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Unified integral associated with the generalized V-function

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Abstract

In this paper, we present two new unified integral formulas involving a generalized V-function. Some interesting special cases of the main results are also considered in the form of corollaries. Due to the general nature of the V-function, several results involving different special functions such as the exponential function, the Mittag-Leffler function, the Lommel function, the Struve function, the Wright generalized Bessel function, the Bessel function and the generalized hypergeometric function are obtained by specializing the parameters in the presented formulas. More results are also discussed in detail.

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1 Introduction and preliminaries

Fractional calculus is as old as the conventional calculus and has been recently applied in various areas of engineering, science, finance, applied mathematics and bioengineering. The V-function is an important special function that provides solutions to a number of problems formulated in terms of fractional order differential, integral and difference equations, therefore it has recently become a subject of interest for many authors in the field of fractional calculus and its applications. In addition, a number of researchers (see [10, 11, 13, 17, 19, 22, 29, 30]) have studied in depth properties, applications and diverse extensions of a range of operators of fractional calculus, this field being very active and extensive around the world. One may refer to the research monographs [12] and [21] for further investigations in the area. Recently, the V-function is defined by Kumar [14] as follows:

$$V(z) = V_n^{a_u, b_v}(l, \mu, \zeta, \delta, m, k_u, A_v, B_w, \eta, \nu, \rho; z) \\ = \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + \nu)^{-\mu} (z/2)^{n\zeta + h\delta + m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]}, \quad (1.1)$$

where

1. $l, \zeta, \delta, m, \nu, \rho, k_u (u = 1, \dots, p), A_v (v = 1, \dots, q), B_w (w = 1, \dots, r)$ are real numbers,

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2. p, q , and r are natural numbers,
3. $a_u, b_v \geq 1 (u = 1, \dots, p; v = 1, \dots, q)$,
4. $\eta > 0, \Re(\mu) > 0, \Re(h) > 0, z$ is a complex variable and ξ is an arbitrary constant,
5. the series on the RHS of (1.1) converges absolutely if $p < q$ or $p = q$ with $|l(z/2)^\zeta| \leq 1$.

For further information on the constraints of the convergence of the RHS of the series (1.1), we refer to Refs. [15, 16]. The V-function defined by (1.1) is of general character as it assimilates a variety of valuable functions such as the MacRobert E-function, the exponential function [4], the generalized Mittag-Leffler function [9, 23, 31, 34], the Lommel function [8], the Struve function [26, 33, 35], the generalized Bessel function [27], the Bessel function [5, 36], the generalized hypergeometric function [4, 32, 37, 38] and the unified Riemann–Zeta function [7].

Some special cases of the V-function (1.1) are as follows:

- (i) For $w = 1, h = 1, p = P, q = Q, l = -2, \mu = 1, \zeta = 1, \delta = 0, m = 0, k_u = 0, A_v = 0, B_1 = -1, \eta = 1, v = -1, \rho = 1$ and $\xi = 1$, the V-function (1.1) turns into a generalized hypergeometric function (see, e.g., [4]),

$$V_n^{a_u, 1, b_v}(-2, 1, 1, 0, 0, 0, 0, -1, 1, -1, 1; z) = {}_pF_Q(a_P; b_Q, z). \quad (1.2)$$

- (ii) For $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, b_2 = 1, l = 1, \mu = 1, \zeta = 2, \delta = 1, m = 0, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = 0, \eta = 1, v = 0, \rho = 1$ and $\xi = 1/(\Gamma(h))$, the V-function (1.1) turns into a Bessel function (see, e.g., [5]),

$$V_n^{1, h, 1, 1}(1, 1, 2, 1, 0, 0, 0, 0, 0, 1, 0, 1; z) = J_h(z). \quad (1.3)$$

- (iii) For $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, b_2 = 1, l = 2, \mu = 1, \zeta = 1, \delta = 1, m = 0, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = 0, \eta = 0, \rho = 1$ and $\xi = 1/(\Gamma(h))$, the V-function (1.1) turns into the Wright generalized Bessel function (see, e.g., [5]),

$$V_n^{1, h, 1, 1}(1, 1, 2, 1, 0, 0, 0, 0, 0, \eta, 0, 1; z) = J_h^\eta(z). \quad (1.4)$$

- (iv) For $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 3/2, b_2 = 1, l = 1, \mu = 1, \zeta = 2, \delta = 1, m = 1, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = 1/2, \eta = 1, v = 1/2, \rho = 1$ and $\xi = 1/\Gamma(h)\Gamma(3/2)$, the V-function (1.1) turns into the Struve function (see, e.g., [5]),

$$V_n^{1, h, 3/2, 1}(1, 1, 2, 1, 1, 0, 0, 0, 1/2, 1, 1/2, 1; z) = H_h(z). \quad (1.5)$$

- (v) For $u = 1, v = 2, w = 1, a_1 = 1, b_1 = (\tau + \epsilon + 3)/2, b_2 = (\tau - \epsilon + 3)/2, l = 1, \mu = 1, \zeta = 2, h = 1, \delta = \tau, m = 1, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = -1, \eta = 1, v = -1, \rho = 1$ and $\xi = 2^{\tau+1}/\{(\tau + \epsilon + 1)(\tau - \epsilon + 1)\}$, the V-function (1.1) turns into the Lommel function (see, e.g., [5]),

$$V_n^{1, 1, (\tau+\epsilon+3)/2, (\tau-\epsilon+3)/2}(1, 1, 2, \tau, 1, 0, 0, 0, -1, 1, 1, 1; z) = S_{\tau, \epsilon}(z). \quad (1.6)$$

- (vi) For $u = 1, v = 1, w = 1, a_1 = 1, b_1 = 1, l = -2, \mu = 1, \zeta = 1, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = -1, \eta = -1, \rho = 1$ and $\xi = 1/(\Gamma(h))$, the V-function (1.1) turns into the Mittag-Leffler function (see, e.g., [9, 23]),

$$V_n^{1, h, 1}(-2, 1, 1, 0, 0, 0, -1, \eta, -1, 1, 1; z) = E_{\eta, h}(z). \quad (1.7)$$

- (vii) For $u = 1, v = 1, w = 1, a_1 = a, b_1 = 1, l = -2, \zeta = 1, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = 0, \eta = 1, \nu = 0, \rho = 0$ and $\xi = 1$, the V-function (1.1) turns into the unified Riemann–Zeta function (see, e.g., [7]),

$$V_n^{a,h,1}(-2, \mu, 1, 0, 0, 0, 0, 0, 1, 0, 0; z) = \phi_a(z, \mu, h). \quad (1.8)$$

- (viii) For $u = 1, v = 1, w = 1, a_1 = 1, b_1 = 1, l = 2, \zeta = 1, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = -1, \eta = 1, \nu = -1, \rho = 1, h = 1, \mu = 1$, and $\xi = 1$, the V-function (1.1) turns into the e^{-z} function,

$$V_n^{1,1,1}(2, 1, 1, 0, 0, 0, 0, -1, 1, -1, 1; z) = e^{-z}. \quad (1.9)$$

- (ix) For $w = 1, p = P, q = Q, l = 2, \mu = 1, \zeta = 1, \delta = 0, m = 0, k_u = 0, A_u = 0, B_1 = -1, \eta = 1, \nu = -1, \rho = 1, h = 1$ and $\xi = \frac{\prod_{u=1}^P \Gamma(a_u)}{\prod_{v=1}^Q \Gamma(b_v)}$, the V-function (1.1) turns into the MacRobert E-function (see, e.g., [4]),

$$V_n^{1,1,1}(2, 1, 1, 0, 0, 1, 0, -1, 1, -1, 1; z) = E[P; (a_P); Q; (b_Q); z^{-1}]. \quad (1.10)$$

- (x) For $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, k_1 = 0, h = 1/2, l = 1, \mu = 1, \zeta = 2, \delta = 0, m = 0, A_1 = 0, A_2 = -1, B_1 = 0, \eta = 1, \nu = -1/2, \rho = 1$ and $\xi = 1$, the V-function (1.1) turns into the $\cos z$ function,

$$V_n^{1,1/2,1,1}(1, 1, 2, 0, 0, 0, 0, -1, 0, 1, -1/2, 1; z) = \cos z. \quad (1.11)$$

- (xi) For $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, k_1 = 0, h = 1/2, l = 1, \mu = 1, \zeta = 2, \delta = 2, m = 0, A_1 = 0, A_2 = -1, B_1 = 0, \eta = 1, \nu = -1/2, \rho = 1$ and $\xi = 1$, the V-function (1.1) turns into the $\sin z$ function,

$$V_n^{1,1/2,1,1}(1, 1, 2, 2, 0, 0, 0, -1, 0, 1, -1/2, 1; z) = \sin z. \quad (1.12)$$

To proceed in our next investigation, we need to recall the following Oberhettinger integral formula [28]:

$$\int_0^\infty z^{\lambda-1} (z + b + \sqrt{z^2 + 2bz})^{-\gamma} dz = 2\gamma b^{-\lambda} \left(\frac{b}{2}\right)^\lambda \frac{\Gamma(2\lambda)\Gamma(\gamma - \lambda)}{\Gamma(1 + \lambda + \gamma)}, \quad (1.13)$$

provided that $0 < \Re(\lambda) < \Re(\gamma)$. Also, we need to recall the Lavoie–Trottier integral formula [18]:

$$\int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} \left(1 - \frac{z}{3}\right)^{2\lambda-1} \left(1 - \frac{z}{4}\right)^{\gamma-1} dz = \left(\frac{2}{3}\right)^{2\gamma} \frac{\Gamma(\gamma)\Gamma(\lambda)}{\Gamma(\lambda + \gamma)}, \quad (1.14)$$

provided that $\Re(\lambda) > 0, \Re(\gamma) > 0$. For further investigations of the function, the reader may be referred to the recent work of [1–3, 20, 24, 25] and the references therein.

However, the main object of this paper is to establish certain new integrals involving the V-function. The results are presented as theorems and corollaries that may potentially be very useful. At last, we establish special cases of our main results connecting various special functions.

2 Main results

In this section, we establish four generalized integral formulas for the V-function. These formulas are given by the following theorems.

Theorem 2.1 Let $\lambda, \gamma, \alpha \in \mathbb{C}$; $l, \zeta, \delta, m, v, \rho, k_u, A_v, B_w \in \Re$; $b, p, q, r \in \mathbb{N}$; $a_u, b_v \geq 1$, $\eta > 0$, $\Re(\mu) > 0$, $\Re(h) > 0$, $\Re(\alpha) > 0$, $z > 0$ and $\xi > 0$ be arbitrary constants, such that $0 < \Re(\lambda) < \Re(\gamma + \alpha(n\zeta + h\delta + m))$. Then the following integral holds true:

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} V_n^{a_u, h, b_v} \left(l, \mu, \zeta, \delta, m, k_u, A_v, B_w, \eta, v, \rho; \frac{y}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \Gamma(2\lambda) \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2b^\alpha)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \frac{\Gamma(\gamma + \alpha(n\zeta + h\delta + m) + 1) \Gamma(\gamma - \lambda + \alpha(n\zeta + h\delta + m))}{\Gamma(\gamma + \lambda + \alpha(n\zeta + h\delta + m) + 1) \Gamma(\gamma + \alpha(n\zeta + h\delta + m))}, \end{aligned} \quad (2.1)$$

where $A = (z + b + \sqrt{z^2 + 2bz})$.

Proof For the convenience of the reader, we denote the left-hand side of (2.1) by \mathfrak{J}_1 . Therefore, by invoking (1.1) in the integrand (2.1) and interchanging the order of integration and summation, which is verified by the uniform convergence of the involved series under the given conditions, we get

$$\begin{aligned} \mathfrak{J}_1 &= \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \int_0^\infty z^{\lambda-1} A^{-(\gamma+\alpha(n\zeta+h\delta+m))} dz. \end{aligned} \quad (2.2)$$

Hence, on applying the integral formula (1.13) for the integral in (2.2), we, under the valid conditions, obtain the following expression:

$$\begin{aligned} \mathfrak{J}_1 &= \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times 2(\gamma + \alpha(n\zeta + h\delta + m)) \left(\frac{b}{2} \right)^\lambda b^{-(\gamma+\alpha(n\zeta+h\delta+m))} \\ & \quad \times \frac{\Gamma(2\lambda) \Gamma(\gamma + \alpha(n\zeta + h\delta + m) - \lambda)}{\Gamma(1 + \gamma + \alpha(n\zeta + h\delta + m) + \lambda)} \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2b^\alpha)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \frac{\Gamma(2\lambda) \Gamma(1 + \gamma + \alpha(n\zeta + h\delta + m)) \Gamma(\gamma + \alpha(n\zeta + h\delta + m) - \lambda)}{\Gamma(\gamma + \alpha(n\zeta + h\delta + m)) \Gamma(1 + \gamma + \alpha(n\zeta + h\delta + m) + \lambda)}, \end{aligned}$$

which is the desired result. \square

Theorem 2.2 Let $\lambda, \gamma, \alpha \in \mathbb{C}$; $l, \zeta, \delta, m, v, \rho, k_u, A_v, B_w \in \Re$; $b, p, q, r \in \mathbb{N}$; $a_u, b_v \geq 1$, $\eta > 0$, $\Re(\mu) > 0$, $\Re(h) > 0$, $\Re(\alpha) > 0$, $z > 0$ and $\xi > 0$ be arbitrary constants, such that $0 < \Re(\lambda) <$

$\Re(\gamma + \alpha(n\zeta + h\delta + m))$. Then the following integral holds true:

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} V_n^{a_u, b_v} \left(l, \mu, \zeta, \delta, m, k_u, A_v, B_w, \eta, v, \rho; \frac{yz^\alpha}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \Gamma(\gamma - \lambda) \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \left(\frac{y}{2^{\alpha+1}} \right)^{n\zeta+h\delta+m} \frac{\Gamma(\gamma + \alpha(n\zeta + h\delta + m) + 1) \Gamma(2\lambda + 2\alpha(n\zeta + h\delta + m))}{\Gamma(\gamma + \lambda + 2\alpha(n\zeta + h\delta + m) + 1) \Gamma(\gamma + \alpha(n\zeta + h\delta + m))}. \end{aligned} \quad (2.3)$$

Proof By following a technique similar to what has already been used in the proof of Theorem 2.1, we can easily prove the integral formula (2.3). Therefore, we omit the detailed proof. \square

Theorem 2.3 Let $\lambda, \gamma, \alpha \in \mathbb{C}$; $l, \zeta, \delta, m, v, \rho, k_u, A_v, B_w \in \Re$; $p, q, r \in \mathbb{N}$; $a_u, b_v \geq 1$, $\eta > 0$, $\Re(\mu) > 0$, $\Re(h) > 0$, $\Re(\alpha) > 0$, $z > 0$ and $\xi > 0$ be arbitrary constants, such that $0 < \Re(\lambda) < \Re(\gamma + \alpha(n\zeta + h\delta + m))$. Then the following integral holds true:

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \\ & \quad \times V_n^{a_u, b_v} (l, \mu, \zeta, \delta, m, k_u, A_v, B_w, \eta, v, \rho; y(1-z)^2 T^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} \Gamma(\lambda) \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \frac{\Gamma(\gamma + \alpha(n\zeta + h\delta + m))}{\Gamma(\lambda + \gamma + \alpha(n\zeta + h\delta + m))}, \end{aligned} \quad (2.4)$$

where $S = (1 - \frac{z}{3})$ and $T = (1 - \frac{z}{4})$.

Proof For more convenience, we denote the left-hand side of (2.4) by \Im_2 . Therefore, by invoking (1.1) in the integral part of (2.4) and interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we obtain

$$\begin{aligned} \Im_2 &= \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \int_0^1 z^{\lambda-1} (1-z)^{2\gamma+2\alpha(n\zeta+h\delta+m)-1} (S)^{2\lambda-1} (T)^{\gamma+\alpha(n\zeta+h\delta+m)-1} dz. \end{aligned} \quad (2.5)$$

Now, upon applying the integral formula (1.14) to the integral part of (2.5) we obtain the following expression under their valid conditions:

$$\begin{aligned} \Im_2 &= \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\zeta+h\delta+m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n \rho + B_w}]} \\ & \quad \times \left(\frac{2}{3} \right)^{2\lambda} \frac{\Gamma(\lambda) \Gamma(\gamma + \alpha(n\zeta + h\delta + m))}{\Gamma(\lambda + \gamma + \alpha(n\zeta + h\delta + m))} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{2}{3}\right)^{2\lambda} \Gamma(\lambda) \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\xi + h\delta + m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n\rho + B_w}]} \\
&\quad \times \frac{\Gamma(\gamma + \alpha(n\xi + h\delta + m))}{\Gamma(\lambda + \gamma + \alpha(n\xi + h\delta + m))},
\end{aligned}$$

which is the desired result. \square

Theorem 2.4 Let $\lambda, \gamma, \alpha \in \mathbb{C}$; $l, \zeta, \delta, m, v, \rho, k_u, A_v, B_w \in \Re$; $p, q, r \in \mathbb{N}$; $a_u, b_v \geq 1$, $\eta > 0$, $\Re(\mu) > 0$, $\Re(h) > 0$, $\Re(\alpha) > 0$, $z > 0$ and $\xi > 0$ be arbitrary constants, such that $0 < \Re(\lambda) < \Re(\gamma + \alpha(n\xi + h\delta + m))$. Then the following integral holds true:

$$\begin{aligned}
&\int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} V_n^{a_u, h, b_v}(l, \mu, \zeta, \delta, m, k_u, A_v, B_w, \eta, v, \rho; yz^\alpha S^{2\alpha}) dz \\
&= \left(\frac{2}{3}\right)^{2\lambda} \Gamma(\gamma) \xi \sum_{n=0}^{\infty} \frac{(-l)^n \prod_{u=1}^p [(a_u)_{n+k_u}] (h + \eta n + v)^{-\mu} (y/2)^{n\xi + h\delta + m}}{\prod_{v=1}^q [(b_v)_{n+A_v}] \prod_{w=1}^r [(h)_{\eta n\rho + B_w}]} \\
&\quad \times \left(\frac{2}{3}\right)^{2\alpha(n\xi + h\delta + m)} \frac{\Gamma(\lambda + \alpha(n\xi + h\delta + m))}{\Gamma(\lambda + \gamma + \alpha(n\xi + h\delta + m))}. \tag{2.6}
\end{aligned}$$

Proof By following the proof of Theorem 2.3, step by step, Eq. (2.6) can easily be obtained. Hence, we omit the detailed proof. \square

3 Special cases

In this section, we aim to present some special cases by adopting certain advisable values of the parameters imposed on Theorems 2.1, 2.2, 2.3 and 2.4. Indeed, such constraints led up to certain interesting results concerning generalized hypergeometric functions, Bessel functions, Wright generalized Bessel functions, Struve functions, Lommel functions, Mittag-Leffler functions, Riemann-Zeta functions, exponential functions and MacRobert E-functions as follows.

(i) By inserting $w = 1, h = 1, p = P, q = Q, l = -2, \mu = 1, \zeta = 1, \delta = 0, m = 0, k_u = 0, A_v = 0, B_1 = -1, \eta = 1, v = -1, \rho = 1$ and $\xi = 1$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns into the generalized hypergeometric function [4]. Moreover, by applying the result ([6], Eq. (2.1.2.2), pp. 34)

$$(\beta)_{\alpha n} = \alpha^{\alpha n} \prod_{i=1}^{\alpha} \left[\frac{\beta + i - 1}{\alpha} \right]_n \tag{3.1}$$

under the assumption that the conditions of Theorems 2.1–2.4 are employed, we, respectively, get

$$\begin{aligned}
&\int_0^\infty z^{\lambda-1} A^{-\gamma} {}_P F_Q \left(a_P; b_Q; \frac{y}{A^\alpha} \right) dz \\
&= 2^{1-\lambda} b^{\lambda-\gamma} \gamma \frac{\Gamma(2\lambda) \Gamma(\gamma - \lambda)}{\Gamma(\lambda + \gamma + 1)} \\
&\quad \times {}_{P+2\alpha} F_{Q+2\alpha} \left[a_1, a_2, \dots, a_P; \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha}, \dots, \frac{\gamma+\alpha}{\alpha}, \right. \\
&\quad \left. b_1, b_2, \dots, b_Q; \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha}, \dots, \frac{\gamma+\alpha-1}{\alpha} \right],
\end{aligned}$$

$$\left[\begin{array}{c} \frac{\gamma-\lambda}{\alpha}, \frac{\gamma-\lambda+1}{\alpha} \dots, \frac{\gamma-\lambda+\alpha-1}{\alpha} \\ \frac{\lambda+\gamma+1}{\alpha}, \frac{\lambda+\gamma+2}{\alpha} \dots, \frac{\lambda+\gamma+\alpha}{\alpha} \end{array} \middle| y b^{-\alpha} \right], \quad (3.2)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} {}_P F_Q \left(a_P; b_Q; \frac{yz^\alpha}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \gamma \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda)}{\Gamma(\lambda+\gamma+1)} \\ & \times {}_{P+3\alpha} F_{Q+3\alpha} \left[\begin{array}{c} a_1, a_2, \dots, a_P; \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha} \\ b_1, b_2, \dots, b_Q; \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha} \end{array} \middle| 2^{-\alpha} y \right], \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} {}_P F_Q (a_P; b_Q; y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} B(\gamma, \lambda) {}_{P+\alpha} F_{Q+\alpha} \left[\begin{array}{c} a_1, a_2, \dots, a_P; \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha} \\ b_1, b_2, \dots, b_Q; \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{array} \middle| y \right], \end{aligned} \quad (3.4)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} {}_P F_Q (a_P; b_Q; y(z)^\alpha S^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} B(\gamma, \lambda) {}_{P+\alpha} F_{Q+\alpha} \left[\begin{array}{c} a_1, a_2, \dots, a_P; \frac{\lambda}{\alpha}, \frac{1+\lambda}{\alpha} \dots, \frac{\lambda+\alpha-1}{\alpha} \\ b_1, b_2, \dots, b_Q; \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{array} \middle| \left(\frac{4}{9} \right)^\alpha y \right]. \end{aligned} \quad (3.5)$$

(ii) Putting $\alpha = 1, u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, b_2 = 1, l = 1, \mu = 1, \zeta = 2, \delta = 1, m = 0, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = 0, \eta = 1, \nu = 0, \rho = 1$ and $\xi = 1/(\Gamma(h))$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns to a Bessel function [5]. By applying the result from ([6], Eq. (2.1.5.3), pp. 38)

$$(\beta)_{2n} = 2^{2n} \left(\frac{\beta}{2} \right)_n \left(\frac{\beta+1}{2} \right)_n \quad (3.6)$$

when the conditions already imposed on Theorems 2.1–2.4 are applied, we, respectively, have

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} J_h \left(\frac{y}{A} \right) dz = 2^{1-\lambda-h} b^{\lambda-h-\gamma} (\gamma+h) y^h \\ & \times \frac{\Gamma(2\lambda)\Gamma(\gamma+h-\lambda)}{\Gamma(\lambda+\gamma+h+1)\Gamma(1+h)} \\ & \times {}_3 F_4 \left[\begin{array}{c} \frac{2+\gamma+h}{2}, \frac{\gamma+h-\lambda}{2}, \frac{1+\gamma+h-\lambda}{2} \\ 1+h, \frac{\gamma+h}{2}, \frac{1+\gamma+h+\lambda}{2}, \frac{2+\gamma+h+\lambda}{2} \end{array} \middle| -\left(\frac{y}{2b} \right)^2 \right], \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} J_h \left(\frac{yz}{A} \right) dz = 2^{-\gamma-2h} b^{\lambda-\gamma} y^h (\gamma+h) \\ & \times \frac{\Gamma(\gamma-h)\Gamma(\lambda+h)\Gamma(\frac{2\lambda+2h+1}{2})}{\Gamma(h+1)(\lambda+\gamma+1)\Gamma(\frac{\lambda+\gamma+2h+1}{2})\Gamma(\frac{\lambda+\gamma+2h+2}{2})} \\ & \times {}_5 F_6 \left[\begin{array}{c} \frac{\gamma+h}{2}, \frac{\gamma+h+2}{2}, \frac{\lambda+h+1}{2}, \frac{2\lambda+2h+1}{4}, \frac{2\lambda+2h+3}{4} \\ h+1, \frac{\gamma+h}{2}, \frac{1+\gamma+\lambda+2h}{4}, \frac{2+\gamma+\lambda+2h}{4}, \frac{3+\gamma+\lambda+2h}{4}, \frac{4+\gamma+\lambda+2h}{4} \end{array} \middle| -\frac{y^2}{16} \right], \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} J_h(y(1-z)^2 T) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} \left(\frac{y}{2}\right)^h \frac{B(\gamma+h, 2\lambda)}{\Gamma(h+1)} {}_2F_3 \left[h+1, \frac{\frac{\gamma+h}{2}, \frac{1+\gamma+h}{2}}{\frac{2\lambda+\gamma+h+1}{2}, \frac{2\lambda+\gamma+h}{2}} \middle| -\frac{y^2}{4} \right], \end{aligned} \quad (3.9)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} J_h(yzS^2) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} \left(\frac{y}{3}\right)^h \frac{B(\gamma, \lambda+h)}{\Gamma(h+1)} {}_2F_3 \left[h+1, \frac{\frac{\lambda+h}{2}, \frac{1+\lambda+h}{2}}{\frac{\lambda+\gamma+h+1}{2}, \frac{\lambda+\gamma+h}{2}} \middle| -\frac{4y^2}{81} \right]. \end{aligned} \quad (3.10)$$

(iii) Putting $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, b_2 = 1, l = 2, \mu = 1, \zeta = 1, \delta = 1, m = 0, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = 0, \nu = 0, \rho = 1$ and $\xi = 1/(\Gamma(h))$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns to a Wright generalized Bessel function [5]. Similarly, upon using the result (3.1) under the assumption that the conditions in Theorems 2.1–2.4 are applied, we, respectively, get

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} J_h^\eta \left(\frac{y}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \gamma \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda)}{\Gamma(h+1)\Gamma(\lambda+\gamma+1)} \\ & \quad \times {}_{2\alpha}F_{2\alpha+\eta} \left[\frac{\frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}}{\frac{h+1}{\eta}, \frac{h+2}{\eta}, \dots, \frac{h+\eta}{\eta}, \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}}, \frac{y}{b^\alpha \eta^\eta} \right], \\ & \quad \frac{\frac{\gamma-\lambda}{\alpha}, \frac{\gamma-\lambda+1}{\alpha} \dots, \frac{\gamma-\lambda+\alpha-1}{\alpha}}{\frac{\lambda+\gamma+1}{\alpha}, \frac{\lambda+\gamma+2}{\alpha} \dots, \frac{\lambda+\gamma+\alpha}{\alpha}} \left| -\frac{y}{b^\alpha \eta^\eta} \right], \end{aligned} \quad (3.11)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} J_h^\eta \left(\frac{yz^\alpha}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \gamma \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda)}{\Gamma(h+1)\Gamma(\lambda+\gamma+1)} \\ & \quad \times {}_{3\alpha}F_{3\alpha+\eta} \left[\frac{\frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}}{\frac{h+1}{\eta}, \frac{h+2}{\eta}, \dots, \frac{h+\eta}{\eta}, \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}}, \frac{y}{2^\alpha \eta^\eta} \right], \\ & \quad \frac{\frac{2\lambda}{2\alpha}, \frac{2\lambda+1}{2\alpha} \dots, \frac{2\lambda+2\alpha-1}{2\alpha}}{\frac{\lambda+\gamma+1}{2\alpha}, \frac{\lambda+\gamma+2}{2\alpha} \dots, \frac{\lambda+\gamma+2\alpha}{2\alpha}} \left| -\frac{y}{2^\alpha \eta^\eta} \right], \end{aligned} \quad (3.12)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} J_h^\eta(y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} \frac{B(\gamma, \lambda)}{\Gamma(h+1)} \\ & \quad \times {}_\alpha F_{\alpha+\eta} \left[\frac{\frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}}{\frac{h+1}{\eta}, \frac{h+2}{\eta}, \dots, \frac{h+\eta}{\eta}, \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha}} \middle| -\frac{y}{\eta^\eta} \right], \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} J_h^\eta(yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} \frac{B(\gamma, \lambda)}{\Gamma(h+1)} \end{aligned}$$

$$\times {}_{\alpha}F_{\alpha+\eta} \left[\begin{matrix} \frac{\lambda}{\alpha}, \frac{1+\lambda}{\alpha}, \dots, \frac{\lambda+\alpha-1}{\alpha} \\ \frac{h+1}{\eta}, \frac{h+2}{\eta}, \dots, \frac{h+\eta}{\eta}, \frac{\lambda+\gamma}{\alpha}, \frac{1+\gamma+\alpha}{\alpha}, \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{matrix} \middle| -\left(\frac{4}{9}\right)^{\alpha} \frac{y}{\eta^{\eta}} \right]. \quad (3.14)$$

(iv) By inserting $u = 1, v = 2, w = 1, \alpha_1 = 1, b_1 = 3/2, b_2 = 1, l = 1, \mu = 1, \zeta = 2, \delta = 1, m = 1, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = 1/2, \eta = 1, \nu = 1/2, \rho = 1$ and $\xi = 1/\Gamma(h)\Gamma(3/2)$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns to a Struve function [5] and, by using the result (3.1) under the assumptions of Theorems 2.1–2.4, we, respectively, arrive at

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} H_h \left(\frac{y}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-(\gamma+\alpha(h+1))} (\gamma + \alpha(h+1)) \left(\frac{y}{2} \right)^{h+1} \\ & \times \frac{\Gamma(2\lambda)\Gamma(\gamma - \lambda + \alpha(h+1))}{\Gamma(1 + \lambda + \gamma + \alpha(h+1))\Gamma(3/2 + h)\Gamma(3/2)} \\ & \times {}_{1+4\alpha}F_{2+4\alpha} \left[\begin{matrix} 1, \frac{1+\gamma+\alpha(h+1)}{2\alpha}, \frac{2+\gamma+\alpha(h+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(h+1)+2\alpha}{2\alpha}, \\ 3/2, h + 3/2, \frac{\gamma+\alpha(h+1)}{2\alpha}, \frac{1+\gamma+\alpha(h+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(h+1)+2\alpha-1}{2\alpha} \end{matrix} \middle| -\frac{y^2}{4b^{2\alpha}} \right], \end{aligned} \quad (3.15)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} H_h \left(\frac{yz^\alpha}{A^\alpha} \right) dz \\ &= 2^{1-(\lambda+\alpha(h+1))} b^{\lambda-\gamma} (\gamma + \alpha(h+1)) \left(\frac{y}{2} \right)^{h+1} \\ & \times \frac{\Gamma(\gamma - \lambda)\Gamma(2\lambda + 2\alpha(h+1))}{\Gamma(\frac{3}{2})\Gamma(h + \frac{3}{2})\Gamma(1 + \lambda + \gamma + 2\alpha(h+1))} \\ & \times {}_{1+4\alpha}F_{2+6\alpha} \left[\begin{matrix} \frac{2\lambda+2\alpha(h+1)}{2\alpha}, \frac{2\lambda+2\alpha(h+1)+1}{2\alpha}, \dots, \frac{2\lambda+2\alpha(h+1)+2\alpha-1}{2\alpha}, \\ 3/2, h + 3/2, \frac{\gamma+\alpha(h+1)}{2\alpha}, \frac{\gamma+\alpha(h+1)+1}{2\alpha}, \dots, \frac{\gamma+\alpha(h+1)+2\alpha-1}{2\alpha} \end{matrix} \middle| -\frac{y^2}{16} \right], \end{aligned} \quad (3.16)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} H_h(y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} \left(\frac{y}{2} \right)^{h+1} \frac{B(\gamma + \alpha(h+1), \lambda)}{\Gamma(h+3/2)\Gamma(3/2)} \\ & \times {}_{1+2\alpha}F_{2+2\alpha} \left[\begin{matrix} 1, \frac{\gamma+\alpha(h+1)}{2\alpha}, \frac{1+\gamma+\alpha(h+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(h+1)+2\alpha-1}{2\alpha} \\ \frac{3}{2}, h + \frac{3}{2}, \frac{\lambda+\gamma+\alpha(h+1)}{2\alpha}, \frac{\lambda+\gamma+\alpha(h+1)+1}{2\alpha}, \dots, \frac{\lambda+\gamma+\alpha(h+1)+2\alpha-1}{2\alpha} \end{matrix} \middle| -\frac{y^2}{4} \right], \end{aligned} \quad (3.17)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} H_h(yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3} \right)^{2(\lambda+\alpha(h+1))} \left(\frac{y}{2} \right)^{h+1} \frac{B(\gamma, \lambda + \alpha(h+1))}{\Gamma(h+3/2)\Gamma(3/2)} \\ & \times {}_{1+2\alpha}F_{2+2\alpha} \left[\begin{matrix} 1, \frac{\lambda+\alpha(h+1)}{2\alpha}, \frac{1+\lambda+\alpha(h+1)}{2\alpha}, \\ \frac{3}{2}, h + \frac{3}{2}, \frac{\lambda+\gamma+\alpha(h+1)}{2\alpha}, \frac{\lambda+\gamma+\alpha(h+1)+1}{2\alpha}, \dots, \frac{\lambda+\gamma+\alpha(h+1)+2\alpha-1}{2\alpha} \end{matrix} \middle| -\frac{y^2}{4} \right], \end{aligned}$$

$$\cdots, \frac{\lambda+\alpha(h+1)+2\alpha-1}{2\alpha} \Big| - \left(\frac{16}{81} \right)^\alpha \frac{y^2}{4} \Bigg]. \quad (3.18)$$

(v) By inserting $u = 1, v = 2, w = 1, a_1 = 1, b_1 = (\tau + \epsilon + 3)/2, b_2 = (\tau - \epsilon + 3)/2, l = 1, \mu = 1, \zeta = 2, h = 1, \delta = \tau, m = 1, k_1 = 0, A_1 = 0, A_2 = 0, B_1 = -1, \eta = 1, \nu = -1, \rho = 1$ and $\xi = 2^{\tau+1}/\{(\tau + \epsilon + 1)(\tau - \epsilon + 1)\}$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function becomes a Lommel function [5]. By using (3.1) with conditions imposed on Theorems 2.1–2.4, we, respectively, obtain

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} S_{\tau,\epsilon}(yA^{-\alpha}) dz \\ &= \frac{2^{1-\lambda} b^{\lambda-(\gamma+\alpha(\tau+1))} y^{\tau+1} (\gamma + \alpha(\tau + 1))}{(\tau - \epsilon + 1)(\tau + \epsilon + 1)} \\ &\times \frac{\Gamma(2\lambda)\Gamma(\gamma + \alpha(\tau + 1) - \lambda)}{\Gamma(\lambda + \gamma + \alpha(\tau + 1) + 1)} \\ &\times {}_{1+4\alpha}F_{2+4\alpha} \left[\begin{matrix} 1, \frac{1+\gamma+\alpha(\tau+1)}{2\alpha}, \frac{2+\gamma+\alpha(\tau+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(\tau+1)+2\alpha}{2\alpha}, \\ \frac{\tau-\epsilon+3}{2}, \frac{\tau+\epsilon+3}{2}, \frac{\gamma+\alpha(\tau+1)}{2\alpha}, \frac{1+\gamma+\alpha(\tau+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(\tau+1)+2\alpha-1}{2\alpha}, \end{matrix} \middle| \frac{y^2}{4b^{\alpha}} \right], \end{aligned} \quad (3.19)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} S_{\tau,\epsilon}(yz^\alpha A^{-\alpha}) dz \\ &= \frac{2^{1-(\lambda+\alpha(\tau+1))} b^{\lambda-\gamma} y^{\tau+1} (\gamma + \alpha(\tau + 1))}{(\tau - \epsilon + 1)(\tau + \epsilon + 1)} \\ &\times \frac{\Gamma(2(\lambda + \alpha(\tau + 1)))\Gamma(\gamma - \lambda)}{\Gamma(\lambda + \gamma + \alpha(\tau + 1) + 1)} \\ &\times {}_{1+6\alpha}F_{2+6\alpha} \left[\begin{matrix} 1, \frac{1+\gamma+\alpha(\tau+1)}{2\alpha}, \frac{2+\gamma+\alpha(\tau+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(\tau+1)+2\alpha}{2\alpha}, \\ \frac{\tau-\epsilon+3}{2}, \frac{\tau+\epsilon+3}{2}, \frac{\gamma+\alpha(\tau+1)}{2\alpha}, \frac{1+\gamma+\alpha(\tau+1)}{2\alpha}, \dots, \frac{\gamma+\alpha(\tau+1)+2\alpha-1}{2\alpha}, \end{matrix} \middle| \frac{y^2}{4a^{\alpha+1}} \right], \end{aligned} \quad (3.20)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} (S)^{2\lambda-1} (T)^{\gamma-1} S_{\tau,\epsilon}(y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} \frac{B(\gamma + \alpha(\tau + 1), \lambda)}{(\tau - \epsilon + 1)(\tau + \epsilon + 1)} \\ &\times {}_{1+2\alpha}F_{2+2\alpha} \left[\begin{matrix} 1, \frac{\gamma+\alpha(\tau+1)}{2\alpha}, \frac{1+\gamma+\alpha(\tau+1)}{2\alpha}, \\ \frac{\tau-\epsilon+3}{2}, \frac{\tau+\epsilon+3}{2}, \frac{\lambda+\gamma+2\alpha(\tau+1)}{2\alpha}, \frac{1+\lambda+\gamma+2\alpha(\tau+1)}{2\alpha}, \end{matrix} \middle| -\frac{y^2}{4} \right], \end{aligned} \quad (3.21)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} S_{\tau,\epsilon}(yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3} \right)^{2(\lambda+\alpha(\tau+1))} \frac{B(\gamma, \lambda + \alpha(\tau + 1))}{(\tau - \epsilon + 1)(\tau + \epsilon + 1)} \end{aligned}$$

$$\begin{aligned} & \times {}_{1+2\alpha}F_{2+2\alpha} \left[\begin{array}{c} 1, \frac{\lambda+\alpha(\tau+1)}{2\alpha}, \frac{1+\lambda+\alpha(\tau+1)}{2\alpha}, \\ \frac{\tau-\epsilon+3}{2}, \frac{\tau+\epsilon+3}{2}, \frac{\lambda+\gamma+\alpha(\tau+1)}{2\alpha}, \frac{1+\lambda+\gamma+\alpha(\tau+1)}{2\alpha}, \\ \dots, \frac{\lambda+\alpha(\tau+1)+2\alpha-1}{2\alpha} \end{array} \middle| -\frac{y^2}{4} \left(\frac{4}{9} \right)^\alpha \right] \\ & \dots, \frac{\lambda+\gamma+\alpha(\tau+1)+2\alpha-1}{2\alpha} \end{aligned} \quad (3.22)$$

(vi) By inserting $u = 1, v = 1, w = 1, a_1 = 1, b_1 = 1, l = -2, \mu = 1, \zeta = 1, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = -1, \nu = -1, \rho = 1$ and $\xi = 1/(\Gamma(h))$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function becomes a Mittag-Leffler function [9, 23]. By using (3.1) with the understanding that the conditions of Theorems 2.1–2.4 are applied, we, respectively, obtain

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} E_{h,\eta} \left(\frac{y}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma}(\gamma) \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda)}{\Gamma(h)\Gamma(\lambda+\gamma+1)} \\ & \times {}_{1+2\alpha}F_{2\alpha+\eta} \left[\begin{array}{c} 1, \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}, \\ \frac{h}{\eta}, \frac{h+1}{\eta}, \dots, \frac{h+\eta-1}{\eta}, \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}, \\ \frac{\gamma-\lambda}{\alpha}, \frac{\gamma-\lambda+1}{\alpha} \dots, \frac{\gamma-\lambda+\alpha-1}{\alpha} \end{array} \middle| \frac{y}{b^\alpha \eta^\eta} \right], \end{aligned} \quad (3.23)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} E_{h,\eta} \left(\frac{yz^\alpha}{A^\alpha} \right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma}(\gamma) \frac{\Gamma(\gamma-\lambda)}{\Gamma(h+1)\Gamma(\lambda+\gamma+1)} \\ & \times {}_{1+3\alpha}F_{3\alpha+\eta} \left[\begin{array}{c} 1, \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}, \\ \frac{h}{\eta}, \frac{h+1}{\eta}, \dots, \frac{h+\eta}{\eta}, \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}, \\ \frac{2\lambda}{2\alpha}, \frac{2\lambda+1}{2\alpha} \dots, \frac{2\lambda+2\alpha-1}{2\alpha} \end{array} \middle| \frac{y}{2^\alpha \eta^\eta} \right], \end{aligned} \quad (3.24)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} E_{h,\eta} (y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} \frac{B(\gamma, \lambda)}{\Gamma(h)} \\ & \times {}_{1+\alpha}F_{\alpha+\eta} \left[\begin{array}{c} 1, \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha} \\ \frac{h}{\eta}, \frac{h+1}{\eta}, \dots, \frac{h+\eta-1}{\eta}, \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{array} \middle| \frac{y}{\eta^\eta} \right], \end{aligned} \quad (3.25)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} E_{h,\eta} (yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} \frac{B(\gamma, \lambda)}{\Gamma(h)} \\ & \times {}_{1+\alpha}F_{\alpha+\eta} \left[\begin{array}{c} 1, \frac{\lambda}{\alpha}, \frac{1+\lambda}{\alpha} \dots, \frac{\lambda+\alpha-1}{\alpha} \\ \frac{h}{\eta}, \frac{h+1}{\eta}, \dots, \frac{h+\eta-1}{\eta}, \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{array} \middle| \left(\frac{4}{9} \right)^\alpha \frac{y}{\eta^\eta} \right]. \end{aligned} \quad (3.26)$$

(vii) Putting $u = 1, v = 1, w = 1, a_1 = a, b_1 = 1, l = -2, \zeta = 1, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = 0, \eta = 1, \nu = 0, \rho = 0$ and $\xi = 1$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns into a unified Riemann–Zeta function [7], and by using (3.1) under the conditions imposed on

Theorems 2.1–2.4, we, respectively, get

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} \phi_a(yA^{-\alpha}, \mu, h) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \\ & \times \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(\gamma - \lambda + \alpha n) \Gamma(2\lambda) \Gamma(\gamma + \alpha n + 1)}{(h+n)^\mu \Gamma(\gamma + \alpha n) \Gamma(\lambda + \gamma + 1 + \alpha n) n!} \left(\frac{y}{b^\alpha}\right)^n, \end{aligned} \quad (3.27)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} \phi_a\left(\frac{yz^\alpha}{A^\alpha}, \mu, h\right) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} \\ & \times \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(\gamma - \lambda) \Gamma(2\lambda + 2\alpha n) \Gamma(\gamma + \alpha n + 1)}{(h+n)^\mu \Gamma(\gamma + \alpha n) \Gamma(\lambda + \gamma + 1 + 2\alpha n) n!} \left(\frac{y}{2}\right)^n, \end{aligned} \quad (3.28)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \phi_a(y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(\gamma + \alpha n) \Gamma(\lambda)}{(h+n)^\mu \Gamma(\gamma + \lambda + \alpha n)} \left(\frac{y^n}{n!}\right), \end{aligned} \quad (3.29)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \phi_a(yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(\gamma) \Gamma(\lambda + \alpha n)}{(h+n)^\mu \Gamma(\gamma + \lambda + \alpha n)} \left(\frac{2}{3}\right)^{\alpha n} \frac{y^n}{n!}. \end{aligned} \quad (3.30)$$

(viii) Putting $u = 1, v = 1, w = 1, a_1 = 1, b_1 = 1, l = 2, \zeta = 1, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = -1, \eta = 1, \nu = -1, \rho = 1, h = 1, \mu = 1$, and $\xi = 1$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns into a e^{-z} function and, by using (3.1) under the conditions imposed on Theorems 2.1–2.4, respectively, we get

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} e^{-(yA^{-\alpha})} dz = 2^{1-\lambda} b^{\lambda-\gamma} (\gamma) \frac{\Gamma(2\lambda) \Gamma(\gamma - \lambda)}{\Gamma(\lambda + \gamma + 1)} \\ & \times {}_{2\alpha}F_{2\alpha} \left[\begin{matrix} \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}, \frac{\gamma-\lambda}{\alpha}, \frac{\gamma-\lambda+1}{\alpha} \dots, \frac{\gamma-\lambda+\alpha-1}{\alpha} \\ \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}, \frac{\lambda+\gamma+1}{\alpha}, \frac{\lambda+\gamma+2}{\alpha} \dots, \frac{\lambda+\gamma+\alpha}{\alpha} \end{matrix} \middle| -yb^{-\alpha} \right], \end{aligned} \quad (3.31)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} e^{-(yz^\alpha A^{-\alpha})} dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma} (\gamma) \frac{\Gamma(2\lambda) \Gamma(\gamma - \lambda)}{\Gamma(\lambda + \gamma + 1)} \\ & \times {}_{3\alpha}F_{3\alpha} \left[\begin{matrix} \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}, \frac{2\lambda}{2\alpha}, \frac{2\lambda+1}{2\alpha} \dots, \frac{2\lambda+2\alpha-1}{2\alpha} \\ \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}, \frac{\lambda+\gamma+1}{2\alpha}, \frac{\lambda+\gamma+2}{2\alpha} \dots, \frac{\lambda+\gamma+2\alpha}{2\alpha} \end{matrix} \middle| -\frac{y}{2^\alpha} \right], \end{aligned} \quad (3.32)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} e^{-(y(1-z)^{2\alpha} T^\alpha)} dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} B(\gamma, \lambda) \end{aligned}$$

$$\times {}_{\alpha}F_{\alpha} \left[\begin{matrix} \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha} \\ \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{matrix} \middle| -y \right], \quad (3.33)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} e^{-(yz^\alpha S^{2\alpha})} dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} B(\gamma, \lambda) \\ & \times {}_{\alpha}F_{\alpha} \left[\begin{matrix} \frac{\lambda}{\alpha}, \frac{1+\lambda}{\alpha} \dots, \frac{\lambda+\alpha-1}{\alpha} \\ \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{matrix} \middle| -\left(\frac{4}{9} \right)^\alpha y \right]. \end{aligned} \quad (3.34)$$

(ix) Putting $w = 1, p = P, q = Q, l = 2, \mu = 1, \zeta = 1, \delta = 0, m = 0, k_u = 0, A_u = 0, B_1 = -1, \eta = 1, v = -1, \rho = 1, h = 1$ and $\xi = \frac{\prod_{u=1}^P \Gamma(a_u)}{\prod_{v=1}^Q \Gamma(b_v)}$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns into a MacRobert E-function [4]. Moreover, by using the result (3.1) with conditions already imposed on Theorems 2.1–2.4, respectively, we get

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} E \left(P, (a_P); Q, (b_Q); \frac{1}{yA^\alpha} \right) dz \\ &= \frac{\prod_{u=1}^P \Gamma(a_u)}{\prod_{v=1}^Q \Gamma(b_v)} \frac{2^{1-\lambda} b^{\lambda-\gamma} \Gamma(\gamma+1) \Gamma(2\lambda) \Gamma(\gamma-\lambda)}{\Gamma(\gamma) \Gamma(\lambda+\gamma+1)} \\ & \times {}_{P+2\alpha}F_{Q+2\alpha} \left[\begin{matrix} a_1, a_2, \dots, a_P; \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}, \\ b_1, b_2, \dots, b_Q, \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}, \end{matrix} \middle| -yb^{-\alpha} \right], \end{aligned} \quad (3.35)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} E \left(P, (a_P); Q, (b_Q); \frac{1}{yz^\alpha A^{-\alpha}} \right) dz \\ &= \frac{\prod_{u=1}^P \Gamma(a_u)}{\prod_{v=1}^Q \Gamma(b_v)} \frac{2^{1-\lambda} b^{\lambda-\gamma} \Gamma(\gamma+1) \Gamma(2\lambda) \Gamma(\gamma-\lambda)}{\Gamma(\gamma) \Gamma(\lambda+\gamma+1)} \\ & \times {}_{P+3\alpha}F_{Q+3\alpha} \left[\begin{matrix} a_1, a_2, \dots, a_P; \frac{1+\gamma}{\alpha}, \frac{2+\gamma}{\alpha} \dots, \frac{\gamma+\alpha}{\alpha}, \\ b_1, b_2, \dots, b_Q; \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha}, \end{matrix} \middle| 2^{-\alpha} y \right], \end{aligned} \quad (3.36)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} E \left(P, (a_P); Q, (b_Q); \frac{1}{y(1-z)^{2\alpha} T^\alpha} \right) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} B(\gamma, \lambda) \frac{\prod_{u=1}^P \Gamma(a_u)}{\prod_{v=1}^Q \Gamma(b_v)} \\ & \times {}_{P+\alpha}F_{Q+\alpha} \left[\begin{matrix} a_1, a_2, \dots, a_P; \frac{\gamma}{\alpha}, \frac{1+\gamma}{\alpha} \dots, \frac{\gamma+\alpha-1}{\alpha} \\ b_1, b_2, \dots, b_Q; \frac{\lambda+\gamma}{\alpha}, \frac{1+\lambda+\gamma}{\alpha} \dots, \frac{\lambda+\gamma+\alpha-1}{\alpha} \end{matrix} \middle| -y \right], \end{aligned} \quad (3.37)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} E \left(P, (a_P); Q, (b_Q); \frac{1}{yz^\alpha S^{2\alpha}} \right) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} B(\gamma, \lambda) \frac{\prod_{u=1}^P \Gamma(a_u)}{\prod_{v=1}^Q \Gamma(b_v)} \end{aligned}$$

$$\times {}_{P+2\alpha}F_{Q+2\alpha} \left[\begin{matrix} a_1, a_2, \dots, a_P; \frac{2\lambda}{2\alpha}, \frac{1+2\lambda}{2\alpha} \dots, \frac{2\lambda+2\alpha-1}{2\alpha} \\ b_1, b_2, \dots, b_Q; \frac{\lambda+\gamma}{2\alpha}, \frac{1+\lambda+\gamma}{2\alpha} \dots, \frac{\lambda+\gamma+2\alpha-1}{2\alpha} \end{matrix} \middle| -\left(\frac{4}{9}\right)^{\alpha} y \right]. \quad (3.38)$$

(x) Inserting $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, k_1 = 0, h = 1/2, l = 1, \mu = 1, \zeta = 2, \delta = 0, m = 0, A_1 = 0, A_2 = -1, B_1 = 0, \eta = 1, \nu = -1/2, \rho = 1$ and $\xi = 1$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns into a $\cos z$ function and, by using (3.1) under the assumptions of Theorems 2.1–2.4, we, respectively, get

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} \cos(yA^{-\alpha}) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma}(\gamma) \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda)}{\Gamma(\lambda+\gamma+1)} \\ & \quad \times {}_{4\alpha}F_{4\alpha+1} \left[\begin{matrix} \frac{1+\gamma}{2\alpha}, \frac{2+\gamma}{2\alpha} \dots, \frac{\gamma+2\alpha}{2\alpha}, \frac{\gamma-\lambda}{2\alpha}, \frac{\gamma-\lambda+1}{2\alpha} \dots, \frac{\gamma-\lambda+2\alpha-1}{2\alpha} \\ \frac{1}{2}, \frac{\gamma}{2\alpha}, \frac{1+\gamma}{2\alpha} \dots, \frac{\gamma+2\alpha-1}{2\alpha}, \frac{\lambda+\gamma+1}{2\alpha}, \frac{\lambda+\gamma+2}{2\alpha} \dots, \frac{\lambda+\gamma+2\alpha}{2\alpha} \end{matrix} \middle| -\frac{y^2}{4b^{2\alpha}} \right], \end{aligned} \quad (3.39)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} \cos(yz^\alpha A^{-\alpha}) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma}(\gamma) \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda)}{\Gamma(\lambda+\gamma+1)} \\ & \quad \times {}_{6\alpha}F_{1+6\alpha} \left[\begin{matrix} \frac{1+\gamma}{2\alpha}, \frac{2+\gamma}{2\alpha} \dots, \frac{\gamma+2\alpha}{2\alpha}, \frac{2\lambda}{4\alpha}, \frac{2\lambda+1}{4\alpha} \dots, \frac{2\lambda+4\alpha-1}{4\alpha} \\ \frac{1}{2}, \frac{\gamma}{2\alpha}, \frac{1+\gamma}{2\alpha} \dots, \frac{\gamma+2\alpha-1}{2\alpha}, \frac{\lambda+\gamma+1}{4\alpha}, \frac{\lambda+\gamma+2}{4\alpha} \dots, \frac{\lambda+\gamma+4\alpha}{4\alpha} \end{matrix} \middle| -\frac{y^2}{4^{1+\alpha}} \right], \end{aligned} \quad (3.40)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \cos(y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} B(\gamma, \lambda) \\ & \quad \times {}_{2\alpha}F_{1+2\alpha} \left[\begin{matrix} \frac{\gamma}{2\alpha}, \frac{1+\gamma}{2\alpha} \dots, \frac{\gamma+2\alpha-1}{2\alpha} \\ \frac{1}{2}, \frac{\lambda+\gamma}{2\alpha}, \frac{1+\lambda+\gamma}{2\alpha} \dots, \frac{\lambda+\gamma+2\alpha-1}{2\alpha} \end{matrix} \middle| -\frac{y^2}{4} \right], \end{aligned} \quad (3.41)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \cos(yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3}\right)^{2\lambda} B(\gamma, \lambda) \\ & \quad \times {}_{2\alpha}F_{1+2\alpha} \left[\begin{matrix} \frac{\lambda}{2\alpha}, \frac{1+\lambda}{2\alpha} \dots, \frac{\lambda+2\alpha-1}{2\alpha} \\ \frac{1}{2}, \frac{\lambda+\gamma}{2\alpha}, \frac{1+\lambda+\gamma}{2\alpha} \dots, \frac{\lambda+\gamma+2\alpha-1}{2\alpha} \end{matrix} \middle| -\left(\frac{16}{81}\right)^\alpha \frac{y^2}{4} \right]. \end{aligned} \quad (3.42)$$

(xi) Putting $u = 1, v = 2, w = 1, a_1 = 1, b_1 = 1, k_1 = 0, h = 1/2, l = 1, \mu = 1, \zeta = 2, \delta = 2, m = 0, A_1 = 0, A_2 = -1, B_1 = 0, \eta = 1, \nu = -1/2, \rho = 1$ and $\xi = 1$ in Eqs. (2.1), (2.3), (2.4) and (2.6), the V-function turns into a $\sin z$ function, then by using (3.1) under the conditions on Theorems 2.1 to 2.4, we, respectively, get

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} \sin(yA^{-\alpha}) dz \\ &= 2^{1-\lambda} b^{\lambda-\gamma+\alpha}(\gamma+\alpha) \frac{\Gamma(2\lambda)\Gamma(\gamma-\lambda+\alpha)}{\Gamma(\lambda+\gamma+\alpha+1)} \\ & \quad \times {}_{4\alpha}F_{4\alpha+1} \left[\begin{matrix} \frac{1+\gamma+\alpha}{2\alpha}, \frac{2+\gamma+\alpha}{2\alpha} \dots, \frac{\gamma+\alpha+2\alpha}{2\alpha}, \\ \frac{3}{2}, \frac{\gamma+\alpha}{2\alpha}, \frac{1+\gamma+\alpha}{2\alpha} \dots, \frac{\gamma+\alpha+2\alpha-1}{2\alpha} \end{matrix} \middle| -\left(\frac{16}{81}\right)^\alpha \frac{y^2}{4} \right], \end{aligned}$$

$$\left[\begin{array}{c} \frac{\gamma-\lambda+\alpha}{2\alpha}, \frac{\gamma-\lambda+\alpha+1}{2\alpha} \dots, \frac{\gamma-\lambda+\alpha+2\alpha-1}{2\alpha} \\ \frac{\lambda+\gamma+\alpha+1}{2\alpha}, \frac{\lambda+\gamma+\alpha+2}{2\alpha} \dots, \frac{\lambda+\gamma+\alpha+2\alpha}{2\alpha} \end{array} \right] - \frac{y^2}{4b^{2\alpha}}, \quad (3.43)$$

$$\begin{aligned} & \int_0^\infty z^{\lambda-1} A^{-\gamma} \sin(yz^\alpha A^{-\alpha}) dz \\ &= 2^{1-\lambda+\alpha} b^{\lambda-\gamma} (\gamma + \alpha) \frac{\Gamma(2\lambda + 2\alpha) \Gamma(\gamma - \lambda)}{\Gamma(\lambda + \gamma + 1 + 2\alpha)} \\ & \times {}_{6\alpha}F_{1+6\alpha} \left[\begin{array}{c} \frac{1+\gamma+\alpha}{2\alpha}, \frac{2+\gamma+\alpha}{2\alpha} \dots, \frac{\gamma+\alpha+2\alpha}{2\alpha}, \\ \frac{3}{2}, \frac{\gamma+\alpha}{2\alpha}, \frac{1+\gamma+\alpha}{2\alpha} \dots, \frac{\gamma+\alpha+2\alpha-1}{2\alpha}, \\ \frac{2\lambda+2\alpha}{4\alpha}, \frac{2\lambda+2\alpha+1}{4\alpha} \dots, \frac{2\lambda+2\alpha+4\alpha-1}{4\alpha} \\ \frac{\lambda+\gamma+2\alpha+1}{4\alpha}, \frac{\lambda+\gamma+2\alpha+2}{4\alpha} \dots, \frac{\lambda+\gamma+2\alpha+4\alpha}{4\alpha} \end{array} \right] - \frac{y^2}{4b^{2\alpha}} \end{aligned} \quad (3.44)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \sin(y(1-z)^{2\alpha} T^\alpha) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda} B(\gamma + \alpha, \lambda) y \\ & \times {}_{2\alpha}F_{1+2\alpha} \left[\begin{array}{c} \frac{\gamma+\alpha}{2\alpha}, \frac{1+\gamma+\alpha}{2\alpha} \dots, \frac{\gamma+\alpha+2\alpha-1}{2\alpha} \\ \frac{3}{2}, \frac{\lambda+\gamma+\alpha}{2\alpha}, \frac{1+\lambda+\gamma+\alpha}{2\alpha} \dots, \frac{\lambda+\gamma+\alpha+2\alpha-1}{2\alpha} \end{array} \right] - \frac{y^2}{4} \end{aligned} \quad (3.45)$$

$$\begin{aligned} & \int_0^1 z^{\lambda-1} (1-z)^{2\gamma-1} S^{2\lambda-1} T^{\gamma-1} \sin(yz^\alpha S^{2\alpha}) dz \\ &= \left(\frac{2}{3} \right)^{2\lambda+2\alpha} B(\gamma, \lambda + \alpha) y \\ & \times {}_{2\alpha}F_{1+2\alpha} \left[\begin{array}{c} \frac{\lambda+\alpha}{2\alpha}, \frac{1+\lambda+\alpha}{2\alpha} \dots, \frac{\lambda+\alpha+2\alpha-1}{2\alpha} \\ \frac{3}{2}, \frac{\lambda+\gamma+\alpha}{2\alpha}, \frac{1+\lambda+\gamma+\alpha}{2\alpha} \dots, \frac{\lambda+\gamma+\alpha+2\alpha-1}{2\alpha} \end{array} \right] - \left(\frac{16}{81} \right)^\alpha \frac{y^2}{4}. \end{aligned} \quad (3.46)$$

4 Concluding remarks

In the present paper, we have investigated some new integral formulas involving the V-function, which are expressed in terms of suitable special functions. Also, we have easily seen that the exponential function, the Mittag-Leffler function, the Lommel function, the Struve function, the Wright generalized Bessel function, the Bessel function and the generalized hypergeometric function are special cases of the V-function. Therefore, the results presented in this paper are easily converted in terms of various special functions after some suitable parametric replacements. The V-functions are associated with a wide range of problems in diverse areas of mathematical physics, for example, neutron physics, plasma physics and radio physics. So the results presented in this paper may be applicable in the theory of mathematical physics as well.

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