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Alternating double t -values and T -values

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Abstract

Recently, Hoffman (Commun. Number Theory Phys. 13:529–567, 2019), Kaneko and Tsumura (Tsukuba J. Math. (in press), 2020) introduced and systematically studied two variants of multiple zeta values of level two, i.e., multiple t -values and multiple T -values, respectively. In this paper, by the contour integration and residue theorem, we establish two general identities, which further reduce to the expressions of the alternating double t -values and T -values. Some examples are also provided.

MSC: 11M32; 11B65

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1 Introduction and notations

For positive integers n and p , let $H_n^{(p)}$ and $\bar{H}_n^{(p)}$ stand for the n th generalized harmonic number and the n th generalized alternating harmonic number defined by

$$H_n^{(p)} := \sum_{k=1}^n \frac{1}{k^p} \quad \text{and} \quad \bar{H}_n^{(p)} := \sum_{k=1}^n \frac{(-1)^{k-1}}{k^p},$$

respectively. If $p > 1$ (or resp. $p > 0$), the generalized harmonic number $H_n^{(p)}$ (or resp. $\bar{H}_n^{(p)}$) converges to the (Riemann) zeta value $\zeta(p)$ (or resp. alternating zeta value $\bar{\zeta}(p)$):

$$\lim_{n \rightarrow \infty} H_n^{(p)} = \zeta(p) \quad \left(\text{or resp. } \lim_{n \rightarrow \infty} \bar{H}_n^{(p)} = \bar{\zeta}(p) \right).$$

When $k = 1$, $H_n^{(1)} \equiv H_n$ (resp. $\bar{H}_n^{(1)} \equiv \bar{H}_n$) is the classical harmonic number (resp. the classical alternating harmonic number). The empty sums $H_0^{(p)}$ and $\bar{H}_0^{(p)}$ are conventionally understood to be zero.

For positive integers p_1, \dots, p_k with $p_1 > 1$, the multiple zeta value (MZV for short) is defined by

$$\zeta(p_1, p_2, \dots, p_k) := \sum_{n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{p_1} n_2^{p_2} \dots n_k^{p_k}}. \quad (1.1)$$

The study of multiple zeta values began in the early 1990s with the works of Hoffman [4] and Zagier [16]. The study of multiple zeta values have attracted numerous research

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interests in the area in the last two decades. For detailed history and applications, please see the book of Zhao [17].

Let $h_n^{(p)}$ be the n th odd harmonic number, which is defined for $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$ by

$$h_n^{(p)} := \sum_{k=1}^n \frac{1}{(k-1/2)^p}, \quad h_0^{(p)} := 0, \quad h_n := h_n^{(1)}. \quad (1.2)$$

If $p > 1$, the generalized harmonic number $h_n^{(p)}$ converges to the \tilde{t} -value:

$$\lim_{n \rightarrow \infty} h_n^{(p)} = \tilde{t}(p) = \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^p}. \quad (1.3)$$

A twin sibling of the odd harmonic number is called alternating odd harmonic number, defined by

$$\bar{h}_n^{(p)} := \sum_{k=1}^n \frac{(-1)^{k-1}}{(k-1/2)^p}, \quad \bar{h}_0^{(p)} := 0, \quad \bar{h}_n := \bar{h}_n^{(1)}, \quad (1.4)$$

which was introduced in [14]. When taking the limit $n \rightarrow \infty$ in above, we get the so-called alternating \bar{t} -value

$$\bar{t}(p) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1/2)^p} \quad (p \geq 1). \quad (1.5)$$

Note that from [3], for nonnegative integer k , we have the generating function of $\bar{t}(2k+1)$

$$\frac{\pi}{\cos(\pi s)} = 2 \sum_{k=0}^{\infty} \bar{t}(2k+1) s^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} \pi^{2k+1}}{(2k)!} s^{2k},$$

where E_{2k} is the Euler number. Thus, we compute

$$\bar{t}(2k+1) = \frac{(-1)^k E_{2k} \pi^{2k+1}}{2(2k)!} \quad (k \geq 0).$$

In a recent paper [5], Hoffman introduced and studied the more general multiple t -values

$$\begin{aligned} t(p_1, p_2, \dots, p_k) &:= \sum_{\substack{n_1 > \dots > n_k \geq 1 \\ n_i \text{ odd}}} \frac{1}{n_1^{p_1} n_2^{p_2} \dots n_k^{p_k}} \\ &= \sum_{n_1 > \dots > n_k \geq 1} \frac{1}{(2n_1-1)^{p_1} (2n_2-1)^{p_2} \dots (2n_k-1)^{p_k}}. \end{aligned} \quad (1.6)$$

As the normalized version,

$$\tilde{t}(p_1, p_2, \dots, p_k) := 2^{p_1 + \dots + p_k} t(p_1, p_2, \dots, p_k), \quad (1.7)$$

we call them multiple \tilde{t} -values.

Kaneko and Tsumura [6, 7] introduced and studied a new kind of multiple zeta values of level two:

$$\begin{aligned} T(p_1, p_2, \dots, p_k) &:= 2^k \sum_{\substack{m_1 > m_2 > \dots > m_k > 0 \\ m_i \equiv k-i+1 \pmod{2}}} \frac{1}{m_1^{p_1} m_2^{p_2} \dots m_k^{p_k}} \\ &= 2^k \sum_{n_1 > n_2 > \dots > n_k > 0} \frac{1}{(2n_1 - k)^{p_1} (2n_2 - k + 1)^{p_2} \dots (2n_k - 1)^{p_k}}, \end{aligned} \quad (1.8)$$

which were called multiple T -values (MTVs). As the normalized version,

$$\tilde{T}(p_1, p_2, \dots, p_k) := 2^{p_1 + \dots + p_k - k} T(p_1, p_2, \dots, p_k), \quad (1.9)$$

we call them multiple \tilde{T} -values.

In (1.1) and (1.6)–(1.9), we put a bar on the top of p_j ($j = 1, \dots, k$) if there is a sign $(-1)^{n_j}$ appearing in the denominator on the right. These with one or more p_j barred are called the alternating MZVs, alternating multiple t -values, alternating multiple \tilde{t} -values, alternating multiple T -values, and alternating multiple \tilde{T} -values, respectively. For example,

$$\begin{aligned} \zeta(p_1, \bar{p}_2, p_3, \bar{p}_4) &= \sum_{n_1 > n_2 > n_3 > n_4 > 0} \frac{(-1)^{n_2 + n_4}}{n_1^{p_1} n_2^{p_2} n_3^{p_3} n_4^{p_4}}, \\ t(\bar{p}_1, \bar{p}_2, p_3, p_4) &= \sum_{n_1 > n_2 > n_3 > n_4 > 0} \frac{(-1)^{n_1 + n_2}}{(2n_1 - 1)^{p_1} (2n_2 - 1)^{p_2} (2n_3 - 1)^{p_3} (2n_4 - 1)^{p_4}}, \\ T(\bar{p}_1, p_2, \bar{p}_3, p_4) &= 2^4 \sum_{n_1 > n_2 > n_3 > n_4 > 0} \frac{(-1)^{n_1 + n_3}}{(2n_1 - 4)^{p_1} (2n_2 - 3)^{p_2} (2n_3 - 2)^{p_3} (2n_4 - 1)^{p_4}}. \end{aligned}$$

In all of these definitions, we call k the “depth” and $p_1 + \dots + p_k$ the “weight”.

The motivation for this paper arises from the results of Flajolet and Salvy’s paper [2] and Wang and Xu’s papers [12, 14]. In [2], Flajolet and Salvy used the methods of contour integration and residue theorem to determine the reducibility of some classical Euler sums. Similarly, in [12, 14], Wang and Xu used the contour integration and residue theorem to evaluate (alternating) Euler sums and Euler T -sums. There have been numerous contributions on the theory of Euler sums in the last two decades, for example, see [1, 8, 9, 11, 13, 15] and the references therein.

The main purpose of this paper is to study the four (alternating) double t -values

$$\tilde{t}(q, p), \tilde{t}(q, \bar{p}), \tilde{t}(\bar{q}, \bar{p}), \tilde{t}(\bar{q}, p)$$

and the four (alternating) double T -values

$$\tilde{T}(q, p), \tilde{T}(q, \bar{p}), \tilde{T}(\bar{q}, \bar{p}), \tilde{T}(\bar{q}, p)$$

by using the methods of contour integration and residue theorem.

2 Double t -values and T -values

In this section, we give explicit evaluations for some (alternating) double t -values and T -values. We will prove these results in Sect. 4.

Theorem 2.1 *For positive integers p and $q > 1$,*

$$\begin{aligned}
 & (1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_{n-1}^{(p)}}{(n-1/2)^q} \\
 &= (1 - (-1)^{p+q}) \tilde{t}(q, p) \\
 &= (-1)^{p+q} \tilde{t}(p+q) - (-1)^p (1 + (-1)^q) \tilde{t}(p) \tilde{t}(q) \\
 &\quad - (-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \zeta(p+q-k-1) \\
 &\quad + 2(-1)^p \sum_{\substack{2k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} \tilde{t}(2k_1) \zeta(k_2+p-1), \tag{2.1}
 \end{aligned}$$

$$\begin{aligned}
 & (1 + (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\tilde{h}_{n-1}^{(p)}}{(n-1/2)^q} \\
 &= -(1 + (-1)^{p+q}) \tilde{t}(q, \bar{p}) \\
 &= -(-1)^{p+q} \tilde{t}(p+q) + (-1)^p (1 + (-1)^q) \tilde{t}(p) \tilde{t}(q) \\
 &\quad - (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \zeta(p+q-k-1) \\
 &\quad - 2(-1)^p \sum_{\substack{2k_1+k_2=q+2, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} \tilde{t}(2k_1-1) \bar{\zeta}(k_2+p-1), \tag{2.2}
 \end{aligned}$$

$$\begin{aligned}
 & (1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\tilde{h}_{n-1}^{(p)}}{(n-1/2)^q} (-1)^{n-1} \\
 &= (1 - (-1)^{p+q}) \tilde{t}(\bar{q}, \bar{p}) \\
 &= (-1)^{p+q} \tilde{t}(p+q) + (-1)^p (1 - (-1)^q) \tilde{t}(p) \tilde{t}(q) \\
 &\quad + (-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \bar{\zeta}(p+q-k-1) \\
 &\quad - 2(-1)^p \sum_{\substack{2k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} \tilde{t}(2k_1) \bar{\zeta}(k_2+p-1), \tag{2.3}
 \end{aligned}$$

$$\begin{aligned}
 & (1 + (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_{n-1}^{(p)}}{(n-1/2)^q} (-1)^{n-1} \\
 &= -(1 + (-1)^{p+q}) \tilde{t}(\bar{q}, p) \\
 &= -(-1)^{p+q} \tilde{t}(p+q) - (-1)^p (1 - (-1)^q) \tilde{t}(p) \tilde{t}(q) \\
 &\quad + (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \bar{\zeta}(p+q-k-1) \\
 &\quad + 2(-1)^p \sum_{\substack{2k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} \binom{p+k_2-2}{p-1} \tilde{t}(2k_1-1) \zeta(p+k_2-1), \tag{2.4}
 \end{aligned}$$

where $\zeta(1) := -2 \log(2)$ and $\tilde{t}(1) := 0$.

Remark 2.2 Note that formulas (2.1) and (2.4) can also be found in Xu and Wang [14].

Example 2.1 We have

$$\begin{aligned}\tilde{t}(3, \bar{1}) &= \frac{1}{2}\bar{t}(4) - \frac{7}{8}\pi\zeta(3) - \frac{1}{4}\pi^3\log(2), \\ \tilde{t}(2, \bar{2}) &= \frac{1}{2}\bar{t}(4) - 3\zeta(2)\bar{t}(2) + \frac{7}{4}\pi\zeta(3), \\ \tilde{t}(\bar{2}, \bar{1}) &= 3\zeta(2)\log(2) - \frac{7}{2}\zeta(3), \\ \tilde{t}(\bar{3}, 1) &= \frac{1}{2}\bar{t}(4) - \frac{1}{2}\log(2)\pi^3 + \frac{7}{8}\pi\zeta(3), \\ \tilde{t}(\bar{3}, \bar{2}) &= -\frac{31}{2}\zeta(5) + \frac{1}{4}\pi^3\bar{t}(2) - \frac{27}{4}\zeta(2)\zeta(3).\end{aligned}$$

Theorem 2.3 For positive integers p and $q > 1$,

$$\begin{aligned}& (1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_n^{(p)}}{n^q} \\&= (1 - (-1)^{p+q}) \tilde{T}(q, p) \\&= -(-1)^p (1 + (-1)^q) \tilde{t}(p) \zeta(q) - (-1)^p \binom{p+q-1}{p-1} \tilde{t}(p+q) \\&\quad - (-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \tilde{t}(p+q-k-1) \\&\quad + (-1)^p \sum_{\substack{k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} (1 + (-1)^{k_1}) \binom{k_2+p-2}{p-1} \zeta(k_1) \tilde{t}(k_2+p-1),\end{aligned}\tag{2.5}$$

$$\begin{aligned}& (1 + (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\bar{h}_n^{(p)}}{n^q} \\&= -(1 + (-1)^{p+q}) \tilde{T}(q, \bar{p}) \\&= (-1)^p (1 + (-1)^q) \bar{t}(p) \zeta(q) + (-1)^p \binom{p+q-1}{p-1} \bar{t}(p+q) \\&\quad - (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \binom{p+q-k-2}{q-1} \bar{t}(k+1) \bar{t}(p+q-k-1) \\&\quad + (-1)^p \sum_{\substack{k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} (1 + (-1)^{k_1}) \binom{k_2+p-2}{p-1} \bar{\zeta}(k_1) \bar{t}(k_2+p-1),\end{aligned}\tag{2.6}$$

$$\begin{aligned}& (1 + (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\bar{h}_n^{(p)}}{n^q} (-1)^{n-1} \\&= -(1 + (-1)^{p+q}) \tilde{T}(\bar{q}, \bar{p}) \\&= (-1)^p (1 + (-1)^q) \bar{t}(p) \bar{\zeta}(q) - (-1)^p \binom{p+q-1}{p-1} \bar{t}(p+q)\end{aligned}$$

$$\begin{aligned}
 & -(-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \tilde{t}(p+q-k-1) \\
 & + (-1)^p \sum_{\substack{k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} (1 + (-1)^{k_1}) \binom{k_2+p-2}{p-1} \zeta(k_1) \tilde{t}(k_2+p-1), \tag{2.7}
 \end{aligned}$$

$$\begin{aligned}
 & (1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_n^{(p)}}{n^q} (-1)^{n-1} \\
 & = (1 - (-1)^{p+q}) \tilde{T}(\bar{q}, p) \\
 & = -(-1)^p (1 + (-1)^q) \tilde{t}(p) \bar{\zeta}(q) + (-1)^p \binom{p+q-1}{p-1} \tilde{t}(p+q) \\
 & - (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \binom{p+q-k-2}{q-1} \tilde{t}(k+1) \tilde{t}(p+q-k-1) \\
 & + (-1)^p \sum_{\substack{k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} (1 + (-1)^{k_1}) \binom{k_2+p-2}{p-1} \bar{\zeta}(k_1) \tilde{t}(k_2+p-1), \tag{2.8}
 \end{aligned}$$

where $\zeta(1) := -2 \log(2)$ and $\tilde{t}(1) := 0$.

Remark 2.4 Note that the explicit evaluation of $T(q, p)$ with odd weight was also proved by Kanenka and Tsumura [6, 7] by another method.

Example 2.2 We have

$$\begin{aligned}
 \tilde{T}(4, 1) &= \frac{31}{2} \zeta(5) - 7 \zeta(2) \zeta(3), \\
 \tilde{T}(3, 2) &= -62 \zeta(5) + 35 \zeta(2) \zeta(3), \\
 \tilde{T}(3, \bar{1}) &= \frac{1}{2} \tilde{t}(4) + \frac{1}{2} \zeta(2) \tilde{t}(2) - \frac{7}{2} \pi \zeta(3), \\
 \tilde{T}(2, \bar{2}) &= -\frac{3}{2} \tilde{t}(4) - \frac{3}{2} \zeta(2) \tilde{t}(2) + 7 \pi \zeta(3), \\
 \tilde{T}(\bar{3}, \bar{1}) &= -\frac{1}{2} \tilde{t}(4) + \zeta(2) \tilde{t}(2), \\
 \tilde{T}(\bar{2}, \bar{2}) &= \frac{3}{2} \tilde{t}(4) - \frac{9}{2} \zeta(2) \tilde{t}(2), \\
 \tilde{T}(\bar{4}, 1) &= \frac{1}{2} \pi \tilde{t}(4) - \frac{31}{2} \zeta(5) - \frac{7}{2} \zeta(2) \zeta(3), \\
 \tilde{T}(\bar{3}, 2) &= -\frac{3}{2} \pi \tilde{t}(4) + 62 \zeta(5) + 7 \zeta(2) \zeta(3).
 \end{aligned}$$

3 Notations and related expansions

In this section, we give some basic notations, definitions, and lemmas. Let $A := \{a_k\}$, $-\infty < k < \infty$ be a sequence of complex numbers with $a_k = o(k^\alpha)$ ($\alpha < 1$) if $k \rightarrow \pm\infty$. For convenience, let A_1 and A_2 denote the constant sequence $\{(1)^k\}$ and the alternating sequence $\{(-1)^k\}$, respectively.

3.1 Notations and definitions

Now, we give three definitions.

Definition 3.1 With A defined above, we define the parametric digamma function $\Phi(-s; A)$ by

$$\Phi(-s; A) := \frac{a_0}{s} + \sum_{k=1}^{\infty} \left(\frac{a_k}{k-1/2} - \frac{a_k}{k-s} \right). \quad (3.1)$$

Definition 3.2 For nonnegative integers $j \geq 1$ and n , we define

$$\begin{aligned} D^{(A)}(j) &:= \sum_{k=1}^{\infty} \frac{a_k}{k^j}, & E_n^{(A)}(j) &:= \sum_{k=1}^n \frac{a_{n-k}}{k^j}, & E_0^{(A)}(j) &:= 0, \\ F_n^{(A)}(j) &:= \begin{cases} \sum_{k=1}^{\infty} \frac{a_{k+n}-a_k}{k}, & j=1, \\ \sum_{k=1}^{\infty} \frac{a_{k+n}}{k^j}, & j>1, \end{cases} & \vec{F}_n^{(A)}(j) &:= \begin{cases} \sum_{k=1}^{\infty} \left(\frac{a_{k+n}}{k} - \frac{a_k}{k-1/2} \right), & j=1, \\ F_n^{(A)}(j), & j>1, \end{cases} \\ \bar{E}_n^{(A)}(j) &:= \sum_{k=1}^n \frac{a_{k-n-1}}{k^j}, & \bar{E}_0^{(A)}(j) &:= 0, & \widehat{E}_n^{(A)}(j) &:= \sum_{k=1}^n \frac{a_{n-k}}{(k-1/2)^j}, & \widehat{E}_0^{(A)}(j) &:= 0, \\ \bar{F}_n^{(A)}(j) &:= \begin{cases} \sum_{k=1}^{\infty} \frac{a_{k-n}-a_k}{k}, & j=1, \\ \sum_{k=1}^{\infty} \frac{a_{k-n}}{k^j}, & j>1, \end{cases} & \widehat{F}_n^{(A)}(j) &:= \begin{cases} \sum_{k=1}^{\infty} \frac{a_{k-n}-a_k}{k-1/2}, & j=1, \\ \sum_{k=1}^{\infty} \frac{a_{k-n}}{(k-1/2)^j}, & j>1, \end{cases} \\ \widetilde{E}_n^{(A)}(j) &:= \sum_{k=1}^n \frac{a_{k-n-1}}{(k-1/2)^j}, & \widetilde{E}_0^{(A)}(j) &:= 0, & \widetilde{F}_n^{(A)}(j) &:= \begin{cases} \sum_{k=1}^{\infty} \frac{a_{k-n}-a_k}{k-1/2}, & j=1, \\ \sum_{k=1}^{\infty} \frac{a_{k-n}}{(k-1/2)^j}, & j>1, \end{cases} \\ G_n^{(A)}(j) &:= E_n^{(A)}(j) - \bar{E}_{n-1}^{(A)}(j) - \frac{a_0}{n^j}, & G_0^{(A)}(j) &:= 0, & L_n^{(A)}(j) &:= F_n^{(A)}(j) + (-1)^j \bar{F}_n^{(A)}(j), \\ M_n^{(A)}(j) &:= E_n^{(A)}(j) + (-1)^j \vec{F}_n^{(A)}(j), & N_n^{(A)}(j) &:= \widehat{E}_n^{(A)}(j) + (-1)^j \widehat{F}_n^{(A)}(j), \\ \bar{N}_n^{(A)}(j) &:= \widetilde{F}_n^{(A)}(j) - \widetilde{E}_{n-1}^{(A)}(j), & R_n^{(A)}(j) &:= G_n^{(A)}(j) + (-1)^j L_n^{(A)}(j), \end{aligned}$$

with $D^{(A)}(1) := \vec{F}_0^{(A)}(1)$. Clearly, $D^{(A_1)}(1) := -2\log(2)$ and $D^{(A_2)}(1) := -\log(2) + \frac{\pi}{2}$.

Remark 3.1 It should be emphasized that many notations in Definition 3.2 were introduced in the reference [12].

Obviously,

$$\vec{F}_n^{(A_1)}(1) = -2\log(2), \quad \vec{F}_n^{(A_2)}(1) = (-1)^{n-1} \log(2) + \frac{\pi}{2}.$$

Clearly, if we let $A = A_1$ or A_2 in Definition 3.2, elementary calculations yield

$$\begin{aligned} M_n^{(A_1)}(j) &= H_n^{(j)} + (-1)^j \begin{cases} -2\log(2), & j=1, \\ \zeta(j), & j>1, \end{cases} \\ M_n^{(A_2)}(j) &= (-1)^{n-1} \bar{H}_n^{(j)} + (-1)^j \begin{cases} (-1)^{n-1} \log(2) + \frac{\pi}{2}, & j=1, \\ (-1)^{n-1} \bar{\zeta}(j), & j>1, \end{cases} \end{aligned}$$

$$\begin{aligned}
 N_n^{(A_1)}(j) &= h_n^{(j)} + (-1)^j \tilde{t}(j), \quad \tilde{t}(1) := 0, \\
 N_n^{(A_2)}(j) &= (-1)^{n-1} \bar{h}_n^{(j)} + (-1)^j \begin{cases} ((-1)^n + 1) \bar{t}(1), & j = 1, \\ (-1)^n \bar{t}(j), & j > 1, \end{cases} \\
 \bar{N}_n^{(A_1)}(j) &= \tilde{t}(j) - h_{n-1}^{(j)}, \quad \tilde{t}(1) := 0, \\
 \bar{N}_n^{(A_2)}(j) &= (-1)^n \bar{h}_{n-1}^{(j)} + \begin{cases} ((-1)^{n-1} + 1) \bar{t}(1), & j = 1, \\ (-1)^{n-1} \bar{t}(j), & j > 1, \end{cases} \\
 R_n^{(A_1)}(j) &= (1 + (-1)^j) \zeta(j), \quad R_n^{(A_2)}(j) = (-1)^{n-1} (1 + (-1)^j) \bar{\zeta}(j).
 \end{aligned}$$

Definition 3.3 ([12, Def. 1.2]) Define the cotangent function with sequence A by

$$\pi \cot(\pi s; A) = \frac{a_0}{s} - 2s \sum_{k=1}^{\infty} \frac{a_k}{k^2 - s^2}. \quad (3.2)$$

It is clear that if we let $A = A_1$ and A_2 in (3.2), respectively, then it becomes

$$\cot(\pi s; A_1) = \cot(\pi s) \quad \text{and} \quad \cot(\pi s; A_2) = \csc(\pi s).$$

3.2 Several identities among Φ -functions

Proposition 3.2 Let $p \geq 1$ and n be nonnegative integers, if $|s - n + 1/2| < 1$, then

$$\frac{\Phi^{(p-1)}(-s; A)}{(p-1)!} = \sum_{j=1}^{\infty} (-1)^{j-1} \binom{j+p-2}{p-1} N_n^{(A)}(j+p-1) (s-n+1/2)^{j-1}. \quad (3.3)$$

Proposition 3.3 Let $p \geq 1$ and n be nonnegative integers, if $|s - n| < 1$ with $s \neq n$, then

$$\frac{\Phi^{(p-1)}(-s; A)}{(p-1)!} = \frac{1}{(s-n)^p} \left\{ a_n - \sum_{j=1}^{\infty} (-1)^j \binom{j+p-2}{p-1} M_n^{(A)}(j+p-1) (s-n)^{j+p-1} \right\}. \quad (3.4)$$

If we set $n = 0$, then for any $|s| < 1$ with $s \neq 0$,

$$\frac{\Phi^{(p-1)}(-s; A)}{(p-1)!} = \frac{a_0}{s^p} + (-1)^p \sum_{j=1}^{\infty} \binom{j+p-2}{j-1} D^{(A)}(j+p-1) s^{j-1}. \quad (3.5)$$

Proposition 3.4 Let p and n be positive integers, if $|s + n - 1/2| < 1$, then

$$\frac{\Phi^{(p-1)}(-s; A)}{(p-1)!} = (-1)^p \sum_{j=1}^{\infty} \binom{j+p-2}{p-1} \bar{N}_n^{(A)}(j+p-1) (s+n-1/2)^{j-1}. \quad (3.6)$$

The method of the proofs of identities (3.3)–(3.6) is completely similar to that in [12, Theorems 2.1–2.3]. Thus, we omit it.

3.3 Lemmas

We define $\tan(s; A) := \cot(\pi/2 - s; A)$. It is clear that $\tan(s; A_1) = \tan(s)$ and $\tan(s; A_2) = \sec(s)$.

Lemma 3.5 ([12, Thm. 2.3]) *With $\cot(\pi s; A)$ defined above, if $|s - n| < 1$ with $s \neq n$ ($n \in \mathbb{Z}$), then*

$$\pi \cot(\pi s; A) = \frac{a_{|n|}}{s - n} - \sum_{j=1}^{\infty} (-\sigma_n)^j R_{|n|}^{(A)}(j) (s - n)^{j-1}, \quad (3.7)$$

where σ_n is defined by the symbol of n , namely

$$\sigma_n := \begin{cases} 1, & n \geq 0, \\ -1, & n < 0. \end{cases}$$

Hence, an elementary calculation yields

$$\pi \tan(\pi s; A) = -\frac{a_{|n-1|}}{s - n + 1/2} + \sum_{j=1}^{\infty} \sigma_{n-1}^j R_{|n-1|}^{(A)}(j) (s - n + 1/2)^{j-1} \quad (3.8)$$

for $|s - n + 1/2| < 1$ with $s \neq n - 1/2$ ($n \in \mathbb{Z}$),

Let $B := \{b_k\}$, $-\infty < k < \infty$ be a sequence of complex numbers with $b_k = o(k^\beta)$ ($\beta < 1$) if $k \rightarrow \pm\infty$. Define a kernel function $\xi(s)$ by the two requirements: 1. $\xi(s)$ is meromorphic in the whole complex plane. 2. $\xi(s)$ satisfies $\xi(s) = o(s)$ over an infinite collection of circles $|s| = \rho_k$ with $\rho_k \rightarrow \infty$. Applying these two conditions of kernel function $\xi(s)$, Flajolet and Salvy showed the following residue theorem.

Lemma 3.6 ([2]) *Let $\xi(s)$ be a kernel function, and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then*

$$\sum_{\alpha \in O} \text{Res} [r(s)\xi(s), s = \alpha] + \sum_{\beta \in S} \text{Res} [r(s)\xi(s), s = \beta] = 0, \quad (3.9)$$

where S is the set of poles of $r(s)$ and O is the set of poles of $\xi(s)$ that are not poles of $r(s)$. Here $\text{Res} [r(s), s = \alpha]$ denotes the residue of $r(s)$ at $s = \alpha$.

4 Two general theorems

In this section, we prove two general theorems which will be used to obtain the explicit evaluations of (alternating) double t -values and (alternating) double T -values.

Theorem 4.1 *For positive integers p and $q > 1$,*

$$\begin{aligned} & - \sum_{n=1}^{\infty} \frac{N_n^{(B)}(p)}{(n-1/2)^q} a_{n-1} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{\bar{N}_n^{(B)}(p)}{(n-1/2)^q} a_n \\ & - (-1)^p \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \binom{p+q-k-2}{q-1} \sum_{n=1}^{\infty} \frac{b_n}{n^{p+q-k-1}} \lim_{s \rightarrow n} \frac{d^k}{ds^k} (\pi \tan(\pi s; A)) \\ & + \text{Res} [f_1(s; A, B), s = 0] = 0, \end{aligned} \quad (4.1)$$

where

$$\frac{d^k}{ds^k}(\pi \tan(\pi s; A)) = \begin{cases} (1 - (-1)^k)k! \tilde{t}(k+1), & A = A_1, \\ (-1)^n(1 + (-1)^k)k! \tilde{t}(k+1), & A = A_2, \end{cases} \quad (4.2)$$

$$\begin{aligned} \text{Res}[f_1(s; A_1, B), s = 0] \\ = b_0(1 + (-1)^{p+q})\tilde{t}(p+q) \\ + 2(-1)^p \sum_{\substack{2k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} \tilde{t}(2k_1)D^{(B)}(k_2+p-1), \end{aligned} \quad (4.3)$$

$$\begin{aligned} \text{Res}[f_1(s; A_2, B), s = 0] \\ = b_0(1 - (-1)^{p+q})\tilde{t}(p+q) \\ + 2(-1)^p \sum_{\substack{2k_1+k_2=q+2, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} \tilde{t}(2k_1-1)D^{(B)}(k_2+p-1). \end{aligned} \quad (4.4)$$

Proof Apply the kernel function

$$\pi \tan(\pi s; A) \frac{\Phi^{(p-1)}(-s; B)}{(p-1)!}$$

to the base function $r(s) = s^{-q}$. Namely, we need to compute the residue of the function

$$f_1(s; A, B) := \pi \tan(\pi s; A) \frac{\Phi^{(p-1)}(-s; B)}{(p-1)!s^q}.$$

Clearly, $f_1(s; A, B)$ only has poles at $s = 0, \pm(n-1/2)$ and n (n is a positive integer). With the help of identities (3.3)–(3.6), we deduce the following residues:

$$\begin{aligned} \text{Res}[f_1, s = n-1/2] &= -\frac{N_n^{(B)}(p)}{(n-1/2)^q} a_{n-1}, \\ \text{Res}[f_1, s = 1/2-n] &= -(-1)^{p+q} \frac{\tilde{N}_n^{(B)}(p)}{(n-1/2)^q} a_n, \\ \text{Res}[f_1, s = n] &= -(-1)^p \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \binom{p+q-k-2}{q-1} \frac{b_n}{n^{p+q-k-1}} \lim_{s \rightarrow n} \frac{d^k}{ds^k}(\pi \tan(\pi s; A)) \end{aligned}$$

and (4.2)–(4.4). Applying Lemma 3.6 yields the desired result. \square

Proof of Theorem 2.1 Setting $A, B \in \{A_1, A_2\}$ in Theorem 4.1 yields the four desired evaluations. \square

Theorem 4.2 For positive integers p and $q > 1$,

$$\begin{aligned} & - \sum_{n=1}^{\infty} \frac{N_n^{(B)}(p)}{n^q} a_{n-1} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{\bar{N}_{n+1}^{(B)}(p)}{n^q} a_{n+1} \\ & - (-1)^p \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \binom{p+q-k-2}{q-1} \sum_{n=0}^{\infty} \frac{b_n}{(n+1/2)^{p+q-k-1}} \lim_{s \rightarrow n} \frac{d^k}{ds^k} (\pi \tan(\pi s; A)) \\ & - (-1)^p a_1 \binom{p+q-1}{p-1} W^{(B)}(p+q) \\ & + (-1)^p \sum_{\substack{k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} Z^{(A)}(k_1) W^{(B)}(k_2+p-1) = 0, \end{aligned} \quad (4.5)$$

where

$$W^{(B)}(j) := \widehat{F}_{-1}^{(B)}(j) = \begin{cases} \sum_{k=1}^{\infty} \frac{a_{k-1}-a_k}{k-1/2}, & j=1, \\ \sum_{k=1}^{\infty} \frac{a_{k-1}}{(k-1/2)^j}, & j>1, \end{cases} \quad (4.6)$$

and

$$Z^{(A)}(j) := (-1)^j R_1^{(A)}(j) = \sum_{k=1}^{\infty} \frac{a_{k+1} + (-1)^j a_{k-1}}{k^j}. \quad (4.7)$$

Proof Apply the kernel function

$$\pi \tan(\pi s; A) \frac{\Phi^{(p-1)}(-s; B)}{(p-1)!}$$

to the base function $r(s) = (s+1/2)^{-q}$. Namely, we need to compute the residues of the function

$$f_2(s; A, B) := \pi \tan(\pi s; A) \frac{\Phi^{(p-1)}(-s; B)}{(p-1)!(s+1/2)^q}.$$

Clearly, $f_2(s; A, B)$ only has poles at 0, n and $\pm(n-1/2)$ (n is a positive integer). With the help of identities (3.3)–(3.6), these residues are

$$\text{Res}[f_2, s = n-1/2] = -\frac{N_n^{(B)}(p)}{n^q} a_{n-1} \quad (n \geq 1),$$

$$\text{Res}[f_2, s = 1/2 - n] = -(-1)^{p+q} \frac{\bar{N}_n^{(B)}(p)}{(n-1)^q} a_n \quad (n \geq 2),$$

$$\text{Res}[f_2, s = n]$$

$$= -(-1)^p \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \binom{p+q-k-2}{q-1} \frac{b_n}{(n+1/2)^{p+q-k-1}} \lim_{s \rightarrow n} \frac{d^k}{ds^k} (\pi \tan(\pi s; A)) \quad (n \geq 0),$$

$$\text{Res}[f_2, s = -1/2]$$

$$= -(-1)^p a_1 \binom{p+q-1}{p-1} W^{(B)}(p+q)$$

$$+ (-1)^p \sum_{\substack{k_1+k_2=q+1, \\ k_1, k_2 \geq 1}} \binom{k_2+p-2}{p-1} Z^{(A)}(k_1) W^{(B)}(k_2+p-1) \\ = 0.$$

Then summing these four contributions and using Lemma 3.6, we may easily deduce the desired evaluation. \square

Proof of Theorem 2.3 Setting $A, B \in \{A_1, A_2\}$ in Theorem 4.2 yields the four desired evaluations. \square

It is possible that closed form representations of some other similar infinite series can be proved using techniques of the present paper.

Remark 4.3 It should be emphasized that Xu [12] defined another parametric digamma function $\Psi(-s; A)$. Very recently, Wang and Xu [10] used the parametric digamma function $\Psi(-s; A)$ to define several new kernel functions. Then they used the methods of contour integration and residue theorem to prove two general theorems (using the two theorems, they obtained Theorems 2.1 and 2.3), which are similar to Theorems 4.1 and 4.2. Moreover, they also showed many other types of results.

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