\kappa}-1}\,\mathrm{d}\lambda$$

and

$$\mathrm{B}_\kappa(z;x,y)=\frac{1}{\kappa}\int_0^z\lambda^{\frac{x}{\kappa}-1}(1-\lambda)^{\frac{b}{y}-1}\,\mathrm{d}\lambda.$$

For some more interesting details and applications about some special functions and their generalizations, see [4, 6, 7, 11–14, 17–19].

Proposition 1.4 *Let $a_i,b_i>0$ for $i=1,2,\dots,n$, then*

$$\sum_{i=1}^n(a_i+b_i)^r\le\sum_{i=1}^n a_i^r+\sum_{i=1}^n b_i^r \tag{1.1}$$

for $0<r<1$.

We now define the class of s-convex functions of Breckner type.

Definition 1.5 ([2]) Let $s \in (0,1]$. A function $\Xi : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an s-convex function, if

$$\Xi\big(\lambda x_1 + (1-\lambda)x_2\big) \le \lambda^s \Xi(x_1) + (1-\lambda)^s \Xi(x_2), \quad \forall x_1, x_2 \in I, \lambda \in [0,1].$$

2 Main results

We now discuss our main results. The first result is the fractional analogue of Hermite–Hadamard's inequality using s-convexity property of the functions involving χ_κ-Hilfer fractional integrals.

Theorem 2.1 *Let $0 \le a_1 < a_2$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a positive function and $\Xi \in L_1[a_1, a_2]$, Also suppose that Ξ is an s-convex function on $[a_1, a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, having a continuous derivative $\chi'(x)$ on (a_1, a_2) and $\alpha \in (0,1)$, then, for $\kappa > 0$, we have*

$$\begin{aligned} &2^{s-1}\Xi\left(\frac{a_1+a_2}{2}\right) \\ &\quad \le \frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big) + {}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right] \\ &\quad \le \left[\frac{3\alpha}{\alpha+ks} - \frac{\alpha}{(\alpha+sk)2^{\frac{\alpha+sk}{\kappa}}}\right]\frac{\Xi(a_1)+\Xi(a_2)}{2}. \end{aligned}$$

Proof Let $x_1, x_2 \in [a_1, a_2]$ and using the s-convexity of Ξ, we have

$$\Xi\left(\frac{x_1+x_2}{2}\right) \le \frac{\Xi(x_1)}{2^s} + \frac{\Xi(x_2)}{2^s}.$$

Let $x_1 = \lambda a_1 + (1-\lambda)a_2$ and $x_2 = (1-\lambda)a_1 + \lambda a_2$, we have

$$2^s\Xi\left(\frac{a_1+a_2}{2}\right) \le \Xi\big(\lambda a_1 + (1-\lambda)a_2\big) + \Xi\big((1-\lambda)a_1 + \lambda a_2\big).$$

Multiplying both sides by $\lambda^{\frac{\alpha}{\kappa}-1}$ and then integrating, we have

$$\frac{2^s\kappa}{\alpha}\Xi\left(\frac{a_1+a_2}{2}\right) \le \int_0^1 \lambda^{\frac{\alpha}{\kappa}-1}\Xi\big(\lambda a_1 + (1-\lambda)a_2\big)\,\mathrm{d}\lambda + \int_0^1 \lambda^{\frac{\alpha}{\kappa}-1}\Xi\big((1-\lambda)a_1 + \lambda a_2\big)\,\mathrm{d}\lambda.$$

Now

$$\begin{aligned} &\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big) +{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right] \\ &\quad = \frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\frac{1}{\kappa\Gamma_\kappa(\alpha)}\left[\int_{\chi^{-1}(a_1)}^{\chi^{-1}(a_2)}\big(a_2-\chi(\nu)\big)^{\frac{\alpha}{\kappa}}(\Xi\circ\chi)(\nu)\chi'(\nu)\,\mathrm{d}\nu \right. \\ &\qquad \left. + \int_{\chi^{-1}(a_1)}^{\chi^{-1}(a_2)}\big(\chi(\nu)-a_1\big)^{\frac{\alpha}{\kappa}}(\Xi\circ\chi)(\nu)\chi'(\nu)\,\mathrm{d}\nu\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{2\kappa}\left[\int_0^1 \lambda^{\frac{\alpha}{\kappa}-1}\Xi\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda+\int_0^1 \lambda^{\frac{\alpha}{\kappa}-1}\Xi\big((1-\lambda)a_1+\lambda a_2\big)\,\mathrm{d}\lambda\right]\\
&\geq 2^{s-1}\Xi\left(\frac{a_1+a_2}{2}\right).
\end{aligned}
$$

To prove the right-hand side, we use the fact that Ξ is an s-convex function, then

$$\Xi\big(\lambda a_1+(1-\lambda)a_2\big)\leq \lambda^s\Xi(a_1)+(1-\lambda)^s\Xi(a_2)$$

and

$$\Xi\big((1-\lambda)a_1+\lambda a_2\big)\leq (1-\lambda)^s\Xi(a_1)+\lambda^s\Xi(a_2).$$

Now

$$\Xi\big(\lambda a_1+(1-\lambda)a_2\big)+\Xi\big((1-\lambda)a_1+\lambda a_2\big)\leq \big(\lambda^s+(1-\lambda)^s\big)\big(\Xi(a_1)+\Xi(a_2)\big).$$

Multiplying both sides by $\lambda^{\frac{\alpha}{\kappa}-1}$ and then integrating with respect to λ on $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 \lambda^{\frac{\alpha}{\kappa}-1}\Xi\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda+\int_0^1 \lambda^{\frac{\alpha}{\kappa}-1}\Xi\big((1-\lambda)a_1+\lambda a_2\big)\,\mathrm{d}\lambda\\
&\quad\leq \left[\frac{3\kappa}{\alpha+ks}-\frac{\kappa}{(\alpha+sk)2^{\frac{\alpha+sk}{\kappa}}}\right]\big[\Xi(a_1)+\Xi(a_2)\big].
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right]\\
&\quad\leq \left[\frac{3\alpha}{\alpha+ks}-\frac{\alpha}{(\alpha+sk)2^{\frac{\alpha+sk}{\kappa}}}\right]\frac{\Xi(a_1)+\Xi(a_2)}{2}.
\end{aligned}
$$

The proof is complete. □

We now prove two new fractional integral identities which will be used as auxiliary results in the development of our next results.

Lemma 2.2 *Let $a_1<a_2$ and $\Xi:[a_1,a_2]\to\mathbb{R}$ be a differentiable mapping on (a_1,a_2). Also suppose that $\Xi\in L[a_1,a_2]$, then, for $\kappa>0$, we have*

$$
\begin{aligned}
&\frac{\Xi(a_1)+\Xi(a_2)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right]\\
&\quad=\frac{a_2-a_1}{2}\int_0^1\left[(1-\lambda)^{\frac{\alpha}{\kappa}}-\lambda^{\frac{\alpha}{\kappa}}\right]\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda.
\end{aligned}
$$

Proof From [1], we have

$$
\begin{aligned}
&\frac{\Xi(a_1)+\Xi(a_2)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right]\\
&\quad=\frac{1}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\int_{\chi^{-1}(a_1)}^{\chi^{-1}(a_2)}\left[\big(\chi(\nu)-a_1\big)^{\frac{\alpha}{\kappa}}-\big(a_2-\chi(\nu)\big)^{\frac{\alpha}{\kappa}}\right]\big(\Xi'\circ\chi\big)(\nu)\chi'(\nu)\,\mathrm{d}\nu
\end{aligned}
$$

$$= \frac{1}{2}\int_{\chi^{-1}(a_1)}^{\chi^{-1}(a_2)}\left[\left(\frac{\chi(\nu)-a_1}{a_2-a_1}\right)^{\frac{\alpha}{\kappa}}-\left(\frac{a_2-\chi(\nu)}{a_2-a_1}\right)^{\frac{\alpha}{\kappa}}\right]\left(\Xi'\circ\chi\right)(\nu)\chi'(\nu)\,\mathrm{d}\nu$$
$$= \frac{a_2-a_1}{2}\int_0^1\left[(1-\lambda)^{\frac{\alpha}{\kappa}}-\lambda^{\frac{\alpha}{\kappa}}\right]\Xi'\left(\lambda a_1+(1-\lambda)a_2\right)\mathrm{d}\lambda.$$

This completes the proof. □

Example 2.3 Let $a_1 = 2$, $a_2 = 3$, $\alpha = \frac{1}{2}$, $\kappa = 2$, $\Xi(x) = x^2$, $\chi(x) = x$. Then all the assumptions in Lemma 2.2 are satisfied.

One can observe that $\frac{\Xi(a_1)+\Xi(a_2)}{2} = \frac{13}{2}$. We have

$$\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]$$
$$= \frac{\Gamma_2(\frac{1}{2})}{8}\left[\frac{1}{\Gamma_2(\frac{1}{2})}\int_2^3\lambda^2(3-\lambda)^{-\frac{3}{4}}\,\mathrm{d}\lambda+\frac{1}{\Gamma_2(\frac{1}{2})}\int_2^3\lambda^2(\lambda-2)^{-\frac{3}{4}}\,\mathrm{d}\lambda\right] = \frac{577}{90}.$$

It follows that

$$\frac{\Xi(a_1)+\Xi(a_2)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]$$
$$= \frac{4}{45}.$$

On the other hand

$$\frac{a_2-a_1}{2}\int_0^1\left[(1-\lambda)^{\frac{\alpha}{\kappa}}-\lambda^{\frac{\alpha}{\kappa}}\right]\Xi'\left(\lambda a_1+(1-\lambda)a_2\right)\mathrm{d}\lambda$$
$$= \frac{1}{2}\int_0^1\left[(1-\lambda)^{\frac{1}{4}}-\lambda^{\frac{1}{4}}\right](6-2\lambda)\,\mathrm{d}\lambda = \frac{4}{45}.$$

Example 2.4 Let $a_1 = 2$, $a_2 = 3$, $\alpha = \frac{1}{2}$, $\kappa = \frac{1}{2}$, $\Xi(x) = x^2$, $\chi(x) = x$. Then all the assumptions in Lemma 2.2 are satisfied.

One can observe that $\frac{\Xi(a_1)+\Xi(a_2)}{2} = \frac{13}{2}$. We have

$$\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]$$
$$= \frac{\Gamma_{\frac{1}{2}}(\frac{1}{2})}{2}\left[\frac{1}{\Gamma_{\frac{1}{2}}(\frac{1}{2})}\int_2^3\lambda^2\,\mathrm{d}\lambda+\frac{1}{\Gamma_{\frac{1}{2}}(\frac{1}{2})}\int_2^3\lambda^2\,\mathrm{d}\lambda\right] = \frac{19}{3}.$$

It follows that

$$\frac{\Xi(a_1)+\Xi(a_2)}{2}-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]$$
$$= \frac{1}{6}.$$

On the other hand

$$\begin{aligned}&\frac{a_2-a_1}{2}\int_0^1\big[(1-\lambda)^{\frac{\alpha}{\kappa}}-\lambda^{\frac{\alpha}{\kappa}}\big]\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda\\&=\frac{1}{2}\int_0^1(1-2\lambda)(6-2\lambda)\,\mathrm{d}\lambda=\frac{1}{6}.\end{aligned}$$

Lemma 2.5 *Let $a_1 < a_2$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping on (a_1, a_2). Also suppose that $\Xi \in L[a_1, a_2]$, then, for $\kappa > 0$, we have*

$$\begin{aligned}&\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\big[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\big]-\Xi\left(\frac{a_1+a_2}{2}\right)\\&=\frac{a_2-a_1}{2}\int_0^1\big[\mu+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\big]\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda,\end{aligned}$$

where

$$\mu=\begin{cases}1, & \text{for } 0\le\lambda<\frac{1}{2},\\ -1, & \text{for } \frac{1}{2}\le\lambda<1.\end{cases}$$

Proof It suffices to show that

$$\begin{aligned}&\frac{a_2-a_1}{2}\int_0^1\mu\,\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda\\&=\frac{a_2-a_1}{2}\int_0^{\frac{1}{2}}\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda-\frac{a_2-a_1}{2}\int_{\frac{1}{2}}^1\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda\\&=\frac{\Xi(a_2)-\Xi(\frac{a_1+a_2}{2})}{2}+\frac{\Xi(a_1)-\Xi(\frac{a_1+a_2}{2})}{2}\\&=\frac{\Xi(a_1)+\Xi(a_2)}{2}-\Xi\left(\frac{a_1+a_2}{2}\right).\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}&\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\big[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\big]-\Xi\left(\frac{a_1+a_2}{2}\right)\\&=\left[\frac{\Xi(a_1)+\Xi(a_2)}{2}-\Xi\left(\frac{a_1+a_2}{2}\right)\right]\\&\quad-\left\{\frac{\Xi(a_1)+\Xi(a_2)}{2}\right.\\&\quad\left.-\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\big[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\big]\right\}\\&=\frac{a_2-a_1}{2}\int_0^1\mu\,\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda\end{aligned}$$

$$
\begin{aligned}
&- \frac{a_2 - a_1}{2} \int_0^1 \bigl[(1-\lambda)^{\frac{\alpha}{\kappa}} - \lambda^{\frac{\alpha}{\kappa}}\bigr] \Xi'\bigl(\lambda a_1 + (1-\lambda)a_2\bigr)\,\mathrm{d}\lambda \\
&= \frac{a_2 - a_1}{2} \int_0^1 \bigl[\mu + \lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\bigr] \Xi'\bigl(\lambda a_1 + (1-\lambda)a_2\bigr)\,\mathrm{d}\lambda.
\end{aligned}
$$

This completes the proof. □

Example 2.6 Let $a_1 = 2$, $a_2 = 3$, $\alpha = \frac{1}{2}$, $\kappa = 2$, $\Xi(x) = x^2$, $\chi(x) = x$. Then all the assumptions in the Lemma 2.5 are satisfied.

One can observe that $\Xi(\frac{a_1+a_2}{2}) = \frac{25}{4}$.

$$
\begin{aligned}
&\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}} \bigl[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\bigl(\chi^{-1}(a_2)\bigr) + {}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\bigl(\chi^{-1}(a_1)\bigr)\bigr] \\
&\quad = \frac{\Gamma_2(\frac{1}{2})}{8}\biggl[\frac{1}{\Gamma_2(\frac{1}{2})}\int_2^3 \lambda^2(3-\lambda)^{-\frac{3}{4}}\,\mathrm{d}\lambda + \frac{1}{\Gamma_2(\frac{1}{2})}\int_2^3 \lambda^2(\lambda-2)^{-\frac{3}{4}}\,\mathrm{d}\lambda\biggr] \\
&\quad = \frac{577}{90}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}} \bigl[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\bigl(\chi^{-1}(a_2)\bigr) + {}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\bigl(\chi^{-1}(a_1)\bigr)\bigr] - \Xi\biggl(\frac{a_1+a_2}{2}\biggr) \\
&\quad = \frac{29}{180}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\frac{a_2 - a_1}{2} \int_0^1 \bigl[\mu + \lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\bigr] \Xi'\bigl(\lambda a_1 + (1-\lambda)a_2\bigr)\,\mathrm{d}\lambda \\
&\quad = \frac{a_2 - a_1}{2} \int_0^1 \mu\,\Xi'\bigl(\lambda a_1 + (1-\lambda)a_2\bigr)\,\mathrm{d}\lambda \\
&\qquad - \frac{a_2 - a_1}{2} \int_0^1 \bigl[\lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\bigr] \Xi'\bigl(\lambda a_1 + (1-\lambda)a_2\bigr)\,\mathrm{d}\lambda = \frac{29}{180}.
\end{aligned}
$$

Example 2.7 Let $a_1 = 2$, $a_2 = 3$, $\alpha = \frac{1}{2}$, $\kappa = \frac{1}{2}$, $\Xi(x) = x^2$, $\chi(x) = x$. Then all the assumptions in Lemma 2.5 are satisfied.

One can observe that $\Xi(\frac{a_1+a_2}{2}) = \frac{25}{4}$. We have

$$
\begin{aligned}
&\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}} \bigl[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\bigl(\chi^{-1}(a_2)\bigr) + {}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\bigl(\chi^{-1}(a_1)\bigr)\bigr] \\
&\quad = \frac{\Gamma_{\frac{1}{2}}(\frac{1}{2})}{2}\biggl[\frac{1}{\Gamma_{\frac{1}{2}}(\frac{1}{2})}\int_2^3 \lambda^2\,\mathrm{d}\lambda + \frac{1}{\Gamma_2(\frac{1}{2})}\int_2^3 \lambda^2\,\mathrm{d}\lambda\biggr] = \frac{19}{3}.
\end{aligned}
$$

It follows that

$$\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\big[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\big]-\Xi\bigg(\frac{a_1+a_2}{2}\bigg)$$
$$=\frac{1}{12}.$$

On the other hand

$$\frac{a_2-a_1}{2}\int_0^1\big[\mu+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\big]\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda$$
$$=\frac{a_2-a_1}{2}\int_0^1\mu\,\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda$$
$$-\frac{a_2-a_1}{2}\int_0^1\big[\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\big]\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\,\mathrm{d}\lambda=\frac{1}{12}.$$

Theorem 2.8 *Let $a_1<a_2$ and $\Xi:[a_1,a_2]\to\mathbb{R}$ be a differentiable mapping on (a_1,a_2). Also suppose that $|\Xi'|$ is s-convex on $[a_1,a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1,a_2]$, having a continuous derivative $\chi'(x)$ on (a_1,a_2) and $\alpha\in(0,1)$, then, for $\kappa>0$, we have*

$$\bigg|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\big[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\big]-\Xi\bigg(\frac{a_1+a_2}{2}\bigg)\bigg|$$
$$\le\frac{a_2-a_1}{2(1+s)}\big(\big|\Xi'(a_1)\big|+\big|\Xi'(a_2)\big|\big).$$

Proof Using Lemma 2.5 and the fact that $|\Xi'|$ is s-convex, we have

$$\bigg|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\big[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\big]-\Xi\bigg(\frac{a_1+a_2}{2}\bigg)\bigg|$$
$$\le\frac{a_2-a_1}{2}\int_0^1\big|\mu+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\big|\big|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\big|\,\mathrm{d}\lambda$$
$$\le\frac{a_2-a_1}{2}\bigg[\int_0^{\frac{1}{2}}\big[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\big]\big[\lambda^s\big|\Xi'(a_1)\big|+(1-\lambda)^s\big|\Xi'(a_2)\big|\big]\,\mathrm{d}\lambda$$
$$+\int_{\frac{1}{2}}^1\big[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\big]\big[\lambda^s\big|\Xi'(a_1)\big|+(1-\lambda)^s\big|\Xi'(a_2)\big|\big]\,\mathrm{d}\lambda\bigg]$$
$$=\frac{a_2-a_1}{2}\bigg\{\big|\Xi'(a_1)\big|\bigg[\int_0^{\frac{1}{2}}\big[\lambda^s-\lambda^s(1-\lambda)^{\frac{\alpha}{\kappa}}+\lambda^{\frac{\alpha}{\kappa}+s}\big]\,\mathrm{d}\lambda$$
$$+\int_{\frac{1}{2}}^1\big[\lambda^s+\lambda^s(1-\lambda)^{\frac{\alpha}{\kappa}}-\lambda^{\frac{\alpha}{\kappa}+s}\big]\,\mathrm{d}\lambda\bigg]$$
$$+\big|\Xi'(a_2)\big|\bigg[\int_0^{\frac{1}{2}}\big[(1-\lambda)^s+\lambda^{\frac{\alpha}{\kappa}}(1-\lambda)^s-(1-\lambda)^{\frac{\alpha}{\kappa}+s}\big]\,\mathrm{d}\lambda$$
$$+\int_{\frac{1}{2}}^1\big[(1-\lambda)^s-\lambda^{\frac{\alpha}{\kappa}}(1-\lambda)^s+(1-\lambda)^{\frac{\alpha}{\kappa}+s}\big]\,\mathrm{d}\lambda\bigg]\bigg\}$$
$$\le\frac{a_2-a_1}{2}\bigg\{\big|\Xi'(a_1)\big|\bigg[\int_0^{\frac{1}{2}}\lambda^s\,\mathrm{d}\lambda+\int_{\frac{1}{2}}^1\lambda^s\,\mathrm{d}\lambda\bigg]$$

$$
\begin{aligned}
&\quad + \left|\Xi'(a_2)\right|\left[\int_0^{\frac{1}{2}}(1-\lambda)^s \,\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 (1-\lambda)^s \,\mathrm{d}\lambda\right]\Bigg\} \\
&= \frac{a_2-a_1}{2}\left\{\left|\Xi'(a_1)\right|\int_0^1 \lambda^s \,\mathrm{d}\lambda + \left|\Xi'(a_2)\right|\int_0^1 (1-\lambda)^s \,\mathrm{d}\lambda\right\} \\
&= \frac{a_2-a_1}{2(1+s)}\left(\left|\Xi'(a_1)\right| + \left|\Xi'(a_2)\right|\right).
\end{aligned}
$$

This completes the proof. □

Theorem 2.9 *Let $a_1 < a_2$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping on (a_1, a_2). Also suppose that $|\Xi'|^q$ is s-convex on $[a_1, a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, having a continuous derivative $\chi'(x)$ on (a_1, a_2) and $\alpha \in (0, 1)$, then, for $\kappa > 0$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right) + {}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right] - \Xi\left(\frac{a_1+a_2}{2}\right)\right| \\
&\quad \le (a_2-a_1)\left(\frac{\kappa}{(\alpha p+\kappa)2^{\frac{\alpha p+\kappa}{\kappa}}}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\left\{\left(\left|\Xi'(a_1)\right|^q + \left(2^{s+1}-1\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\left(2^{s+1}-1\right)\left|\Xi'(a_1)\right|^q + \left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}
\tag{2.1}
$$

where $p^{-1} + q^{-1} = 1$

Proof Using Lemma 2.5, Hölder's integral inequality and the fact that $|\Xi'|^q$ is s-convex, we have

$$
\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right) + {}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right] - \Xi\left(\frac{a_1+a_2}{2}\right)\right| \\
&\quad \le \frac{a_2-a_1}{2}\int_0^1\left|\mu + \lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\right|\left|\Xi'\left(\lambda a_1 + (1-\lambda)a_2\right)\right| \,\mathrm{d}\lambda \\
&\quad \le \frac{a_2-a_1}{2}\left[\int_0^{\frac{1}{2}}\left[1 + \lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\left(\lambda a_1 + (1-\lambda)a_2\right)\right| \,\mathrm{d}\lambda\right. \\
&\quad \left. + \int_{\frac{1}{2}}^1\left[1 - \lambda^{\frac{\alpha}{\kappa}} + (1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\left(\lambda a_1 + (1-\lambda)a_2\right)\right| \,\mathrm{d}\lambda\right] \\
&\quad \le \frac{a_2-a_1}{2}\left[\left(\int_0^{\frac{1}{2}}\left[1 + \lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\right]^p \,\mathrm{d}\lambda\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Xi'\left(\lambda a_1 + (1-\lambda)a_2\right)\right|^q \,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1\left[1 - \lambda^{\frac{\alpha}{\kappa}} + (1-\lambda)^{\frac{\alpha}{\kappa}}\right]^p \,\mathrm{d}\lambda\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Xi'\left(\lambda a_1 + (1-\lambda)a_2\right)\right|^q \,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right] \\
&\quad \le \frac{a_2-a_1}{2}\left(\int_0^{\frac{1}{2}}\left[1 + \lambda^{\frac{\alpha}{\kappa}} - (1-\lambda)^{\frac{\alpha}{\kappa}}\right]^p \,\mathrm{d}\lambda\right)^{\frac{1}{p}} \\
&\quad \times \left\{\left(\int_0^{\frac{1}{2}}\left(\lambda^s\left|\Xi'(a_1)\right|^q + (1-\lambda)^s\left|\Xi'(a_2)\right|^q\right) \,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1\left(\lambda^s\left|\Xi'(a_1)\right|^q + (1-\lambda)^s\left|\Xi'(a_2)\right|^q\right) \,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a_2-a_1}{2}\left(\int_0^{\frac{1}{2}}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]^p\,\mathrm{d}\lambda\right)^{\frac{1}{p}} \\
&\quad\times\left\{\left(\frac{1}{(s+1)2^{s+1}}\left|\Xi'(a_1)\right|^q+\frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\left|\Xi'(a_1)\right|^q+\frac{1}{(s+1)2^{s+1}}\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right\} \\
&\leq \frac{a_2-a_1}{2}\left(2^p\int_0^{\frac{1}{2}}\lambda^{\frac{\alpha p}{\kappa}}\,\mathrm{d}\lambda\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\left\{\left(\left|\Xi'(a_1)\right|^q+\left(2^{s+1}-1\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(2^{s+1}-1\right)\left|\Xi'(a_1)\right|^q+\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\Big\} \\
&= (a_2-a_1)\left(\frac{\kappa}{(\alpha p+\kappa)2^{\frac{\alpha p+\kappa}{\kappa}}}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\left\{\left(\left|\Xi'(a_1)\right|^q+\left(2^{s+1}-1\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(2^{s+1}-1\right)\left|\Xi'(a_1)\right|^q+\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\Big\}.
\end{aligned}
$$

This completes the proof. □

Corollary 2.10 *Let $a_1 < a_2$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping on (a_1, a_2). Also suppose that $|\Xi'|^q, q > 1$, is s-convex on $[a_1, a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, having a continuous derivative $\chi'(x)$ on (a_1, a_2) and $\alpha \in (0, 1)$, then, for $\kappa > 0$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]-\Xi\left(\frac{a_1+a_2}{2}\right)\right| \\
&\quad\leq (a_2-a_1)\left(\frac{\kappa}{(\alpha p+\kappa)2^{\frac{\alpha p+\kappa}{\kappa}}}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\
&\quad\quad\times\left(1+\left(2^{s+1}-1\right)^{\frac{1}{q}}\right)\left(\left|\Xi'(a_1)\right|^q+\left|\Xi'(a_2)\right|^q\right),
\end{aligned}
$$

where $p^{-1} + q^{-1} = 1$.

Proof Use (2.1) and let $a_1 = |\Xi'(a_1)|^q$, $b_1 = (2^{s+1} - 1)|\Xi'(a_2)|^q$, $a_2 = (2^{s+1} - 1)|\Xi'(a_1)|^q$, $b_2 = |\Xi'(a_2)|^q$. Here $0 < \frac{1}{q} < 1$ for $q > 1$. Then using (1.1), we obtain the required result. □

Theorem 2.11 *Let $a_1 < a_2$, $q > 1$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping on (a_1, a_2). Also suppose that $|\Xi'|^q$ is s-convex on $[a_1, a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, having a continuous derivative $\chi'(x)$ on (a_1, a_2) and $\alpha \in (0, 1)$, then, for $\kappa > 0$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]-\Xi\left(\frac{a_1+a_2}{2}\right)\right| \\
&\quad\leq \frac{a_2-a_1}{2}\left(\frac{\kappa}{\alpha+\kappa}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-\kappa}{2\kappa}+\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\
&\quad\quad\times\left\{\left(\left|\Xi'(a_1)\right|^q+\left(2^{s+1}-1\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}+\left(2^{s+1}-1\right)\left|\Xi'(a_1)\right|^q+\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right\}. \quad (2.2)
\end{aligned}
$$

Proof Using Lemma 2.5, the power mean integral inequality and the fact that $|\Xi'|^q$ is s-convex, we have

$$\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right]-\Xi\left(\frac{a_1+a_2}{2}\right)\right|\\
&\quad\le\frac{a_2-a_1}{2}\int_0^1\left|\mu+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right|\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|\,\mathrm{d}\lambda\\
&\quad\le\frac{a_2-a_1}{2}\left[\int_0^{\frac12}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|\,\mathrm{d}\lambda\right.\\
&\qquad\left.+\int_{\frac12}^1\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|\,\mathrm{d}\lambda\right]\\
&\quad\le\frac{a_2-a_1}{2}\left[\left(\int_0^{\frac12}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\mathrm{d}\lambda\right)^{1-\frac1q}\right.\\
&\qquad\times\left(\int_0^{\frac12}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda\right)^{\frac1q}\\
&\qquad+\left(\int_{\frac12}^1\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\mathrm{d}\lambda\right)^{1-\frac1q}\\
&\qquad\left.\times\left(\int_{\frac12}^1\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda\right)^{\frac1q}\right]\\
&\quad\le\frac{a_2-a_1}{2}\left(\int_0^{\frac12}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\mathrm{d}\lambda\right)^{1-\frac1q}\\
&\qquad\times\left\{\left(\int_0^{\frac12}\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\big(\lambda^s\left|\Xi'(a_1)\right|^q+(1-\lambda)^s\left|\Xi'(a_2)\right|^q\big)\,\mathrm{d}\lambda\right)^{\frac1q}\right.\\
&\qquad\left.+\left(\int_{\frac12}^1\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\big(\lambda^s\left|\Xi'(a_1)\right|^q+(1-\lambda)^s\left|\Xi'(a_2)\right|^q\big)\,\mathrm{d}\lambda\right)^{\frac1q}\right\}\\
&\quad\le\frac{a_2-a_1}{2}\left(\frac{\kappa}{\alpha+\kappa}\right)^{1-\frac1q}\left(\frac{\alpha-\kappa}{2\kappa}+\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)^{1-\frac1q}\\
&\qquad\times\left\{\left(\left|\Xi'(a_1)\right|^q\int_0^{\frac12}\left[\lambda^s-\lambda^s(1-\lambda)^{\frac{\alpha}{\kappa}}+\lambda^{\frac{\alpha}{\kappa}+s}\right]\mathrm{d}\lambda\right.\right.\\
&\qquad\left.+\left|\Xi'(a_2)\right|^q\int_0^{\frac12}\left[(1-\lambda)^s+\lambda^{\frac{\alpha}{\kappa}}(1-\lambda)^s-(1-\lambda)^{\frac{\alpha}{\kappa}+s}\right]\mathrm{d}\lambda\right)^{\frac1q}\\
&\qquad+\left(\left|\Xi'(a_1)\right|^q\int_{\frac12}^1\left[\lambda^s+\lambda^s(1-\lambda)^{\frac{\alpha}{\kappa}}-\lambda^{\frac{\alpha}{\kappa}+s}\right]\mathrm{d}\lambda\right.\\
&\qquad\left.\left.+\left|\Xi'(a_2)\right|^q\int_{\frac12}^1\left[(1-\lambda)^s-\lambda^{\frac{\alpha}{\kappa}}(1-\lambda)^s+(1-\lambda)^{\frac{\alpha}{\kappa}+s}\right]\mathrm{d}\lambda\right)^{\frac1q}\right\}\\
&\quad\le\frac{a_2-a_1}{2}\left(\frac{\kappa}{\alpha+\kappa}\right)^{1-\frac1q}\left(\frac{\alpha-\kappa}{2\kappa}+\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)^{1-\frac1q}\\
&\qquad\times\left\{\left(\left|\Xi'(a_1)\right|^q\int_0^{\frac12}\lambda^s\,\mathrm{d}\lambda+\left|\Xi'(a_2)\right|^q\int_0^{\frac12}(1-\lambda)^s\,\mathrm{d}\lambda\right)^{\frac1q}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left|\Xi'(a_1)\right|^q\int_{\frac{1}{2}}^1\lambda^s\,\mathrm{d}\lambda+\left|\Xi'(a_2)\right|^q\int_{\frac{1}{2}}^1(1-\lambda)^s\,\mathrm{d}\lambda\right)^{\frac{1}{q}}\Bigg\}\\
&=\frac{a_2-a_1}{2}\left(\frac{\kappa}{\alpha+\kappa}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-\kappa}{2\kappa}+\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)^{1-\frac{1}{q}}\\
&\quad\times\Bigg\{\left(\frac{1}{(s+1)2^{s+1}}\left|\Xi'(a_1)\right|^q+\frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\left|\Xi'(a_1)\right|^q+\frac{1}{(s+1)2^{s+1}}\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\Bigg\}\\
&=\frac{a_2-a_1}{2}\left(\frac{\kappa}{\alpha+\kappa}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-\kappa}{2\kappa}+\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\\
&\quad\times\left\{\left(\left|\Xi'(a_1)\right|^q+\left(2^{s+1}-1\right)\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}+\left(2^{s+1}-1\right)\left|\Xi'(a_1)\right|^q+\left|\Xi'(a_2)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This completes the proof. □

Corollary 2.12 *Let $a_1 < a_2$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping on (a_1, a_2). Also suppose that $|\Xi'|^q, q > 1$ is s-convex on $[a_1, a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, having a continuous derivative $\chi'(x)$ on (a_1, a_2) and $\alpha \in (0, 1)$, then, for $\kappa > 0$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]-\Xi\left(\frac{a_1+a_2}{2}\right)\right|\\
&\quad\le\frac{a_2-a_1}{2}\left(\frac{\kappa}{\alpha+\kappa}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-\kappa}{2\kappa}+\frac{1}{2^{\frac{\alpha}{\kappa}}}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\\
&\quad\quad\times\left(1+\left(2^{s+1}-1\right)^{\frac{1}{q}}\right)\left(\left|\Xi'(a_1)\right|^q+\left|\Xi'(a_2)\right|^q\right).
\end{aligned}
$$

Proof Using the same technique as in the proof of Corollary 2.10, the proof is complete. □

Theorem 2.13 *Let $a_1 < a_2$ and $\Xi : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping on (a_1, a_2). Also suppose that $|\Xi'|^q$ is s-concave on $[a_1, a_2]$, $\chi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, having a continuous derivative $\chi'(x)$ on (a_1, a_2) and $\alpha \in (0, 1)$, then, for $\kappa > 0$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\left(\chi^{-1}(a_2)\right)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\left(\chi^{-1}(a_1)\right)\right]-\Xi\left(\frac{a_1+a_2}{2}\right)\right|\\
&\quad\le(a_2-a_1)\left(\frac{\kappa}{(\alpha p+\kappa)2^{\alpha p+\kappa}}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\left|\Xi'\left(\frac{a_1+3a_2}{4}\right)\right|+\left|\Xi'\left(\frac{3a_1+a_2}{4}\right)\right|\right),
\end{aligned}
$$

where $p^{-1} + q^{-1} = 1$.

Proof Using Lemma 2.5, Hölder's integral inequality and the fact that $|\Xi'|^q$ is s-concave, we have

$$\begin{aligned}
&\left|\frac{\Gamma_\kappa(\alpha+\kappa)}{2(a_2-a_1)^{\frac{\alpha}{\kappa}}}\left[{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_1)^+}(\Xi\circ\chi)\big(\chi^{-1}(a_2)\big)+{}_\kappa I^{\alpha:\chi}_{\chi^{-1}(a_2)^-}(\Xi\circ\chi)\big(\chi^{-1}(a_1)\big)\right]-\Xi\left(\frac{a_1+a_2}{2}\right)\right|\\
&\quad\le\frac{a_2-a_1}{2}\int_0^1\left|\mu+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right|\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|\,\mathrm{d}\lambda\\
&\quad\le\frac{a_2-a_1}{2}\left[\int_0^{\frac{1}{2}}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|\,\mathrm{d}\lambda\right.\\
&\quad\quad\left.+\int_{\frac{1}{2}}^1\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|\,\mathrm{d}\lambda\right]\\
&\quad\le\frac{a_2-a_1}{2}\left[\left(\int_0^{\frac{1}{2}}\left[1+\lambda^{\frac{\alpha}{\kappa}}-(1-\lambda)^{\frac{\alpha}{\kappa}}\right]^p\,\mathrm{d}\lambda\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right.\\
&\quad\quad\left.+\left(\int_{\frac{1}{2}}^1\left[1-\lambda^{\frac{\alpha}{\kappa}}+(1-\lambda)^{\frac{\alpha}{\kappa}}\right]^p\,\mathrm{d}\lambda\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right]\\
&\quad=(a_2-a_1)\left(\frac{\kappa}{(\alpha p+\kappa)2^{\alpha p+\kappa}}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda\right)^{\frac{1}{q}}\\
&\quad\quad+\left(\int_{\frac{1}{2}}^1\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda\right)^{\frac{1}{q}}\Big].
\end{aligned}$$

Since $|\Xi'|^q$ is a concave function on $[a_1,a_2]$ and using the Jensen integral inequality, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}}\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda&\le\left(\int_0^{\frac{1}{2}}\lambda^*\,\mathrm{d}\lambda\right)\left|\Xi'\left(\frac{(\int_0^{\frac{1}{2}}(\lambda a_1+(1-\lambda)a_2)\,\mathrm{d}\lambda)}{(\int_0^{\frac{1}{2}}\lambda^*\,\mathrm{d}\lambda)}\right)\right|\\
&\le\frac{1}{2}\left|\Xi'\left(\frac{a_1+3a_2}{4}\right)\right|.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_{\frac{1}{2}}^1\left|\Xi'\big(\lambda a_1+(1-\lambda)a_2\big)\right|^q\,\mathrm{d}\lambda&\le\left(\int_{\frac{1}{2}}^1\lambda^*\,\mathrm{d}\lambda\right)\left|\Xi'\left(\frac{(\int_{\frac{1}{2}}^1(\lambda a_1+(1-\lambda)a_2)\,\mathrm{d}\lambda)}{(\int_0^{\frac{1}{2}}\lambda^*\,\mathrm{d}\lambda)}\right)\right|\\
&\le\frac{1}{2}\left|\Xi'\left(\frac{3a_1+a_2}{4}\right)\right|.
\end{aligned}$$

This completes the proof. □

Acknowledgements
The second author is thankful to HEC, Pakistan. The authors are thankful to the editor and reviewers for their valuable comments and suggestions.

Funding
This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11701176, 11626101, 11601485).

Availability of data and materials
No data were used to support this study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Huzhou University, Huzhou, China. [2]Department of Mathematics, Government College University, Faisalabad, Pakistan. [3]Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 4 August 2020 Accepted: 14 October 2020 Published online: 21 October 2020

References
1. Awan, M.U., Talib, S., Chu, Y.M., Noor, M.A., Noor, K.I.: Some new refinements of Hermite–Hadamard-type inequalities involving ψ_k-Riemann–Liouville fractional integrals and applications. Math. Probl. Eng. **2020**, Article ID 3051920 (2020)
2. Breckner, W.W.: Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen. Publ. Inst. Math. **23**, 13–20 (1978)
3. Dragomir, S.S., Pearce, C.E.M.: Selected topics on Hermite–Hadamard inequalities and applications, Victoria University, Australia (2000)
4. Ghaffar, A., Ali, A., Ahmed, S., Akram, S., Junjua, M-.U-.D., Baleanu, D., Nisar, K.S.: A novel analytical technique to obtain the solitary solutions for nonlinear evolution equation of fractional order. Adv. Differ. Equ. **2020**, 308 (2020)
5. Hilfer, R. (ed.): Applications of Fractional Calculus in Physics World Scientific, Singapore (2000)
6. Huang, C.-J., Rahman, G., Nisar, K.S., Ghaffar, A., Qi, F.: Some inequalities of the Hermite–Hadamard type for *k*-fractional conformable integrals. Aust. J. Math. Anal. Appl. **16**(1), Article 7 (2019)
7. Jothimani, K., Kaliraj, K., Hammouch, Z., Ravichandran, C.: New results on controllability in the framework of fractional integrodifferential equations with nondense domain. Eur. Phys. J. Plus **134**, 441 (2019). https://doi.org/10.1140/epjp/i2019-12858-8
8. Kilbas, A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
9. Liu, K., Wang, J.-R., O'Regan, D.: On the Hermite–Hadamard type inequality for ψ-Riemann–Liouville fractional integrals via convex functions. J. Inequal. Appl. **2019**, 27 (2019)
10. Mubeen, S., Habibullah, G.M.: *k*-Fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(2), 89–94 (2012)
11. Qi, F., Rahman, G., Hussain, S.M., Du, W.-S., Nisar, K.S.: Some inequalities of Cebysev type for conformable *k*-fractional integral operators. Symmetry **10**, 614 (2018). https://doi.org/10.3390/sym10110614
12. Rahman, G., Mubeen, S., Nisar, K.S.: On generalized *k*-fractional derivative operator. AIMS Math. **5**(3), 1936–1945 (2020)
13. Rahman, G., Nisar, K.S., Ghaffar, A., Qi, F.: Some inequalities of the Gruss type for conformable *k*-fractional integral operators. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**, 9 (2020)
14. Rashid, S., Hammouch, Z., Ashraf, R., Baleanu, D., Nisar, K.S.: New quantum estimates in the setting of fractional calculus theory. Adv. Differ. Equ. **2020**, 383 (2020)
15. Sarikaya, M.Z., Karaca, A.: On the *k*-Riemann–Liouville fractional integral and applications. Int. J. Stat. Math. **1**(3), 33–43 (2014)
16. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403–2407 (2013)
17. Subashini, R., Jothimani, K., Nisar, K.S., Ravichandran, C.: New results on nonlocal functional integro-differential equations via Hilfer fractional derivative. Alex. Eng. J. (2020). https://doi.org/10.1016/j.aej.2020.01.055
18. Valliammal, N., Ravichandran, C., Hammouch, Z., Baskonus, H.M.: A new investigation on fractional-ordered neutral differential systems with state-dependent delay. Int. J. Nonlinear Sci. Numer. Simul. **20**(7–8), 803–809 (2019)
19. Valliammal, N., Ravichandran, C., Nisar, K.S.: Solutions to fractional neutral delay differential nonlocal systems. Chaos Solitons Fractals **138**, 109912 (2020)
20. Zhao, Y., Sang, H., Xiong, W., Cui, Z.: Hermite–Hadamard-type inequalities involving ψ-Riemann–Liouville fractional integrals via *s*-convex functions. J. Inequal. Appl. **2020**, 128 (2020)