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# Some identities of Lah–Bell polynomials

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## Abstract

Recently, the  $n$ th Lah–Bell number was defined as the number of ways a set of  $n$  elements can be partitioned into nonempty linearly ordered subsets for any nonnegative integer  $n$ . Further, as natural extensions of the Lah–Bell numbers, Lah–Bell polynomials are defined. We study Lah–Bell polynomials with and without the help of umbral calculus. Notably, we use three different formulas in order to express various known families of polynomials such as higher-order Bernoulli polynomials and poly-Bernoulli polynomials in terms of the Lah–Bell polynomials. In addition, we obtain several properties of Lah–Bell polynomials.

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**Keywords:** Lah–Bell polynomials; Higher-order Bernoulli polynomials; Poly-Bernoulli polynomials; Umbral calculus

## 1 Introduction

The Stirling number of the second kind  $S_2(n, k)$  is the number of ways to partition a set with  $n$  elements into  $k$  nonempty subsets. Thus  $B_n = \sum_{k=0}^n S_2(n, k)$ , which is known as the  $n$ th Bell number, is the number of ways to partition a set with  $n$  elements into nonempty subsets. Further, the Bell polynomials  $B_n(x)$  are natural extensions of the Bell numbers.

The Lah number  $L(n, k)$  counts the number of ways a set of  $n$  elements can be partitioned into  $k$  nonempty linearly ordered subsets. So  $B_n^L = \sum_{k=0}^n L(n, k)$ , which was recently defined as the  $n$ th Lah–Bell number (see [8]), counts the number of ways a set of  $n$  elements can be partitioned into nonempty linearly ordered subsets. In addition, the Lah–Bell polynomials  $B_n^L(x)$  are also defined as natural extensions of the Lah–Bell numbers.

The aim of this paper is to study some properties of Lah–Bell polynomials with and without the help of umbral calculus. In particular, we represent several known families of polynomials in terms of the Lah–Bell polynomials, and vice versa. This has been done by using three different means, namely by using a formula derived from the definition of Sheffer polynomials (see Theorem 1), the transfer formula (see (29)), and the general formula expressing one Sheffer polynomial in terms of other Sheffer polynomial (see (12)). In more detail, we express Bernoulli polynomials, powers of  $x$ , poly-Bernoulli polynomials, and higher-order Bernoulli polynomials in terms of the Lah–Bell polynomials. In addition, we represent the Lah–Bell polynomials in terms of powers of  $x$  and of falling factorials. In addition, we obtain several properties of Lah–Bell polynomials. For the rest of this section, we recall some necessary facts that are needed throughout this paper and briefly review

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basic facts about umbral calculus. For more details on umbral calculus, we refer the reader to [13].

We recall from [8] that Lah–Bell polynomials  $B_n^L(x)$  are given by

$$e^{x(\frac{1}{1-t}-1)} = \sum_{n=0}^{\infty} B_n^L(x) \frac{t^n}{n!}, \tag{1}$$

and the Lah–Bell numbers are given by  $B_n^L = B_n^L(1)$ .

For  $r \in \mathbb{N}$ , the higher-order Bernoulli polynomials are given by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1, 3, 4, 13]}). \tag{2}$$

We note that  $B_n^{(r)} = B_n^{(r)}(0)$  ( $n \geq 0$ ) are called the higher-order Bernoulli numbers.

For  $k \in \mathbb{Z}$ , the polylogarithm function is defined by

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (\text{see [10]}). \tag{3}$$

Bayad and Hamahata [2] considered the poly-Bernoulli polynomials (of index  $k$ ) given by

$$\frac{Li_k(1 - e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \beta_n^{(k)}(x) \frac{t^n}{n!}. \tag{4}$$

For  $x = 0$ ,  $\beta_n^{(k)} = \beta_n^{(k)}(0)$  are called the poly-Bernoulli numbers (of index  $k$ ) (see [7]). More precisely, the  $n$ th poly-Bernoulli polynomials of index  $k$  are defined as  $\beta_n^{(k)}(x + 1)$  in [2] and the  $n$ th poly-Bernoulli numbers of index  $k$  are defined as  $\beta_n^{(k)}(1)$  in [7].

From (1), we note that

$$\begin{aligned} e^{(x+y)(\frac{1}{1-t}-1)} &= e^{x(\frac{1}{1-t}-1)} \cdot e^{y(\frac{1}{1-t}-1)} \\ &= \sum_{l=0}^{\infty} B_l^L(x) \frac{t^l}{l!} \sum_{m=0}^{\infty} B_m^L(y) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} B_l^L(x) B_{n-l}^L(y) \right) \frac{t^n}{n!}. \end{aligned} \tag{5}$$

By (1) and (5), we get

$$B_n^L(x + y) = \sum_{l=0}^n \binom{n}{l} B_l^L(x) B_{n-l}^L(y) \quad (n \geq 0) \quad (\text{see [8]}).$$

Let  $\mathbb{C}$  be the field of complex numbers, and let  $\mathcal{F}$  be the set of all power series in the variable  $t$  over  $\mathbb{C}$  given by

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{6}$$

Let  $\mathbb{P} = \mathbb{C}[x]$ , and let  $\mathbb{P}^*$  be the vector space of all linear functionals on  $\mathbb{P}$ .

For  $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ , we define the linear functional on  $\mathbb{P}$  by

$$\langle f(t) | x^n \rangle = a_n, \quad \text{for all } n \geq 0 \text{ (see [5, 6, 8–13]).} \tag{7}$$

Thus, by (7), we get

$$\langle t^k | x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0) \text{ (see [5, 6, 8–13]),} \tag{8}$$

where  $\delta_{n,k}$  is the Kronecker’s symbol.

By (7) and (8), we easily get  $\langle e^{yt} | x^n \rangle = y^n$ , and so  $\langle e^{yt} | P(x) \rangle = P(y)$ . The order  $o(f(t))$  of a power series  $f(t) (\neq 0)$  is the smallest integer  $k$  for which the coefficient of  $t^k$  does not vanish. If  $f(t)$  is a series with  $o(f(t)) = 1$ , then  $f(t)$  is called a delta series.

If  $f(t)$  is a series with  $o(f(t)) = 0$ , then  $f(t)$  is called an invertible series. For  $f(t), g(t) \in \mathcal{F}$  with  $o(f(t)) = 1, o(g(t)) = 0$ , there exists a unique sequence  $s_n(x)$  of polynomials such that

$$\langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k} \quad (n, k \geq 0) \text{ (see [13]).} \tag{9}$$

The sequence  $s_n(x)$  is called the Sheffer sequence for the pair  $(g(t), f(t))$ , which is denoted by  $s_n(x) \sim (g(t), f(t))$ .

It is well known that  $s_n(x) \sim (g(t), f(t))$  if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} \frac{s_n(x)}{n!} t^n \quad \text{(see [11, 13]),} \tag{10}$$

for all  $x \in \mathbb{C}$  where  $\bar{f}(t)$  is the compositional inverse of  $f(t)$  such that  $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ .

Let  $s_n(x) \sim (g(t), f(t))$  and  $r_n(x) \sim (h(t), g(t))$  ( $n \geq 0$ ). Then we have

$$s_n(x) = \sum_{m=0}^n A_{n,m} r_m(x) \quad (n \geq 0), \tag{11}$$

where

$$A_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (l(\bar{f}(t)))^m | x^n \right\rangle \quad \text{(see [13]).} \tag{12}$$

### 2 Some identities of Lah–Bell polynomials

Here we represent several known families of polynomials in terms of the Lah–Bell polynomials, and vice versa. This will be done by using three different means, namely by using a formula derived from the definition of Sheffer polynomials (see Theorem 1), the transfer formula (see (29)), and the general formula expressing one Sheffer polynomial in terms of other Sheffer polynomial (see (12)).

From (1) and (10), we note that

$$B_n^L(x) \sim \left( 1, 1 - \frac{1}{1+t} \right), \tag{13}$$

and

$$\sum_{n=0}^{\infty} B_n^L(x) \frac{t^n}{n!} = e^{x(\frac{1}{1-t}-1)} = \sum_{l=0}^{\infty} \frac{x^l}{l!} \left( \frac{1}{1-t} - 1 \right)^l$$

$$= \sum_{l=0}^{\infty} x^l \sum_{n=l}^{\infty} L(n, l) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^n x^l L(n, l) \right) \frac{t^n}{n!}, \tag{14}$$

where  $L(n, l) = \frac{n!}{l!} \binom{n-1}{l-1}$  are the Lah numbers given by

$$\frac{1}{l!} \left( \frac{1}{1-t} - 1 \right)^l = \frac{1}{l!} \left( \frac{t}{1-t} \right)^l = \sum_{n=l}^{\infty} L(n, l) \frac{t^n}{n!}. \tag{15}$$

Here the generating function of the Lah numbers in (15) can be easily derived either from power series expansion of the left-hand side of (15) or from the identity

$$\exp\left(\frac{ut}{1-t}\right) = \sum_{n=0}^{\infty} \sum_{l=0}^n L(n, l) \frac{u^l t^n}{n!}, \tag{16}$$

which is stated on [4, p. 156]. It is not difficult to show that

$$\begin{aligned} e^{x(\frac{1}{1-t}-1)} &= e^{-x} e^{\frac{x}{1-t}} = e^{-x} \sum_{n=0}^{\infty} x^n \frac{1}{n!} (1-t)^{-n} \\ &= e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \sum_{k=0}^{\infty} \frac{\langle l \rangle_k}{k!} t^k \\ &= \sum_{k=0}^{\infty} \left( e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \langle l \rangle_k \right) \frac{t^k}{k!}, \end{aligned} \tag{17}$$

where  $\langle x \rangle_0 = 1, \langle x \rangle_n = x(x+1) \cdots (x+n-1), n \geq 1$ .

Thus, we have

$$B_k^L(x) = e^{-x} \sum_{l=0}^{\infty} \frac{\langle l \rangle_k}{l!} x^l \quad (k \geq 0) \text{ (see [8]).} \tag{18}$$

For  $n \in \mathbb{N}$ , by (18), we get

$$\begin{aligned} x \sum_{k=1}^n \binom{n-1}{k-1} B_{k-1}^L(x) &= x \sum_{k=1}^n \binom{n-1}{k-1} e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \langle l \rangle_{k-1} \\ &= x e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \sum_{k=0}^{n-1} \binom{n-1}{k} \langle l \rangle_k. \end{aligned} \tag{19}$$

Let

$$\mathbb{P}_n = \{P(x) \in \mathbb{C}[x] \mid \deg P(x) \leq n\} \quad (n \geq 0).$$

Then  $\mathbb{P}_n$  is an  $(n + 1)$ -dimensional vector space over  $\mathbb{C}$ . For  $P(x) \in \mathbb{P}_n$ , with  $P(x) = \sum_{m=0}^n A_m B_m^L(x)$ , we have

$$\left\langle \left( \frac{t}{1+t} \right)^m \middle| P(x) \right\rangle = \sum_{l=0}^n A_l \left\langle \left( \frac{t}{1+t} \right)^m \middle| B_l^L(x) \right\rangle$$

$$= \sum_{l=0}^n A_l l! \delta_{m,l} = m! A_m. \tag{20}$$

By (20), we have

$$A_m = \frac{1}{m!} \left\langle \left( \frac{t}{1+t} \right)^m \middle| P(x) \right\rangle \quad (n \geq 0).$$

Therefore, we obtain the following theorem.

**Theorem 1** For  $P(x) \in \mathbb{P}_n$ , we have

$$P(x) = \sum_{m=0}^n A_m B_m^L(x) \quad (n \geq 0),$$

where

$$A_m = \frac{1}{m!} \left\langle \left( \frac{t}{1+t} \right)^m \middle| P(x) \right\rangle.$$

When  $r = 1$  in (2),  $B_n(x) = B_n^{(1)}(x)$  ( $n \geq 0$ ) are called the ordinary Bernoulli polynomials. Let us take  $x = 0$ . Then  $B_n = B_n(0)$  ( $n \geq 0$ ) are called the ordinary Bernoulli numbers.

From (2), we note that

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} B_{n-l} x^l \in \mathbb{P}_n. \tag{21}$$

For  $P(x) = B_n(x) \in \mathbb{P}_n$ , we have

$$B_n(x) = \sum_{m=0}^n A_m B_m^L(x) \quad (n \geq 0), \tag{22}$$

where

$$A_m = \frac{1}{m!} \left\langle \left( \frac{t}{1+t} \right)^m \middle| B_n(x) \right\rangle. \tag{23}$$

From (8), we easily note that

$$\langle t^k | P(x) \rangle = P^{(k)}(0), \quad \text{where } P^{(k)}(0) = \left. \frac{d^k}{dx^k} P(x) \right|_{x=0} \quad (\text{see [9, 11, 13]}). \tag{24}$$

By (23), we get

$$\begin{aligned} \frac{1}{m!} \left\langle \left( \frac{t}{1+t} \right)^m \middle| B_n(x) \right\rangle &= \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^l \left\langle \left( \frac{1}{1+t} \right)^l \middle| B_n(x) \right\rangle \\ &= \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^l \sum_{k=0}^n \binom{l+k-1}{k} (-1)^k \langle t^k | B_n(x) \rangle \end{aligned}$$

$$= \frac{1}{m!} \sum_{l=0}^m \sum_{k=0}^m \binom{m}{l} \binom{l+k-1}{k} (-1)^{l+k} k! \binom{n}{k} B_{n-k}. \tag{25}$$

Therefore, by (22), (23), and (25), we obtain the following theorem.

**Theorem 2** For  $n \geq 0$ , we have

$$B_n(x) = \sum_{m=0}^n \left( \sum_{k=0}^n \sum_{l=0}^m \frac{k!}{m!} \binom{m}{l} \binom{l+k-1}{k} \binom{n}{k} (-1)^{l+k} B_{n-k} \right) B_n^L(x).$$

Let us take  $P(x) = x^n \in \mathbb{P}_n$ . Then, by Theorem 1, we have

$$x^n = \sum_{k=0}^n A_k B_k^L(x), \tag{26}$$

where

$$\begin{aligned} A_k &= \frac{1}{k!} \left\langle \left( \frac{t}{1+t} \right)^k \middle| x^n \right\rangle = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^l \left\langle \left( \frac{1}{1+t} \right)^l \middle| x^n \right\rangle \\ &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^l \sum_{m=0}^n \binom{l+m-1}{m} (-1)^m \langle t^m | x^n \rangle \\ &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{l+n} \binom{l+n-1}{n} n! \end{aligned} \tag{27}$$

Therefore, by (26) and (27), we obtain the following theorem.

**Theorem 3** For  $n \geq 0$ , we have

$$x^n = \sum_{k=0}^n \left\{ \frac{n!}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{l+n} \binom{l+n-1}{n} \right\} B_k^L(x).$$

For each nonnegative integer  $k$ , the differential operator  $t^k$  on  $\mathbb{P}$  is defined by

$$t^k x^n = \begin{cases} (n)_k x^{n-k} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases} \tag{28}$$

Here  $(x)_k$  is the falling factorial given by  $(x)_0 = 1, (x)_k = x(x-1) \cdots (x-k+1), k \geq 1$ .

Extending this linearly, any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

gives a differential operator on  $\mathbb{P}$  defined by

$$f(t)x^n = \sum_{k=0}^n \binom{n}{k} a_k x^{n-k} \quad (n \geq 0).$$

For  $p_n(x) \sim (1, f(t)), q_n(x) \sim (1, g(t))$ , we have the transfer formula given by

$$q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 0) \text{ (see [13]).} \tag{29}$$

We consider the following two Sheffer sequences:

$$B_n^L(x) \sim \left( 1, \frac{t}{1+t} \right), \quad x^n \sim (1, t) \quad (n \geq 0). \tag{30}$$

From (28), (29), and (30), we note that

$$\begin{aligned} B_n^L(x) &= x \left( \frac{1}{1+t} \right)^n x^{-1} x^n = x \sum_{l=0}^{\infty} \binom{n+l-1}{l} (-1)^l t^l x^{n-1} \\ &= x \sum_{l=0}^{n-1} \binom{n+l-1}{l} (-1)^l \binom{n-1}{l} l! x^{n-1-l} \\ &= \sum_{l=0}^{n-1} \binom{n+l-1}{l} \binom{n-1}{l} l! (-1)^l x^{n-l}. \end{aligned} \tag{31}$$

Therefore, we obtain the following theorem.

**Theorem 4** For  $n \in \mathbb{N}$ , we have

$$B_n^L(x) = \sum_{l=0}^{n-1} \binom{n+l-1}{l} \binom{n-1}{l} l! (-1)^l x^{n-l}$$

Let us consider the following two Sheffer sequences:

$$B_n^L(x) \sim \left( 1, \frac{t}{1+t} \right) \tag{32}$$

and

$$\beta_n^{(k)}(x) \sim \left( \frac{e^t - 1}{\text{Li}_k(1 - e^{-t})}, t \right). \tag{33}$$

From (11) and (12), we note that

$$\beta_n^{(k)}(x) = \sum_{m=0}^n A_{n,m} B_m^L(x), \tag{34}$$

where

$$\begin{aligned} A_{n,m} &= \frac{1}{m!} \left\langle \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} \left( \frac{t}{1+t} \right)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{j+l-1}{j} (-1)^{l+j} \left\langle \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} t^j \middle| x^n \right\rangle \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{m!} \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{j+l-1}{j} (-1)^{l+j} \binom{n}{j} j! \left\langle \frac{\text{Li}_k(1-e^{-t})}{e^t-1} \middle| x^{n-j} \right\rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{j+l-1}{j} (-1)^{l+j} \binom{n}{j} j! \beta_{n-j}^{(k)}.
 \end{aligned} \tag{35}$$

Therefore, by (34) and (35), we obtain the following theorem.

**Theorem 5** For  $n \geq 0$ , we have

$$\beta_n^{(k)}(x) = \sum_{m=0}^n \left\{ \frac{1}{m!} \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{j+l-1}{j} (-1)^{l+j} \binom{n}{j} j! \beta_{n-j}^{(k)} \right\} B_m^L(x).$$

For the following two Sheffer sequences:

$$B_n^L(x) \sim \left( 1, \frac{t}{1+t} \right), \quad (x)_n \sim (1, e^t - 1) \quad (n \geq 0),$$

we have

$$B_n^L(x) = \sum_{m=0}^n A_{n,m}(x)_m, \tag{36}$$

where

$$\begin{aligned}
 A_{n,m} &= \frac{1}{m!} \langle (e^{\frac{1}{1-t}} - 1)^m | x^n \rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \langle e^{l \frac{1}{1-t}} | x^n \rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} B_n^L(l).
 \end{aligned} \tag{37}$$

Therefore, by (36) and (37), we obtain the following theorem.

**Theorem 6** For  $n \geq 0$ , we have

$$\begin{aligned}
 B_n^L(x) &= \sum_{m=0}^n \left\{ \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} B_n^L(l) \right\} (x)_m \\
 &= \sum_{m=0}^n \binom{x}{m} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} B_n^L(l).
 \end{aligned}$$

Finally, we consider the following two Sheffer sequences:

$$B_n^L(x) \sim \left( 1, \frac{t}{1+t} \right), \quad B_n^{(r)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^r, t \right) \quad (r \in \mathbb{N}).$$

From (11) and (12), we have

$$B_n^{(r)}(x) = \sum_{m=0}^n A_{n,m} B_m^L(x), \tag{38}$$

where

$$\begin{aligned}
 A_{n,m} &= \frac{1}{m!} \left\langle \left( \frac{t}{e^t - 1} \right)^r \left( \frac{t}{1+t} \right)^m \middle| x^n \right\rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^l \left\langle \left( \frac{t}{e^t - 1} \right)^r \left( \frac{1}{1+t} \right)^l \middle| x^n \right\rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^l \sum_{j=0}^n \binom{l+j-1}{j} (-1)^j \left\langle \left( \frac{t}{e^t - 1} \right)^r t^j \middle| x^n \right\rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{l+j-1}{j} \binom{n}{j} j! (-1)^{l+j} \left\langle \left( \frac{t}{e^t - 1} \right)^r \middle| x^{n-j} \right\rangle \\
 &= \frac{1}{m!} \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{l+j-1}{j} \binom{n}{j} j! (-1)^{l+j} B_{n-j}^{(r)}. \tag{39}
 \end{aligned}$$

Therefore, by (38) and (39), we obtain the following theorem.

**Theorem 7** For  $n \geq 0$ , we have

$$B_n^{(r)} = \sum_{m=0}^n \frac{1}{m!} \left\{ \sum_{l=0}^m \sum_{j=0}^n \binom{m}{l} \binom{l+j-1}{j} \binom{n}{j} j! (-1)^{l+j} B_{n-j}^{(r)} \right\} B_m^L(x).$$

### 3 Conclusion

The Lah number  $L(n, k)$  counts the number of ways a set of  $n$  elements can be partitioned into  $k$  nonempty linearly ordered subsets. Then  $B_n^L = \sum_{k=0}^n L(n, k)$ , which was recently defined as the  $n$ th Lah–Bell number, counts the number of ways a set of  $n$  elements can be partitioned into nonempty linearly ordered subsets. In addition, the Lah–Bell polynomials  $B_n^L(x)$  are also defined as natural extensions of the Lah–Bell numbers.

In this paper, we studied some properties of Lah–Bell polynomials with and without the help of umbral calculus. Among other things, we represented several known families of polynomials in terms of the Lah–Bell polynomials, and vice versa, by using three different means, namely by using a formula derived from the definition of Sheffer polynomials (see Theorem 1), the transfer formula (see (29)), and the general formula expressing one Sheffer polynomial in terms of other Sheffer polynomial (see (12)). In more detail, we expressed Bernoulli polynomials, powers of  $x$ , poly-Bernoulli polynomials, and higher-order Bernoulli polynomials in terms of the Lah–Bell polynomials. In addition, we represented the Lah–Bell polynomials in terms of powers of  $x$  and of falling factorials. In addition, we obtained several properties of Lah–Bell polynomials.

It is one of our future projects to continue exploring some special numbers and polynomials, and also their degenerate versions, as well as to find their applications in physics, science and engineering, as well as in mathematics.

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**Consent for publication**

All authors want to publish this paper in this journal.

**Authors' contributions**

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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