Lou et al. *Advances in Difference Equations* (2020) 2020:446
https://doi.org/10.1186/s13662-020-02902-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

# *Iq*-Calculus and *Iq*-Hermite–Hadamard inequalities for interval-valued functions

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**Abstract**

In this paper, we introduce the *Iq*-derivative and *Iq*-integral for interval-valued functions and give their basic properties. As a promotion of *q*-Hermite–Hadamard inequalities, we also give the *Iq*-Hermite-Hadamard inequalities for interval-valued functions. At the same time, we give some examples to illustrate the results.

**MSC:** 26D15; 26E25; 26A33

**Keywords:** Interval-valued functions; *Iq*-Derivative; *Iq*-Integral; *Iq*-Hermite–Hadamard inequalities

## 1 Introduction

Quantum calculus is a type of calculus without limits, sometimes called $q$-calculus. At the beginning of the twentieth century, Jackson first defined and studied $q$-calculus in a systematic manner, which can be tracked back to the time of Euler and Jacobi. Based on the works of Jackson, $q$-calculus continued to play a critical role in other areas such as quantum mechanics, fluid mechanics, and combinatorics, which attracted a sea of scholars to devote themselves to the research of this kind of calculus. In 2002, Kac and Cheung [1] introduced some knowledge about $q$-calculus in detail. Afterwards, some scholars have continued to extend it. In 2013, Tariboon and Ntouyas [2] promoted the concepts of $q$-calculus over finite intervals, discussed their properties, and gave applications in impulsive difference equations. Shortly after, Alp [3] obtained some $q$-Hermite–Hadamard-type inequalities. Regarding the development and promotion of $q$-calculus, we recommend [4–17] and the references cited therein to interested readers. In addition, the development of the $q$-fractional calculus can be found in [18–22].

On the other hand, the book written by Moore [23] described a method where an uncertain variable is replaced by an interval of real numbers and used interval arithmetic, which plays a great role in improving the reliability of the calculation results and making error analysis automatically. In recent years, it has been widely used in solving some uncertain problems in many fields. Bede and Stefanini [24]proposed the concepts of $gH$-difference and $gH$-derivative, which overcome the major shortcomings of $H$-derivative. Since then, the theory of interval analysis has gradually developed in the past ten years. For example, Lupulescu [25] developed a theory of fractional calculus for interval-valued functions.

Chalco-Cano et al. [26] dealt with the algebra of $gH$-differentiable interval-valued functions. More details can be founded in [27–29]. Particularly, in the field of inequalities, in 2017, Costa [30] presented the notions of convexity and gave the Jensen inequality for interval-valued functions. Based on this, some scholars combined classical inequalities with interval-valued functions to obtain some integral inequalities; see [31–33].

Motivated by the works mentioned, in this paper, we discuss the quantum calculus for interval-valued functions (shortly, $Iq$-calculus). Firstly, we give the concepts of $Iq$-calculus and define the $Iq$-derivative and $Iq$-integral. We also give some basic properties. Moreover, we generalize some $q$-Hermite–Hadamard-type inequalities. Since quantum calculus is a particular case of time-scale calculus (Bohner and Peterson [34, 35]), the results of this paper are helpful for future research on integral inequalities for interval-valued functions on time-scales. At the same time, the results of this paper can be used as a powerful tool in fuzzy analysis, interval optimization, and interval-valued differential equations.

The paper is organized as follows. We review some basic properties of interval analysis in Sect. 2. In Sect. 3, we put forward the concepts of $Iq$-derivative and give some properties. Similarly, we present the concepts of $Iq$-integral and some properties in Sect. 4. In Sect. 5, we give some new $Iq$-Hermite–Hadamard-type inequalities. Finally, Sect. 6 contains some conclusions. We give several examples to illustrate the statements.

## 2 Preliminaries

First, let $\mathcal{K}_c = \{U = [u^-, u^+] | u^-, u^+ \in \mathbb{R}, u^- \leq u^+\}$ be the set of all closed intervals. The length of an interval $[u^-, u^+] \in \mathcal{K}_c$ is denoted by $\ell(U) := u^+ - u^-$. Moreover, we say that $U$ is positive if $u^- > 0$, and we denote by $\mathcal{K}_c^+$ all positive intervals belonging to $\mathcal{K}_c$.

For any $U = [u^-, u^+]$, $V = [v^-, v^+] \in \mathcal{K}_c$, and $\alpha \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$U + V = \left[u^-, u^+\right] + \left[v^-, v^+\right] = \left[u^- + v^-, u^+ + v^+\right]$$

and

$$\alpha U = \alpha\left[u^-, u^+\right] = \begin{cases} [\alpha u^-, \alpha u^+] & \text{if } \alpha > 0, \\ \{0\} & \text{if } \alpha = 0, \\ [\alpha u^+, \alpha u^-] & \text{if } \alpha < 0. \end{cases}$$

**Definition 2.1** ([36]) For any $U, V \in \mathcal{K}_c$, we define the $gH$-difference of $U$ and $V$ as the set $W \in \mathcal{K}_c$ such that

$$U \ominus_g V = W \quad \Longleftrightarrow \quad \begin{cases} & \text{(a) } U = V + W, \\ \text{or} & \text{(b) } V = U + (-W). \end{cases} \tag{2.1}$$

Clearly,

$$U \ominus_g V = \begin{cases} [u^- - v^-, u^+ - v^+] & \text{if } \ell(U) \geq \ell(V), \\ [u^+ - v^+, u^- - v^-] & \text{if } \ell(U) < \ell(V). \end{cases} \tag{2.2}$$

In particular, if $V = v \in \mathbb{R}$ is a constant, then

$$U \ominus_g V = \left[u^- - v, u^+ - v\right].$$

The relationship "$\subseteq$" between $U$ and $V$ can be defined as

$$U \subseteq V \quad \text{if} \quad v^- \leq u^- \quad \text{and} \quad u^+ \leq v^+. \tag{2.3}$$

The Hausdorff–Pompeiu distance $\mathcal{H} : \mathcal{K}_c \times \mathcal{K}_c \to [0,\infty)$ between $U$ and $V$ is defined by $\mathcal{H}(U,V) = \max\{|u^- - v^-|, |u^+ - v^+|\}$. Subsequently, $(\mathcal{K}_c, \mathcal{H})$ is a complete metric space (see [37]).

**Definition 2.2** $F : [s,t] \to \mathcal{K}_c$ is said to be continuous at $x_0 \in [s,t]$ if

$$\mathcal{H}\big(F(x), F(x_0)\big) \to 0 \quad \text{as } x \to x_0.$$

We denote by $C([s,t],\mathcal{K}_c)$ and $C([s,t],\mathbb{R})$ the sets of all continuous interval-valued functions and real-valued functions on $[s,t]$, respectively.

For more basic notations from interval analysis, see [24, 36, 38].

In this paper, we use the symbols $F$ and $G$ for interval-valued functions. For any $F : [s,t] \to \mathcal{K}_c$ such that $F = [f^-, f^+]$, we say that $F$ is $\ell$-increasing (or $\ell$-decreasing) on $[s,t]$ if $\ell(F) : [s,t] \to [0,\infty)$ is increasing (or decreasing) on $[s,t]$. If $\ell(F)$ is monotone on $[s,t]$, then we say that $F$ is $\ell$-monotone on $[s,t]$.

## 3 *Iq*-Derivative for interval-valued functions

In this section, we present the concepts of $Iq$-derivative and give some properties. Firstly, let us recall the definition of $q$-derivative. Let $0 < q < 1$ be any constant.

**Definition 3.1** ([2]) Let $f \in C([s,t],\mathbb{R})$. The $q$-derivative of $f$ at $x \in [s,t]$ is defined by

$${}_sD_qf(x) = \frac{f(x) - f(qx + (1-q)s)}{(1-q)(x-s)}, \quad x \neq s, \qquad {}_sD_qf(s) = \lim_{x \to s^+} D_qf(x). \tag{3.1}$$

If ${}_sD_qf(x)$ exists for all $x \in [s,t]$, then $f$ is called $q$-differentiable on $[s,t]$. Note that if $s = 0$ in (3.1), then ${}_0D_qf = D_qf$, where $D_q$ is the well-known $q$-Jackson derivative of the function $f$ defined by

$$D_qf(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [2].

Now we introduce the $Iq$-derivative and some corresponding properties.

**Definition 3.2** Let $F \in C([s,t],\mathcal{K}_c)$. The $Iq$-derivative of $F$ at $x \in [s,t]$ is defined by

$${}_sD_qF(x) = \frac{F(x) \ominus_g F(qx + (1-q)s)}{(1-q)(x-s)}, \quad x \neq s, \qquad {}_sD_qF(s) = \lim_{x \to s^+} D_qF(x), \tag{3.2}$$

where $D_qF$ is called the $Iq$-Jackson derivative of $F$ defined by

$$D_qF(x) = \frac{F(x) \ominus_g F(qx)}{(1-q)x}.$$

If ${}_sD_qF(x)$ exists for all $x \in [s,t]$, then $F$ is called $Iq$-differentiable on $[s,t]$.

**Theorem 3.3** *A function $F : [s,t] \to \mathcal{K}_c$ is $Iq$-differentiable at $x \in [s,t]$ if and only if $f^-$ and $f^+$ are $q$-differentiable at $x \in [s,t]$, and*

$$ {}_sD_qF(x) = \left[\min\left\{{}_sD_qf^-(x), {}_sD_qf^+(x)\right\}, \max\left\{{}_sD_qf^-(x), {}_sD_qf^+(x)\right\}\right]. \tag{3.3}$$

*Proof* Suppose $F$ is $Iq$-differentiable at $x$. Then there exist $g^-(x), g^+(x)$ such that ${}_sD_qF(x) = [g^-(x), g^+(x)]$. According to Definition 3.2,

$$ g^-(x) = \min\left\{\frac{f^-(x) - f^-(qx + (1-q)s)}{(1-q)(x-s)}, \frac{f^+(x) - f^+(qx + (1-q)s)}{(1-q)(x-s)}\right\}$$

and

$$ g^+(x) = \max\left\{\frac{f^-(x) - f^-(qx + (1-q)s)}{(1-q)(x-s)}, \frac{f^+(x) - f^+(qx + (1-q)s)}{(1-q)(x-s)}\right\}$$

exist. Then ${}_sD_qf^-(x)$ and ${}_sD_qf^+(x)$ exist, and (3.3) is feasible.

Conversely, suppose $f^-$ and $f^+$ are $q$-differentiable at $x$.

If ${}_sD_qf^-(x) \le {}_sD_qf^+(x)$, then

$$\begin{aligned} &\left[{}_sD_qf^-(x), {}_sD_qf^+(x)\right] \\ &\quad = \left[\frac{f^-(x) - f^-(qx + (1-q)s)}{(1-q)(x-s)}, \frac{f^+(x) - f^+(qx + (1-q)s)}{(1-q)(x-s)}\right] \\ &\quad = \frac{f(x) \ominus_g f(qx + (1-q)s)}{(1-q)(x-s)} \\ &\quad = {}_sD_qF(x). \end{aligned}$$

So $F$ is $Iq$-differentiable at $x$. Similarly, if ${}_sD_qf^-(x) \ge {}_sD_qf^+(x)$, then ${}_sD_qF(x) = [{}_sD_qf^+(x), {}_sD_qf^-(x)]$. □

We illustrate this result by the following example.

*Example* 3.4 Consider $F : [s,t] \to \mathcal{K}_c$ given by $F(x) = [-|x|, |x|]$. It follows that $F(x)$ is $Iq$-differentiable. By Definition 3.2, for $s < 0$, we have

$$\begin{aligned} {}_sD_qF(0) &= \frac{[0,0] \ominus_g [(1-q)s, -(1-q)s]}{(1-q)(-s)} \\ &= \left[\min\left\{\frac{0-(1-q)s}{(1-q)(-s)}, \frac{0+(1-q)s}{(1-q)(-s)}\right\}, \max\left\{\frac{0-(1-q)s}{(1-q)(-s)}, \frac{0+(1-q)s}{(1-q)(-s)}\right\}\right], \\ &= [-1, 1], \end{aligned}$$

and if $s = 0$, then

$$
\begin{aligned}
{}_0D_qF(0) &= \lim_{x\to 0^+} \frac{[-|x|,|x|] \ominus_g [-q|x|,q|x|]}{(1-q)x} \\
&= \Bigg[\min\Bigg\{\lim_{x\to 0^+}\frac{-|x|+q|x|}{(1-q)x}, \lim_{x\to 0^+}\frac{|x|-q|x|}{(1-q)x}\Bigg\}, \\
&\qquad \max\Bigg\{\lim_{x\to 0^+}\frac{-|x|+q|x|}{(1-q)x}, \lim_{x\to 0^+}\frac{|x|-q|x|}{(1-q)x}\Bigg\}\Bigg] \\
&= [-1,1].
\end{aligned}
$$

Meanwhile, we know that $f^-(x) = -|x|$ and $f^+(x) = |x|$ are $q$-differentiable at 0. Similarly, if $s < 0$, then we have

$$
\begin{aligned}
{}_sD_qf^-(0) &= \frac{0-(1-q)s}{(1-q)(-s)} = 1, \\
{}_sD_qf^+(0) &= \frac{0+(1-q)s}{(1-q)(-s)} = -1;
\end{aligned}
$$

and if $s = 0$, then

$$
\begin{aligned}
{}_0D_qf^-(0) &= \lim_{x\to 0^+}\frac{-|x|+q|x|}{(1-q)x} = -1, \\
{}_0D_qf^+(0) &= \lim_{x\to 0^+}\frac{|x|-q|x|}{(1-q)x} = 1.
\end{aligned}
$$

This shows that ${}_sD_qF(0) = [{}_sD_qf^+(0), {}_sD_qf^-(0)]$ if $s < 0$ and ${}_0D_qF(0) = [{}_0D_qf^-(0), {}_0D_qf^+(0)]$ if $s = 0$.

To illustrate the nature of the derivatives more clearly, we give the following results.

**Theorem 3.5** *Let $F : [s,t] \to \mathcal{K}_c$. If $F$ is Iq-differentiable on $[s,t]$, then we have*:
(i) ${}_sD_qF(x) = [{}_sD_qf^-(x), {}_sD_qf^+(x)]$ *for all $x \in [s,t]$ if $F$ is $\ell$-increasing*;
(ii) ${}_sD_qF(x) = [{}_sD_qf^+(x), {}_sD_qf^-(x)]$ *for all $x \in [s,t]$ if $F$ is $\ell$-decreasing.*

*Proof* First, suppose $F$ is $\ell$-increasing and $Iq$-differentiable on $[s,t]$. For any $x \in [s,t]$, we have $x > qx + (1-q)s$. Since $\ell(F) = f^+ - f^-$ is increasing, we have

$$
\begin{aligned}
&\big[f^+(x) - f^-(x)\big] - \big[f^+\big(qx+(1-q)s\big) - f^-\big(qx+(1-q)s\big)\big] > 0, \\
&f^+(x) - f^+\big(qx+(1-q)s\big) > f^-(x) - f^-\big(qx+(1-q)s\big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
{}_sD_qF(x) &= \frac{[f^-(x),f^+(x)] \ominus_g [f^-(qx+(1-q)s),f^+(qx+(1-q)s)]}{(1-q)(x-s)} \\
&= \left[\frac{f^-(x)-f^-(qx+(1-q)s)}{(1-q)(x-s)}, \frac{f^+(x)-f^+(qx+(1-q)s)}{(1-q)(x-s)}\right] \\
&= \big[{}_sD_qf^-(x), {}_sD_qf^+(x)\big].
\end{aligned}
$$

The other condition can be similarly proved. □

*Remark* 3.6 Let $c \in (s,t)$ be a given point. If $F$ is $\ell$-increasing on $[s,c)$ and $\ell$-decreasing on $(c,t]$, then ${}_sD_qF = [{}_sD_qf^-, {}_sD_qf^+]$ on $[s,c)$ and ${}_sD_qF = [{}_sD_qf^+, {}_sD_qf^-]$ on $(c,t]$.

*Example* 3.7 Let $F : [0,1] \to \mathcal{K}_c$ be given by $F(x) = [-x^2-1, x^2-2x]$. Since $\ell(F) = 2x^2-2x+1$, it follows that $F$ is $\ell$-decreasing on $[0,\frac{1}{2})$ and $\ell$-increasing on $(\frac{1}{2},1]$. Since $f^-(x) = -x^2-1$ and $f^+(x) = x^2-2x$ are $q$-differentiable on $[0,1]$, we have

$$\begin{aligned}
{}_0D_qf^-(x) &= \frac{-x^2-1+(qx)^2+1}{(1-q)x} = -(1+q)x, \\
{}_0D_qf^+(x) &= \frac{x^2-2x-(qx)^2+2(qx)}{(1-q)x} = (1+q)x-2, \\
{}_{\frac{1}{2}}D_qf^-(x) &= \frac{-x^2-1-[-(qx+\frac{1}{2}(1-q)^2)-1]}{(1-q)(x-\frac{1}{2})} \\
&= -(1+q)x-\frac{1}{2}(1-q), \\
{}_{\frac{1}{2}}D_qf^+(x) &= \frac{x^2-2x-(qx+\frac{1}{2}(1-q))^2+2(qx+\frac{1}{2}(1-q))}{(1-q)(x-\frac{1}{2})} \\
&= (1+q)x-2+\frac{1}{2}(1-q),
\end{aligned}$$

and

$$\lim_{x\to\frac{1}{2}} D_qf^-(x) = -1; \qquad \lim_{x\to\frac{1}{2}} D_qf^+(x) = -1.$$

Therefore

$${}_sD_qF(x) = \begin{cases} [-(1+q)x, (1+q)x-2] & \text{if } x \in [0,\frac{1}{2}), \\ \{-1\} & \text{if } x = \frac{1}{2}, \\ [(1+q)x-2+\frac{1}{2}(1-q), -(1+q)x-\frac{1}{2}(1-q)] & \text{if } x \in (\frac{1}{2},1]. \end{cases}$$

**Theorem 3.8** *Let $F : [s,t] \to \mathcal{K}_c$ be Iq-differentiable on $[s,t]$. Then for all $C = [c^-,c^+] \in \mathcal{K}_c$ and $\alpha \in \mathbb{R}$, the functions $F + C$ and $\alpha F$ are Iq-differentiable on $[s,t]$, and ${}_sD_q(F+C) = {}_sD_qF$ and ${}_sD_q(\alpha F) = \alpha\,{}_sD_qF$.*

*Proof* For any $x \in [s,t]$,

$$\begin{aligned}
\text{(i)}\quad {}_sD_q\big(F(x)+C\big) &= \frac{(F(x)+C) \ominus_g (F(qx+(1-q)s)+C)}{(1-q)(x-s)} \\
&= \frac{F(x) \ominus_g F(qx+(1-q)s)}{(1-q)(x-s)} \\
&= {}_sD_qF(x), \\
\text{(ii)}\quad {}_sD_q\big(\alpha F(x)\big) &= \frac{\alpha F(x) \ominus_g \alpha F(qx+(1-q)s)}{(1-q)(x-s)} \\
&= \alpha\frac{F(x) \ominus_g F(qx+(1-q)s)}{(1-q)(x-s)} \\
&= \alpha\,{}_sD_qF(x).
\end{aligned}$$

□

**Theorem 3.9** *Let $F:[s,t]\to\mathcal{K}_c$ be Iq-differentiable on $[s,t]$. For $C=[c^-,c^+]\in\mathcal{K}_c$, if $\ell(F)-\ell(C)$ has a constant sign on $[s,t]$, then the function $F\ominus_g C$ is Iq-differentiable on $[s,t]$, and ${}_sD_q(F\ominus_g C)={}_sD_qF$.*

*Proof* For any $x\in[s,t]$,

$$\begin{aligned}{}_sD_q\bigl(F(x)\ominus_g C\bigr)&=\frac{(F(x)\ominus_g C)\ominus_g(F(qx+(1-q)s)\ominus_g C)}{(1-q)(x-s)}\\&=\frac{F(x)\ominus_g F(qx+(1-q)s)}{(1-q)(x-s)}\\&={}_sD_qF(x).\end{aligned}$$

□

**Theorem 3.10** *Let $F,G:[s,t]\to\mathcal{K}_c$. If $F$, $G$ are Iq-differentiable on $[s,t]$, then the sum $F+G:[s,t]\to\mathcal{K}_c$ is Iq-differentiable on $[s,t]$, and one of the following cases holds:*

(a) *If $F$, $G$ are equally $\ell$-monotonic on $[s,t]$, then for all $x\in[s,t]$,*

$$ {}_sD_q\bigl(F(x)+G(x)\bigr)={}_sD_qF(x)+{}_sD_qG(x). \tag{3.4}$$

(b) *If $F$ and $G$ are differently $\ell$-monotonic on $[s,t]$, then for all $x\in[s,t]$,*

$$ {}_sD_q(F+G)(x)={}_sD_qF(x)\ominus_g(-1){}_sD_qG(x). \tag{3.5}$$

*Moreover, in all cases, we have*

$$ {}_sD_q\bigl(F(x)+G(x)\bigr)\subseteq{}_sD_qF(x)+{}_sD_qG(x). \tag{3.6}$$

*Proof* (a) Suppose $F$, $G$ are $Iq$-differentiable and $\ell$-increasing on $[s,t]$. Then $f^-,f^+,g^-$, and $g^+$ are $q$-differentiable, and

$$ {}_sD_qf^-\le{}_sD_qf^+,\qquad {}_sD_qg^-\le{}_sD_qg^+.$$

Then $f^-+g^-$ and $f^++g^+$ are $q$-differentiable functions on $[s,t]$, and thus $F+G$ is $Iq$-differentiable on $[s,t]$, and

$$\begin{aligned}{}_sD_q(F+G)&=\bigl[\min\bigl\{{}_sD_qf^-+{}_sD_qg^-,{}_sD_qf^++{}_sD_qg^+\bigr\},\\&\qquad\max\bigl\{{}_sD_qf^-+{}_sD_qg^-,{}_sD_qf^++{}_sD_qg^+\bigr\}\bigr]\\&=\bigl[{}_sD_qf^-+{}_sD_qg^-,{}_sD_qf^++{}_sD_qg^+\bigr]\\&={}_sD_qF+{}_sD_qG.\end{aligned}\tag{3.7}$$

Similarly, we can prove that both $F$ and $G$ are $\ell$-decreasing.

(b) Suppose $F$ is $\ell$-increasing and $G$ is $\ell$-decreasing. Then

$$ {}_sD_qf^-\le{}_sD_qf^+,\qquad {}_sD_qg^-\ge{}_sD_qg^+.$$

On the one hand,

$$
{}_sD_q(F+G) = \bigl[\min\bigl\{{}_sD_qf^- + {}_sD_qg^-, {}_sD_qf^+ + {}_sD_qg^+\bigr\}, \\
\max\bigl\{{}_sD_qf^- + {}_sD_qg^-, {}_sD_qf^+ + {}_sD_qg^+\bigr\}\bigr]. \tag{3.8}
$$

On the other hand,

$$
\begin{aligned}
{}_sD_qF \ominus_g (-1){}_sD_qG &= \bigl[{}_sD_qf^-, {}_sD_qf^+\bigr] \ominus_g (-1)\bigl[{}_sD_qg^+, {}_sD_qg^-\bigr] \\
&= \bigl[{}_sD_qf^-, {}_sD_qf^+\bigr] \ominus_g \bigl[-{}_sD_qg^-, -{}_sD_qg^+\bigr] \\
&= \bigl[\min\bigl\{{}_sD_qf^- + {}_sD_qg^-, {}_sD_qf^+ + {}_sD_qg^+\bigr\}, \\
&\quad \max\bigl\{{}_sD_qf^- + {}_sD_qg^-, {}_sD_qf^+ + {}_sD_qg^+\bigr\}\bigr]. 
\end{aligned} \tag{3.9}
$$

Comparing (3.8) with (3.9), we get (3.5). Further,

$$
{}_sD_qF + {}_sD_qG = \bigl[{}_sD_qf^- + {}_sD_qg^+, {}_sD_qf^+ + {}_sD_qg^-\bigr].
$$

So if $F+G$ is $\ell$-increasing or $\ell$-decreasing, we get

$$
{}_sD_q\bigl(F(x)+G(x)\bigr) \subseteq {}_sD_qF(x) + {}_sD_qG(x). \tag{3.10}
$$

The opposite case can be similarly proved. □

**Theorem 3.11** *Let $F,G:[s,t]\to\mathcal{K}_c$. If $F$, $G$ are Iq-differentiable and $\ell(F)-\ell(G)$ has a constant sign on $[s,t]$, then the function $F\ominus_g G:[s,t]\to\mathcal{K}_c$ is Iq-differentiable on $[s,t]$, and one of the following cases holds*:

(a) *If $F,G$ are equally $\ell$-monotonic on $[s,t]$, then for all $x\in[s,t]$,*

$$
{}_sD_q\bigl(F(x)\ominus_g G(x)\bigr) = {}_sD_qF(x) \ominus_g {}_sD_qG(x). \tag{3.11}
$$

(b) *If $F$ and $G$ are differently $\ell$-monotonic on $[s,t]$, then for all $x\in[s,t]$,*

$$
{}_sD_q(F\ominus_g G)(x) = {}_sD_qF(x) + (-1){}_sD_qG(x). \tag{3.12}
$$

*Proof* We now assume that $\ell(F)\geq\ell(G)$ on $[s,t]$ and $F\ominus_g G=[f^- - g^-, f^+ - g^+]$.

(a) Suppose $F$, $G$ are $\ell$-increasing on $[s,t]$. Since $F$, $G$ are $Iq$-differentiable, we have that $f^-, f^+, g^-$, and $g^+$ are $q$-differentiable and

$$
{}_sD_qf^- \leq {}_sD_qf^+, \qquad {}_sD_qg^- \leq {}_sD_qg^+.
$$

Then $f^- - g^-$ and $f^+ - g^+$ are $q$-differentiable functions on $[s,t]$. So $F\ominus_g G$ is $Iq$-differentiable on $[s,t]$, and

$$
\begin{aligned}
{}_sD_q(F\ominus_g G) &= \bigl[\min\bigl\{{}_sD_qf^- - {}_sD_qg^-, {}_sD_qf^+ - {}_sD_qg^+\bigr\}, \\
&\quad \max\bigl\{{}_sD_qf^- - {}_sD_qg^-, {}_sD_qf^+ - {}_sD_qg^+\bigr\}\bigr] \\
&= \bigl[{}_sD_qf^-, {}_sD_qf^+\bigr] \ominus_g \bigl[{}_sD_qg^-, {}_sD_qg^+\bigr] \\
&= {}_sD_qF \ominus_g {}_sD_qG.
\end{aligned} \tag{3.13}
$$

The case where $F$ and $G$ are both $\ell$-decreasing can be similarly proved.

(b) Suppose $F$ is $\ell$-increasing and $G$ is $\ell$-decreasing. From (a) we have that

$$_sD_qf^- \le {}_sD_qf^+, \qquad {}_sD_qg^- \ge {}_sD_qg^+.$$

Since $\ell(F) \ge \ell(G)$, on one hand,

$$\begin{aligned} {}_sD_q(F \ominus_g G) &= \bigl[\min\bigl\{{}_sD_qf^- - {}_sD_qg^-, {}_sD_qf^+ - {}_sD_qg^+\bigr\}, \\ &\quad \max\bigl\{{}_sD_qf^- - {}_sD_qg^-, {}_sD_qf^+ - {}_sD_qg^+\bigr\}\bigr] \\ &= \bigl[{}_sD_qf^- - {}_sD_qg^-, {}_sD_qf^+ - {}_sD_qg^+\bigr]. \end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned} {}_sD_qF + (-1){}_sD_qG &= \bigl[{}_sD_qf^-, {}_sD_qf^+\bigr] + (-1)\bigl[{}_sD_qg^+, {}_sD_qg^-\bigr] \\ &= \bigl[{}_sD_qf^-, {}_sD_qf^+\bigr] + \bigl[-{}_sD_qg^-, -{}_sD_qg^+\bigr] \\ &= \bigl[{}_sD_qf^- - {}_sD_qg^-, {}_sD_qf^+ - {}_sD_qg^+\bigr]. \end{aligned} \tag{3.15}$$

Comparing (3.14) with (3.15), we get (3.12). The opposite case can be similarly proved. □

*Example* 3.12 Let $F, G : [0, 2] \to \mathcal{K}_c$ be given by $F(x) = [0, -x^2 + 2x]$ and $G(x) = [0, 2x^2 - 4x + 3]$. Since $\ell(F(x)) = -x^2 + 2x$ and $\ell(G(x)) = 2x^2 - 4x + 3$, $\ell(F(x)) \le \ell(G(x))$ for all $x \in [0, 2]$. We have that $F(x)$ is $\ell$-increasing on $[0, 1]$ and $\ell$-decreasing on $[1, 2]$; $G(x)$ is $\ell$-decreasing on $[0, 1]$ and $\ell$-increasing on $[1, 2]$.

Further, we have that $F(x) + G(x) = [0, x^2 - 2x + 3]$ and $F(x) \ominus_g G(x) = [-3x^2 + 6x - 3, 0]$. Since $\ell(F(x) + G(x)) = x^2 - 2x + 3$ and $\ell(F(x) \ominus_g G(x)) = 3x^2 - 6x + 3$, $F(x) + G(x)$ and $F(x) \ominus_g G(x)$ are $\ell$-decreasing on $[0, 1]$ and $\ell$-increasing on $[1, 2]$. For all $x \in [0, 1]$, we get

$$\begin{aligned} {}_0D_qF(x) &= \bigl[{}_0D_qf^-(x), {}_0D_qf^+(x)\bigr] = \bigl[0, -(1+q)x + 2\bigr], \\ {}_0D_qG(x) &= \bigl[{}_0D_qg^+(x), {}_0D_qg^-(x)\bigr] = \bigl[2(1+q)x - 4, 0\bigr], \\ {}_0D_q\bigl(F(x) + G(x)\bigr) &= \bigl[{}_0D_q\bigl(f^+(x) + g^+(x)\bigr), {}_0D_q\bigl(f^-(x) + g^-(x)\bigr)\bigr] \\ &= \bigl[(1+q)x - 2, 0\bigr], \\ {}_0D_q\bigl(F(x) \ominus_g G(x)\bigr) &= \bigl[{}_0D_q\bigl(f^-(x) - g^-(x)\bigr), {}_0D_q\bigl(f^+(x) - g^+(x)\bigr)\bigr] \\ &= \bigl[0, -3(1+q)x + 6\bigr]. \end{aligned}$$

Then from (3.9) and (3.15) we have

$$\begin{aligned} {}_0D_qF(x) \ominus_g (-1){}_0D_qG(x) &= \bigl[0, -(1+q)x + 2\bigr] \ominus_g (-1)\bigl[2(1+q)x - 4, 0\bigr] \\ &= \bigl[0, -(1+q)x + 2\bigr] \ominus_g \bigl[0, -2(1+q)x + 4\bigr] \\ &= \bigl[\min\bigl\{0, (1+q)x - 2\bigr\}, \max\bigl\{0, (1+q)x - 2\bigr\}\bigr] \\ &= \bigl[(1+q)x - 2, 0\bigr], \end{aligned}$$

$$\begin{aligned}
{}_0D_qF(x)+(-1){}_0D_qG(x)&=\bigl[0,-(1+q)x+2\bigr]+(-1)\bigl[2(1+q)x-4,0\bigr]\\
&=\bigl[0,-(1+q)x+2\bigr]+\bigl[0,-2(1+q)x+4\bigr]\\
&=\bigl[0,-3(1+q)x+6\bigr].
\end{aligned}$$

Further, for all $x\in[1,2]$, we similarly obtain

$$\begin{aligned}
{}_1D_qF(x)&=\bigl[{}_1D_qf^+(x),{}_1D_qf^-(x)\bigr]=\bigl[-(1+q)x-(1-q)+2,0\bigr],\\
{}_1D_qG(x)&=\bigl[{}_1D_qg^-(x),{}_1D_qg^+(x)\bigr]=\bigl[0,2(1+q)x+2(1-q)-4\bigr],\\
{}_1D_q\bigl(F(x)+G(x)\bigr)&=\bigl[{}_1D_q\bigl(f^-(x)+g^-(x)\bigr),{}_1D_q\bigl(f^+(x)+g^+(x)\bigr)\bigr]\\
&=\bigl[0,(1+q)x+(1-q)-2\bigr],\\
{}_1D_q\bigl(F(x)\ominus_g G(x)\bigr)&=\bigl[{}_1D_q\bigl(f^+(x)-g^+(x)\bigr),{}_1D_q\bigl(f^-(x)-g^-(x)\bigr)\bigr]\\
&=\bigl[-3(1+q)x-3(1-q)+6,0\bigr],
\end{aligned}$$

and

$$\begin{aligned}
&{}_0D_qF(x)\ominus_g(-1){}_0D_qG(x)\\
&\quad=\bigl[-(1+q)x-(1-q)+2,0\bigr]\ominus_g(-1)\bigl[0,2(1+q)x+2(1-q)-4\bigr]\\
&\quad=\bigl[-(1+q)x-(1-q)+2,0\bigr]\ominus_g\bigl[-2(1+q)x-2(1-q)+4,0\bigr]\\
&\quad=\bigl[\min\bigl\{(1+q)x+(1-q)-2,0\bigr\},\max\bigl\{(1+q)x+(1-q)-2,0\bigr\}\bigr]\\
&\quad=\bigl[0,(1+q)x+(1-q)-2\bigr],\\
&{}_0D_qF(x)+(-1){}_0D_qG(x)\\
&\quad=\bigl[-(1+q)x-(1-q)+2,0\bigr]+(-1)\bigl[0,2(1+q)x+2(1-q)-4\bigr]\\
&\quad=\bigl[-(1+q)x-(1-q)+2,0\bigr]+\bigl[-2(1+q)x-2(1-q)+4,0\bigr]\\
&\quad=\bigl[-3(1+q)x-3(1-q)+6,0\bigr].
\end{aligned}$$

Obviously, we see that ${}_sD_q(F+G)(x)={}_sD_qF(x)\ominus_g(-1){}_sD_qG(x)$ and ${}_sD_q(F\ominus_g G)(x)={}_sD_qF(x)+(-1){}_sD_qG(x)$.

## 4 *Iq*-Integral for interval-valued functions

In this section, we present the concepts of $Iq$-integral and give some properties. Firstly, let us recall the definition of $q$-integral.

**Definition 4.1** ([2]) Let $f\in C([s,t],\mathbb{R})$. Then the $q$-integral is defined by

$$\int_s^{\xi}f(x)\,{}_sd_qx=(1-q)(\xi-s)\sum_{n=0}^{\infty}q^nf\bigl(q^n\xi+\bigl(1-q^n\bigr)s\bigr)\tag{4.1}$$

for all $\xi \in [s,t]$. Additionally, if $c \in (s,\xi)$, then the definite $q$-integral on $[s,t]$ is defined by

$$\begin{aligned}\int_c^\xi f(x)\,{}_s d_q x &= \int_s^\xi f(x)\,{}_s d_q x - \int_s^c f(x)\,{}_s d_q x \\ &= (1-q)(\xi - s)\sum_{n=0}^{\infty} q^n f\bigl(q^n\xi + \bigl(1-q^n\bigr)s\bigr) \\ &\quad - (1-q)(c-s)\sum_{n=0}^{\infty} q^n f\bigl(q^n c + \bigl(1-q^n\bigr)s\bigr).\end{aligned} \tag{4.2}$$

Note that if $s = 0$, then (4.1) reduces to the classical $q$-Jackson integral of a function $f$ defined by $\int_0^\xi f(x)\,{}_0 d_q x = (1-q)\xi\sum_{n=0}^{\infty} q^n f(q^n\xi)$ for $x \in [0,\infty)$. For more details, see [2].

Next, we give the concept of the $Iq$-integral and discuss some basic properties.

**Definition 4.2** Let $F \in C([s,t],\mathcal{K}_c)$. Then the $Iq$-integral is defined by

$$\int_s^\xi F(x)\,{}_s d_q^I x = (1-q)(\xi - s)\sum_{n=0}^{\infty} q^n F\bigl(q^n\xi + \bigl(1-q^n\bigr)s\bigr) \tag{4.3}$$

for all $\xi \in [s,t]$.

**Theorem 4.3** *Let $F \in C([s,t],\mathcal{K}_c)$. If $c \in (s,\xi)$, then we have that*

$$\int_s^c F(x)\,{}_s d_q^I x + \int_c^\xi F(x)\,{}_s d_q^I x = \int_s^\xi F(x)\,{}_s d_q^I x. \tag{4.4}$$

*Proof*

$$\begin{aligned}&\int_s^c F(x)\,{}_s d_q^I x + \int_c^\xi F(x)\,{}_s d_q^I x \\ &= (1-q)(c-s)\sum_{n=0}^{\infty} q^n F\bigl(q^n c + \bigl(1-q^n\bigr)s\bigr) \\ &\quad + (1-q)(\xi - c)\sum_{n=0}^{\infty} q^n F\bigl(q^n\xi + \bigl(1-q^n\bigr)c\bigr) \\ &= \Biggl[(1-q)(c-s)\sum_{n=0}^{\infty} q^n f^-\bigl(q^n c + \bigl(1-q^n\bigr)s\bigr), \\ &\quad (1-q)(c-s)\sum_{n=0}^{\infty} q^n f^+\bigl(q^n c + \bigl(1-q^n\bigr)s\bigr)\Biggr] \\ &\quad + \Biggl[(1-q)(\xi - c)\sum_{n=0}^{\infty} q^n f^-\bigl(q^n\xi + \bigl(1-q^n\bigr)c\bigr), \\ &\quad (1-q)(\xi - c)\sum_{n=0}^{\infty} q^n f^+\bigl(q^n\xi + \bigl(1-q^n\bigr)c\bigr)\Biggr] \\ &= \Biggl[(1-q)(\xi - s)\sum_{n=0}^{\infty} q^n f^-\bigl(q^n\xi + \bigl(1-q^n\bigr)s\bigr),\end{aligned}$$

$$
\left.(1-q)(\xi-s)\sum_{n=0}^{\infty}q^{n}f^{+}\left(q^{n}\xi+\left(1-q^{n}\right)s\right)\right]
$$
$$
=(1-q)(\xi-s)\sum_{n=0}^{\infty}q^{n}F\left(q^{n}\xi+\left(1-q^{n}\right)s\right)=\int_{s}^{\xi}F(x)\,{}_{s}d_{q}^{I}x. \qquad \square
$$

**Theorem 4.4** *Let $F:[s,t]\to\mathcal{K}_c$. If $F\in C([s,t],\mathcal{K}_c)$, then $F$ is Iq-integrable if and only if $f^-$ and $f^+$ are q-integrable over $[s,t]$. Moreover,*

$$
\int_{s}^{\xi}F(x)\,{}_{s}d_{q}^{I}x=\left[\int_{s}^{\xi}f^{-}(x)\,{}_{s}d_{q}x,\int_{s}^{\xi}f^{+}(x)\,{}_{s}d_{q}x\right]. \tag{4.5}
$$

*Proof* The proof can be obtained by combining Definitions 4.1 and 4.2 and hence is omitted. $\square$

*Example* 4.5 Let $F:[0,1]\to\mathcal{K}_c$ be given by $F(x)=[x^2,x]$. For $0<q<1$, we have

$$
\begin{aligned}
\int_{0}^{1}F(x)\,{}_{0}d_{q}^{I}x&=\left[\int_{0}^{1}x^{2}\,{}_{0}d_{q}x,\int_{0}^{1}x\,{}_{0}d_{q}x\right]\\
&=\left[(1-q)\sum_{n=0}^{\infty}q^{3n},(1-q)\sum_{n=0}^{\infty}q^{2n}\right]\\
&=\left[\frac{1}{1+q+q^{2}},\frac{1}{1+q}\right].
\end{aligned}
$$

**Theorem 4.6** *Let $F,G:[s,t]\to\mathcal{K}_c$, and let $\alpha\in\mathbb{R}$. If $F,G\in C([s,t],\mathcal{K}_c)$, then for $x\in[s,t]$, we have*:

(i) $\int_{s}^{\xi}[F(x)+G(x)]\,{}_{s}d_{q}x=\int_{s}^{\xi}F(x)\,{}_{s}d_{q}x+\int_{s}^{\xi}G(x)\,{}_{s}d_{q}x$;
(ii) $\int_{s}^{\xi}\alpha F(x)\,{}_{s}d_{q}x=\alpha\int_{s}^{\xi}F(x)\,{}_{s}d_{q}x$.

*Proof* From Definition 4.2 we have:

$$
\begin{aligned}
\text{(i)}\quad &\int_{s}^{\xi}\left[F(x)+G(x)\right]{}_{s}d_{q}^{I}x\\
&=(1-q)(\xi-s)\sum_{n=0}^{\infty}q^{n}\left[F\left(q^{n}\xi+\left(1-q^{n}\right)s\right)+G\left(q^{n}\xi+\left(1-q^{n}\right)s\right)\right]\\
&=(1-q)(\xi-s)\sum_{n=0}^{\infty}q^{n}F\left(q^{n}\xi+\left(1-q^{n}\right)s\right)\\
&\quad+(1-q)(\xi-s)\sum_{n=0}^{\infty}q^{n}G\left(q^{n}\xi+\left(1-q^{n}\right)s\right)\\
&=\int_{s}^{\xi}F(x)\,{}_{s}d_{q}x+\int_{s}^{\xi}G(x)\,{}_{s}d_{q}x;\\
\text{(ii)}\quad &\int_{s}^{\xi}\alpha F(x)\,{}_{s}d_{q}^{I}x=(1-q)(\xi-s)\sum_{n=0}^{\infty}q^{n}\alpha F\left(q^{n}\xi+\left(1-q^{n}\right)s\right)\\
&=\alpha\int_{s}^{\xi}F(x)\,{}_{s}d_{q}x. \qquad \square
\end{aligned}
$$

**Theorem 4.7** *If $F, G \in C([s,t], \mathcal{K}_c)$, then*

$$\int_s^\xi F(x)\,{}_s d_q^I x \ominus_g \int_s^\xi G(x)\,{}_s d_q^I x \subseteq \int_s^\xi F(x) \ominus_g G(x)\,{}_s d_q^I x.$$

*Moreover, if $\ell(F) - \ell(G)$ has a constant sign on $[s,t]$, then*

$$\int_s^\xi F(x)\,{}_s d_q^I x \ominus_g \int_s^\xi G(x)\,{}_s d_q^I x = \int_s^\xi F(x) \ominus_g G(x)\,{}_s d_q^I x.$$

*Proof* First, we have

$$\begin{aligned}
&\int_s^\xi \min\{f^- - g^-, f^+ - g^+\}\,{}_s d_q x \\
&\quad \le \min\left\{\int_s^\xi (f^- - g^-)\,{}_s d_q x, \int_s^\xi (f^+ - g^+)\,{}_s d_q x\right\} \\
&\quad \le \max\left\{\int_s^\xi (f^- - g^-)\,{}_s d_q x, \int_s^\xi (f^+ - g^+)\,{}_s d_q x\right\} \\
&\quad \le \int_s^\xi \max\{f^- - g^-, f^+ - g^+\}\,{}_s d_q x.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_s^\xi F\,{}_s d_q^I x \ominus_g \int_s^\xi G\,{}_s d_q^I x \\
&\quad = \left[\min\left\{\int_s^\xi (f^- - g^-)\,{}_s d_q x, \int_s^\xi (f^+ - g^+)\,{}_s d_q x\right\},\right. \\
&\qquad \left.\max\left\{\int_s^\xi (f^- - g^-)\,{}_s d_q x, \int_s^\xi (f^+ - g^+)\,{}_s d_q x\right\}\right] \\
&\quad \subseteq \left[\int_s^\xi \min\{f^- - g^-, f^+ - g^+\}\,{}_s d_q x, \int_s^\xi \max\{f^- - g^-, f^+ - g^+\}\,{}_s d_q x\right] \\
&\quad = \int_s^\xi F \ominus_g G\,{}_s d_q^I x.
\end{aligned}$$

Moreover, $F \ominus_g G = [f^- - g^-, f^+ - g^+]$ if $\ell(F) \ge \ell(G)$, or $F \ominus_g G = [f^+ - g^+, f^- - g^-]$ if $\ell(F) \le \ell(G)$. We now assume that $\ell(F) \ge \ell(G)$ on $[s,t]$ and $F \ominus_g G = [f^- - g^-, f^+ - g^+]$. So we have $\int_s^\xi (f^- - g^-)\,{}_s d_q x \le \int_s^\xi (f^+ - g^+)\,{}_s d_q x$. This implies that

$$\begin{aligned}
\int_s^\xi F \ominus_g G\,{}_s d_q^I x &= \left[\int_s^\xi \min\{f^- - g^-, f^+ - g^+\}\,{}_s d_q x,\right. \\
&\qquad \left.\int_s^\xi \max\{f^- - g^-, f^+ - g^+\}\,{}_s d_q x\right] \\
&= \left[\int_s^\xi f^-\,{}_s d_q x, \int_s^\xi f^+\,{}_s d_q x\right] \ominus_g \left[\int_s^\xi g^-\,{}_s d_q x, \int_s^\xi g^+\,{}_s d_q x\right] \\
&= \int_s^\xi F\,{}_s d_q^I x \ominus_g \int_s^\xi G\,{}_s d_q^I x.
\end{aligned}$$

□

**Theorem 4.8** *Let $F : [s,t] \to \mathcal{K}_c$. If $F$ is Iq-differentiable on $[s,t]$, then ${}_sD_qF$ is Iq-integrable. Moreover, if $F$ is $\ell$-monotone on $[s,t]$, then*

$$F(x) \ominus_g F(c) = \int_c^x {}_sD_qF(\xi)\,{}_sd_q^I\xi \quad \textit{for all } c \in [s,x]. \tag{4.6}$$

*Proof* If $F$ is $Iq$-differentiable on $[s,t]$, then from Theorem 3.3 it follows that $f^-$ and $f^+$ are $q$-differentiable. Hence ${}_sD_qf^-$ and ${}_sD_qf^+$ exist on $[s,t]$. Meanwhile, ${}_sD_qf^-$ and ${}_sD_qf^+$ are $q$-integrable. Therefore Theorem 4.4 imply that ${}_sD_qF$ is $Iq$-integrable. If $F$ is $\ell$-increasing on $[s,t]$, then ${}_sD_qF(x) = [{}_sD_qf^-(x), {}_sD_qf^+(x)]$ for all $x \in [s,t]$. Then we have that

$$\begin{aligned} f^-(x) - f^-(c) &= \int_c^x {}_sD_qf^-(\xi)\,{}_sd_q\xi, \\ f^+(x) - f^+(c) &= \int_c^x {}_sD_qf^+(\xi)\,{}_sd_q\xi. \end{aligned}$$

It follows that

$$F(x) = F(c) + \int_c^x {}_sD_qF(\xi)\,{}_sd_q^I\xi.$$

Since $F$ is $\ell$-increasing on $[s,t]$, by (2.1) we have

$$F(x) \ominus_g F(c) = \int_c^x {}_sD_qF(\xi)\,{}_sd_q^I\xi.$$

If $F$ is $\ell$-decreasing on $[s,t]$, then ${}_sD_qF(x) = [{}_sD_qf^+(x), {}_sD_qf^-(x)]$ for all $c \in [s,x]$. Then we get that

$$\begin{aligned} \int_c^x {}_sD_qF(\xi)\,{}_sd_q^I\xi &= \left[\int_c^x {}_sD_qf^+(\xi)\,{}_sd_q\xi, \int_c^x {}_sD_qf^-(\xi)\,{}_sd_q\xi\right] \\ &= \left[f^+(x) - f^+(c), f^-(x) - f^-(c)\right] \\ &= \left[f^-(x), f^+(x)\right] \ominus_g \left[f^-(c), f^+(c)\right] \\ &= F(x) \ominus_g F(c). \end{aligned}$$

□

*Remark* 4.9 We remark that if $F$ is $\ell$-increasing on $[s,t]$, then (4.6) is equivalent with

$$F(x) = F(c) + \int_c^x {}_sD_qF(\xi)\,{}_sd_q^I\xi,$$

and if $F$ is $\ell$-decreasing on $[s,t]$, then (4.6) is equivalent with

$$F(x) = F(c) \ominus_g (-1)\int_c^x {}_sD_qF(\xi)\,{}_sd_q^I\xi$$

for all $x \in [s,t]$. Also, we remark that relation (4.6) can be false if $F$ is not $\ell$-monotone on $[s,t]$. Indeed, let $F : [0,2] \to \mathcal{K}_c$ be given by $F(x) = [0, -x^2 + 2x]$. For $c \in (0,1)$ and $x \in (1,2)$, we have that (see Example 3.12)

$$\begin{aligned}\int_c^x {}_0D_qF(\xi)\,{}_sd_q^I\xi &= \int_c^1 {}_0D_qF(\xi)\,{}_sd_q^I\xi + \int_1^x {}_1D_qF(\xi)\,{}_sd_q^I\xi \\ &= \big[0, c^2-2c+1\big] + \big[-x^2+2x-1, 0\big] \\ &= \big[-x^2+2x-1, c^2-2c+1\big].\end{aligned}$$

Then we get that

$$\begin{aligned}F(x)\ominus_g F(c) &= \big[\min\big\{0,(c-x)(c+x-2)\big\}, \max\big\{0,(c-x)(c+x-2)\big\}\big] \\ &\neq \int_c^x {}_0D_qF(\xi)\,{}_sd_q^I\xi.\end{aligned}$$

Therefore (4.6) is not true for all $x \in [0,2]$.

*Example* 4.10 Let $F:[0,2]\to\mathcal{K}_c$ be given by $F(x)=[0,x^2]$. Since $F(x)$ is $Iq$-differentiable and $\ell$-increasing on $[0,2]$, ${}_sD_qF(x)$ is $Iq$-integrable, and ${}_sD_qF(x)=[0,(1+q)x]$. Let $c=1\in[0,x]$. Then

$$F(x)\ominus_g F(1) = \big[0, x^2-1\big],$$

and

$$\begin{aligned}\int_1^x {}_0D_qF(\xi)\,{}_0d_q^I\xi &= \left[0, \int_1^x (1+q)\xi\,{}_0d_q\xi\right] \\ &= \left[0, \int_0^x (1+q)\xi\,{}_0d_q\xi - \int_0^1 (1+q)\xi\,{}_0d_q\xi\right] \\ &= \big[0, x^2-1\big].\end{aligned}$$

## 5 *Iq*-Hermite–Hadamard inequalities for interval-valued functions

Now we review the definition and properties of convex interval-valued functions.

**Definition 5.1** ([31]) Let $F:[s,t]\to\mathcal{K}_c$. We say that $F$ is convex if for all $x,y\in[s,t]$ and $\xi\in[0,1]$, we have

$$F\big(\xi x+(1-\xi)y\big) \supseteq \xi F(x)+(1-\xi)F(y). \tag{5.1}$$

We denote by $SX([s,t],\mathcal{K}_c)$ the set of all convex interval-valued functions.

**Theorem 5.2** ([31]) *Let $F:[s,t]\to\mathcal{K}_c^+$. Then $F$ is convex if and only if $f^-$ is convex and $f^+$ is concave on $[s,t]$.*

**Theorem 5.3** *Let $F:[s,t]\to\mathcal{K}_c^+$. If $F\in SX([s,t],\mathcal{K}_c)$ and $F$ is Iq-differentiable on $[s,t]$, then*

$$F\left(\frac{qs+t}{1+q}\right) \supseteq \frac{1}{t-s}\int_s^t F(x)\,{}_sd_q^I x \supseteq \frac{qF(s)+F(t)}{1+q}. \tag{5.2}$$

*Proof* According to the *Iq*-differentiability of $F$ on $[s,t]$, there are two tangents at the point $\frac{qs+t}{1+q} \in (s,t)$, and their equations are

$$h_1^-(x) = f^-\left(\frac{qs+t}{1+q}\right) + {}_sD_qf^-\left(\frac{qs+t}{1+q}\right)\left(x - \frac{qs+t}{1+q}\right)$$

and

$$h_1^+(x) = f^+\left(\frac{qs+t}{1+q}\right) + {}_sD_qf^+\left(\frac{qs+t}{1+q}\right)\left(x - \frac{qs+t}{1+q}\right).$$

Since $F \in SX([s,t],\mathcal{K}_c)$, we have

$$H_1(x) \supseteq F(x)$$

for all $x \in [s,t]$. By *Iq*-integrating this inequality with respect to $x$ on $[s,t]$ we obtain

$$\begin{aligned}
&\int_s^t H_1(x)\,{}_sd_q^Ix \\
&= \int_s^t \left[F\left(\frac{qs+t}{1+q}\right) + {}_sD_qF\left(\frac{qs+t}{1+q}\right)\left(x - \frac{qs+t}{1+q}\right)\right]{}_sd_q^Ix \\
&= (t-s)F\left(\frac{qs+t}{1+q}\right) + {}_sD_qF\left(\frac{qs+t}{1+q}\right)\left(\int_s^t x\,{}_sd_qx - (t-s)\frac{qs+t}{1+q}\right) \\
&= (t-s)F\left(\frac{qs+t}{1+q}\right) + {}_sD_qF\left(\frac{qs+t}{1+q}\right) \\
&\quad \times \left((1-q)(t-s)\sum_{n=0}^{\infty} q^n\big((1-q^n)s + q^nt\big) - (t-s)\frac{qs+t}{1+q}\right) \\
&= (t-s)F\left(\frac{qs+t}{1+q}\right) + {}_sD_qF\left(\frac{qs+t}{1+q}\right) \\
&\quad \times \left((1-q)(t-s)\left[\left(\frac{1}{1-q} - \frac{1}{1-q^2}\right)s + \frac{1}{1-q^2}t\right] - (t-s)\frac{qs+t}{1+q}\right) \\
&= (t-s)F\left(\frac{qs+t}{1+q}\right) + {}_sD_qF\left(\frac{qs+t}{1+q}\right)\left((t-s)\frac{qs+t}{1+q} - (t-s)\frac{qs+t}{1+q}\right) \\
&= (t-s)F\left(\frac{qs+t}{1+q}\right) \supseteq \int_s^t F(x)\,{}_sd_q^Ix.
\end{aligned} \tag{5.3}$$

Further, the straight line through the points $(s,f^-(s))$ and $(t,f^-(t))$ can be expressed by the linear equation

$$y^-(x) = f^-(s) + \frac{f^-(t) - f^-(s)}{t-s}(x-s),$$

and through the points $(s,f^+(s))$ and $(t,f^+(t))$ by the linear equation

$$y^+(x) = f^+(s) + \frac{f^+(t) - f^+(s)}{t-s}(x-s).$$

Since $F \in SX([s,t],\mathcal{K}_c)$, we have

$$Y(x) \subseteq F(x)$$

for all $x \in [s,t]$. By $Iq$-integrating this inequality with respect to $x$ on $[s,t]$ we get

$$\begin{aligned}
&\int_s^t Y(x)\,{}_s d_q^I x \\
&\quad = \int_s^t \left(F(s) + \frac{F(t)-F(s)}{t-s}(x-s)\right){}_s d_q^I x \\
&\quad = (t-s)F(s) + \frac{F(t)-F(s)}{t-s}\int_s^t (x-s)\,{}_s d_q x \\
&\quad = (t-s)F(s) + \frac{F(t)-F(s)}{t-s}\left(\int_s^t x\,{}_s d_q x - s(t-s)\right) \\
&\quad = (t-s)F(s) + \frac{F(t)-F(s)}{t-s}\left((1-q)(t-s)\sum_{n=0}^{\infty} q^n\big((1-q^n)s + q^n t\big) - s(t-s)\right) \\
&\quad = (t-s)F(s) + \frac{F(t)-F(s)}{t-s}\Bigg((1-q)(t-s) \\
&\qquad \times \left[\left(\frac{1}{1-q} - \frac{1}{1-q^2}\right)s + \frac{1}{1-q^2}t\right] - s(t-s)\Bigg) \\
&\quad = (t-s)F(s) + \big(F(t)-F(s)\big)\left(\frac{qs+t}{1+q} - s\right) \\
&\quad = (t-s)F(s) + (t-s)\frac{F(t)-F(s)}{1+q} \\
&\quad = (t-s)\frac{qF(s)+F(t)}{1+q} \subseteq \int_s^t F(x)\,{}_s d_q^I x.
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4), we come to the following conclusion. □

**Theorem 5.4** *Let $F : [s,t] \to \mathcal{K}_c^+$. If $F \in SX([s,t],\mathcal{K}_c)$ and $F$ is $Iq$-differentiable on $[s,t]$, then*

$$\begin{aligned}
F\left(\frac{s+qt}{1+q}\right) + \frac{(1-q)(t-s)}{1+q}\,{}_s D_q F\left(\frac{s+qt}{1+q}\right) &\supseteq \frac{1}{t-s}\int_s^t F(x)\,{}_s d_q x \\
&\supseteq \frac{qF(s)+F(t)}{1+q}.
\end{aligned} \tag{5.5}$$

*Proof* According to the $Iq$-differentiability of $F$ on $[s,t]$, there are two tangents at the point $\frac{s+qt}{1+q} \in (s,t)$, and their equations are

$$h_2^-(x) = f^-\left(\frac{s+qt}{1+q}\right) + {}_s D_q f^-\left(\frac{s+qt}{1+q}\right)\left(x - \frac{s+qt}{1+q}\right)$$

and

$$h_2^+(x) = f^+\left(\frac{s+qt}{1+q}\right) + {}_s D_q f^+\left(\frac{s+qt}{1+q}\right)\left(x - \frac{s+qt}{1+q}\right).$$

Since $F \in SX([s,t],\mathcal{K}_c)$, we have

$$H_2(x) \supseteq F(x)$$

for all $x \in [s,t]$. By $Iq$-integrating this inequality with respect to $x$ on $[s,t]$ we have

$$\begin{aligned}
&\int_s^t H_2(x)\,{}_sd_q^Ix \\
&= \int_s^t \left[F\left(\frac{s+qt}{1+q}\right) + {}_sD_qF\left(\frac{s+qt}{1+q}\right)\left(x-\frac{s+qt}{1+q}\right)\right]{}_sd_q^Ix \\
&= (t-s)F\left(\frac{s+qt}{1+q}\right) + {}_sD_qF\left(\frac{s+qt}{1+q}\right)\left(\int_s^t x\,{}_sd_qx-(t-s)\frac{s+qt}{1+q}\right) \\
&= (t-s)F\left(\frac{s+qt}{1+q}\right) + {}_sD_qF\left(\frac{s+qt}{1+q}\right) \\
&\quad\times\left((1-q)(t-s)\sum_{n=0}^{\infty}q^n\bigl((1-q^n)s+q^nt\bigr)-(t-s)\frac{s+qt}{1+q}\right) \\
&= (t-s)F\left(\frac{s+qt}{1+q}\right) + {}_sD_qF\left(\frac{s+qt}{1+q}\right) \\
&\quad\times\left((1-q)(t-s)\left[\left(\frac{1}{1-q}-\frac{1}{1-q^2}\right)s+\frac{1}{1-q^2}t\right]-(t-s)\frac{qs+t}{1+q}\right) \\
&= (t-s)F\left(\frac{s+qt}{1+q}\right) + {}_sD_qF\left(\frac{s+qt}{1+q}\right)\left((t-s)\frac{qs+t}{1+q}-(t-s)\frac{s+qt}{1+q}\right) \\
&= (t-s)F\left(\frac{qs+t}{1+q}\right) + {}_sD_qF\left(\frac{s+qt}{1+q}\right)\frac{(t-s)^2(1-q)}{1+q} \supseteq \int_s^t F(x)\,{}_sd_q^Ix. \qquad (5.6)
\end{aligned}$$

Combining (5.6) and (5.4), we come to the following conclusion. □

**Theorem 5.5** *Let $F:[s,t]\to\mathcal{K}_c^+$. If $F \in SX([s,t],\mathcal{K}_c)$ and $F$ is $Iq$-differentiable on $[s,t]$, then*

$$\begin{aligned}
F\left(\frac{s+t}{2}\right) + \frac{(1-q)(t-s)}{2(1+q)}{}_sD_qF\left(\frac{s+t}{2}\right) &\supseteq \frac{1}{t-s}\int_s^t F(x)\,{}_sd_qx \\
&\supseteq \frac{qF(s)+F(t)}{1+q}. \qquad (5.7)
\end{aligned}$$

*Proof* According to the $Iq$-differentiability of $F$ on $[s,t]$, there are two tangents at the point $\frac{s+t}{2} \in (s,t)$, and their equations

$$h_3^-(x) = f^-\left(\frac{s+t}{2}\right) + {}_sD_qf^-\left(\frac{s+t}{2}\right)\left(x-\frac{s+t}{2}\right)$$

and

$$h_3^+(x) = f^+\left(\frac{s+t}{2}\right) + {}_sD_qf^+\left(\frac{s+t}{2}\right)\left(x-\frac{s+t}{2}\right).$$

Since $F \in SX([s,t],\mathcal{K}_c)$, we have

$$H_3(x) \supseteq F(x)$$

for all $x \in [s,t]$. By $Iq$-integrating this inequality with respect to $x$ on $[s,t]$ we have

$$\begin{aligned}
&\int_s^t H_3(x)\,{}_s d_q^I x \\
&= \int_s^t \left[F\left(\frac{s+t}{2}\right) + {}_sD_qF\left(\frac{s+t}{2}\right)\left(x - \frac{s+t}{2}\right)\right]{}_s d_q^I x \\
&= (t-s)F\left(\frac{s+t}{2}\right) + {}_sD_qF\left(\frac{s+t}{2}\right)\left(\int_s^t x\,{}_s d_q x - (t-s)\frac{s+t}{2}\right) \\
&= (t-s)F\left(\frac{s+t}{2}\right) + {}_sD_qF\left(\frac{s+t}{2}\right) \\
&\quad \times \left((1-q)(t-s)\sum_{n=0}^{\infty} q^n\bigl((1-q^n)s + q^n t\bigr) - (t-s)\frac{s+t}{2}\right) \\
&= (t-s)F\left(\frac{s+t}{2}\right) + {}_sD_qF\left(\frac{s+t}{2}\right) \\
&\quad \times \left((1-q)(t-s)\left[\left(\frac{1}{1-q} - \frac{1}{1-q^2}\right)s + \frac{1}{1-q^2}t\right] - (t-s)\frac{s+t}{2}\right) \\
&= (t-s)F\left(\frac{s+t}{2}\right) + {}_sD_qF\left(\frac{s+t}{2}\right)\left((t-s)\frac{qs+t}{1+q} - (t-s)\frac{s+t}{2}\right) \\
&= (t-s)F\left(\frac{s+t}{2}\right) + {}_sD_qF\left(\frac{s+t}{2}\right)\frac{(t-s)^2(1-q)}{2(1+q)} \supseteq \int_s^t F(x)\,{}_s d_q^I x.
\end{aligned} \tag{5.8}$$

Combining (5.8) and (5.4), we come to the following conclusion. □

**Theorem 5.6** *Let $F : [s,t] \to \mathcal{K}_c^+$. If $F \in SX([s,t],\mathcal{K}_c)$ and $F$ is $Iq$-differentiable on $[s,t]$, then*

$$\max\{\triangle_1, \triangle_2, \triangle_3\} \supseteq \frac{1}{t-s}\int_s^t F(x)\,{}_s d_q x \supseteq \frac{qF(s)+F(t)}{1+q}, \tag{5.9}$$

*where*

$$\begin{aligned}
\triangle_1 &= F\left(\frac{qs+t}{1+q}\right), \\
\triangle_2 &= F\left(\frac{s+qt}{1+q}\right) + \frac{(1-q)(t-s)}{1+q}\,{}_sD_qF\left(\frac{s+qt}{1+q}\right), \\
\triangle_3 &= F\left(\frac{s+t}{2}\right) + \frac{(1-q)(t-s)}{2(1+q)}\,{}_sD_qF\left(\frac{a+b}{2}\right).
\end{aligned}$$

*Proof* Combining (5.3), (5.6), (5.8), and (5.4) proves the conclusion. □

*Example* 5.7 Let $F:[0,1]\to\mathcal{K}_c$ be given by $F(x)=[x^2,-x^2+4]$. It is obvious that $F(x)$ is $Iq$-differentiable on $[0,1]$. For $q=\frac{1}{2}$, we have

$$F\left(\frac{qs+t}{1+q}\right)=F\left(\frac{2}{3}\right)=\left[\frac{4}{9},\frac{32}{9}\right],$$
$$\frac{1}{t-s}\int_s^t F(x)\,{}_sd_q^Ix=\left[\int_0^1 x^2\,{}_0d_qx,\int_0^1\left(-x^2+4\right){}_0d_qx\right]=\left[\frac{4}{7},\frac{24}{7}\right],$$

and

$$\frac{qF(s)+F(t)}{1+q}=\left[\frac{2}{3},\frac{10}{3}\right].$$

Since

$$\left[\frac{4}{9},\frac{32}{9}\right]\supseteq\left[\frac{4}{7},\frac{24}{7}\right]\supseteq\left[\frac{2}{3},\frac{10}{3}\right],$$

Theorem 5.3 is verified.

Since $\ell(F(x))=-2x^2+4$, it follows that $F$ is $\ell$-decreasing on $[0,1]$. Then by Theorem 3.5 we obtain that ${}_sD_qF=[-(1+q)x,(1+q)x]$ and

$$F\left(\frac{s+qt}{1+q}\right)+\frac{(1-q)(t-s)}{1+q}{}_sD_qF\left(\frac{s+qt}{1+q}\right)=F\left(\frac{1}{3}\right)+\frac{1}{3}{}_0D_qF\left(\frac{1}{3}\right)=\left[-\frac{1}{18},\frac{73}{18}\right],$$
$$F\left(\frac{s+t}{2}\right)+\frac{(1-q)(t-s)}{2(1+q)}{}_sD_qF\left(\frac{s+t}{2}\right)=F\left(\frac{1}{2}\right)+\frac{1}{6}{}_0D_qF\left(\frac{1}{2}\right)=\left[\frac{1}{8},\frac{31}{8}\right].$$

Since

$$\left[-\frac{1}{18},\frac{73}{18}\right]\supseteq\left[\frac{4}{7},\frac{24}{7}\right]\supseteq\left[\frac{2}{3},\frac{10}{3}\right]$$

and

$$\left[\frac{1}{8},\frac{31}{8}\right]\supseteq\left[\frac{4}{7},\frac{24}{7}\right]\supseteq\left[\frac{2}{3},\frac{10}{3}\right],$$

Theorems 5.4 and 5.5 are verified.

## 6 Conclusions

In this work, we introduced the concepts of $Iq$-derivative and $Iq$-integral and discussed some basic their properties. Furthermore, we established some new $Iq$-Hermite–Hadamard-type inequalities. In the field of quantum calculus and time-scale calculus, our results are more applicable than ever. In the future, we intend to study some applications in interval optimizations by using $Iq$-calculus. Meanwhile, we may apply $Iq$-calculus to other fields, such as the integral inequalities and fractional calculus.

**Acknowledgements**

We are very grateful to the anonymous referees for several valuable and helpful comments, suggestions, and questions, which helped us to improve the paper into the present form.

**Funding**
This work was supported in part by the Fundamental Research Funds for Central Universities (2019B44914), Special Soft Science Research Projects of Technological Innovation in Hubei Province (2019ADC146), Key Projects of Educational Commission of Hubei Province of China (D20192501), the Natural Science Foundation of Jiangsu Province (BK20180500), and the National Key Research and Development Program of China (2018YFC1508100).

**Abbreviations**
Not applicable.

**Availability of data and materials**
Not applicable.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Each of the authors contributed to each part of this study equally. All authors read and approved the final manuscript.

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## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 May 2020 Accepted: 16 August 2020 Published online: 27 August 2020

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